The multiplicative constant for the Meijer-G kernel determinant

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Abstract

We compute the multiplicative constant in the large gap asymptotics of the Meijer-G point process. This point process generalizes the Bessel point process and appears at the hard edge of Cauchy–Laguerre multi-matrix models and of certain product random matrix ensembles.

1 Introduction and main results

The Meijer-G point process is a determinantal point process whose kernel is built out of Meijer G-functions. It appears in the study of the smallest squared singular values of certain product random matrices [32, 28] and in Cauchy multi-matrix models [11, 7] in the limit of large matrix dimension (see below for more details). Of particular interest is the distribution of the smallest particle, or equivalently, the probability of finding no particle in the interval $[0, s]$, $s > 0$. In some particular cases, this distribution is related to a system of partial differential equations [37, 36]. In this work, we are interested in the tail behavior of this distribution as $s \to +\infty$, known as the large gap asymptotics. The study of such asymptotics was initiated in [16], where the first two terms in the asymptotic expansion were found, and then pursued in [15], where the third term was obtained. The purpose of this paper is to obtain an explicit expression for the next term in the expansion, which is the multiplicative constant.

Meijer-G point process. Given $m, n, q, p \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $a_1, \ldots, a_p, b_1, \ldots, b_q \in \mathbb{C}$ such that

$$0 \le m \le q, \quad 0 \le n \le p \qquad \text{and} \qquad a_k - b_j \neq 1,\, 2,\, 3,\, \ldots \qquad \text{for all } 1 \le k \le n, \text{ and } 1 \le j \le m,$$

the Meijer G-function is defined by [34, Eq. 16.17.1]

$$G^{m,n}_{p,q}\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - t) \prod_{j=1}^n \Gamma(1 - a_j + t)}{\prod_{j=m+1}^q \Gamma(1 - b_j + t) \prod_{j=n+1}^p \Gamma(a_j - t)} z^t dt, \tag{1.1}$$

where $z \in \mathbb{C}$ and $z^t = |z|^t e^{it \arg z}$ with $\arg z \in (-\pi, \pi)$. The integration contour L separates the poles of $\Gamma(b_j - t)$, $j = 1, \ldots, m$, from those of $\Gamma(1 - a_j + t)$, $j = 1, \ldots, n$. Furthermore, L is unbounded, oriented upwards, and goes to ∞ in sectors of the complex plane such that the integral in (1.1) converges. For example, we can choose L as follows:

- If $|\arg z| < \frac{\pi}{2}(2m + 2n - q - p)$, then L can be chosen to start at $-i\infty$ and end at $+i\infty$.
- If $q > p$, then L can be a loop starting and ending at $+\infty$.
- If $q < p$, then L can be a loop starting and ending at $-\infty$.

Let $r > q \ge 0$ be fixed integers and let

$$\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q > -1. \tag{1.2}$$

The Meijer-G point process is a determinantal point process on $\mathbb{R}^+ = (0, +\infty)$ whose kernel is given by

$$\mathbb{K}(x, y) = \int_0^1 G^{1,q}_{q,r+1}\left(\begin{array}{c} -\mu_1, \ldots, -\mu_q \\ 0, -\nu_1, \ldots, -\nu_r \end{array} \middle| tx \right) G^{r,0}_{q,r+1}\left(\begin{array}{c} \mu_1, \ldots, \mu_q \\ \nu_1, \ldots, \nu_r, 0 \end{array} \middle| ty \right) dt. \tag{1.3}$$

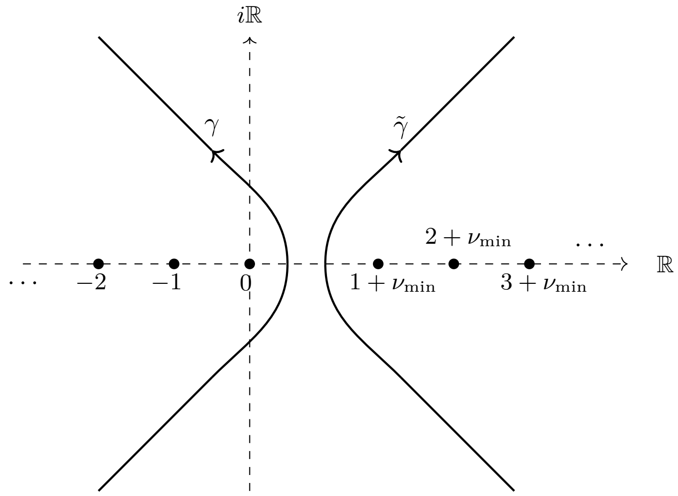

Figure 1: The contours γ and $\tilde{\gamma}$, and $\nu_{\min} = \min\{\nu_1, \dots, \nu_r, \mu_1, \dots, \mu_q\}$.

The kernel (1.3) can equivalently be written as [16, Eq. (1.14) and Proposition A.2]

$$\mathbb{K}(x,y) = \int_\gamma \frac{du}{2\pi i} \int_{\tilde{\gamma}} \frac{dv}{2\pi i} \frac{F(u)}{F(v)} \frac{x^{-u} y^{v-1}}{v-u}, \tag{1.4}$$

where F is given in terms of the Gamma function Γ by

$$F(z) = \frac{\Gamma(z) \prod_{j=1}^q \Gamma(1+\mu_j - z)}{\prod_{j=1}^r \Gamma(1+\nu_j - z)}. \tag{1.5}$$

The contours γ and $\tilde{\gamma}$ in (1.4) are disjoint, oriented upwards, and tend to infinity in sectors lying strictly in the left and right half-planes, respectively. Furthermore, they separate the poles of $\Gamma(z)$ from the poles of $\prod_{j=1}^q \Gamma(1+\mu_j - z) \prod_{j=1}^r \Gamma(1+\nu_j - z)$, see Figure 1.

The Meijer-G point process generalizes in a natural way the Bessel point process, which is the point process most commonly encountered at hard edges for random matrix ensembles. In fact, if $q = 0$, $r = 1$ and $\nu_1 = \nu > -1$, then the kernel (1.3) can be written as (see e.g. [32, Section 5.3])

$$\mathbb{K}(x,y) = \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \int_0^1 J_\nu(2\sqrt{tx}) J_\nu(2\sqrt{ty}) dt = 4 \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \mathbb{K}_{\mathrm{Be}}(4x, 4y), \tag{1.6}$$

where $\mathbb{K}_{\mathrm{Be}}$ is the kernel of the Bessel point process

$$\mathbb{K}_{\mathrm{Be}}(x,y) = \frac{J_\nu(\sqrt{x})\sqrt{y} J_\nu'(\sqrt{y}) - \sqrt{x} J_\nu'(\sqrt{x}) J_\nu(\sqrt{y})}{2(x-y)}, \tag{1.7}$$

and J_ν is the Bessel function of the first kind of order ν.

Gap probability. We consider the probability to observe a gap on $[0,s]$ in the Meijer-G point process. It follows from the general theory of determinantal point processes [35] that this probability can be written as a Fredholm determinant:

$$\mathbb{P}(\text{gap on } [0,s]) = \det\left(1 - \mathbb{K}\big|_{[0,s]}\right). \tag{1.8}$$

It is well-known [37] that the distribution of the smallest particle in the Bessel point process (which corresponds to (1.8) with $q = 0$ and $r = 1$) is naturally expressed in terms of the solution of a Painlevé V equation. For $q = 0$, $r \geq 2$ and integer values of $\nu_1, \dots, \nu_r$, the Fredholm determinant (1.8) is instead related to a more involved system of partial differential equations [36].

Large gap asymptotics. Asymptotics for (1.8) as $s \to +\infty$ are referred to as *large gap asymptotics*. For the Bessel point process, these asymptotics are known and given by:

$$\det\left(1 - \mathbb{K}|_{[0,s]}\right) = \frac{G(1+\nu)}{(2\pi)^{\frac{\nu}{2}}} \exp\left(- s + 2\nu\sqrt{s} - \frac{\nu^2}{4}\ln(4s) + \mathcal{O}\left(s^{-1/2}\right)\right), \quad \text{as } s \to +\infty, \tag{1.9}$$

where G denotes Barnes' G-function and $\mathbb{K}$ is as in (1.6). The asymptotics (1.9) were first conjectured by Tracy and Widom in [37], then proved by Ehrhardt [25] for all complex values of ν in the strip $-1 < \operatorname{Re}\nu < 1$, and finally proved by Deift, Krasovsky and Vasilevska [17] for $-1 < \operatorname{Re}\nu$. Note that, because of the rescaling (1.6), the left-hand-side in (1.9) is equal to $\det\left(1 - \mathbb{K}_{\mathrm{Be}}|_{[0,4s]}\right)$.

The study of the large gap asymptotics of the Meijer-G point process (for general real values of the parameters) has been initiated by Claeys, Girotti and Stivigny in [16]. They proved that there exist real constants ρ, a, b, c and C such that

$$\det\left(1 - \mathbb{K}|_{[0,s]}\right) = C\exp\left(- as^{2\rho} + bs^{\rho} + c\ln s + \mathcal{O}\left(s^{-\rho}\right)\right), \qquad \text{as } s \to +\infty, \tag{1.10}$$

and derived the following explicit expressions for ρ, a and b:[1]

$$\rho = \frac{1}{1+r-q}, \qquad a = \frac{(r-q)^{\frac{1-r+q}{1+r-q}}(r-q+1)^2}{4}, \tag{1.11}$$

$$b = (1+r-q)(r-q)^{-\frac{r-q}{1+r-q}}\left[\sum_{j=1}^{r}\nu_j - \sum_{k=1}^{q}\mu_k\right]. \tag{1.12}$$

These asymptotics have been extended to all orders in [15], where it was shown that, for any $N \in \mathbb{N}$, there exist constants $C_1, \ldots, C_N \in \mathbb{R}$ such that

$$\det\left(1 - \mathbb{K}|_{[0,s]}\right) = C\exp\left(- as^{2\rho} + bs^{\rho} + c\ln s + \sum_{j=1}^{N} C_j s^{-j\rho} + \mathcal{O}\left(s^{-(N+1)\rho}\right)\right), \quad \text{as } s \to +\infty. \tag{1.13}$$

Furthermore, the following explicit expression for c was obtained (see [15, Remark 5.2]):

$$c = \frac{r-q-1}{12(r-q+1)} - \frac{1}{2(r-q+1)}\left(\sum_{j=1}^{r}\nu_j^2 - \sum_{k=1}^{q}\mu_k^2\right). \tag{1.14}$$

Note that C is a *multiplicative* constant in (1.13), which means that there is no precise description of the large gap asymptotics without its explicit expression.

The problem of determining multiplicative constants in the asymptotics of Toeplitz, Hankel, and Fredholm determinants is a notoriously difficult problem with a long history. The first constant factor problem was solved in Onsager's 1948 computation for the spontaneous magnetization of the square-lattice Ising model (see [5, 6] or Yang's publication [39] on the same subject). The constant in the gap expansion of the sine kernel is known as the Dyson or Widom–Dyson constant (see e.g. [21, 24]); its occurrence in the asymptotic expansion was first established in 1973 by des Cloizeaux and Mehta [22], and its value was first determined (non-rigorously) by Dyson [23] building on earlier work of Widom [38]. Three different rigorous derivations of the Dyson constant were eventually presented in relatively rapid succession starting with the derivation by Krasovsky [29] in 2004, followed by the derivations in [24] and [21]; these results also apply to certain related Toeplitz determinants. Pioneering works where multiplicative constants are found for other determinants include [19, 4] for the Airy Fredholm determinant, [25, 17] for the Bessel Fredholm determinant, and [30, 20] for Hankel and Toeplitz determinants with Fisher-Hartwig singularities.

In this paper, we derive an explicit expression for the constant C in (1.13) for general values of the parameters. The exact expression of C combined with (1.11), (1.12) and (1.14) allows for an accurate description of the large gap asymptotics, in the sense that the ratio

$$\frac{\det\left(1 - \mathbb{K}|_{[0,s]}\right)}{C\exp(-as^{2\rho} + bs^{\rho} + c\ln s)}$$

tends to 1 as $s \to +\infty$. The following theorem is our main result.

[1] Three point processes were considered in [16], and the corresponding quantities were denoted with an upperscript (j), $j = 1, 2, 3$. The coefficients (1.11)-(1.12) correspond to [16, Theorem 1.2] with $j = 2$.

Theorem 1.1 (Explicit expression for the constant C)**.** *Let $r > q \geq 0$ be fixed integers and let $\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q > -1$. The constant C that appears in the asymptotic formula (1.10) is given by*

$$C = \frac{\prod_{j=1}^{r} G(1+\nu_j) \prod_{k=1}^{q} (2\pi)^{\frac{\mu_k}{2}}}{\prod_{k=1}^{q} G(1+\mu_k) \prod_{j=1}^{r} (2\pi)^{\frac{\nu_j}{2}}} \exp\left\{(1-r+q)\zeta'(-1)\right\}$$
$$\times \exp\Bigg\{\Bigg(\frac{1+r-q-(r-q)^2}{2(1+r-q)}\Bigg[\sum_{\ell=1}^{r}\nu_\ell^2 - \sum_{k=1}^{q}\mu_k^2\Bigg] + \frac{-2+(r-q)^2(r-q-1)}{24(1+r-q)}$$
$$+ \sum_{1\leq j<k\leq r} \nu_j\nu_k + \sum_{1\leq j<k\leq q}\mu_j\mu_k - \sum_{j=1}^{r}\nu_j\sum_{k=1}^{q}\mu_k + \sum_{k=1}^{q}\mu_k^2\Bigg)\ln(r-q)\Bigg\}$$
$$\times \exp\Bigg\{\Bigg(-\frac{2-r+q}{2}\Bigg[\sum_{\ell=1}^{r}\nu_\ell^2 - \sum_{k=1}^{q}\mu_k^2\Bigg] - \frac{(r-q-1)^2}{24}$$
$$- \sum_{1\leq j<k\leq r} \nu_j\nu_k - \sum_{1\leq j<k\leq q}\mu_j\mu_k + \sum_{j=1}^{r}\nu_j\sum_{k=1}^{q}\mu_k - \sum_{k=1}^{q}\mu_k^2\Bigg)\ln(1+r-q)\Bigg\}, \tag{1.15}$$

where G denotes Barnes' G-function and $\zeta'(-1)$ the derivative of Riemann's zeta function evaluated at -1.

