Mathematical properties of the position-dependent, steady-state water age in a shallow reservoir

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1. Background

As was pointed out by Cammasio (2014), "shallow rectangular reservoirs are common structures in urban hydraulics and river engineering. Despite their simple geometries, complex symmetric and asymmetric flow fields develop in such reservoirs, depending on their expansion ratio and length to width ratio."

Due to their pervasiveness, the above-mentioned phenomena have been studied extensively in the field, in the laboratory and by means of numerical simulations. However, thus far, simple and quantitative diagnostic quantities of them have remained rather elusive. For instance, in his MSc thesis, Adam (2017) suggested that the water's transit time (i.e. the time needed to travel from the inlet to the outlet) may be much greater than that derived from the widely-used rule of thumb, i.e. the ratio of the water volume to the volumetric flow rate.

Clearly, timescale-based diagnoses such as the age (i.e. the time elapsed since entering the reservoir) need to be developed and investigated in depth. The objective of the present working note is to establish the properties of the age and age-related variables in a shallow reservoir.

The aspect ratio[1] of the reservoirs under study is sufficiently small that it is appropriate to deal only with depth-averaged variables. The hydrodynamic variables (i.e. water height, horizontal velocity and diffusivity tensor used to represent the impact of subgrid-scale processes) are time-independent. Obviously, these variables do not depend on the water age, which is a mere diagnostic variable. The latter assumption underlies most of the mathematical developments performed below.

Density ρ of the water is assumed to be constant (Boussinesq approximation). The depth-averaged water velocity and the water column height are denoted $\mathbf{u}$ and H, respectively. They depend only on the position vector $\mathbf{x}=(x,y)$, where x and y are horizontal, Cartesian coordinates.

Let ∇ represent the horizontal del operator. As is well known, the fluid velocity satisfies (steady-state) continuity equation

$$\nabla \bullet (H\mathbf{u}) = 0 \tag{1.1}$$

at any point of the domain of interest (the reservoir under study), hereinafter denoted Ω. Its boundary, Γ, is made up of three parts (Figure 1): the incoming, outgoing and impermeable boundaries, which are referred to as Γ^{in}, Γ^{out} and Γ^{imp}, respectively. On

[1] The aspect ratio is the ratio of the vertical length scale to the horizontal one.

these boundaries, the velocity satisfies

$$[\mathbf{u} \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma^{in}} \le 0 \tag{1.2}$$

$$[\mathbf{u} \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma^{out}} \ge 0 \tag{1.3}$$

and

$$[\mathbf{u} \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma^{imp}} = 0 \tag{1.4}$$

where unit vector $\mathbf{n}$ is the outward normal on the domain boundary Γ. From here on, horizontal velocity $\mathbf{u}$ and water column height H are assumed to be known functions of the horizontal coordinates (in fact, they are simulated numerically) and satisfy constraints (1.1)-(1.4).

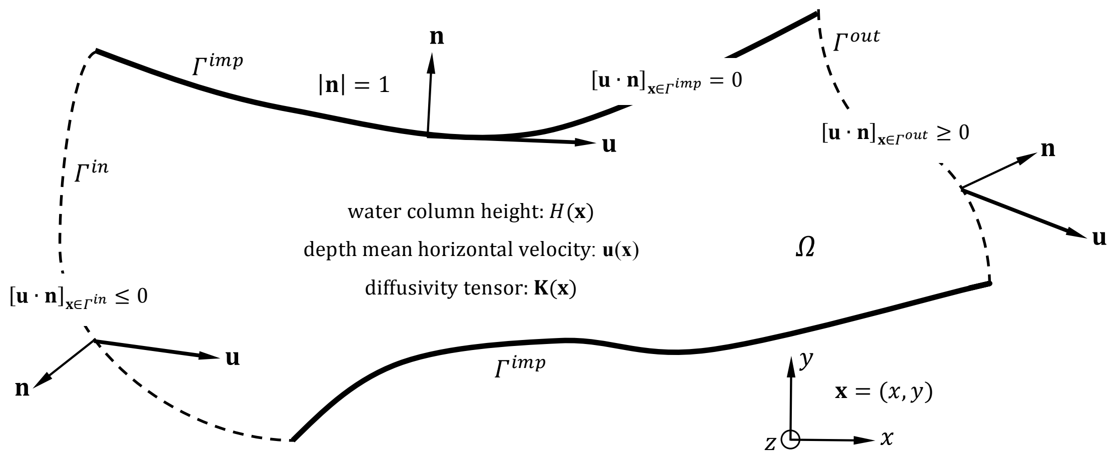

Figure 1. Top view of the domain of interest (Ω), which is limited by open boundaries, Γ^{in} and Γ^{out}, i.e. the incoming and outgoing boundaries, and Γ^{imp}, the impermeable part of the domain boundary.

Hereinafter, we will achieve the following objectives:

1. Put the age-related equations dealt with on a sound footing;
2. Lay out arguments supporting the existence of a steady-state water age[2];
3. Rigorously derive the properties of the water age and related variables;
4. Investigate the mean water age in an elongated reservoir (i.e. a reservoir for which a one-dimensional approach is acceptable), including the derivation of analytical solutions.

[2] The existence of a steady-state water age is not an automatic consequence of the fact that the hydrodynamics under consideration is time-independent as is exemplified in Deleersnijder (2019).

2. Steady-state water age distribution function

In this section, we will establish the equation governing water age distribution function $c(\mathbf{x},\tau)$, where independent variable $\tau \in [0,\infty[$ denotes the age. In addition, we will recall the "initial" ($\tau = 0$) and boundary conditions under which $c(\mathbf{x},\tau)$ is to be evaluated. We will also derive key properties of the distribution function, viz. $c(\mathbf{x},\tau)$ is non negative and $\tau\, c(\mathbf{x},\tau)$ tends to zero in the limit $\tau \to \infty$.

