Surface water age and radio-age in a steady-state, water column model

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Abstract. *We aim to evaluate analytically the age of the surface water (i.e., the time elapsed since last touching the water-air interface) in a steady-state, water column model with constant (vertical) diffusivity. We calculate CART's*[1] *passive tracer or surface water age and a radio-age, which is obtained from the concentration of a passive tracer and a decaying one (first order decay). The previous age is greater than the latter, except at the sea surface, where all ages are prescribed to be zero. The difference between these ages increases as the rate of decay increases. The radio-age is asymptotic to CART's age for small values of the Damköhler number, which is the ratio of the diffusion timescale to the decay timescale.*

Motivation

As was suggested by Deleersnijder et al. (2001), most, if not all well thought out methods for estimating an age, i.e., a measure of an elapsed time, are likely to provide the same value at any time and position if diffusive processes are neglected. However, if this simplifying hypothesis does not hold valid, then all the particles present in an elemental control volume are likely to have ended up in it through different trajectories and, hence, are likely to have different ages. Therefore, an averaging method must be introduced. Accordingly, Deleersnijder et al. (2001) used a mass-weighted arithmetic mean of the ages, which is also resorted to, albeit implicitly, in some other approaches (e.g., Holzer and Hall 2000).

There is no similar age-averaging method for carbon-14 like ages, which I usually call radio-ages. For ventilation studies, the age is generally defined as the time elapsed since leaving the water-air interface[2]. Then, the radio-age deduced from the concentration of a passive tracer and a decaying one is smaller at any time and position than the age of the passive tracer tagging the water particles that have touched the surface and, what is worse, the (absolute value of the) difference increases as the rate of decay increases (Delhez et al. 2003).

A one-dimensional, steady-state version of Delhez et al. (2003) is presented hereinafter. In the present model, the only transport process taken into consideration is (vertical) diffusion. As a consequence, the results highlight the impact of diffusion on the difference between CART's passive tracer or surface water age and the radio-age meant to approximately evaluate the very same timescale.

Steady-state, water column model

Let z denote the vertical coordinate, with $z=-h$ at the seabed and $z=0$ at the surface (i.e.,

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[1] CART = Constituent-oriented Age and Residence time Theory (http://www.climate.be/cart)

[2] Several studies estimated the age as the time leaving the sea surface (e.g., England 1995, Meier 2005, Delhez et al. 2003, Bendtsen et al. 2009, Hong et al. 2016, Xiong and Shen 2022) in order to assess some form of "ventilation rate". On the other hand, White and Deleersnijder (2007) computed an age defined as the time elapsed since leaving the seabed.

the water-air interface). The concentration, $C(z)$, of the passive tracer tagging water particles having touched the surface is the solution of

$$D\frac{d^2C}{dz^2} = 0 \ , \quad C(0)=1 \ , \quad \left[D\frac{dC}{dz}\right]_{z=-h} = 0 \tag{1}$$

where positive constant D is the diffusivity. The related age concentration, $\alpha(z)$, obeys

$$D\frac{d^2\alpha}{dz^2}+C = 0 \ , \quad \alpha(0)=0 \ , \quad \left[D\frac{d\alpha}{dz}\right]_{z=-h} = 0 \tag{2}$$

To estimate the radio-age, a tracer decaying at rate m (>0) needs to be introduced. Its concentration, $C_r(z)$, is obtained by solving the following differential problem:

$$D\frac{d^2C_r}{dz^2}-mC_r = 0 \ , \quad C_r(0)=1 \ , \quad \left[D\frac{dC_r}{dz}\right]_{z=-h} = 0 \tag{3}$$

Clearly, in the present one-dimensional model, the seabed is assumed to be impermeable, which is why no-flux boundary conditions are prescribed on this boundary.

CART's age, $a(z)$, and the radio-age, $r(z)$, are obtained from

$$a(z)=\frac{\alpha(z)}{C(z)} \ , \quad r(z)=\frac{1}{m}\ln\frac{C(z)}{C_r(r)} \tag{4}$$

Timescales and sensitivity analysis

A diffusion timescale and a reactive one are readily defined:

$$\text{diffusion timescale} = T_d = \frac{h^2}{D} \ , \quad \text{reaction timescale} = T_r = \frac{1}{m} \tag{5}$$

Their ratio, which is sometimes called Damköhler number, reads

$$\mu = \frac{\text{diffusion timescale}}{\text{reaction timescale}} = \frac{T_d}{T_r} = \frac{h^2 m}{D} \tag{6}$$

indicating that the smaller the reaction rate, the larger the reaction timescale and, hence, the smaller the Damköhler number.