The proof of Theorem 1.1 will be given in Section 6 after considerable preparations have been carried out in Sections 2-5.

The value of C was previously known for some very particular choices of the parameters. In Section 1.2, we recall these expressions and show that Theorem 1.1 is consistent with them. At the end of Section 1.2, we also provide some numerical checks of Theorem 1.1 for certain values of the parameters.

1.1 Applications in random matrix theory

The Meijer-G point process appears at the hard edge scaling limit of several random matrix ensembles as the size of the matrices becomes large. In what follows, we explain how Theorem 1.1, combined with the results of [16, 15], can be used to obtain information on the large gap asymptotics at the hard edge for these models.

The Cauchy–Laguerre multi-matrix model. The Cauchy two-matrix model has been introduced in [10] and is a model for two positive definite Hermitian matrices coupled in a chain. The probability density function is defined over all pairs (M_1, M_2) of two $n\times n$ positive definite Hermitian matrices, and it takes the form

$$\frac{1}{Z_n}\frac{\det(M_1)^{\alpha_1}\det(M_2)^{\alpha_2}}{\det(M_1+M_2)^n}e^{-\mathrm{Tr}(V_1(M_1)+V_2(M_2))}dM_1dM_2, \tag{1.16}$$

where Z_n is the normalization constant, the two scalar potentials $V_1(x)$, $V_2(x)$ grow sufficiently fast as $x \to +\infty$, and the parameters α_1 and α_2 satisfy $\alpha_1 > -1$, $\alpha_2 > -1$, and $\alpha_1+\alpha_2 > -1$. The eigenvalues $x_1, \ldots, x_n$ of M_1 together with the eigenvalues $y_1, \ldots, y_n$ of M_2 form a two-level determinantal point process. The Meijer-G kernel (1.3) was first discovered by Bertola, Gekhtman and Szmigielski in [11], where they considered the special case of $V_1(x) = V_2(x) = x$, known as the Cauchy-Laguerre two-matrix model. If we consider the point process involving only the eigenvalues of M_1, the associated limiting kernel as $n \to +\infty$ in the hard edge scaling limit, denoted by $\mathcal{G}_{01}$ in [11], is given by $(\frac{x}{y})^{\alpha_1}\mathbb{K}(x,y)$, where $\mathbb{K}$ is given by (1.3) with $r = 2$, $q = 0$, $\nu_1 = \alpha_1+\alpha_2$ and $\nu_2 = \alpha_1$. Letting $x_{\min} := \min\{x_1, \ldots, x_n\}$, we have (see [16, Appendix])

$$\lim_{n\to+\infty}\mathbb{P}(x_{\min} > \tfrac{s}{n}) = \det\left(1-\mathbb{K}|_{[0,s]}\right), \tag{1.17}$$

and we can obtain the tail behavior as $s \to +\infty$ up to and including the constant for the right-hand-side of (1.17) by combining (1.13), (1.11), (1.12), (1.14) and (1.15) with $r = 2$, $q = 0$, $\nu_1 = \alpha_1+\alpha_2$ and $\nu_2 = \alpha_1$.

The Cauchy two-matrix model has been generalized to an arbitrary number r of matrices in [7], and similar results have been obtained at the hard edge as in the case of two matrices. Large gap asymptotics up to and including the constant can also be obtained using (1.11), (1.12), (1.14) and Theorem 1.1, with $q = 0$ but general values of r, $\nu_1, \dots, \nu_r$.

Products of Ginibre matrices. A complex Ginibre matrix is a random matrix whose entries are independent and identically distributed complex Gaussian variables. Let $G_1, \dots, G_r$ be independent complex standard Ginibre matrices of size $(n + \nu_j) \times (n + \nu_{j-1})$, where $\nu_0 = 0$ and $\nu_1, \dots, \nu_r$ are non-negative integers, and consider the product $G_r \cdots G_1$. If $r = 1$, the squared singular values are well-studied and behave statistically as the eigenvalues of the Laguerre Unitary Ensemble, which are distributed according to a determinantal point process whose limiting kernel as $n \to +\infty$ in the hard edge scaling limit is given by the Bessel kernel (1.7) with $\nu = \nu_1$. For general $r \geq 2$, it is known from Akemann, Kieburg and Wei [3] that the squared singular values of $G_r \cdots G_1$ still form a determinantal point process, and from Kuijlaars and Zhang [32] that the limiting kernel in the hard edge scaling limit is the Meijer-G kernel (1.3) with $q = 0$ (see also [2] for a detailed overview). If $x_{\min}$ denotes the smallest squared singular value of $G_r \cdots G_1$, then the limit (1.17) holds, and we can obtain the tail behavior as $s \to +\infty$ up to and including the constant for the right-hand-side of (1.17) by combining (1.13), (1.11), (1.12), (1.14) and (1.15), and setting $q = 0$.

Products of truncated unitary matrices. Let $U_1, \dots, U_r$ be r independent Haar distributed unitary matrices of size $\ell_1 \times \ell_1, \dots, \ell_r \times \ell_r$, respectively, and let T_j be the upper left $(n+\nu_j) \times (n+\nu_{j-1})$ truncation of U_j, $j = 1, \dots, r$. The parameters $\ell_1, \dots, \ell_r$ are positive integers, $\nu_0 = 0$ and $\nu_1, \dots, \nu_r$ are non-negative integers. Furthermore, assume that $\ell_1 \geq 2n+\nu_1$ and $\ell_j \geq n+\nu_j+1$ for $j \geq 2$. The squared singular values of the product $T_r \cdots T_1$ behave statistically as a determinantal point process [28]. Taking $n \to +\infty$, we simultaneously have to let $\ell_j \to +\infty$ for $j = 1, \dots, r$. We choose a subset J of indices

$$J = \{j_1, \dots, j_q\} \subset \{2, \dots, r\}, \qquad \text{with } 0 \leq q = |J| < r$$

and integers $\mu_1, \dots, \mu_q$ with $\mu_k \geq \nu_k + 1$, and assume that

$$\begin{aligned} &\ell_j - n \to +\infty, && \text{for } j \in \{1, \dots, r\} \setminus J, \\ &\ell_{j_k} - n = \mu_k, && \text{for } j_k \in J. \end{aligned}$$

Let $x_{\min}$ denote the smallest squared singular value of $T_r \cdots T_1$. It follows from [28, Theorem 2.8] and [16, Appendix] that

$$\lim_{n \to +\infty} \mathbb{P}(x_{\min} > \tfrac{s}{c_n}) = \det\big(1 - \mathbb{K}|_{[0,s]}\big), \qquad \text{with} \quad c_n = n \prod_{j \notin J} (\ell_j - n), \tag{1.18}$$

and as in (1.17) we can obtain asymptotics as $s \to +\infty$ for (1.18) up to and including the constant by combining (1.13), (1.11), (1.12), (1.14) and (1.15). Note that in the above two models of product random matrices, we only need to utilize Theorem 1.1 for integer values of the parameters.

1.2 Consistency checks of Theorem 1.1 and numerical confirmations

We provide three different consistency checks of Theorem 1.1; the first two verify consistency with known results in the literature for special choices of the parameters (for the constants ρ, a, b and c, these checks were already carried out in [16, 15]), while the third verifies consistency under the transformation $(r, q) \to (r + 1, q + 1)$ whenever $\nu_{r+1} = \mu_{q+1}$. For ease of explanation, we sometimes indicate the dependence of ρ, a, b, c, C on the parameters r, $\nu_1, \dots, \nu_r$, q and $\mu_1, \dots, \mu_q$ explicitly, e.g. for C we write

$$C\big((r; \nu_1, \dots, \nu_r), (q; \mu_1, \dots, \mu_q)\big).$$

We also provide numerical confirmations of Theorem 1.1 at the end of this section.

Consistency with the large gap asymptotics of the Bessel point process. Since our proof of Theorem 1.1 uses the asymptotic formula (1.9) for the Bessel kernel determinant, see Section 1.3, our work does not provide an independent derivation of the constant factor in (1.9); nevertheless, it is instructive to verify that Theorem 1.1 is consistent with (1.9). We verify from (1.11), (1.12), (1.14) and (1.15) that

$$\rho\big((1;\nu),(0;-)\big) = \frac{1}{2}, \qquad a\big((1;\nu),(0;-)\big) = 1, \qquad b\big((1;\nu),(0;-)\big) = 2\nu, \qquad c\big((1;\nu),(0;-)\big) = -\frac{\nu^2}{4},$$

and

$$C\big((1;\nu),(0;-)\big) = \frac{G(1+\nu)}{(2\pi)^{\frac{\nu}{2}}} 2^{-\frac{\nu^2}{2}},$$

which is indeed consistent with (1.9).

Consistency with known results for the Muttalib–Borodin ensembles. The Muttalib–Borodin ensembles [33] with a Laguerre weight are joint probability density functions of the form

$$\frac{1}{Z_n} \prod_{1\le j<k\le n} (x_k - x_j)(x_k^\theta - x_j^\theta) \prod_{j=1}^n x_j^\alpha e^{-x_j}\, dx_j, \tag{1.19}$$

where the n points $x_1, \ldots, x_n$ belong to the interval $[0,+\infty)$, Z_n is a normalization constant and $\theta > 0$ and $\alpha > -1$ are two parameters of the model. These points behave statistically as a determinantal point process whose hard edge limiting kernel $\mathbb{K}^{\mathrm{MB}}$ can be written in terms of Wright's generalized Bessel functions [14]. The corresponding large gap asymptotics are of the form [16]

$$\det\left(1 - \mathbb{K}^{\mathrm{MB}}|_{[0,s]}\right) = C^{\mathrm{MB}} \exp\Big(- a^{\mathrm{MB}} s^{2\rho^{\mathrm{MB}}} + b^{\mathrm{MB}} s^{\rho^{\mathrm{MB}}} + c^{\mathrm{MB}} \ln s + \mathcal{O}\big(s^{-\rho^{\mathrm{MB}}}\big)\Big), \qquad \text{as } s \to +\infty.$$

The constants ρ^{MB}, a^{MB} and b^{MB} have been obtained in [16], and c^{MB} and C^{MB} in [15]. For $q = 0$ and certain particular choices of the parameters r, $\nu_1, \ldots, \nu_r$, α and θ, the kernels $\mathbb{K}$ and $\mathbb{K}^{\mathrm{MB}}$ define the same point process (up to rescaling), see [31, Theorem 5.1]. More precisely, if $r \ge 1$ is an integer, $\alpha > -1$ and

$$\theta = \frac{1}{r}, \qquad \nu_j = \alpha + \frac{j-1}{r}, \qquad j = 1, \ldots, r, \tag{1.20}$$

then the kernels $\mathbb{K}$ and $\mathbb{K}^{\mathrm{MB}}$ are related by

$$\left(\frac{x}{y}\right)^{\alpha} \mathbb{K}(x,y) = r^r \mathbb{K}^{\mathrm{MB}}(r^r x, r^r y).$$

Therefore, if the parameters satisfy (1.20), we obtain the following relations:

$$\rho = \rho^{\mathrm{MB}}, \qquad a = a^{\mathrm{MB}} r^{2r\rho}, \qquad b = b^{\mathrm{MB}} r^{r\rho}, \qquad c = c^{\mathrm{MB}}, \tag{1.21}$$

$$C = r^{rc} C^{\mathrm{MB}}. \tag{1.22}$$

The relations (1.21) were already verified in [15, Remark 5.2], and we now verify that (1.22) holds. Let us explicitly write the dependence of C^{MB} on θ and α. From [15, Theorem 1.1], we have

$$\begin{aligned} C^{\mathrm{MB}}(\theta,\alpha) = {} & \frac{G(1+\alpha)}{(2\pi)^{\frac{\alpha}{2}}} \exp\big(d(1,\alpha) - d(\theta,\alpha)\big) \exp\left(\frac{24\alpha(\alpha+2) + 15 + 3\theta + 4\theta^2}{24(1+\theta)} \ln\theta\right) \\ & \times \exp\left(\frac{6\alpha\theta - 6\alpha(1+\alpha) - (\theta-1)^2}{12\theta} \ln(1+\theta)\right), \end{aligned} \tag{1.23}$$

where $d(\theta,\alpha)$ is a regularized sum, see [15, Eq. (1.12)]. If θ is rational, it follows from [15, Proposition 1.4] that $d(\theta,\alpha)$ can be expressed in terms of $\zeta'(-1)$ and Barnes' G-function evaluated at certain points.

By specializing [15, Proposition 1.4] for $\theta = \frac{1}{r}$, we obtain

$$d(\tfrac{1}{r},\alpha) = r\zeta'(-1) + \frac{1+(1+2\alpha)r}{4}\ln(2\pi) - \frac{1}{12}\left(3+\frac{1}{r}+r+6\alpha(1+r+\alpha r)\right)\ln r - \sum_{k=1}^{r}\ln G\left(1+\alpha+\frac{k}{r}\right). \tag{1.24}$$

Substituting (1.24) into (1.23) with $\theta = \frac{1}{r}$, we obtain after a direct computation that

$$\begin{aligned} r^{rc}C^{\mathrm{MB}}(\tfrac{1}{r},\alpha) = {} & \frac{G(1+\alpha)\prod_{j=1}^{r}G\left(1+\alpha+\frac{k}{r}\right)}{G(2+\alpha)(2\pi)^{\frac{r-1+2r\alpha}{4}}}\exp\{(1-r)\zeta'(-1)\} \\ & \times\exp\left\{\frac{1-r(5+12\alpha)+2r^2(1+6\alpha+6\alpha^2)}{24(1+r)}\ln r\right\} \\ & \times\exp\left\{-\frac{1-2r(1+3\alpha)+r^2(1+6\alpha+6\alpha^2)}{12r}\ln(1+r)\right\}. \end{aligned} \tag{1.25}$$

On the other hand, by substituting the particular values of ν_j given by (1.20) into (1.15), another long but straightforward computation shows that

$$C\big((r;\alpha,\alpha+\tfrac{1}{r},\ldots,\alpha+\tfrac{r-1}{r}),(0,-)\big)$$

is also given by the right-hand-side of (1.25), which proves (1.22).