2.1. Governing equation and boundary conditions

Let z denotes the vertical coordinate, with $z=0$ and $z=H(\mathbf{x})$ at the bottom of the domain (which is flat and horizontal) and at the free surface, respectively. Then, define four-dimensional elemental control domain $\delta\tilde{\Omega}$,

$$\delta\tilde{\Omega} = \overbrace{\left[x-\frac{\delta x}{2}, x+\frac{\delta x}{2}\right]\times\left[y-\frac{\delta y}{2}, y+\frac{\delta y}{2}\right]\times\left[0, H(x,y)\right]}^{=\delta\Omega}\times\left[\tau-\frac{\delta\tau}{2}, \tau+\frac{\delta\tau}{2}\right] \qquad (2.1.1)$$

$$(\delta x, \delta y, \delta\tau \to 0)$$

where δx and δy are horizontal space increments; $\delta\Omega$ denotes the part of $\delta\tilde{\Omega}$ that belongs to the physical space; $\delta\tau$ is the appropriate age increment. The volume of $\delta\Omega$ is $\delta V(\mathbf{x}) = \delta x\, \delta y\, H(\mathbf{x})$.

Water age distribution function $c(\mathbf{x},\tau)$, is defined as follows:

> In $\delta\Omega$, the mass of the water parcels whose age lies in the interval $[\tau, \tau+\delta\tau]$ tends to $\rho\, c(\mathbf{x},\tau)\delta V\, \delta\tau$ in the limit $\delta V, \delta\tau \to 0$.

All variables are time-independent. Therefore, the mass fluxes crossing the boundary of elemental control domain $\delta\tilde{\Omega}$ must balance:

$$\delta y\, \delta\tau \left[H\phi_x\right]_{x-\delta x/2}^{x+\delta x/2} + \delta x\, \delta\tau \left[H\phi_y\right]_{y-\delta y/2}^{y+\delta y/2} + \delta x\, \delta y \left[H\phi_\tau\right]_{\tau-\delta\tau/2}^{\tau+\delta\tau/2} = 0 \qquad (2.1.2)$$

where $\phi_x(\mathbf{x},\tau)$, $\phi_y(\mathbf{x},\tau)$ and $\phi_\tau(\mathbf{x},\tau)$ are the fluxes in the direction of the axis associated with the subscripts x, y and τ, respectively. Since the lower and upper boundaries of the water column are impermeable, no vertical fluxes are to be taken into account in (2.1.2).

The horizontal mass fluxes are due to the combined effects of advection and diffusion. The diffusive flux is parameterised *à la Fourier-Fick* with the help of diffusivity tensor

$$\mathbf{K} = \begin{pmatrix} K_{xx} & K_{xy} \\ K_{yz} & K_{yy} \end{pmatrix} \qquad (2.1.3)$$

Accordingly, the horizontal mass fluxes read

$$\phi_x = \rho c u - \rho\left(K_{xx}\frac{\partial c}{\partial x} + K_{xy}\frac{\partial c}{\partial y}\right) \qquad (2.1.4)$$

and

$$\phi_y = \rho c v - \rho\left(K_{yx}\frac{\partial c}{\partial x} + K_{yy}\frac{\partial c}{\partial y}\right) \tag{2.1.5}$$

The mass flux in the age direction is due to ageing, i.e. the fact that the age of every water parcel tends to increase at the same pace as time progresses. Therefore, the corresponding flux is tantamount to advection with a unit velocity, leading to

$$\phi_\tau = \rho c \tag{2.1.6}$$

Combining (2.1.2)-(2.1.6) and taking the limit $\delta x, \delta y, \delta\tau \to 0$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial x}\left[\rho H c u - \rho H\left(K_{xx}\frac{\partial c}{\partial x} + K_{xy}\frac{\partial c}{\partial y}\right)\right] \\ &\quad + \frac{\partial}{\partial y}\left[\rho H c v - \rho H\left(K_{yx}\frac{\partial c}{\partial x} + K_{yy}\frac{\partial c}{\partial y}\right)\right] + \frac{\partial}{\partial \tau}(\rho H c) = 0\end{aligned} \tag{2.1.7}$$

which may be rewritten as an "evolution" equation (in which age τ plays a role equivalent to that of time in a classical advection-diffusion equation) in vector form as follows:

$$\frac{\partial (Hc)}{\partial \tau} = -\nabla \bullet (Hc\mathbf{u} - H\mathbf{K} \bullet \nabla c) \tag{2.1.8}$$

Diffusivities must be positive, otherwise the associated fluxes would not tend to have a homogenising effect. Similarly, the diffusivity tensor must be positive definite (e.g. Deleersnijder 2012). Furthermore, the diffusivity tensor must be symmetric. If **K** were no symmetric, the divergence of the flux associated with its antisymmetric part, $\mathbf{K}_a = (\mathbf{K} - \mathbf{K}^T)/2$, would be tantamount to an advective term:

$$\nabla \bullet (H\mathbf{K}_a \bullet \nabla c) = \nabla \bullet (Hc\mathbf{v}) \tag{2.1.9}$$

with

$$H\,\mathbf{v} = \frac{1}{2}\left[\frac{\partial}{\partial y}[H(K_{yx} - K_{xy})]\,,\ \frac{\partial}{\partial x}[H(K_{xy} - K_{yx})]\right] \tag{2.1.10}$$

and $\nabla \bullet (H\mathbf{v}) = 0$. Accordingly, the components of the diffusivity tensor must satisfy the following constraints:

$$K_{xy} = K_{yx}\ ,\quad K_{xx}K_{yy} - (K_{xy})^2 > 0 \tag{2.1.11}$$

The "initial" and boundary conditions are as follows:

$$c(\mathbf{x},0) = 0 \tag{2.1.12}$$

$$\left[(Hc\mathbf{u} - H\mathbf{K} \bullet \nabla c) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} = \left[H(\mathbf{u} \bullet \mathbf{n})\delta(\tau - 0)\right]_{\mathbf{x}\in\Gamma^{in}} \tag{2.1.13}$$

$$\left[(H\mathbf{K} \bullet \nabla c) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^{out}\cup\Gamma^{imp}} = 0 \tag{2.1.14}$$

where $\delta(\tau - 0)$ is the Dirac delta function. According to (2.1.13), the age of the water parcels entering the domain through Γ^{in} is prescribed to be zero, implying that the age is defined as the time elapsed since crossing boundary Γ^{in}. Therefore, as a finite amount of time is needed to travel from the domain entrance to a given point in Ω, no water parcel present in the domain of interest can have a zero age, hence "initial" condition (2.1.12). The outgoing boundary is a spillway, which is why the outgoing diffusive flux is assumed to be zero, leading to boundary condition. On the impermeable part of the domain boundary, the advective flux is zero and so is the diffusive one. These considerations lead to (2.1.14).