Using dimensionless vertical coordinate $\sigma=(z+h)/h$, which is equal to zero (resp. unity) at the seabed (resp. sea surface), CART's concentration and age concentration are (Figure 1)

$$C(\sigma)=1 \ , \quad \alpha(\sigma)=\frac{1-\sigma^2}{2}T_d \tag{7}$$

Then, the decaying tracer concentration is (Figure 1)

$$C_r(\sigma)=\frac{\cosh\left(\sigma\sqrt{\mu}\right)}{\cosh\left(\sqrt{\mu}\right)} \tag{8}$$

The CART age, i.e., the (surface) water age, and the radio age are (Figure 2)

$$a(\sigma) = \frac{1-\sigma^2}{2}T_d \ , \quad r(\sigma) = \frac{1}{\mu}\ln\left[\frac{\cosh\left(\sqrt{\mu}\right)}{\cosh\left(\sigma\sqrt{\mu}\right)}\right]T_d \tag{9}$$

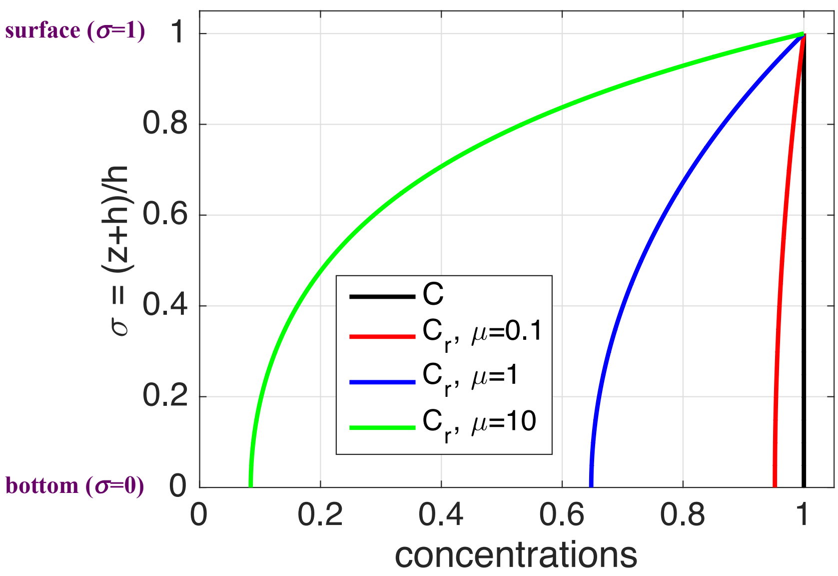

Figure 1. Steady-state concentration of the passive tracer tagging the surface water (C) and concentration of a decaying tracer (C_r) for various values of the Damköhler number (μ). These concentrations are depicted as functions of dimensionless vertical coordinate σ.

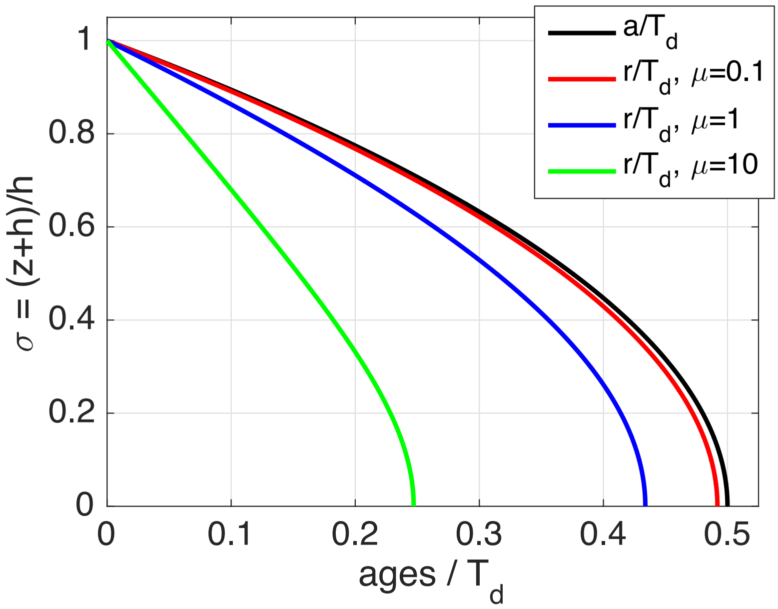

Figure 2. Steady-state normalised surface water age (a/T_d) and normalised radio-ages (r/T_d) for various values of the Damköhler number (μ). These ages are depicted as functions of dimensionless vertical coordinate σ.