Poles-zeros cancellation. If one increases simultaneously r and q by 1, with ν_{r+1} and μ_{q+1} such that $\nu_{r+1} = \mu_{q+1}$, it is easy to see from (1.5) that F (and henceforth the kernel $\mathbb{K}$) remains unchanged. We verify directly from (1.15) that

$$C\big((r+1;\nu_1,\ldots,\nu_r,\nu_{r+1}),(q+1;\mu_1,\ldots,\mu_q,\nu_{r+1})\big) = C\big((r;\nu_1,\ldots,\nu_r),(q;\mu_1,\ldots,\mu_q)\big),$$

which is consistent with this observation.

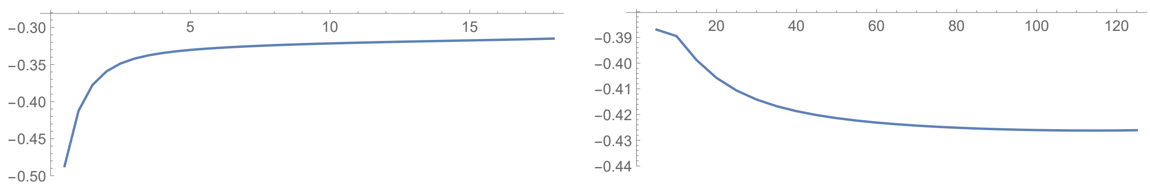

Figure 2: Numerical confirmations of Theorem 1.1.

Numerical confirmations. The all-order expansion (1.13) implies in particular that the function

$$s \mapsto s^{\rho}\Big(\ln\det\big(1-\mathbb{K}|_{[0,s]}\big) - \Big[-as^{2\rho}+bs^{\rho}+c\ln s+\ln C\Big]\Big) \tag{1.26}$$

converges to a constant C_1 as $s \to +\infty$. Figure 2 shows the graph of the function (1.26) in the following two cases:

$$\begin{aligned} &\text{Left:} && r=3, && q=2, && \nu_1=1.31, && \nu_2=2.15, && \nu_3=3.19, && \mu_1=1.87, && \mu_2=2.61, \\ &\text{Right:} && r=4, && q=1, && \nu_1=1.31, && \nu_2=2.15, && \nu_3=2.61, && \nu_4=3.19, && \mu_1=1.87, \end{aligned}$$

which correspond to the coefficients

$$\begin{aligned} &\text{Left:} && \rho=\frac{1}{2}, && a=1, && b=4.34, && c\approx-1.551, && \ln C\approx-2.963, \\ &\text{Right:} && \rho=\frac{1}{4}, && a=\frac{4}{\sqrt{3}}, && b\approx 12.97, && c\approx-2.437, && \ln C\approx-10.097. \end{aligned}$$

For these two choices of the parameters, we observe in Figure 2 that the function (1.26) seems indeed to converge to a constant as $s \to +\infty$. These observations support the validity of (1.11), (1.12), (1.14) and Theorem 1.1, and even slightly more than that, since they also suggest that the first error term in the large s asymptotics of $\ln \det \left(1 - \mathbb{K}|_{[0,s]}\right)$ is of order $s^{-\rho}$, which is consistent with (1.13). It also follows from (1.13) that

$$\widehat{C}(s) := \det \left(1 - \mathbb{K}|_{[0,s]}\right) e^{-(-as^{2\rho}+bs^{\rho}+c\ln s)} \to C, \qquad \text{as } s \to +\infty.$$

We collect here the numerical values of $\ln C$ and $\ln \widehat{C}(s)$ for five choices of the parameters.

$\ln C$	$\ln \widehat{C}(s)$	parameters
≈ -2.96	≈ -3.03	$r = 3,\ q = 2,\ \nu_1 = 1.31,\ \nu_2 = 2.15,\ \nu_3 = 3.19,\ \mu_1 = 1.87,\ \mu_2 = 2.61,\ s = 18$
≈ -10.097	≈ -10.226	$r = 4,\ q = 1,\ \nu_1 = 1.31,\ \nu_2 = 2.15,\ \nu_3 = 2.61,\ \nu_4 = 3.19,\ \mu_1 = 1.87, s = 120$
≈ -4.877	≈ -4.863	$r = 4,\ q = 1,\ \nu_1 = 0.9,\ \nu_2 = 1.17,\ \nu_3 = 1.44,\ \nu_4 = 1.77,\ \mu_1 = 1.29, s = 40$
≈ -3.693	≈ -3.684	$r = 3,\ q = 1,\ \nu_1 = 0.81,\ \nu_2 = 1.45,\ \nu_3 = 1.71,\ \mu_1 = 1.17, s = 32$
≈ -3.919	≈ -3.972	$r = 2,\ q = 1,\ \nu_1 = 1.21,\ \nu_2 = 3.45,\ \mu_1 = 1.87, s = 30$

These numerical checks have been made using the linear algebra package for Fredholm determinants of Bornemann [13].

1.3 Outline of the proof

The expressions (1.11), (1.12) and (1.14) for the coefficients ρ, a, b and c, as well as the all-order expansion (1.13), were obtained in [16, 15] via a method that we briefly explain here. There is a standard procedure, named after Its, Izergin, Korepin and Slavnov (IIKS) [27], which expresses the logarithmic derivative[2] of the Fredholm determinant of an integrable kernel in terms of the solution of a Riemann–Hilbert (RH) problem. The kernel $\mathbb{K}$ given in (1.4) is not integrable (in general), but it was shown in [16] that the IIKS procedure can still be applied (this fact is far from obvious—it is based on ideas from [8, 9] and uses the Mellin transform, see [16] for details). Using the IIKS procedure, the authors of [16] were able to express

$$\partial_s \ln \det \left(1 - \mathbb{K}|_{[0,s]}\right) \tag{1.27}$$

in terms of the solution Y of a 2×2 matrix Riemann–Hilbert (RH) problem. A Deift/Zhou [18] nonlinear steepest descent analysis of this RH problem yields an all-order expansion of Y as $s \to +\infty$ [16, 15], and hence also of (1.27). The constants ρ, a, b and c of (1.11), (1.12) and (1.14) were then obtained after integrating in s the large s asymptotics of (1.27). However, with this method, C appears as an integration constant, and therefore remains undetermined.

The considerations above show that the determination of C is an essentially different problem than finding the other constants in the large s asymptotics of $\det(1 - \mathbb{K}|_{[0,s]})$. The general strategy of the present work consists of using different, and more complicated, differential identities than (1.27). Similar strategies were used in some of the works mentioned below (1.14), and also in [26, 12] (although in [26, 12] the multiplicative constants were not determined).

As mentioned, we need to use different differential identities than (1.27) to evaluate C (i.e. differential identities with respect to other parameters than s). The large gap asymptotics for the Bessel point process is known up to and including the constant (see (1.9)), so the idea is to find a path in the set of parameters r, q, $\nu_1, \ldots, \nu_r$, $\mu_1, \ldots, \mu_q$ which interpolates smoothly between the Bessel kernel and $\mathbb{K}$. The existence of such a path is a priori not clear, since the parameters r and q are integers. The simple, but central idea of this paper is to first set the parameters $\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q$ associated to $\mathbb{K}$ equal to $\nu_{\min}$, where

$$\nu_{\min} = \min\{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\},$$

and "smooth" the product of Gamma functions in (1.5) by considering the following kernel:

$$\mathbb{K}_r(x,y) = \int_\gamma \frac{du}{2\pi i} \int_{\tilde{\gamma}} \frac{dv}{2\pi i} \frac{F_r(u)}{F_r(v)} \frac{x^{-u} y^{v-1}}{v-u}, \tag{1.28}$$

[2]The derivative can be taken with respect to any given parameter of the associated kernel, as long as the kernel depends smoothly on this parameter.

where $r \geq 1$ and $\nu > -1$ are real-valued parameters and

$$F_r(z) = \frac{\Gamma(z)}{\Gamma(1+\nu-z)^r}, \tag{1.29}$$

and where we choose the branch such that $F_r(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, 0]$. Starting with $r = 1$ (note that $\mathbb{K}_r$ reduces to the Bessel kernel for $r = 1$), we first increase r from 1 to $r - q$ and then successively move each of the remaining parameters from $\nu_{\min}$ to its desired value. The process relies on the successive integration of appropriate differential identities for the following quantities:

$$\partial_r \ln\det\left(1 - \mathbb{K}_r|_{[0,s]}\right), \tag{1.30}$$
$$\partial_{\nu_\ell} \ln\det\left(1 - \mathbb{K}|_{[0,s]}\right), \quad \ell \in \{1, \ldots, r\}, \tag{1.31}$$
$$\partial_{\mu_\ell} \ln\det\left(1 - \mathbb{K}|_{[0,s]}\right), \quad \ell \in \{1, \ldots, q\}. \tag{1.32}$$

More precisely, let us define $\mathbb{K}^{(\ell)}$, $\ell \in \{0, 1, \ldots, r, r+1, \ldots, r+q\}$ by

$$\mathbb{K}^{(\ell)} = \begin{cases} \mathbb{K}|_{\nu_{\ell+1}=\ldots=\nu_r=\mu_1=\ldots=\mu_q=\nu_{\min}}, & \text{if } \ell \in \{0, ..., r-1\}, \\ \mathbb{K}|_{\mu_{\ell-r+1}=\ldots=\mu_q=\nu_{\min}}, & \text{if } \ell \in \{r, ...r+q\}. \end{cases} \tag{1.33}$$

Note that $\mathbb{K}^{(0)} = \mathbb{K}_{r-q}$, where $\mathbb{K}_{r-q}$ is defined by (1.28) (with r replaced by $r-q$ and ν by $\nu_{\min}$), and $\mathbb{K}^{(r+q)} = \mathbb{K}$. By integrating successively (1.30), (1.31) and (1.32), we obtain

$$\begin{aligned}
&\ln\det\left(1 - \mathbb{K}_{r-q}|_{[0,s]}\right) = \ln\det\left(1 - \mathbb{K}_{r=1}|_{[0,s]}\right) + \int_1^{r-q} \partial_{r'} \ln\det\left(1 - \mathbb{K}_{r'}|_{[0,s]}\right) dr', \\
&\ln\det\left(1 - \mathbb{K}^{(\ell)}|_{[0,s]}\right) = \ln\det\left(1 - \mathbb{K}^{(\ell-1)}|_{[0,s]}\right) + \int_{\nu_{\min}}^{\nu_\ell} \partial_{\nu_\ell'} \ln\det\left(1 - \mathbb{K}^{(\ell)}|_{[0,s]}\right) d\nu_\ell', \quad \ell = 1, ..., r, \\
&\ln\det\left(1 - \mathbb{K}^{(r+\ell)}|_{[0,s]}\right) = \ln\det\left(1 - \mathbb{K}^{(r+\ell-1)}|_{[0,s]}\right) + \int_{\nu_{\min}}^{\mu_\ell} \partial_{\mu_\ell'} \ln\det\left(1 - \mathbb{K}^{(r+\ell)}|_{[0,s]}\right) d\mu_\ell', \quad \ell = 1, ..., q.
\end{aligned} \tag{1.34}$$

Since the large s asymptotics of $\ln\det\left(1 - \mathbb{K}_{r=1}|_{[0,s]}\right)$ are known up to and including the constant term, see (1.9), this method allows us to obtain C by keeping track of the term of order 1 in the identities (1.34).

Remark 1.2. *In this work, we focus on proving the expression* (1.15) *for* C*. We could also have computed the coefficients* ρ*,* a*,* b *and* c *with the same method (this would have provided an alternative proof and another consistency check for these constants), but these constants are already known [16, 15], and in order to limit the complexity and length of the paper, we have decided to not pursue this direction.*

1.4 Organization of the paper

By employing the IIKS procedure, we will express the quantities (1.30), (1.31) and (1.32) in terms of Y. Since we use the same RH problem as in [16, 15], we can recycle some of the analysis of these papers. We present the necessary material from [16, 15] in Section 2. In Section 3, we express the quantities (1.30), (1.31) and (1.32) in terms of Y. Section 4 is devoted to the first differential identity (1.30). More precisely, we compute the large s asymptotics of (1.30) up to and including the constant term and then perform the integration with respect to r' in (1.34). In Section 5, we proceed similarly with the differential identities with respect to ν_i, $i = 1, ..., r$ and μ_j, $j = 1, ...q$. The proof of Theorem 1.1 is completed in Section 6.

Acknowledgements

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2 Background from [16, 15]

In this section we recall some results from [16, 15] that will be used throughout the paper. The notation adopted in this paper is the same as in [15], and is also almost identical to the notation used in [16]. The only difference with [16] is that the function denoted by $\mathcal{G}$ in this paper and in [15] is instead denoted by G in [16]. In this paper, as well as in [15], G denotes Barnes' G-function.[3]

We start by stating the RH problem for Y.

RH problem for Y

(a) $Y : \mathbb{C} \setminus (\gamma \cup \tilde{\gamma}) \to \mathbb{C}^{2\times 2}$ is analytic, where γ and $\tilde{\gamma}$ are the contours shown in Figure 1.

(b) Y has continuous boundary values Y_+ and Y_- on $\gamma \cup \tilde{\gamma}$ from the left $(+)$ side and right $(-)$ side of $\gamma \cup \tilde{\gamma}$, respectively, and obeys the jump relations

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & -s^{-z}F(z) \\ 0 & 1 \end{pmatrix}, \qquad \text{if } z \in \gamma, \tag{2.1}$$

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & 0 \\ s^{z}F(z)^{-1} & 1 \end{pmatrix}, \qquad \text{if } z \in \tilde{\gamma}, \tag{2.2}$$

where F is defined as in (1.5).

(c) Y admits an expansion of the form

$$Y(z) = I + \frac{Y_1(s)}{z} + \mathcal{O}\big(z^{-2}\big), \qquad \text{as } z \to \infty, \tag{2.3}$$

where Y_1 is a 2×2 matrix that depends on s but not on z.