2.2. Sign of the water age distribution function

The negative part of the water age distribution functions is defined as follows:

$$c^-(\mathbf{x},\tau) = \frac{c(\mathbf{x},\tau)-|c(\mathbf{x},\tau)|}{2} = \begin{cases} 0\,, \text{ if } 0 \le c(\mathbf{x},\tau) \\ c(\mathbf{x},\tau)\,, \text{ if } c(\mathbf{x},\tau) < 0 \end{cases} \tag{2.2.1}$$

Clearly, $c^-(\mathbf{x},\tau)$ is, by definition, non-positive, i.e. $c^-(\mathbf{x},\tau) \le 0$.

Multiplying governing equation (2.1.8) by $c^-(\mathbf{x},\tau)$, using (1.1-2) and taking into account boundary conditions (2.1.13-14), we obtain after some calculations

$$\begin{aligned} \frac{d}{d\tau} \overbrace{\int_\Omega H(c^-)^2\, d\Omega}^{\ge 0} &= - \overbrace{\int_{\Gamma^{in}} H(c^-)^2 (-\mathbf{u}\bullet\mathbf{n})\, d\Gamma^{in}}^{\ge 0,\ \text{since } (-\mathbf{u}\bullet\mathbf{n})\ge 0 \text{ on } \Gamma^{in}} \\ &- 2 \underbrace{\int_{\Gamma^{in}} H c^-(\mathbf{u}\bullet\mathbf{n})\, d\Gamma^{in}}_{\substack{\ge 0,\ \text{since} \\ c^-\le 0 \text{ and } (\mathbf{u}\bullet\mathbf{n})\ge 0 \text{ on } \Gamma^{in}}} \delta(\tau-0) - 2 \underbrace{\int_\Omega H \nabla c^- \bullet \mathbf{K} \bullet \nabla c^-\, d\Omega}_{\substack{\ge 0,\ \text{since } \mathbf{K} \text{ is} \\ \text{positive definite}}} \end{aligned} \tag{2.2.2}$$

which simplifies to

$$\frac{d}{d\tau} \overbrace{\int_\Omega H(c^-)^2\, d\Omega}^{\ge 0} \le 0 \tag{2.2.3}$$

The integral over the domain of interest of non-negative function $H(c^-)^2$ cannot increase as τ increases. Since this function is identically zero at $\tau = 0$, $c^-(\mathbf{x},\tau)$ remains zero at any location and for any value of age τ. Therefore, the water age distribution function is non negative:

$$c(\mathbf{x},\tau) \ge 0 \tag{2.2.4}$$

The mathematical developments leading to (2.2.2) require the use of the divergence and Reynolds' transport theorems. They are inspired by Lewandowski (1997), Appendix C of Deleersnijder et al. (2001) and de Brye et al. (2012).

2.3. The water age distribution function in the limit $\tau \to \infty$

For any positive value of the age ($0 < \tau$), the boundary condition related to the incoming boundary (2.1.13) transforms to

$$\left[(Hc\mathbf{u} - H\mathbf{K}\bullet\nabla c)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} = 0 \tag{2.3.1}$$

This is because Dirac delta function $\delta(\tau-0)$ is zero for $0<\tau$.

Then is convenient to first introduce the following non-negative integrals:

$$I(\tau) = \int_{\Omega} H c^2 \, d\Omega \geq 0 \tag{2.3.2}$$

and

$$J(\tau) = \overbrace{\int_{\Gamma^{in}} H c[-(\mathbf{K}\bullet\nabla c)\bullet\mathbf{n}]\, d\Gamma^{in}}^{\substack{\geq 0, \text{ since } (\mathbf{K}\bullet\nabla c)\bullet\mathbf{n}=c\mathbf{u}\bullet\mathbf{n}\leq 0 \\ \text{on } \Gamma^{in}\text{, see (1.2) and (2.3.1)}}} + \underbrace{\int_{\Gamma^{out}} H c^2 (\mathbf{u}\bullet\mathbf{n})\, d\Gamma^{out}}_{\geq 0, \text{ since } \mathbf{u}\bullet\mathbf{n}\geq 0 \text{ on } \Gamma^{out}} + 2\underbrace{\int_{\Omega} H \nabla c \bullet \mathbf{K} \bullet \nabla c \, d\Omega}_{\substack{\geq 0\text{, since } \mathbf{K} \text{ is} \\ \text{positive definite}}} \geq 0 \tag{2.3.3}$$

Both integrals are positive as long as the age distribution function is not zero at every point of Ω. If $c(\mathbf{x},\tau)$ is zero in the whole domain, then both integrals are zero.

Multiplying governing equation (2.1.8) by $\tau^{2n} c(\mathbf{x},\tau)$ (n=0,1,2...), using continuity equation (1.1), the divergence theorem and the relevant boundary conditions, we obtain after some manipulations

$$\frac{d}{d\tau} \int_{\Omega} H (\tau^n c)^2 \, d\Omega = \tau^{2n} \left[\frac{2n}{\tau} I(\tau) - J(\tau) \right], \quad n=1,2,3... \tag{2.3.4}$$

For $n=0$, the right-hand side of the above relation is negative as long as $c(\mathbf{x},\tau)$ is positive. Thus, the integral over the domain of interest of Hc^2 will decrease monotonically until the water age distribution function is zero at any position so that $c(\mathbf{x},\tau)\to 0$ as $\tau\to\infty$. For $n\geq 1$, since integrals $I(\tau)$ and $J(\tau)$ both scale as c^2, they can be assumed to exhibit the same asymptotic behaviour as $\tau\to\infty$, hence the second term in the right-hand side of (2.3.4) will dominate the first one as τ increases. This implies that $\tau^n c(\mathbf{x},\tau)\to 0$ in the limit $\tau\to\infty$. The limit valid for $n=0$ and that associated to $n\geq 1$ can be cast into the following form

$$\lim_{\tau\to\infty} \tau^n c(\mathbf{x},\tau) = 0 \ , \quad n=0,1,2... \tag{2.3.5}$$

Obviously, the larger the value of n, the slower the convergence to zero of product $\tau^n c(\mathbf{x},\tau)$ as τ increases. This is in agreement with elementary physical intuition.