In accordance with Delhez et al. (2003), the surface water or passive tracer age is larger than the radio-age, except at the sea surface, where all ages are zero. Both ages are asymptotically equivalent for small values of the Damköhler number:

$$r(\sigma) \sim a(\sigma) - \frac{1-\sigma^4}{12}\mu\, T_d \ , \quad \mu \to 0 \tag{10}$$

In addition, it is readily seen that the larger the Damköhler number (i.e., the faster the decay), the larger the difference between the aforementioned ages:

$$\frac{a-r}{T_d} = \int_\sigma^1 \left[\frac{\tanh\left(\sigma\sqrt{\mu}\right)}{\sigma\sqrt{\mu}} - 1\right]\sigma\, d\sigma \quad \Rightarrow \quad \begin{cases} a-r \geq 0 \\ \dfrac{\partial(a-r)}{\partial\mu} \geq 0 \end{cases} \tag{11}$$

Conclusion

In a steady-state water column model, the radio-age may be considered a valuable approximation of CART's surface water age if and only if the Damköhler number is sufficiently small. The analytical solutions obtained above may be used as an elementary test case for the numerical implementation of diagnostic timescales.

It was seen that evaluating CART's surface water age or the corresponding radio-age requires the solution of two scalar transport problems, i.e., calculating the concentration and age concentration in the framework of CART and calculating two concentrations in order to derive the radio-age. These considerations also hold valid for multi-dimensional, time-dependent age computations in which advection and diffusion are taken into account.

References

Bendtsen J., K.E. Gustafsson, J. Söderkvist and J.L.S. Hansen, 2009, Ventilation of bottom water in the North Sea - Baltic Sea transition zone, *Journal of Marine Systems*, 75, 138-149, https://dx.doi.org/10.1016/j.jmarsys.2008.08.006

Deleersnijder E., J.-M. Campin and E.J.M. Delhez, 2001, The concept of age in marine modelling: I. Theory and preliminary model results, *Journal of Marine Systems*, 28, 229-267, https://www.sciencedirect.com/science/article/pii/S0924796301000264

Delhez E.J.M., E. Deleersnijder, A. Mouchet, and J.-M. Beckers, 2003, A note on the age of radioactive tracers, *Journal of Marine Systems*, 38, 277-286, https://www.sciencedirect.com/science/article/pii/S0924796302002452

England M.H., 1995, The age of water and ventilation timescales in a Global Ocean model, *Journal of Physical Oceanography*, 25, 2756-2777

Holzer M. and T.M. Hall, 2000, Transit-time and tracer-age distributions in geophysical flows, *Journal of the Atmospheric Sciences*, 57, 3539-3558

Hong B., W. Gong, S. Peng, Q. Xie, D. Wang, H. Li and H. Xu, 2016, Characteristics of vertical exchange process in the Pearl River estuay, *Aquatic Ecosystem Health & Management*, 19, 286-295, https://dx.doi.org/10.1080/14634988.2016.1205438

Meier H.E.M., 2005, Modeling the age of Baltic Seawater masses: Quantification and steady state sensitivity experiments, *Journal of Geophysical Research*, 110, C02006, https://dx.doi.org/10.1029/2004JC002607

White L. and E. Deleersnijder, 2007, Diagnoses of vertical transport in a three-dimensional finite element model of the tidal circulation around an island, *Estuarine, Coastal and Shelf Science*, 74, 655-669, https://doi.org/10.1016/j.ecss.2006.07.014

Xiong J.and J. Shen, 2022, Vertical transport timescale of surface-produced particulate material in the Chesapeake Bay, *Journal of Geophysical Research Oceans*, 127, e2021JC017592, https://doi.org/10.1029/2021JC017592