It is shown in [16] that the solution of the RH problem for Y exists and is unique for all $s \in \mathbb{R}$. In the steepest descent analysis of the RH problem, the authors of [16] introduce a sequence of transformations $Y \mapsto U \mapsto T \mapsto S \mapsto R$, where R is the solution of a small norm RH problem. We only recall here what is needed for the proof of Theorem 1.1, and refer to [16, 15] for more details. Let us choose the branch for $\ln F$ such that

$$\ln F(z) = \ln\Gamma(z) + \sum_{k=1}^{q} \ln\Gamma(1+\mu_k - z) - \sum_{j=1}^{r} \ln\Gamma(1+\nu_j - z), \tag{2.4}$$

and the branches on the right-hand-side of (2.4) are the principal ones. Then $z \mapsto \ln F(z)$ is analytic for $z \in \mathbb{C} \setminus \big((-\infty, 0] \cup [1+\nu_{\min}, +\infty)\big)$. The first transformation $Y \mapsto U$ involves the change of variables

$$z(\zeta) = is^{\rho}\zeta + \frac{1+\nu_{\min}}{2},$$

where we recall that $\nu_{\min} = \min\{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\}$. The asymptotics of $\ln F(z(\zeta))$ as $s^{\rho}\zeta \to \infty$ are

$$\begin{aligned} \ln F\left(is^{\rho}\zeta + \frac{1+\nu_{\min}}{2}\right) &= is^{\rho}\zeta\ln(s) + is^{\rho}\big(c_1\zeta\ln(i\zeta) + c_2\zeta\ln(-i\zeta) + c_3\zeta\big) \\ &\quad + c_4\ln(s) + c_5\ln(i\zeta) + c_6\ln(-i\zeta) + c_7 + \frac{c_8}{is^{\rho}\zeta} + \mathcal{O}\left(\frac{1}{s^{2\rho}\zeta^2}\right), \end{aligned}$$

where the constants $c_1, \ldots, c_8$ are given by

$$\begin{aligned} & c_1 = 1, \qquad c_2 = r-q, \qquad c_3 = -(r-q+1) \qquad c_4 = \frac{\nu_{\min}}{2} + \frac{1}{1+r-q}\left(\sum_{k=1}^{q}\mu_k - \sum_{j=1}^{r}\nu_j\right), \\ & c_5 = \frac{\nu_{\min}}{2}, \qquad c_6 = (r-q)\frac{\nu_{\min}}{2} + \sum_{k=1}^{q}\mu_k - \sum_{j=1}^{r}\nu_j, \qquad c_7 = \frac{1+q-r}{2}\ln(2\pi), \\ & c_8 = \frac{1+r-q}{8}\left(\nu_{\min}^2 - \frac{1}{3}\right) - \frac{1}{2}\left(\sum_{k=1}^{q}\mu_k^2 - \sum_{j=1}^{r}\nu_j^2\right) + \frac{\nu_{\min}}{2}\left(\sum_{k=1}^{q}\mu_k - \sum_{j=1}^{r}\nu_j\right). \end{aligned} \tag{2.5}$$

[3] If G has subscripts and superscripts, such as $G^{m,n}_{p,q}$, then it denotes the Meijer G-function.

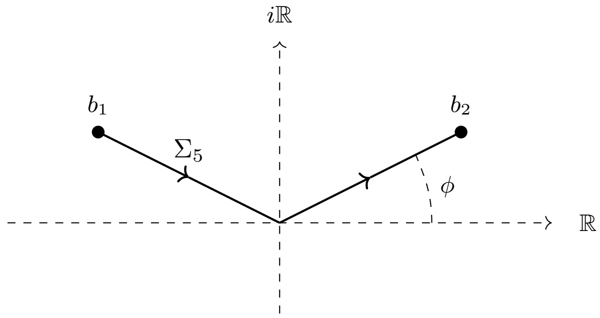

Figure 3: The points $b_2 = |b_2|e^{i\phi}$ and $b_1 = -\overline{b_2}$ and the contour $\Sigma_5 = [b_1, 0] \cup [0, b_2]$.

We define the function $\mathcal{G}$ by

$$\mathcal{G}(\zeta) = F\left(is^\rho\zeta + \frac{1+\nu_{\min}}{2}\right)e^{-is^\rho(\zeta \ln s - h(\zeta))}, \tag{2.6}$$

where $h(\zeta) = -\zeta(c_1 \ln(i\zeta) + c_2 \ln(-i\zeta) + c_3)$. Note that the function $\mathcal{G}(\zeta)$ also depends on the parameters s, r, q, ν_j, and μ_k, although this is not indicated in the notation. The $T \mapsto S$ transformation utilizes a g-function $\zeta \mapsto g(\zeta)$ which is analytic for $\zeta \in \mathbb{C} \setminus \Sigma_5$, where Σ_5 is the union of two segments

$$\Sigma_5 = [b_1, 0] \cup [0, b_2], \tag{2.7}$$

oriented from left to right, see Figure 3. The points b_1 and b_2 which characterize Σ_5 are defined by

$$b_2 = -\overline{b_1} = |b_2|e^{i\phi}, \quad \phi \in \left[0, \tfrac{\pi}{2}\right), \tag{2.8}$$

where

$$\begin{aligned} \operatorname{Re} b_2 = -\operatorname{Re} b_1 &= 2\left(\frac{c_2}{c_1}\right)^{\frac{c_1-c_2}{2(c_1+c_2)}} e^{-\frac{c_1+c_2+c_3}{c_1+c_2}} = 2(r-q)^{\frac{1-r+q}{2(1+r-q)}}, \\ \sin\phi &= \frac{c_2-c_1}{c_2+c_1} = \frac{r-q-1}{r-q+1} \in [0,1). \end{aligned}$$

The g-function is defined via its second derivative given by

$$g''(\zeta) = -i\frac{c_1+c_2}{2}\left(\frac{1}{\zeta} - \frac{1}{r(\zeta)} + \frac{i\operatorname{Im} b_2}{\zeta r(\zeta)}\right), \tag{2.9}$$

where $\zeta \mapsto r(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \Sigma_5$ and is defined by

$$r(\zeta) = \sqrt{(\zeta-b_1)(\zeta-b_2)}, \tag{2.10}$$

where the branch is fixed such that $r(\zeta) \sim \zeta$ as $\zeta \to \infty$. For $\zeta \in \Sigma_5$, one has $r_+(\zeta) + r_-(\zeta) = 0$. It can be shown that $g''(\zeta) = \mathcal{O}(\zeta^{-3})$ as $\zeta \to \infty$, and then g' and g are defined by

$$g'(\zeta) = \int_\infty^\zeta g''(\xi)d\xi, \qquad g(\zeta) = \int_\infty^\zeta g'(\xi)d\xi, \tag{2.11}$$

where the path of integration lies in $\mathbb{C} \setminus \Sigma_5$. The RH problem for S has exponentially decaying jumps outside Σ_5, and one needs to construct approximations to S in different regions of the complex plane. Let $\mathbb{D}_\delta(b_1)$ and $\mathbb{D}_\delta(b_2)$ denote two disks of sufficiently small radius $\delta > 0$ centered at b_1 and b_2, respectively. For $\zeta \in \mathbb{C} \setminus (\mathbb{D}_\delta(b_1) \cup \mathbb{D}_\delta(b_2))$, S is approximated by a so-called global parametrix $P^{(\infty)}$, while for $\zeta \in \mathbb{D}_\delta(b_1) \cup \mathbb{D}_\delta(b_2)$, S is approximated by local parametrices P that are defined in terms of Airy functions. We omit the exact definition of P here, the interested reader can find it in [16, 15]. The construction of $P^{(\infty)}$ is given in terms of a function p that will be important for us and which is defined by

$$p(\zeta) = -\frac{r(\zeta)}{2\pi i}\int_{\Sigma_5} \frac{\ln \mathcal{G}(\xi)}{r_+(\xi)} \frac{d\xi}{\xi-\zeta}, \tag{2.12}$$

where the branch of $\ln \mathcal{G}$ is such that

$$\ln \mathcal{G}(\zeta) = \ln F\left(is^{\rho}\zeta + \frac{1+\nu_{\min}}{2}\right) - is^{\rho}(\zeta \ln s - h(\zeta)).$$

The function p has the following jumps across Σ_5:

$$p_+(\zeta) + p_-(\zeta) = -\ln \mathcal{G}(\zeta), \qquad \zeta \in \Sigma_5, \tag{2.13}$$

and the following asymptotics as $\zeta \to \infty$:

$$p(\zeta) = p_0 + \mathcal{O}(\zeta^{-1}), \qquad \text{where} \qquad p_0 = \frac{1}{2\pi i}\int_{\Sigma_5} \frac{\ln \mathcal{G}(\xi)}{r_+(\xi)} d\xi.$$

The global parametrix $P^{(\infty)}(\zeta)$ is defined by [16, Eq. (3.51)]

$$P^{(\infty)}(\zeta) = e^{-p_0\sigma_3} Q^{(\infty)}(\zeta) e^{p(\zeta)\sigma_3} \quad \text{with} \quad Q^{(\infty)}(\zeta) = \frac{1}{2}\begin{pmatrix} \frac{\gamma(\zeta)+\gamma(\zeta)^{-1}}{2} & \frac{\gamma(\zeta)-\gamma(\zeta)^{-1}}{2i} \\ \frac{\gamma(\zeta)-\gamma(\zeta)^{-1}}{-2i} & \frac{\gamma(\zeta)+\gamma(\zeta)^{-1}}{2} \end{pmatrix}, \tag{2.14}$$

where the branch of the function

$$\gamma(\zeta) = \left(\frac{\zeta - b_1}{\zeta - b_2}\right)^{\frac{1}{4}}$$

is chosen such that $\gamma(\zeta)$ is analytic on $\mathbb{C} \setminus \Sigma_5$ and $\gamma(\zeta) \sim 1$ as $\zeta \to \infty$. The solution of the RH problem for R is then given by [16] (see also [15, Eq. (2.32)])

$$R(\zeta) = e^{p_0\sigma_3} S(\zeta) \times \begin{cases} P(\zeta)^{-1} e^{-p_0\sigma_3}, & \zeta \in \mathbb{D}_\delta(b_1) \cup \mathbb{D}_\delta(b_2), \\ P^{(\infty)}(\zeta)^{-1} e^{-p_0\sigma_3}, & \zeta \in \Gamma_R \setminus (\mathbb{D}_\delta(b_1) \cup \mathbb{D}_\delta(b_2)). \end{cases} \tag{2.15}$$

Let $\{\Sigma_j\}_1^4$ denote the contours

$$\Sigma_2 = -\overline{\Sigma_1} = b_2 + e^{i(\phi+\epsilon)}\mathbb{R}_{\geq 0}, \qquad \Sigma_4 = -\overline{\Sigma_3} = b_2 + e^{-i\epsilon}\mathbb{R}_{\geq 0}$$

for some fixed $\epsilon \in (0, \frac{\pi}{10})$, with the orientation from left to right, and define

$$\tilde{\Sigma}_j = \Sigma_j \setminus (\mathbb{D}_\delta(b_1) \cup \mathbb{D}_\delta(b_2)), \qquad j = 1, ..., 5. \tag{2.16}$$

The function R defined in (2.15) is analytic for $\zeta \in \mathbb{C} \setminus \Gamma_R$, where

$$\Gamma_R = \big(\partial\mathbb{D}_\delta(b_1) \cup \partial\mathbb{D}_\delta(b_2)\big) \cup \bigcup_{j=1}^{5} \tilde{\Sigma}_j,$$

and $\partial\mathbb{D}_\delta(b_1)$ and $\partial\mathbb{D}_\delta(b_2)$ are oriented clockwise, see Figure 4.

Remark 2.1. *The jumps for Y depend on F. If F is replaced by F_r in (2.1)-(2.2), then the steepest descent analysis of Y has not been carried out in [16]. However, it is not hard to see that the same analysis applies also in this case (with $\nu_{\min} = \nu$); the only difference is that the coefficients (2.5) are replaced by*

$$c_1 = 1, \qquad c_2 = r, \qquad c_3 = -(r+1), \qquad c_4 = \nu\left(\frac{1}{2} - \frac{r}{1+r}\right),$$
$$c_5 = \frac{\nu}{2}, \qquad c_6 = -\frac{r\nu}{2}, \qquad c_7 = \frac{1-r}{2}\ln(2\pi), \qquad c_8 = \frac{1+r}{8}\left(\nu^2 - \frac{1}{3}\right). \tag{2.17}$$

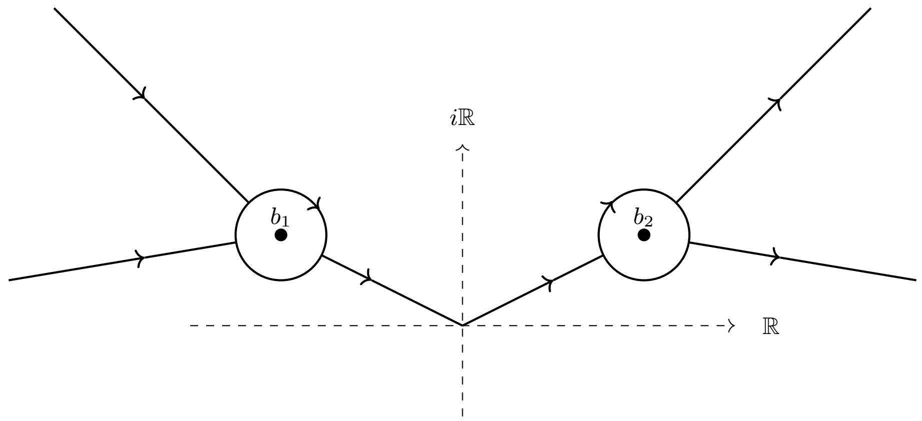

Figure 4: The contour Γ_R.