In the right-hand side of (2.3.4), the first term is zero for $n=0$, which is because $\tau^0 c = c$ is associated with the water mass budget. For $n\geq 1$, the aforementioned term is positive, for it is related to ageing. The second term in the right hand side of (2.3.4) is negative — unless the age distribution function is zero at any location. This term is associated with all the processes that tend to cap the water age, i.e. the influx of water parcels with zero age through Γ^{in}, the outgoing water flux through Γ^{out} and diffusion, which prevents the occurrence of stagnation points, for the diffusivity tensor is positive definite.

3. Water concentration

The concentration of the water, $C(\mathbf{x})$, defined as a mass fraction, must be equal to unity at any location in the domain of interest. Accordingly, the water age distribution function must satisfy the following integral constraint (Delhez et al. 1999)

$$C(\mathbf{x}) = \int_0^\infty c(\mathbf{x},\tau)\, d\tau = 1 \tag{3.1}$$

To prove that this relation holds valid, we will first establish the partial differential problem obeyed by $C(\mathbf{x})$. To this end, we will integrate over the age all the relations from which $c(\mathbf{x},\tau)$ is derived, bearing in mind the fact that the water height, the velocity and the diffusivity tensor are independent of age τ.

Integrating over the age governing equation (2.1.8) yields

$$\int_0^\infty \frac{\partial(Hc)}{\partial\tau}\, d\tau = -\nabla\bullet(HC\mathbf{u} - H\mathbf{K}\bullet\nabla C) \tag{3.2}$$

The left-hand side of the above relation simplifies to

$$\int_0^\infty \frac{\partial(Hc)}{\partial\tau}\, d\tau = \underbrace{H(\mathbf{x})c(\mathbf{x},\infty)}_{=0,\ \text{see (2.3.5)}} - \underbrace{H(\mathbf{x})c(\mathbf{x},0)}_{=0,\ \text{see (2.1.12)}} = 0 \tag{3.3}$$

Combining (3.2) and (3.3) leads to steady-state advection-diffusion equation

$$\nabla\bullet(H\mathbf{K}\bullet\nabla C - HC\mathbf{u}) = 0 \tag{3.4}$$

which must be solved under boundary conditions

$$\left[(HC\mathbf{u} - H\mathbf{K}\bullet\nabla C)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} = \left[H\,\mathbf{u}\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} \tag{3.5}$$

and

$$\left[(H\mathbf{K}\bullet\nabla C)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{out}\cup\Gamma^{imp}} = 0 \tag{3.6}$$

which are obtained by integrating (2.1.13-14) over τ.

Obviously, $C(\mathbf{x})=1$ is a solution to partial differential problem (3.4)-(3.6). However, it is desirable to show that this solution is unique. To achieve this goal, we first assume that a different solution, $\tilde{C}(\mathbf{x})$, exists. Then, the difference between these solutions

$$\hat{C}(\mathbf{x}) = \tilde{C}(\mathbf{x}) - C(\mathbf{x}) = \tilde{C}(\mathbf{x}) - 1 \tag{3.7}$$

is readily seen to be the solution of the following partial differential problem:

$$\nabla\bullet(H\mathbf{K}\bullet\nabla\hat{C} - H\hat{C}\mathbf{u}) = 0 \tag{3.8}$$

$$\left[(H\hat{C}\mathbf{u} - H\mathbf{K}\bullet\nabla\hat{C})\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} = \tag{3.9}$$

and

$$\left[(H\mathbf{K}\bullet\nabla\hat{C})\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{out}\cup\Gamma^{imp}} = 0 \tag{3.10}$$

Multiplying (3.8) by $\hat{C}(\mathbf{x})$, integrating over the domain, using the divergence theorem and boundary conditions (3.9-10), we obtain

$$2\underbrace{\int_{\Omega} H\,\nabla\hat{C}\bullet\mathbf{K}\bullet\nabla\hat{C}\,d\Omega}_{\substack{\geq 0,\ \text{since } \mathbf{K} \text{ is} \\ \text{positive definite}}} + \underbrace{\int_{\Gamma^{in}} H\,(\hat{C})^2\,(-\mathbf{u}\bullet\mathbf{n})\,d\Gamma^{in}}_{\geq 0,\ \text{since } (-\mathbf{u}\bullet\mathbf{n})\geq 0 \text{ on } \Gamma^{in}} = 0 \tag{3.11}$$

The left hand-side of (3.11) exhibits two non-negative integrals. Since their sum must be zero, each of these integrals must be zero. As a result, $\hat{C}(\mathbf{x})$ must be zero. Therefore, solution $C(\mathbf{x})=1$ is unique. QED.

4. Mean age

4.1. Governing equations and boundary conditions

In accordance with Delhez et al. (1999) and the age-averaging hypothesis (Deleersnijder et al. 2001), the mean age of the water parcels present in an elemental control volume is the mass-weighted arithmetic mean of their ages. As a result, at location **x**, mean age $a(\mathbf{x})$ is the ratio of the first-order moment of the age concentration function to the zero-th order one, i.e.

$$a(\mathbf{x}) = \frac{\int_0^\infty \tau\,c(\mathbf{x},\tau)\,d\tau}{\int_0^\infty c(\mathbf{x},\tau)\,d\tau} \tag{4.1.1}$$

The denominator of the right-hand side of (4.1.1) is the water concentration, which has been seen to be equal to unity at any position. Thus, the numerator, which is called age concentration (Delhez et al. 1999), is equal to the mean age:

$$a(\mathbf{x}) = \int_0^\infty \tau\,c(\mathbf{x},\tau)\,d\tau \tag{4.1.2}$$

To derive the equation governing the mean age, we multiply (2.1.8) by τ and integrate over the age. The left-hand side leads to

$$\begin{aligned}\int_0^\infty \tau\frac{\partial(Hc)}{\partial\tau}\,d\tau &= \int_0^\infty\left[\frac{\partial}{\partial\tau}(\tau Hc)-Hc\right]d\tau \\ &= \underbrace{\left[\tau Hc\right]_{\tau=0}^{\tau=\infty}}_{\substack{=0,\text{ see} \\ (2.1.12)\text{ and }(2.3.5)}} - H\underbrace{\int_0^\infty c\,d\tau}_{\substack{=1,\text{ see} \\ (3.1)}} = -H\end{aligned} \tag{4.1.3}$$

whilst the right-hand side easily yields

$$-\int_0^\infty [\tau\nabla \bullet (Hc\mathbf{u} - H\mathbf{K} \bullet \nabla c)]\, d\tau \;=\; -\nabla \bullet (Ha\mathbf{u} - H\mathbf{K} \bullet \nabla a) \tag{4.1.4}$$

Then, combining (4.1.3) and (4.1.4), we obtain the equation satisfied by the mean age:

$$\nabla \bullet (H\mathbf{K} \bullet \nabla a - Ha\mathbf{u}) \;+\; H \;=\; 0 \tag{4.1.5}$$

The relevant boundary conditions are established by evaluating the first-order moment of (2.1.13) and (2.1.14) and taking into account (4.1.2), i.e.

$$[(Ha\mathbf{u} - H\mathbf{K} \bullet \nabla A) \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma^{in}} \;=\; 0 \tag{4.1.6}$$

$$[(H\mathbf{K} \bullet \nabla a) \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma^{out} \cup \Gamma^{imp}} \;=\; 0 \tag{4.1.7}$$

By having recourse to developments similar to those that allowed us to establish the unicity of the solution to the partial differential problem obeyed by the water concentration, it is possible to show that solution to (4.1.5)-(4.1.7) is unique, as it should be.

The value of the age on the incoming boundary is unlikely to be zero, for, on Γ^{in}, there is a mixture of water parcels that are entering the domain (whose age is zero) and particles that have been moving for some time in the domain (whose age is positive), resulting in a positive mean age. Clearly, it is diffusion that can bring back to the incoming boundary water parcels that were inside the domain at an earlier time. Thus, as is exemplified in Section (6.2), the smaller the relative impact of diffusion the smaller the value of the mean water age on Γ^{in} is likely to be.

4.2. Sign of the mean age

Relation (4.1.2) indicates that mean age $a(\mathbf{x})$ must be non-negative. It is, however, appropriate to show that the solution to differential problem (4.1.5)-(4.1.7) actually satisfies this constraint.

Demonstrating that the age is non-negative is tantamount to showing that the negative part of the age

$$a^-(\mathbf{x}) \;=\; \frac{a(\mathbf{x}) - |a(\mathbf{x})|}{2} \;=\; \begin{cases} 0\,, \text{ if } 0 \le a(\mathbf{x}) \\ a(\mathbf{x})\,, \text{ if } a(\mathbf{x}) < 0 \end{cases} \tag{4.2.1}$$

is zero at every location. This is achieved by having recourse to a method similar to that used in Section (2.2). Accordingly, we multiply (4.1.5) by $a^-(\mathbf{x})$. Then, the resulting equation is integrated over the domain of interest, the boundary conditions are used, and, after some manipulations, we obtain

$$2\overbrace{\int_{\Omega} H\nabla a^{-}\bullet\mathbf{K}\bullet\nabla a^{-}\,d\Omega}^{\substack{\geq 0,\text{ since }\mathbf{K}\text{ is}\\ \text{positive definite}}} + \overbrace{\int_{\Gamma^{in}} H(a^{-})^2(-\mathbf{u}\bullet\mathbf{n})\,d\Gamma^{in}}^{\geq 0,\text{ since }(-\mathbf{u}\bullet\mathbf{n})\geq 0\text{ on }\Gamma^{in}}$$
$$+\underbrace{\int_{\Gamma^{out}} H(a^{-})^2\mathbf{u}\bullet\mathbf{n}\,d\Gamma^{out}}_{\geq 0,\text{ since }\mathbf{u}\bullet\mathbf{n}\geq 0\text{ on }\Gamma^{out}} + 2\underbrace{\int_{\Omega} H(-a^{-})\,d\Omega}_{\substack{\geq 0,\text{ since }(-a^{-})\\ \text{is positive definite}}} = 0 \tag{4.2.2}$$

Each of the integrals above is non-negative. For their sum to be zero, each of them must be zero, which is possible if and only if $a^{-}(\mathbf{x})$ is zero at every location in the domain. As a consequence, the age is non negative, i.e.

$$a(\mathbf{x}) \geq 0 \tag{4.2.3}$$

4.3. Location of the maximum of the age

Let $a_{\max}^{out}$ represent the largest value of the age on the outgoing boundary. Then, the deviation of the age with respect to this value is introduced:

$$\hat{a}(\mathbf{x}) = a(\mathbf{x}) - a_{\max}^{out} \tag{4.3.1}$$

Substituting (.4.3) into (4.1.5)-(4.1.7) and using (1.1), we obtain the differential problem obeyed by $\hat{a}(\mathbf{x})$:

$$\nabla\bullet(H\mathbf{K}\bullet\nabla\hat{a} - H\hat{a}\mathbf{u}) + H = 0 \tag{4.3.2}$$

$$\left[(H\hat{a}\mathbf{u} - H\mathbf{K}\bullet\nabla\hat{a})\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} = \left[-Ha_{\max}^{out}\mathbf{u}\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} \tag{4.3.3}$$

$$\left[(H\mathbf{K}\bullet\nabla\hat{a})\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{out}\cup\Gamma^{imp}} = 0 \tag{4.3.4}$$

If the maximum of the age is located inside the domain (rather than on the outgoing boundary), then the positive part of $\hat{a}(\mathbf{x})$

$$\hat{a}^{+}(\mathbf{x}) = \frac{\hat{a}(\mathbf{x})+|\hat{a}(\mathbf{x})|}{2} = \begin{cases} 0\,, \text{ if } \hat{a}(\mathbf{x})\leq 0 \\ \hat{a}(\mathbf{x})\,, \text{ if } 0<\hat{a}(\mathbf{x}) \end{cases} \tag{4.3.5}$$

must be non zero in a portion of the domain of interest.