3 Differential identities in r, ν_ℓ, and μ_ℓ

In this section, we express the logarithmic derivatives (1.30), (1.31) and (1.32) in terms of the RH problem for Y via the IIKS procedure. By combining [16, Propositions 2.1 and 2.2] and [9, Theorem 2.1], we obtain

$$\partial_r \ln\det\big(1-\mathbb{K}_r|_{[0,s]}\big) = \int_{\gamma\cup\tilde\gamma} \mathrm{Tr}[Y_-^{-1}(z)Y_-'(z)\partial_r J(z)J^{-1}(z)]\frac{dz}{2\pi i}, \tag{3.1}$$

$$\partial_{\nu_\ell} \ln\det\big(1-\mathbb{K}|_{[0,s]}\big) = \int_{\gamma\cup\tilde\gamma} \mathrm{Tr}[Y_-^{-1}(z)Y_-'(z)\partial_{\nu_\ell} J(z)J^{-1}(z)]\frac{dz}{2\pi i}, \quad \ell\in\{1,\dots,r\}, \tag{3.2}$$

$$\partial_{\mu_\ell} \ln\det\big(1-\mathbb{K}|_{[0,s]}\big) = \int_{\gamma\cup\tilde\gamma} \mathrm{Tr}[Y_-^{-1}(z)Y_-'(z)\partial_{\mu_\ell} J(z)J^{-1}(z)]\frac{dz}{2\pi i}, \quad \ell\in\{1,\dots,q\}, \tag{3.3}$$

where the RH solution Y in (3.1) has the jumps (2.1)-(2.2) with F replaced by F_r (defined in (1.29)). The quantity $(\partial_r J)J^{-1}$ in (3.1) is given by

$$\partial_r J(z)J^{-1}(z) = \ln\Gamma(1+\nu-z)\big(J(z)-I\big)\sigma_3. \tag{3.4}$$

In (3.2)-(3.3), Y satisfies the jumps (2.1)-(2.2) with F given by (1.5), and we have

$$\partial_{\nu_\ell} J(z)J^{-1}(z) = \psi(1+\nu_\ell-z)\big(J(z)-I\big)\sigma_3, \qquad \ell\in\{1,\dots,r\}, \tag{3.5}$$

$$\partial_{\mu_\ell} J(z)J^{-1}(z) = -\psi(1+\mu_\ell-z)\big(J(z)-I\big)\sigma_3, \qquad \ell\in\{1,\dots,q\}, \tag{3.6}$$

where $\psi=(\ln\Gamma)'$ denotes the di-gamma function. The same arguments as in the proof of [15, Lemma 6.1] apply here (so we do not provide details), and we obtain

$$\partial_r \ln\det\big(1-\mathbb{K}_r|_{[0,s]}\big) = \frac{1}{2}\int_{\gamma\cup\tilde\gamma} \ln\Gamma(1+\nu-z)\,\mathrm{Tr}[Y_+^{-1}(z)Y_+'(z)\sigma_3 - Y_-^{-1}(z)Y_-'(z)\sigma_3]\frac{dz}{2\pi i}, \tag{3.7}$$

$$\partial_{\nu_\ell} \ln\det\big(1-\mathbb{K}|_{[0,s]}\big) = \frac{1}{2}\int_{\gamma\cup\tilde\gamma} \psi(1+\nu_\ell-z)\,\mathrm{Tr}[Y_+^{-1}(z)Y_+'(z)\sigma_3 - Y_-^{-1}(z)Y_-'(z)\sigma_3]\frac{dz}{2\pi i}, \tag{3.8}$$

$$\partial_{\mu_\ell} \ln\det\big(1-\mathbb{K}|_{[0,s]}\big) = -\frac{1}{2}\int_{\gamma\cup\tilde\gamma} \psi(1+\mu_\ell-z)\,\mathrm{Tr}[Y_+^{-1}(z)Y_+'(z)\sigma_3 - Y_-^{-1}(z)Y_-'(z)\sigma_3]\frac{dz}{2\pi i}, \tag{3.9}$$

where $\ell\in\{1,\dots,r\}$ in (3.8) and $\ell\in\{1,\dots,q\}$ in (3.9).

Using the chain of transformations $Y\mapsto U\mapsto T\mapsto S\mapsto R$ in the steepest descent analysis of [16], we rewrite the differential identities (3.7), (3.8), and (3.9) in a way that is more convenient for the asymptotic analysis as $s\to+\infty$. Recall that the transformation $Y\mapsto U$ involves the change of variables $z=i\zeta s^\rho+(1+\nu_{\min})/2$ (and that $\nu_{\min}=\nu$ in (3.7), see also Remark 2.1). The functions

$$\ln\Gamma\left(\frac{1+\nu}{2}-i\zeta s^\rho\right), \quad \psi\left(\frac{1+2\nu_\ell-\nu_{\min}}{2}-i\zeta s^\rho\right), \quad \text{and} \quad \psi\left(\frac{1+2\mu_\ell-\nu_{\min}}{2}-i\zeta s^\rho\right) \tag{3.10}$$

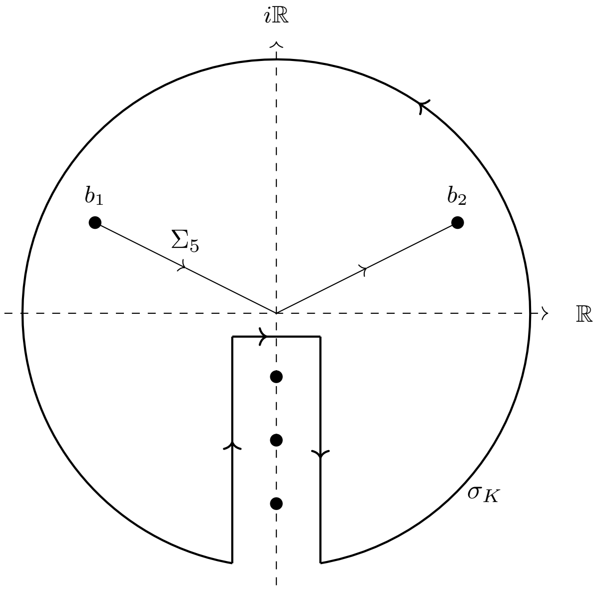

Figure 5: The contour σ_K surrounds Σ_5 but does not enclose any of the poles of the functions in (3.10).

appearing in (3.7), (3.8) and (3.9) have infinitely many poles on $i\mathbb{R}^-$. These poles depend on s and approach 0 as $s \to +\infty$. For example, the left-most function in (3.10) has simple poles at $\{\zeta_j\}_{j=0}^{+\infty} \subset i\mathbb{R}^-$, where

$$\zeta_j = \frac{1}{is^\rho}\left(\frac{1+\nu}{2} + j\right), \qquad j = 0, 1, \ldots$$

Following [15], we define for $K > |b_2|$ the contour σ_K as given in Figure 5. The contour σ_K surrounds Σ_5 in the positive direction in such a way that the poles of (3.10) lie in the region exterior to σ_K. The circular part of σ_K has radius K. We choose the contour σ_K to cross the imaginary axis at the point $\zeta_0/2$ and to have a horizontal part of constant length as s changes. Note that, since the poles of (3.10) approach 0 as $s \to +\infty$, the contour σ_K depends on s, even if K is independent of s. We define $\sigma = \sigma_{2|b_2|}$. Furthermore, we define the contour

$$\tilde{\Sigma}_K = \bigcup_{j=1}^{4} \tilde{\Sigma}_j \setminus \{|\zeta| \le K\},$$

where the contours $\tilde{\Sigma}_j$ are defined by (2.16).

Lemma 3.1 (Differential identities)**.** *Let K be such that $K > 2|b_2|$. Then the following statements hold:*

(a) Let $r \geq 1$ and $\nu = \nu_{\min} > -1$. Then

$$\partial_r \ln\det\left(1 - \mathbb{K}_r|_{[0,s]}\right) = I_{1,r} + I_{2,r} + I_{3,r}(K) + I_{4,r}(K), \tag{3.11}$$

where

$$I_{1,r} = -s^\rho \int_\sigma \ln\Gamma\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)\, g'(\zeta)\frac{d\zeta}{2\pi i}, \tag{3.12}$$

$$I_{2,r} = -\frac{1}{2}\int_\sigma \ln\Gamma\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)\,\mathrm{Tr}\left[P^{(\infty)}(\zeta)^{-1}P^{(\infty)}(\zeta)'\sigma_3\right]\frac{d\zeta}{2\pi i}, \tag{3.13}$$

$$I_{3,r}(K) = -\frac{1}{2}\int_{\sigma_K} \ln\Gamma\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)\,\mathrm{Tr}\left[P^{(\infty)}(\zeta)^{-1}e^{-p_0\sigma_3}R^{-1}(\zeta)R'(\zeta)e^{p_0\sigma_3}P^{(\infty)}(\zeta)\sigma_3\right]\frac{d\zeta}{2\pi i}, \tag{3.14}$$

$$I_{4,r}(K) = \frac{1}{2}\int_{\tilde{\Sigma}_K} \ln\Gamma\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)\times$$
$$\mathrm{Tr}\left[P^{(\infty)}(\zeta)^{-1}e^{-p_0\sigma_3}\big(R_+^{-1}(\zeta)R_+'(\zeta) - R_-^{-1}(\zeta)R_-'(\zeta)\big)e^{p_0\sigma_3}P^{(\infty)}(\zeta)\sigma_3\right]\frac{d\zeta}{2\pi i}. \tag{3.15}$$

(b) Let $r > q \geq 0$ be integers. Let $\ell \in \{1,\ldots,r\}$ and $k \in \{1,\ldots,q\}$. Then

$$\partial_{\nu_\ell} \ln\det\left(1 - \mathbb{K}|_{[0,s]}\right) = I_{1,\nu_\ell} + I_{2,\nu_\ell} + I_{3,\nu_\ell}(K) + I_{4,\nu_\ell}(K), \tag{3.16}$$

$$\partial_{\mu_k} \ln\det\left(1 - \mathbb{K}|_{[0,s]}\right) = -I_{1,\mu_k} - I_{2,\mu_k} - I_{3,\mu_k}(K) - I_{4,\mu_k}(K), \tag{3.17}$$

where, for $\alpha \in \{\nu_\ell, \mu_k\}$,

$$I_{1,\alpha} = -\,s^\rho\int_\sigma \psi\Big(\frac{1+2\alpha-\nu_{\min}}{2} - is^\rho\zeta\Big)\, g'(\zeta)\frac{d\zeta}{2\pi i}, \tag{3.18}$$

$$I_{2,\alpha} = -\,\frac{1}{2}\int_\sigma \psi\Big(\frac{1+2\alpha-\nu_{\min}}{2} - is^\rho\zeta\Big)\,\mathrm{Tr}\left[P^{(\infty)}(\zeta)^{-1}P^{(\infty)}(\zeta)'\sigma_3\right]\frac{d\zeta}{2\pi i}, \tag{3.19}$$

$$I_{3,\alpha}(K) = -\,\frac{1}{2}\int_{\sigma_K} \psi\Big(\frac{1+2\alpha-\nu_{\min}}{2} - is^\rho\zeta\Big)$$
$$\times\,\mathrm{Tr}\left[P^{(\infty)}(\zeta)^{-1}e^{-p_0\sigma_3}R^{-1}(\zeta)R'(\zeta)e^{p_0\sigma_3}P^{(\infty)}(\zeta)\sigma_3\right]\frac{d\zeta}{2\pi i}, \tag{3.20}$$

$$I_{4,\alpha}(K) = \frac{1}{2}\int_{\tilde{\Sigma}_K} \psi\Big(\frac{1+2\alpha-\nu_{\min}}{2} - is^\rho\zeta\Big)\,\mathrm{Tr}\Big[P^{(\infty)}(\zeta)^{-1}e^{-p_0\sigma_3}$$
$$\times\,\big(R_+^{-1}(\zeta)R_+'(\zeta) - R_-^{-1}(\zeta)R_-'(\zeta)\big)e^{p_0\sigma_3}P^{(\infty)}(\zeta)\sigma_3\Big]\frac{d\zeta}{2\pi i}. \tag{3.21}$$

Proof. The proof is analogous to the proof of [15, Lemma 6.2] and consists of implementing the chain of transformations $Y \mapsto U \mapsto T \mapsto S \mapsto R$ in (3.7), (3.8), and (3.9), and performing a contour deformation. □

Sections 4 and 5 are devoted to the computation of the constant terms in the large s asymptotics of the right-hand sides of (3.11), (3.16)–(3.17) and to the integration of these identities. The proof of Theorem 1.1 is then given in Section 6.

4 Asymptotics of the differential identity in r

In this section, we compute the large s asymptotics of the four quantities $I_{1,r}$, $I_{2,r}$, $I_{3,r}(K)$, and $I_{4,r}(K)$ appearing on the right-hand side of the differential identity (3.11) in r. By integrating the resulting asymptotics with respect to r, we obtain the constant term in the large s asymptotics of $\ln\det\big(1 - \mathbb{K}_r|_{[0,s]}\big)$.

Throughout this section, we assume that $r \geq 1$ and $\nu > -1$. The quantities $c_1,\ldots,c_8$ and b_1, b_2 are defined by (2.17) and (2.8), respectively. As mentioned in Remark 1.2, we focus in this work on proving the expression (1.15) for C, because the coefficients ρ, a, b and c are already known from [16, 15]. Therefore, to avoid unnecessary computations, we introduce the notation Ω.