First, we multiply (4.3.2) by $\hat{a}^{+}(\mathbf{x})$. Then, we perform manipulations rather similar to those that led to (4.2.2), eventually yielding

$$2\overbrace{\int_{\Omega} H\nabla\hat{a}^{+}\bullet\mathbf{K}\bullet\nabla\hat{a}^{+}\,d\Omega}^{\substack{\geq 0,\text{ since }\mathbf{K}\text{ is}\\ \text{positive definite}}} + \overbrace{\int_{\Gamma^{in}} H(\hat{a}^{+})^2(-\mathbf{u}\bullet\mathbf{n})\,d\Gamma^{in}}^{\geq 0,\text{ since }(-\mathbf{u}\bullet\mathbf{n})\geq 0\text{ on }\Gamma^{in}} + \overbrace{\int_{\Gamma^{out}} H(\hat{a}^{+})^2(\mathbf{u}\bullet\mathbf{n})\,d\Gamma^{out}}^{\geq 0,\text{ since }(\mathbf{u}\bullet\mathbf{n})\geq 0\text{ on }\Gamma^{out}}$$
$$= 2\underbrace{\int_{\Gamma^{in}} H\hat{a}^{+}a_{\max}^{out}(-\mathbf{u}\bullet\mathbf{n})\,d\Gamma^{in}}_{\geq 0,\text{ since }(-\mathbf{u}\bullet\mathbf{n})\geq 0\text{ on }\Gamma^{in}} + 2\underbrace{\int_{\Omega} H\,\hat{a}^{+}\,d\Omega}_{\substack{\geq 0,\text{ since }\hat{a}^{+}\text{ is}\\ \text{positive definite}}} \tag{4.3.6}$$

Clearly, $\hat{a}^{+}(\mathbf{x})$ is not necessarily zero at any point of Ω. Thus, the maximum of the age may be located inside the domain of interest. At first glance, this may be regarded as counterintuitive, but a plausible explanation may be constructed rather easily (e.g. Deleersnijder 2020).

4.4. Transit time

According to Zimmerman (1976), the transit time in semi-enclosed marine or estuarine domains is the time needed to travel from the inlet to the outlet. Clearly, this definition is applicable to the present study: the age on the outgoing boundary may be regarded as the transit time[3], i.e.

$$\xi(\mathbf{x}) = \left[a(\mathbf{x})\right]_{\mathbf{x}\in\Gamma^{out}} \tag{4.4.1}$$

Its average (over the outgoing boundary) reads

$$\overline{\xi} = \frac{1}{S^{out}} \int_{\Gamma^{out}} H(\mathbf{x})\xi(\mathbf{x})\, d\Gamma^{out} \tag{4.4.2}$$

As will be seen, the transit times satisfies a remarkable scaling property. To establish it, we must first evaluate the volume of the domain of interest,

$$V = \int_{\Omega} H(\mathbf{x})\, d\Omega \tag{4.4.2}$$

the area of the outgoing boundary,

$$S^{out} = \int_{\Gamma^{out}} H(\mathbf{x})\, d\Gamma^{out} \tag{4.4.3}$$

and the outgoing volumetric flow rate (m^3/s),

$$Q = \int_{\Gamma^{out}} H\, \mathbf{u} \bullet \mathbf{n}\, d\Gamma^{out} \tag{4.4.4}$$

which is equal to the incoming one since the flow is at a steady state.

Integrating over the domain of interest mean age equation (4.1.5), using the divergence theorem and boundary conditions (4.1.6-7), we obtain

$$-\int_{\Gamma^{out}} H\,\xi\, \mathbf{u} \bullet \mathbf{n}\, d\Gamma^{out} + \underbrace{\int_{\Omega} H\, d\Omega}_{\substack{=V, \\ \text{see (4.4.2)}}} = 0 \tag{4.4.5}$$

Finally, combining (4.4.3)-(4.4.5) yields

[3] Strictly speaking, the transit time can be defined at any point in the domain: it is the sum of the age and the residence time (i.e. the time needed to leave the domain). See, for instance, http://www.climate.be/cart. It is only on the outgoing boundary that the transit time is equal to the age. Since the age is better suited to the needs of the present study, the residence time is not evaluated herein. Therefore, although it is somewhat of a misuse of words, it is convenient to call "transit time" the value of the age on the outlet of the domain of interest.

$$\overline{\xi} = \frac{V}{Q} - \int_{\Gamma^{out}} H\,\hat{\xi}\,\mathbf{u}\cdot\mathbf{n}\,d\Gamma^{out} \tag{4.4.6}$$

with $\hat{\xi}(\mathbf{x}) = \xi(\mathbf{x}) - \overline{\xi}$. This result underscores the importance of ratio V/Q . The latter should be regarded as the order of magnitude of the transit time, which can be estimated without explicitly calculating the age.

5. Computing the age

To obtain age distribution function $c(\mathbf{x},\tau)$, partial differential problem (2.1.8)-(2.1.12-14) must be solved numerically. Unfortunately, as the forcing function is the Dirac delta impulse prescribed on the incoming boundary, straightforwardly tackling this problem is unlikely to be easy. This is the reason why it is desirable to work out a more tractable approach.

The first step consists in introducing the Laplace transform of the age distribution function:

$$\varsigma(\mathbf{x},s) = \int_0^\infty e^{-s\tau}\,c(\mathbf{x},\tau)\,d\tau \tag{5.1}$$

Taking the Laplace transform of (2.1.8) and (.2.13-14), using (2.1.12), we readily obtain

$$Hs\varsigma = -\nabla\cdot\left(H\varsigma\mathbf{u} - H\mathbf{K}\cdot\nabla\varsigma\right) \tag{5.2}$$

$$\left[(H\varsigma\mathbf{u} - H\mathbf{K}\cdot\nabla\varsigma)\cdot\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} = \left[H\,\mathbf{u}\cdot\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} \tag{5.3}$$

$$\left[(H\mathbf{K}\cdot\nabla\varsigma)\cdot\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{out}\cup\Gamma^{imp}} = 0 \tag{5.4}$$

Then, introducing function $\eta(\mathbf{x},s) = \varsigma(\mathbf{x},s)/s$ allows us to transform (5.2)-(5.4) to

$$Hs\eta = -\nabla\cdot\left(H\eta\mathbf{u} - H\mathbf{K}\cdot\nabla\eta\right) \quad . \tag{5.5}$$

$$\left[(H\eta\mathbf{u} - H\mathbf{K}\cdot\nabla\eta)\cdot\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} = \frac{1}{s}\left[H\,\mathbf{u}\cdot\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} \tag{5.6}$$

$$\left[(H\mathbf{K}\cdot\nabla\eta)\cdot\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{out}\cup\Gamma^{imp}} = 0 \tag{5.7}$$

It is readily seen that that original of $\eta(\mathbf{x},s)$, which will be denoted $b(\mathbf{x},\tau)$ below, is the solution of the following partial differential problem:

$$\frac{\partial(Hb)}{\partial\tau} = -\nabla\cdot\left(Hb\mathbf{u} - H\mathbf{K}\cdot\nabla b\right) \tag{5.8}$$

$$b(\mathbf{x},0) = 0 \tag{5.9}$$

$$\left[(Hb\mathbf{u} - H\mathbf{K}\cdot\nabla b)\cdot\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} = \left[H\,\mathbf{u}\cdot\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{in}} \tag{5.10}$$

$$\left[(H\mathbf{K}\cdot\nabla b)\cdot\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{out}\cup\Gamma^{imp}} = 0 \; . \tag{5.11}$$

Problem (5.8)-(5.11) is a rather classical advection-diffusion problem (in which τ plays a role equivalent to that of the time) and is much easier to deal with numerically than problem (2.1.8)-(2.1.12-14). This is because the incoming flux is independent of τ rather than a Dirac delta function of age τ.

Finally, owing to definition $\xi(\mathbf{x},s) = \varsigma(\mathbf{x},s)/s$, the age distribution function can be obtained by evaluating the following derivative

$$c(\mathbf{x},\tau) = \frac{\partial}{\partial\tau} b(\mathbf{x},\tau) \quad . \tag{5.12}$$

This simply is an application of a well-known result of the theory of linear system dynamics: the impulse response is the derivative of the step response. In the present context, the former and the latter responses are $c(\mathbf{x},\tau)$ and $b(\mathbf{x},\tau)$, respectively.

It can be easily proved that $b(\mathbf{x},\tau)$ satisfies

$$\lim_{\tau\to\infty} b(\mathbf{x},\tau) = 1 \tag{5.13}$$

Then, integrating (5.12) over the age and using (5.9) and (5.13) lead to the water concentration:

$$C(\mathbf{x}) = \underbrace{\int_0^\infty c(\mathbf{x},\tau)\, d\tau}_{\substack{= C(\mathbf{x}), \\ \text{see (3.1)}}} = \int_0^\infty \frac{\partial}{\partial\tau} b(\mathbf{x},\tau)\, d\tau = \underbrace{b(\mathbf{x},\infty)}_{\substack{=1,\text{ see} \\ (5.13)}} - \underbrace{b(\mathbf{x},0)}_{\substack{=0,\text{ see} \\ (5.9)}} \tag{5.14}$$

which simplifies to $C(\mathbf{x}) = 1$, which is agreement with (3.1), as it should be.

6. Elongated basins

In this section, we will consider only basins for which the length (distance from inlet to outlet) may be unambiguously defined. If, in addition, the width of the reservoir is much smaller than its length, then it may be deemed appropriate to have recourse to a one-dimensional approach, relying only on cross section-averaged variables.

6.1. Reservoirs with variable cross-sectional area

Let x, L, $S(x)$ and $U(x)$ denote the longitudinal coordinate (distance to the incoming boundary), the length of the reservoir ($0 \le x \le L$), its cross-sectional area and the average of the longitudinal velocity over a cross section, respectively (Figure S2). Since the flow is a steady state, the (longitudinal) volumetric flow rate must be constant:

$$Q = S(x)U(x) \tag{6.1.1}$$

The unresolved processes are parameterised by means of longitudinal diffusivity $K(x)$, which is assumed to be positive.

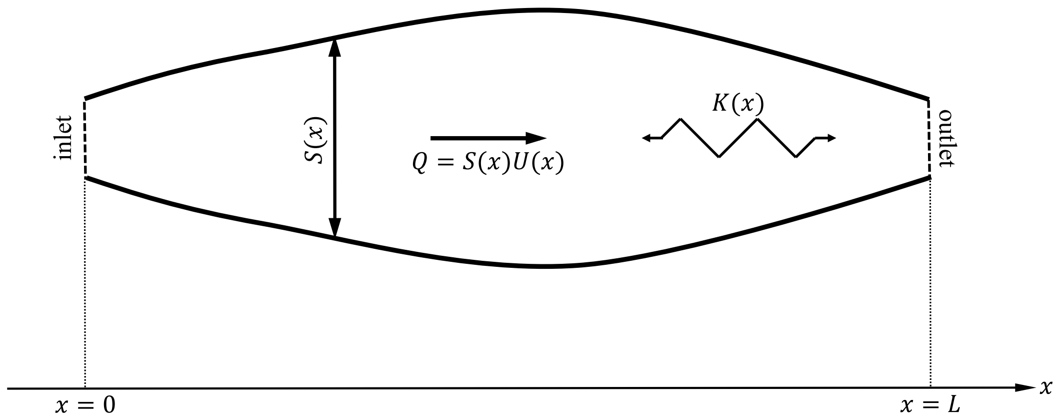

Figure 2. Schematic representation of an elongated reservoir. The longitudinal coordinate is denoted x, whilst $S(x)$, $U(x)$, and $K(x)$ represent the cross-sectional area, the velocity and the relevant diffusivity, respectively. The volumetric flow rate, Q, is a constant.

Cross section-averaged age $a(x)$ is the solution of equation

$$\frac{d}{dx}\left(SK\frac{da}{dx} - Qa \right) + S = 0 \tag{6.1.2}$$

which is to be solved under boundary conditions

$$\left[SK\frac{da}{dx} - Qa \right]_{x=0} = 0 \tag{6.1.3}$$

$$\left[SK\frac{da}{dx} \right]_{x=L} = 0 \tag{6.1.4}$$

Obviously, (6.1.2)-(6.1.4) are the one-dimensional counterpart of depth-averaged equations (4.1.5)-(4.1.7).