Notation. Let $t \in \mathbb{R}$ and $f, g : (t, \infty) \to \mathbb{C}$. The notation

$$f(s) = \Omega\big(g(s)\big), \qquad \text{as } s \to +\infty,$$

means that either $f \equiv 0$ or that there exist $c > 0$ and $s_0 > 0$ independent of s such that

$$|f(s)| \geq c|g(s)|, \qquad \text{for all } s \geq s_0.$$

4.1 Asymptotics of $I_{1,r}$

Proposition 4.1. *Let $\nu > -1$ and let $I_{1,r}$ be the function defined by (3.11). Then*

$$I_{1,r} = \Omega(\ln(s^\rho)) + I_{1,r}^{(c)} + \mathcal{O}\bigg(\frac{\ln(s^\rho)}{s^\rho}\bigg) \tag{4.1}$$

as $s \to +\infty$ uniformly for r in compact subsets of $[1, +\infty)$, where

$$\begin{aligned} I_{1,r}^{(c)} = {} & \frac{c_1 + c_2}{2}\bigg\{\frac{1 - 3\nu^2}{24} + \zeta'(-1) - \ln G\bigg(\frac{\nu+1}{2}\bigg) + \frac{\nu - 1}{2}\ln\Gamma\bigg(\frac{\nu+1}{2}\bigg)\bigg\}\bigg(1 - \frac{\operatorname{Im} b_2}{|b_2|}\bigg) \\ & + \frac{c_1 + c_2}{48}(1 - 3\nu^2)\bigg(\bigg(1 + \frac{\operatorname{Im} b_2}{|b_2|}\bigg)\ln\bigg(\frac{|b_2| + \operatorname{Im} b_2}{2}\bigg) - 2\frac{\operatorname{Im} b_2}{|b_2|}\ln|b_2|\bigg). \end{aligned} \tag{4.2}$$

Proof. We define the function

$$\Psi(\zeta) = s^\rho \int_0^\zeta \ln\Gamma\bigg(\frac{1+\nu}{2} - is^\rho\xi\bigg)d\xi. \tag{4.3}$$

Then $\Psi(\zeta)$ is analytic on σ and an integration by parts and (2.9) yield

$$I_{1,r} = \int_\sigma \Psi(\zeta)g''(\zeta)\frac{d\zeta}{2\pi i} = i\frac{c_1 + c_2}{2}\int_\sigma \Psi(\zeta)\bigg(1 - i\frac{\operatorname{Im} b_2}{\zeta}\bigg)\frac{1}{r(\zeta)}\frac{d\zeta}{2\pi i},$$

where we have used the fact that $\frac{1}{\zeta}\Psi(\zeta)$ has no pole at $\zeta = 0$. We first collapse the contour σ onto Σ_5. Second, recalling that $r_+(\zeta) + r_-(\zeta) = 0$ for $\zeta \in \Sigma_5$, we rewrite the resulting integral only in terms of r_+. Finally, we deform the contour on the $+$ side of Σ_5 into another contour $\gamma_{b_2b_1}$, which is the part of the counterclockwise oriented circle with radius $|b_2|$ centered at the origin going from b_2 to b_1. We note again that, since $\frac{1}{\zeta}\Psi(\zeta)$ is analytic at $\zeta = 0$, there is no residue at $\zeta = 0$ during this contour deformation. This gives

$$I_{1,r} = i(c_1 + c_2)\int_{\gamma_{b_2b_1}} \Psi(\zeta)\bigg(1 - i\frac{\operatorname{Im} b_2}{\zeta}\bigg)\frac{1}{r(\zeta)}\frac{d\zeta}{2\pi i}. \tag{4.4}$$

It remains to compute the large s asymptotics of $\Psi(\zeta)$ uniformly for $\zeta \in \gamma_{b_2b_1}$. The following formula is useful for us (cf. [34, Eq. 5.17.4])

$$\int_1^z \ln\Gamma(z')dz' = \frac{z-1}{2}\ln(2\pi) - \frac{(z-1)z}{2} + (z-1)\ln\Gamma(z) - \ln G(z), \tag{4.5}$$

where G is Barnes' G-function. By applying (4.5) twice with

$$z = \frac{1+\nu}{2} - is^\rho\zeta \qquad \text{and} \qquad z = \frac{1+\nu}{2}$$

in (4.3), we obtain

$$\begin{aligned} \Psi(\zeta) = {} & i\int_{\frac{1+\nu}{2}}^{\frac{1+\nu}{2} - is^\rho\zeta} \ln\Gamma(z')dz' \\ = {} & \frac{i}{2}\bigg[s^{2\rho}\zeta^2 + is^\rho\zeta\big(\nu - \ln(2\pi)\big) - 2\ln\frac{G\big(\frac{1+\nu}{2} - is^\rho\zeta\big)}{G\big(\frac{1+\nu}{2}\big)} - (1-\nu)\ln\frac{\Gamma\big(\frac{1+\nu}{2} - is^\rho\zeta\big)}{\Gamma\big(\frac{1+\nu}{2}\big)} \\ & - 2is^\rho\zeta\ln\Gamma\bigg(\frac{1+\nu}{2} - is^\rho\zeta\bigg)\bigg]. \end{aligned} \tag{4.6}$$

The large z asymptotics of $\ln\Gamma(z)$ and $\ln G(z)$ are given by (cf. [34, Eqs. 5.11.1 and 5.17.5])

$$\ln G(z+1) = \frac{z^2}{4} + z\ln\Gamma(z+1) - \left(\frac{z(z+1)}{2} + \frac{1}{12}\right)\ln z - \frac{1}{12} + \zeta'(-1) + O(z^{-2}),$$
$$\ln\Gamma(z) = (z-\tfrac{1}{2})\ln z - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} + O(z^{-3}) \tag{4.7}$$

as $z\to\infty$ uniformly for $|\arg z| < \pi - \epsilon$ for some $\epsilon > 0$. This implies

$$\Psi(\zeta) = \Omega\big(\ln(s^\rho)\big) - \frac{i}{24}\bigg\{1 - 3\nu^2 - \frac{i\pi}{2} + 24\zeta'(-1) + \frac{3i\pi\nu^2}{2} - 24\ln G\Big(\frac{\nu+1}{2}\Big)$$
$$+ 12(\nu-1)\ln\Gamma\Big(\frac{\nu+1}{2}\Big) + (1-3\nu^2)\ln\zeta\bigg\} + \mathcal{O}\big(s^{-\rho}\big) \tag{4.8}$$

as $s\to+\infty$ uniformly for $\zeta\in\gamma_{b_2b_1}$ and r in compact subsets of $(0,+\infty)$. Substituting (4.8) into (4.4) gives (4.1) and, in particular,

$$I_{1,r}^{(c)} = (c_1+c_2)\bigg\{\frac{1-3\nu^2}{24}\Big(1 - \frac{\pi i}{2}\Big) + \zeta'(-1) - \ln G\Big(\frac{\nu+1}{2}\Big) + \frac{\nu-1}{2}\ln\Gamma\Big(\frac{\nu+1}{2}\Big)\bigg\}$$
$$\times\int_{\gamma_{b_2b_1}}\Big(1 - \frac{i\mathrm{Im}\,b_2}{\zeta}\Big)\frac{1}{r(\zeta)}\frac{d\zeta}{2\pi i} + \frac{c_1+c_2}{24}(1-3\nu^2)\int_{\gamma_{b_2b_1}}\ln(\zeta)\Big(1 - \frac{i\mathrm{Im}\,b_2}{\zeta}\Big)\frac{1}{r(\zeta)}\frac{d\zeta}{2\pi i}.$$

It was shown in [15, Lemma 7.2] that

$$2\int_{\gamma_{b_2b_1}}\frac{1}{r(\zeta)}\frac{d\zeta}{2\pi i} = 1, \qquad 2\int_{\gamma_{b_2b_1}}\frac{\ln(\zeta)}{r(\zeta)}\frac{d\zeta}{2\pi i} = \ln\big(i(|b_2| + \mathrm{Im}\,b_2)\big) - \ln(2),$$
$$2\int_{\gamma_{b_2b_1}}\frac{1}{\zeta r(\zeta)}\frac{d\zeta}{2\pi i} = -\frac{i}{|b_2|}, \qquad 2\int_{\gamma_{b_2b_1}}\frac{\ln(\zeta)}{\zeta r(\zeta)}\frac{d\zeta}{2\pi i} = \frac{\ln\Big(\frac{2i|b_2|^2}{|b_2|+\mathrm{Im}\,b_2}\Big)}{i|b_2|}, \tag{4.9}$$

which proves (4.2) and the proposition. □

4.2 Asymptotics of $I_{2,r}$

In this subsection we compute large s asymptotics for $I_{2,r}$, which we recall is given by

$$I_{2,r} = -\frac{1}{2}\int_\sigma \ln\Gamma\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)\,\mathrm{Tr}\Big[P^{(\infty)}(\zeta)^{-1}P^{(\infty)}(\zeta)'\sigma_3\Big]\frac{d\zeta}{2\pi i}.$$

Recalling the definition (2.14) of the global parametrix $P^{(\infty)}(\zeta)$, a straightforward calculation yields

$$\mathrm{Tr}\Big[P^{(\infty)}(\zeta)^{-1}P^{(\infty)}(\zeta)'\sigma_3\Big] = \mathrm{Tr}\Big[Q^{(\infty)}(\zeta)^{-1}Q^{(\infty)\prime}(\zeta)\sigma_3\Big] + \mathrm{Tr}[p'(\zeta)I] = 2p'(\zeta). \tag{4.10}$$

Therefore, integrating by parts, using the jump condition (2.13) of $p(\zeta)$, and then collapsing the contour σ onto Σ_5, we obtain

$$I_{2,r} = -is^\rho\int_\sigma\psi\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)p(\zeta)\frac{d\zeta}{2\pi i} = is^\rho\int_{\Sigma_5}\psi\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)\big(p_+(\zeta) - p_-(\zeta)\big)\frac{d\zeta}{2\pi i} = Z_r + X_r,$$

where

$$Z_r = -2is^\rho\int_{\gamma_{b_2b_1}}\psi\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)p(\zeta)\frac{d\zeta}{2\pi i},$$
$$X_r = is^\rho\int_{\Sigma_5}\psi\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)\ln\mathcal{G}(\zeta)\frac{d\zeta}{2\pi i}. \tag{4.11}$$

It remains to find the large s asymptotics of Z_r and X_r.

4.2.1 Asymptotics of X_r

The large s asymptotics of X_r are described in terms of the Hurwitz zeta function $\zeta(z,u)$ which is defined for $\mathrm{Re}\, z > 1$ and $u \neq 0, -1, -2, \dots$ by

$$\zeta(z,u) = \sum_{n=0}^{\infty} \frac{1}{(n+u)^z}.$$

Proposition 4.2. *Let $\nu \geq 0$ and X_r be defined by (4.11). Then*

$$X_r = \Omega\big(\ln(s^\rho)\big) + X_{1,r}^{(c)} + X_{2,r}^{(c)} + X_{3,r}^{(c)} + \mathcal{O}\left(\frac{\ln(s^\rho)}{s^\rho}\right) \tag{4.12}$$

as $s \to +\infty$ uniformly for r in compact subsets of $[1,+\infty)$, where

$$\begin{aligned} X_{1,r}^{(c)} = \frac{i}{48\pi}\Big\{ & 6r\nu^2\big[\ln^2(-ib_1) - \ln^2(-ib_2)\big] + 6\nu\ln(2\pi)\big[\ln(ib_2) - \ln(ib_1)\big] \\ & + \big[\ln(-ib_2) - \ln(-ib_1)\big]\Big((1+r)(1-3\nu^2) + 6\nu(1-2r)\ln(2\pi)\Big) \\ & + 6\nu^2\Big(\ln(-ib_2)\ln(ib_2) - \ln(-ib_1)\ln(ib_1)\Big)\Big\}, \end{aligned} \tag{4.13}$$

$$X_{2,r}^{(c)} = \ln G(1+\nu) - \zeta'(-1) - \frac{\nu\ln(2\pi)}{4} + \frac{3\nu^2-1}{24} - \zeta'\Big(-1, \frac{1+\nu}{2}+1\Big) + \frac{\nu+1}{2}\ln\Big(\frac{\nu+1}{2}\Big), \tag{4.14}$$

$$X_{3,r}^{(c)} = \frac{1+r+3(r-3)\nu^2}{24}\frac{\ln(b_1/b_2)\ln(-b_1b_2)}{4\pi i} - \frac{(1-3\nu^2)(1+r) - 6(r-1)\nu\ln(2\pi)}{24}\frac{\ln(b_1/b_2)}{2\pi i}, \tag{4.15}$$

where $\zeta'(z,u) = \partial_z\zeta(z,u)$ denotes the derivative in the z-variable of the Hurwitz zeta function.

Proof. We first rewrite the integral X_r in a convenient way. Performing an integration by parts yields

$$X_r = -\frac{\ln\Gamma\big(\frac{1+\nu}{2} - is^\rho\zeta\big)\ln\mathcal{G}(\zeta)}{2\pi i}\bigg|_{\zeta=b_1}^{b_2} + \int_{\Sigma_5}\ln\Gamma\Big(\frac{1+\nu}{2} - is^\rho\zeta\Big)\frac{\mathcal{G}'(\zeta)}{\mathcal{G}(\zeta)}\frac{d\zeta}{2\pi i}. \tag{4.16}$$

By the definition (2.6) of $\mathcal{G}(\zeta)$ (with F replaced by F_r) and the identity $\rho^{-1} = c_1 + c_2$, it holds that

$$\frac{\mathcal{G}'(\zeta)}{\mathcal{G}(\zeta)} = is^\rho\Big\{\psi\Big(\frac{1+\nu}{2} + is^\rho\Big) - \ln(is^\rho\zeta) + r\psi\Big(\frac{1+\nu}{2} - is^\rho\Big) - r\ln(-is^\rho\zeta)\Big\}. \tag{4.17}$$

By substituting (4.17) into (4.16) and using the change of variables $w = is^\rho\zeta$, we split X_r as

$$X_r = X_{1,r} + X_{2,r} + X_{3,r}, \tag{4.18}$$

where $X_{1,r}$, $X_{2,r}$ and $X_{3,r}$ are given by

$$\begin{aligned} X_{1,r} &= -\frac{\ln\Gamma\big(\frac{1+\nu}{2} - is^\rho\zeta\big)\ln\mathcal{G}(\zeta)}{2\pi i}\bigg|_{\zeta=b_1}^{b_2}, \\ X_{2,r} &= \int_{is^\rho\Sigma_5}\ln\Gamma\Big(\frac{1+\nu}{2} - w\Big)\Big\{\psi\Big(\frac{1+\nu}{2} + w\Big) - \ln(w) - f(w)\Big\}\frac{dw}{2\pi i}, \\ X_{3,r} &= \int_{is^\rho\Sigma_5}\ln\Gamma\Big(\frac{1+\nu}{2} - w\Big)\Big\{r\psi\Big(\frac{1+\nu}{2} - w\Big) - r\ln(-w) + f(w)\Big\}\frac{dw}{2\pi i}, \end{aligned}$$

with

$$f(w) = \frac{\nu}{2}\Big(\frac{1}{w-m} - \frac{m}{(w-m)^2}\Big) + \frac{1-3\nu^2}{24(w-m)^2} \qquad \text{and} \qquad m = \frac{1+\nu}{2}. \tag{4.19}$$

We have added and subtracted the term $f(w)$ in order to make the integrand of $X_{2,r}$ vanish as $w^{-2}\ln w$ as $w \to \infty$. Indeed, from (4.7) and the asymptotic formula [34, Eq 5.11.2] of the di-gamma function given by

$$\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \mathcal{O}\Big(\frac{1}{z^4}\Big), \qquad z \to \infty, \tag{4.20}$$

for $|\arg z| < \pi - \delta$ with some fixed $\delta > 0$, we have

$$\psi\Big(\frac{1+\nu}{2} + w\Big) = \ln w + \frac{\nu}{2w} + \frac{1-3\nu^2}{24w^2} + \mathcal{O}\Big(\frac{1}{w^3}\Big), \tag{4.21}$$

$$= \ln w + f(w) + \mathcal{O}\Big(\frac{1}{w^3}\Big), \qquad w \to \infty, \tag{4.22}$$

where $|\arg w| < \pi - \delta$ with some fixed $\delta > 0$. Note that the integral $X_{2,r}$ is convergent as long as $m \notin is^\rho\Sigma_5$; the choice $m = \frac{1+\nu}{2}$ is made because it makes the upcoming computations easier. The remainder of the proof consists of computing the large s asymptotics of $X_{1,r}$, $X_{2,r}$ and $X_{3,r}$.