To formulate the solution of this ordinary differential problem, it is convenient to introduce auxiliary functions

$$\Xi(x) = \int_0^x S(x')\,dx' \quad , \tag{6.1.5}$$

$$\Pi(x) = \frac{U(x)L}{K(x)} = \frac{QL}{S(x)K(x)} \quad , \tag{6.1.6}$$

$$\lambda(x) = \exp\left[-\frac{1}{L}\int_x^L \Pi(x')\,dx' \right] \tag{6.1.7}$$

and

$$\zeta(x) = \int_x^L \frac{\Xi(x')\,dx'}{S(x')K(x')\lambda(x')} \quad . \tag{6.1.8}$$

The volume V of the domain is $\Xi(L) = V$ and $\Pi(x)$ may be viewed as a local (i.e. position-dependent) Peclet number. The mean water age is

$$\boxed{a(x) = \frac{V}{Q}\lambda(x) + \lambda(x)\zeta(x)} \tag{6.1.9}$$

In contrast with the age of the two-dimensional, depth-averaged model, the maximum of age (6.1.9) occurs on the outgoing boundary. This is because there is only one path from the inlet to the inlet. Since $\lambda(L) = 1$ and $\zeta(L) = 0$, the age on the outgoing boundary is

$$\max_{0 \le x \le L} a(x) = a(L) = \frac{V}{Q}\underbrace{\lambda(L)}_{=1} + \underbrace{\lambda(L)}_{=1}\underbrace{\zeta(L)}_{=0} = \frac{V}{Q} \tag{6.1.10}$$

The maximum value of the age is V/Q irrespective of the diffusivity and cross-sectional area profiles. This remarkable result can also be obtained by integrating over the length of the channel the differential equation obeyed by the age, yielding

$$\begin{aligned} 0 &= \int_0^L \left[\frac{d}{dx}\left(SK\frac{da}{dx} - Qa\right) + S\right]dx = \left[SK\frac{da}{dx} - Qa\right]_{x=0}^{x=L} + \int_0^L S\,dx \\ &= \underbrace{\left[SK\frac{da}{dx} - Qa\right]_{x=L}}_{=-Q\tilde{a}(L),\ \text{see (6.1.4)}} - \underbrace{\left[SK\frac{da}{dx} - Qa\right]_{x=0}}_{=0,\ \text{see (6.1.3)}} + \underbrace{\int_0^L S\,dx}_{=V} = -Q\,a(L) + V \end{aligned} \tag{6.1.11}$$

which is equivalent to (6.1.10).

6.2. Reservoirs with constant cross-sectional area

If the cross-sectional area and the along-flow diffusivity are constant, then auxiliary functions (6.1.5)-(6.1.8) simplify to

$$\begin{cases} \Xi(x) = Sx\,, \quad \Pi(x) = \dfrac{UL}{K} = Pe\,, \quad \lambda(x) = e^{-Pe(1-x/L)} \\ \zeta(x) = \dfrac{e^{Pe(1-x/L)}(1 + Pe\,x/L) - (1+Pe)}{Pe}\,\dfrac{L}{U} \end{cases} \tag{6.2.1}$$

where constant $Pe = UL/K$ denotes the Peclet number. Substituting (6.2.1) into (6.1.9) yields the mean water age (Figure 3)

$$a(x) = \frac{L}{U}\left(\frac{x}{L} + \frac{1 - e^{-Pe(1-x/L)}}{Pe}\right) = \frac{V}{Q}\left(\frac{x}{L} + \frac{1 - e^{-Pe(1-x/L)}}{Pe}\right) \tag{6.2.2}$$

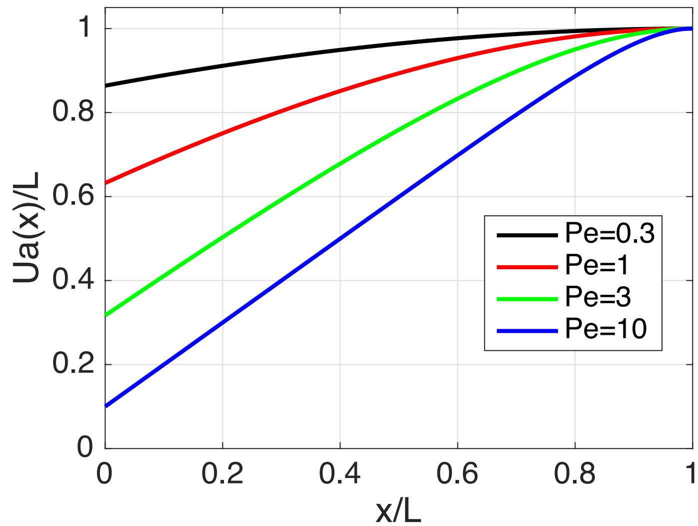

Figure 3. Mean water age (6.2.2) for various values of the Peclet number, Pe. Dimensionless age $Ua(x)/L$ is represented as a function of x/L, where L is the length of the domain. This approach is valid for an elongated reservoir (width << length): all variables are cross section-averaged.

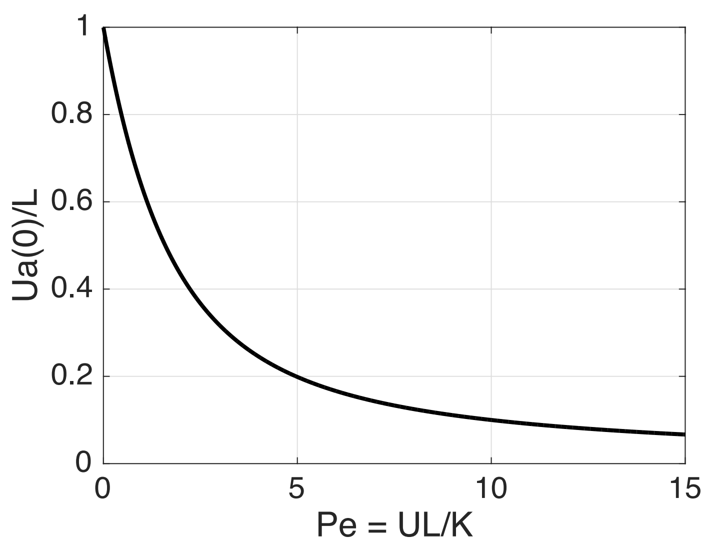

Figure 4. Mean water age (6.2.3) on the inlet of an elongated reservoir. Dimensionless age $Ua(0)/L$ is represented as a function of Peclet number Pe.

As expected, the transit time is $a(L) = L/U = V/Q$. It is worth underscoring that the age on the inlet is non-zero,

$$a(0) = \frac{L}{U}\frac{1-e^{-Pe}}{Pe} = \frac{V}{Q}\frac{1-e^{-Pe}}{Pe} \tag{6.2.3}$$

and, unsurprisingly, diminishes as the Peclet number increases (Figure 4). This is in agreement with the following asymptotic expansions:

$$a(0) \sim \frac{L}{U}\left(1-\frac{Pe}{2}\right) , \quad Pe \to 0 \tag{6.2.4}$$

$$a(0) \sim \frac{L}{U}\frac{1}{Pe} , \quad Pe \to \infty \tag{6.2.5}$$

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