Asymptotics of $X_{1,r}$. From [16, Eq. (3.15)], we have

$$\ln \mathcal{G}(\zeta) = c_4 \ln s + c_5 \ln(i\zeta) + c_6 \ln(-i\zeta) + c_7 + \frac{c_8}{is^\rho \zeta} + \mathcal{O}\Big(\frac{1}{s^{2\rho}\zeta^2}\Big), \qquad s^\rho\zeta \to \infty. \tag{4.23}$$

The asymptotic formula (4.23) is in particular valid for $\zeta = b_1$ and $\zeta = b_2$. By combining these asymptotics together with (4.7) and (2.17), we obtain

$$X_{1,r} = \Omega\big(\ln(s^\rho)\big) + X_{1,r}^{(c)} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big) \qquad \text{as } s \to +\infty, \tag{4.24}$$

where $X_{1,r}^{(c)}$ is given by (4.13).

Asymptotics of $X_{2,r}$. Recall that $X_{2,r}$ is given by

$$X_{2,r} = \int_{is^\rho\Sigma_5} \ln\Gamma\Big(\frac{1+\nu}{2} - w\Big)\Big\{\psi\Big(\frac{1+\nu}{2} + w\Big) - \ln(w) - f(w)\Big\}\frac{dw}{2\pi i},$$

and that the integrand is $\mathcal{O}(w^{-2}\ln w)$ as $w \to \infty$. Thus we have

$$X_{2,r} = \int_{\gamma_\infty} \ln\Gamma\Big(\frac{1+\nu}{2} - w\Big)\Big\{\psi\Big(\frac{1+\nu}{2} + w\Big) - \ln(w) - f(w)\Big\}\frac{dw}{2\pi i} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big)$$
$$=: X_{2,r}^{(c)} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big)$$

as $s \to +\infty$, where $f(w)$ is defined by (4.19) and where the contour γ_∞ is a line oriented upwards and approaching infinity which crosses the real line between the origin and $m = \frac{1+\nu}{2}$ (see Figure 6). We will compute $X_{2,r}$ by integration and then contour deformation. However, we first need to add and subtract the term $\frac{\nu}{2w}$ in the integrand and to split $X_{2,r}$ into two parts as follows:

$$X_{2,r}^{(c)} = \int_{\gamma_\infty} \ln\Gamma\Big(\frac{1+\nu}{2} - w\Big)\Big\{\psi\Big(\frac{1+\nu}{2} + w\Big) - \ln(w) - \frac{\nu}{2w} - \frac{1-3\nu^2}{24(w-m)^2}\Big\}\frac{dw}{2\pi i}$$
$$+ \int_{\gamma_\infty} \ln\Gamma\Big(\frac{1+\nu}{2} - w\Big)\Big\{\frac{\nu}{2w} - \frac{\nu}{2}\Big(\frac{1}{w-m} - \frac{m}{(w-m)^2}\Big)\Big\}\frac{dw}{2\pi i}.$$

Now, we integrate by parts the first integral, while the second integral can be evaluated explicitly by deforming the contour to infinity on the left half-plane (there is only a residue at $w = 0$). This gives

$$X_{2,r}^{(c)} = \int_{\gamma_\infty} \psi\Big(\frac{1+\nu}{2} - w\Big)\Big\{\ln\Gamma\Big(\frac{1+\nu}{2} + w\Big) + w(1 - \ln(w))$$
$$- \frac{\nu}{2}\ln(w) + \frac{1-3\nu^2}{12(2w-\nu-1)} - \frac{\ln(2\pi)}{2}\Big\}\frac{dw}{2\pi i} + \frac{\nu}{2}\ln\Gamma\Big(\frac{1+\nu}{2}\Big).$$

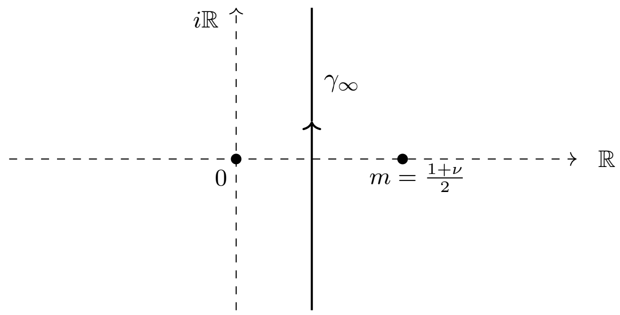

Figure 6: The contour γ_∞.

For the remaining integral, we deform the contour to infinity in the right half-plane. Note that this would not have been possible without adding and substracting the term $\frac{\nu}{2w}$. Since $\psi(\frac{1+\nu}{2} - w)$ has simple poles with residue 1 at $w = m + n$, $n = 0, 1, 2, ...$, we pick up the following residue contributions

$$-\left\{ \ln\Gamma\big(1+\nu+n\big) + (n+m)\big(1-\ln(n+m)\big) - \frac{\nu}{2}\ln(n+m) + \frac{1-3\nu^2}{24n} - \frac{\ln(2\pi)}{2} \right\}$$

at the points $n + m$ for $n = 1, 2, \ldots$, and

$$-\left\{ \ln\Gamma\big(1+\nu\big) + m\big(1-\ln(m)\big) - \frac{\nu}{2}\ln(m) - \frac{\ln(2\pi)}{2} \right\} + \frac{\gamma_{\mathrm{E}}(1-3\nu^2)}{24}$$

at $w = m$, where γ_{E} is Euler's gamma constant. This yields

$$X_{2,r}^{(c)} = -\sum_{n=0}^{\infty} \left\{ \ln\Gamma\big(1+\nu+n\big) + (n+m)\big(1-\ln(n+m)\big) - \frac{\nu}{2}\ln(n+m) + \frac{1-3\nu^2}{24(n+1)} - \frac{\ln(2\pi)}{2} \right\} + \frac{\gamma_{\mathrm{E}}(1-3\nu^2)}{24} + \frac{\nu}{2}\ln\Gamma\left(\frac{1+\nu}{2}\right).$$

The series is convergent since it arises from a convergent integral. We rewrite Euler's gamma constant (see [34, Eq. 5.2.3]) as

$$\gamma_E = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \ln\left(1+\frac{1}{n}\right) \right], \tag{4.25}$$

which implies

$$X_{2,r}^{(c)} = -\sum_{n=0}^{\infty} \left\{ \ln\Gamma\big(1+\nu+n\big) + (n+m)\big(1-\ln(n+m)\big) + \frac{1-3\nu^2}{24}\ln\left(1+\frac{1}{n+1}\right) - \frac{\nu}{2}\ln(n+m) - \frac{\ln(2\pi)}{2} \right\} + \frac{\nu}{2}\ln\Gamma\left(\frac{1+\nu}{2}\right). \tag{4.26}$$

This series can be computed explicitly. From the formula (cf. [34, Eq. 5.17.1])

$$G(z+1) = \Gamma(z)G(z),$$

we deduce

$$-\sum_{n=0}^{N} \ln\Gamma\big(1+\nu+n\big) = -\ln G\big(2+\nu+N\big) + \ln G\big(1+\nu\big).$$

The asymptotic formula (4.7) then implies that

$$-\sum_{n=0}^{N} \ln\Gamma\big(1+\nu+n\big) = \Omega(\ln N) - \zeta'(-1) - \frac{\ln(2\pi)}{2}(1+\nu) + \ln G\big(1+\nu\big) + \mathcal{O}\big(N^{-1}\big) \tag{4.27}$$

as $N \to +\infty$, where $\zeta'(-1)$ denotes the derivative of the Riemann zeta function evaluated at -1. Furthermore, from [15, Eq. (10.11)] with $\theta = 1$, we have

$$\sum_{n=0}^{N} \left(\frac{1+\nu}{2}+n\right)\ln\left(\frac{1+\nu}{2}+n\right) = \frac{1+\nu}{2}\ln\left(\frac{1+\nu}{2}\right) + \Omega(\ln N) \\ + \frac{3(1+\nu^2)+8+12\nu}{24} - \zeta'\Big(-1, \frac{1+\nu}{2}+1\Big) + \mathcal{O}\big(N^{-1}\big), \qquad N \to +\infty, \tag{4.28}$$

where we recall that $\zeta(z,u)$ is the Hurwitz zeta function and $\zeta'(-1,m+1) = \partial_z \zeta(z,m+1)|_{z=-1}$. Also, it is easy to verify from $\Gamma(z+1) = z\Gamma(z)$ that

$$\sum_{n=0}^{N} \ln\left(\frac{1+\nu}{2}+n\right) = \ln\Gamma\Big(\frac{1+\nu}{2}+N+1\Big) - \ln\Gamma\Big(\frac{1+\nu}{2}\Big),$$

and thus, by (4.7),

$$\frac{\nu}{2}\sum_{n=0}^{N} \ln\left(\frac{1+\nu}{2}+n\right) = \Omega(\ln N) + \frac{\nu}{4}\ln(2\pi) - \frac{\nu}{2}\ln\Gamma\Big(\frac{1+\nu}{2}\Big) + \mathcal{O}\big(N^{-1}\big), \qquad N \to +\infty. \tag{4.29}$$

Another straightforward calculation shows that

$$\sum_{n=0}^{N} \ln\left(1+\frac{1}{n+1}\right) = \ln(N+2) = \ln N + \mathcal{O}\big(N^{-1}\big), \qquad N \to +\infty. \tag{4.30}$$

Substituting (4.27), (4.28), (4.29), and (4.30) into (4.26) gives

$$X_{2,r} = X_{2,r}^{(c)} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big), \qquad s \to +\infty, \tag{4.31}$$

where $X_{2,r}^{(c)}$ is given by (4.14).

Asymptotics of $X_{3,r}$. The integrand of $X_{3,r}$ is analytic on the left of $is^\rho\Sigma_5$. By deforming the contour $is^\rho\Sigma_5$ to $is^\rho\gamma_{b_2b_1}$, where $\gamma_{b_2b_1}$ is defined as in (4.4), we rewrite $X_{3,r}$ as follows:

$$X_{3,r} = -\int_{is^\rho\gamma_{b_2b_1}} \ln\Gamma\Big(\frac{1+\nu}{2}-w\Big)\Big\{r\psi\Big(\frac{1+\nu}{2}-w\Big) - r\ln(-w) + f(w)\Big\}\frac{dw}{2\pi i} \\ = -\int_{is^\rho\gamma_{b_2b_1}} \ln\Gamma\Big(\frac{1+\nu}{2}-w\Big)\Big\{r\psi\Big(\frac{1+\nu}{2}-w\Big) - r\ln(-w) + \frac{\nu}{2w} + \frac{1-3\nu^2}{24w^2}\Big\}\frac{dw}{2\pi i} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big)$$

as $s \to +\infty$. Using the asymptotic formulas (4.7) and (4.20) for the integrand of $X_{3,r}$, we find

$$X_{3,r} = -\int_{is^\rho\gamma_{b_2b_1}} \Big\{\frac{\nu}{2}(r-1)\big(\ln(-w)-1\big) \\ + \frac{1+r-3\nu^2(1+r)-6(r-1)\nu\ln(2\pi) - (1+r+3(r-3)\nu^2)\ln(-w)}{24w} \\ + \mathcal{O}\big(w^{-2}\ln(-w)\big)\Big\}\frac{dw}{2\pi i} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big), \qquad s \to +\infty.$$

After the change of variables $w = is^\rho\zeta$, we obtain

$$X_{3,r} = \Omega\big(\ln(s^\rho)\big) + \frac{1+r+3(r-3)\nu^2}{24}\int_{\gamma_{b_2b_1}} \frac{\ln(-i\zeta)}{\zeta}\frac{d\zeta}{2\pi i} \\ - \frac{1+r-3\nu^2(1+r)-6(r-1)\nu\ln(2\pi)}{24}\int_{\gamma_{b_2b_1}} \frac{1}{\zeta}\frac{d\zeta}{2\pi i} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big) \tag{4.32}$$

as $s \to +\infty$. Since

$$\int_{\gamma_{b_2 b_1}} \frac{1}{\zeta} \frac{d\zeta}{2\pi i} = \frac{\ln(b_1/b_2)}{2\pi i}, \qquad \int_{\gamma_{b_2 b_1}} \frac{\ln(-i\zeta)}{\zeta} \frac{d\zeta}{2\pi i} = \frac{\ln(b_1/b_2)\ln(-b_1 b_2)}{4\pi i}, \tag{4.33}$$

the asymptotics (4.32) can be rewritten as

$$X_{3,r} = \Omega\big(\ln(s^\rho)\big) + X_{3,r}^{(c)} + \mathcal{O}\left(\frac{\ln(s^\rho)}{s^\rho}\right), \qquad s \to +\infty, \tag{4.34}$$

where $X_{3,r}^{(c)}$ is given by (4.15).

Asymptotics of X_r. By substituting (4.24), (4.31), and (4.34) into (4.18), we obtain (4.12). This completes the proof of Proposition 4.2. □

4.2.2 Asymptotics of Z_r

In this subsection, we compute the asymptotics of

$$Z_r = -2is^\rho \int_{\gamma_{b_2 b_1}} \psi\Big(\frac{1+\nu}{2} - is^\rho \zeta\Big) p(\zeta) \frac{d\zeta}{2\pi i}, \tag{4.35}$$

where $p(\zeta)$ is defined by (2.12). Some of the following computations are similar to those performed in [15, Section 8]. In particular, the quantity $\hat{f}_1$ in [15, Eq (3.15)] is in our case equal to

$$\hat{f}_1 = -i\frac{3\nu^2 - 1}{24}$$

and appears naturally in the asymptotics of Z_r.

Proposition 4.3. *Let $\nu > -1$ and Z_r be defined by (4.35). Then*

$$Z_r = \Omega\big(\ln(s^\rho)\big) + Z_r^{(c)} + \mathcal{O}\left(\frac{\ln s^\rho}{s^\rho}\right) \tag{4.36}$$

as $s \to +\infty$ uniformly for r in compact subsets of $[1, +\infty)$, where

$$\begin{aligned} Z_r^{(c)} = -2i\Bigg\{ &\frac{1}{4}\left(\frac{c_8(\arg b_1 - \arg b_2)}{2} + \frac{\pi(c_8 - 2i\hat{f}_1)}{2}\left(1 - \frac{\operatorname{Im} b_2}{|b_2|}\right)\right) + \frac{1}{2\pi i}\left(\frac{ic_8}{4}\big((\ln b_1)^2 - (\ln b_2)^2\big)\right. \\ &+ \frac{\pi(c_8 - 2i\hat{f}_1)}{2|b_2|}\left[|b_2| - \operatorname{Im} b_2 - |b_2| \ln\left(\frac{2i|b_2|^2}{|b_2| + \operatorname{Im} b_2}\right) + \operatorname{Im} b_2 \ln\left(\frac{i(|b_2| + \operatorname{Im} b_2)}{2}\right)\right]\Bigg) \\ &+ \frac{\nu}{16\pi}\Bigg(- 4i\pi \ln(|b_2|)(c_5 - c_6) + 4i\pi c_6 \ln\left(\frac{2}{|b_2| + \operatorname{Im} b_2}\right) + \ln b_1(-2c_7 + i\pi(3c_5 + c_6)) \\ &- (c_5 + c_6)\ln^2(b_1) + \ln(b_2)(2c_7 + i\pi(c_5 - c_6)) + (c_5 + c_6)\ln^2(b_2) + 2\pi^2 c_5\Bigg)\Bigg\}. \end{aligned} \tag{4.37}$$

Proof. By (4.20), we see that

$$\psi\left(\frac{1+\nu}{2} - is^\rho\zeta\right) = \ln(s^\rho) + \ln(-i\zeta) - \frac{\nu}{2is^\rho\zeta} + \mathcal{O}\big(s^{-2\rho}\big), \qquad \text{as } s \to +\infty, \tag{4.38}$$

uniformly for $\zeta \in \gamma_{b_2 b_1}$. From [15, beginning of Section 8.3 and Eq. (3.38)] with the coefficients $c_1, \ldots, c_8$ given by (2.17), we have

$$p(\zeta) = -\frac{c_4}{2\rho}\ln(s^\rho) + \frac{\mathcal{B}(\zeta)}{2} + \frac{\mathcal{A}(\zeta)}{s^\rho} + \mathcal{O}\big(s^{-2\rho}\big) \tag{4.39}$$

as $s \to +\infty$ uniformly for $\zeta \in \gamma_{b_2 b_1}$ and r in compact subsets of $[1,+\infty)$, where

$$\mathcal{B}(\zeta) = -c_5 r(\zeta) \int_0^{i\infty} \frac{d\xi}{r(\xi)(\xi-\zeta)} - c_6 r(\zeta) \int_0^{-i\infty} \frac{d\xi}{r(\xi)(\xi-\zeta)} - c_5 \ln(i\zeta) - c_6 \ln(-i\zeta) - c_7, \tag{4.40}$$

$$\mathcal{A}(\zeta) = \frac{ic_8}{2\zeta} + r(\zeta) \frac{c_8 - \frac{3\nu^2-1}{12}}{2\zeta|b_2|}. \tag{4.41}$$

After substituting (4.38) and (4.39) into the definition (4.35) of Z_r, we write

$$Z_r = \Omega\big(\ln(s^\rho)\big) - 2i\bigg(\int_{\gamma_{b_2 b_1}} \mathcal{A}(\zeta)\big(\ln(\zeta) - \tfrac{\pi i}{2}\big)\frac{d\zeta}{2\pi i} - \frac{\nu}{4i}\int_{\gamma_{b_2 b_1}} \frac{\mathcal{B}(\zeta)}{\zeta}\frac{d\zeta}{2\pi i}\bigg) + \mathcal{O}\bigg(\frac{\ln s^\rho}{s^\rho}\bigg) \tag{4.42}$$

as $s \to +\infty$. It was shown in [15, Lemma 8.4] that

$$\int_{\gamma_{b_2 b_1}} \mathcal{A}(\zeta) d\zeta = -\frac{c_8(\arg b_1 - \arg b_2)}{2} - \frac{\pi\big(c_8 - \frac{3\nu^2-1}{12}\big)}{2}\bigg(1 - \frac{\operatorname{Im} b_2}{|b_2|}\bigg), \tag{4.43}$$

$$\begin{aligned} \int_{\gamma_{b_2 b_1}} \ln(\zeta)\mathcal{A}(\zeta)d\zeta = {} & \frac{ic_8}{4}\big((\ln b_1)^2 - (\ln b_2)^2\big) + \frac{\pi\big(c_8 - \frac{3\nu^2-1}{12}\big)}{2|b_2|}\bigg\{|b_2| - \operatorname{Im} b_2 - |b_2| \ln\bigg(\frac{2i|b_2|^2}{|b_2| + \operatorname{Im} b_2}\bigg) \\ & + (\operatorname{Im} b_2)\ln\bigg(\frac{i(|b_2| + \operatorname{Im} b_2)}{2}\bigg)\bigg\}, \end{aligned} \tag{4.44}$$

$$\begin{aligned} \int_{\gamma_{b_2 b_1}} \frac{\mathcal{B}(\zeta)}{\zeta} d\zeta = \frac{1}{2}\bigg\{ & -4i\pi \ln(|b_2|)(c_5 - c_6) + 4i\pi c_6 \ln\bigg(\frac{2}{|b_2| + \operatorname{Im}(b_2)}\bigg) \\ & + \ln(b_1)(-2c_7 + i\pi(3c_5 + c_6)) - (c_5 + c_6)(\ln b_1)^2 \\ & + \ln(b_2)(2c_7 + i\pi(c_5 - c_6)) + (c_5 + c_6)(\ln b_2)^2 + 2\pi^2 c_5 \bigg\}. \end{aligned} \tag{4.45}$$

By substituting (4.43), (4.44), and (4.45) into (4.42), we obtain (4.36) after a long computation. This completes the proof of the proposition. ☐

4.3 Asymptotics of $I_{3,r}(K)$ and $I_{4,r}(K)$

In this section we compute the large s asymptotics of $I_{3,r}(K)$ and $I_{4,r}(K)$ defined in (3.14) and (3.15).

Proposition 4.4. *Let $\nu > -1$ and $K = s^\rho$. Then*

$$I_{3,r}(K) = \Omega\big(\ln(s^\rho)\big) + I_{3,r}^{(c)} + \mathcal{O}\bigg(\frac{\ln(s^\rho)}{s^\rho}\bigg) \tag{4.46}$$

as $s \to +\infty$ uniformly for r in compact subsets of $[1,+\infty)$, where

$$I_{3,r}^{(c)} = \frac{-2(1+r) + 3r(r^2-1)\nu^2 + 2r(1+3r\nu^2)\ln(r)}{24r(1+r)^2}.$$

Proof. Proceeding as in the proof of [15, Proposition 9.1], one obtains

$$I_{3,r}(K) = -\frac{1}{2s^\rho}\int_{\tilde{\sigma}} \ln\Gamma\bigg(\frac{1+\nu}{2} - is^\rho\zeta\bigg) W(\zeta)\frac{d\zeta}{2\pi i} + \mathcal{O}\bigg(\frac{\ln(s^\rho)}{s^\rho}\bigg), \qquad s \to +\infty,$$

where $\tilde{\sigma}$ surrounds the horizontal segment $[b_1, b_2]$ but does not surround the origin, and is oriented counterclockwise. The function $W(\zeta)$ is defined by

$$\begin{aligned} W(\zeta) = \frac{1}{\tilde{r}(\zeta)} \operatorname{Tr}\bigg[& \bigg(-\frac{A}{(\zeta-b_1)^2} - \frac{2B}{(\zeta-b_1)^3} + \frac{\bar{A}}{(\zeta-b_2)^2} - \frac{2\bar{B}}{(\zeta-b_2)^3}\bigg) \\ & \times \begin{pmatrix} \zeta - i\operatorname{Im} b_2 & i\operatorname{Re} b_2 \\ i\operatorname{Re} b_2 & i\operatorname{Im} b_2 - \zeta \end{pmatrix}\bigg], \end{aligned} \tag{4.47}$$

where $\tilde{r}(\zeta) = \sqrt{(\zeta - b_1)(\zeta - b_2)}$ has a branch cut on $[b_1, b_2]$, such that $\tilde{r}(\zeta) \sim \zeta$ as $\zeta \to \infty$. The matrices A and B denote the coefficients appearing in the large s asymptotics of R (cf. [15, Proposition 4.1]) and are given by

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}, \qquad B = -\frac{5b_1}{48(c_1 + c_2)} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \tag{4.48}$$

with

$$\begin{aligned}
A_{1,1} &= \frac{3\mathrm{Im}\, b_2 + 2i\mathrm{Re}\, b_2 - 12(|b_2|(c_5 - c_6)(c_5 + c_6) + (c_5^2 + c_6^2)\mathrm{Im}\, b_2 + 2ic_5c_6\mathrm{Re}\, b_2)}{48(c_1 + c_2)\mathrm{Re}\, b_2}, \\
A_{1,2} &= \frac{4i(3|b_2|(c_5 - c_6)(1 + c_5 + c_6) + \mathrm{Im}\, b_2 + 3(c_5 + c_5^2 + c_6 + c_6^2)\mathrm{Im}\, b_2)}{48(c_1 + c_2)\mathrm{Re}\, b_2} \\
&\quad - \frac{(5 + 12c_6 + 12c_5(1 + 2c_6))}{48(c_1 + c_2)}, \\
A_{2,1} &= \frac{12i|b_2|(c_5 - c_6)(-1 + c_5 + c_6) + 4i(1 + 3(c_5 - 1)c_5 + 3(c_6 - 1)c_6)\mathrm{Im}\, b_2}{48(c_1 + c_2)\mathrm{Re}\, b_2} \\
&\quad + \frac{-5 + 12(c_5 + c_6 - 2c_5c_6)}{48(c_1 + c_2)}, \\
A_{2,2} &= -A_{1,1}.
\end{aligned}$$

Using the asymptotics (4.7) of $\ln \Gamma$, we obtain

$$I_{3,r}(K) = -\frac{i}{2} \int_{\tilde{\sigma}} \big(1 - \ln(-is^\rho \zeta)\big)\zeta W(\zeta) \frac{d\zeta}{2\pi i} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big), \qquad s \to +\infty. \tag{4.49}$$

From (4.47), we see that

$$W(\zeta) = -\frac{\mathrm{Tr}[(A - \bar{A})\sigma_3]}{\zeta^2} + \mathcal{O}(\zeta^3) = -i\frac{1 - 12c_5c_6}{6(c_1 + c_2)\zeta^2} + \mathcal{O}(\zeta^3) \qquad \text{as } \zeta \to \infty.$$

Thus, after splitting the leading term in (4.49) into two parts, we obtain

$$I_{3,r}(K) = -i\frac{1 - 12c_5c_6}{6(c_1 + c_2)} \frac{-i}{2}\Big(1 - \ln(s^\rho)\Big) + \frac{i}{2}\int_{\tilde{\sigma}} \ln(-i\zeta)\zeta W(\zeta)\frac{d\zeta}{2\pi i} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big) \tag{4.50}$$

as $s \to +\infty$, where we have deformed $\tilde{\sigma}$ to infinity for the first part. The last integral in (4.50) can be evaluated as follows:

$$\begin{aligned}
\frac{i}{2}\int_{\tilde{\sigma}} \ln(-i\zeta)\zeta W(\zeta)\frac{d\zeta}{2\pi i} &= \lim_{R\to\infty}\Big\{\frac{i}{2}\int_{C_R} \ln(-i\zeta)\zeta W(\zeta)\frac{d\zeta}{2\pi i} + \frac{i}{2}\int_{-iR}^0 \zeta W(\zeta)d\zeta\Big\} \\
&= \lim_{R\to\infty}\Big\{-\frac{i}{2}\mathrm{Tr}[(A - \bar{A})\sigma_3]\int_{C_R} \frac{\ln(-i\zeta)}{\zeta}\frac{d\zeta}{2\pi i} + \frac{i}{2}\int_{-R}^0 itW(it)idt\Big\},
\end{aligned}$$

where C_R is the circle centered at the origin of radius R oriented positively. Since

$$\begin{aligned}
&\int_{C_R} \frac{\ln(-i\zeta)}{\zeta}\frac{d\zeta}{2\pi i} = \ln(R), \\
&\tilde{r}(it) = -i\sqrt{(t - \mathrm{Im}\, b_2)^2 + (\mathrm{Re}\, b_2)^2}, \qquad \text{for } t < 0,
\end{aligned}$$

we compute the integral $\int_{-iR}^0 \zeta W(\zeta)d\zeta$ by a rather long primitive calculation, which uses the definition (4.47) of $W(\zeta)$. Then, after substituting the expressions (2.5) and (2.8), we obtain

$$\frac{i}{2}\int_{\tilde{\sigma}} \ln(-i\zeta)\zeta W(\zeta)\frac{d\zeta}{2\pi i} = \frac{(1 + r)\big(-2 + r(2 + (9r - 3)\nu^2)\big) + 2r(1 + 3r\nu^2)\ln(r)}{24r(1 + r)^2}. \tag{4.51}$$

Substituting (4.51) into (4.50) and using again (2.17), we obtain (4.46), which finishes the proof. □

Proposition 4.5. *Let $\nu > -1$ and $K = s^\rho$. Then, for any integer $N \geq 1$,*

$$I_{4,r}(K) = \mathcal{O}\big(s^{-N\rho}\big) \tag{4.52}$$

as $s \to +\infty$ uniformly for r in compact subsets of $[1, +\infty)$.

Proof. The proof is analogous to the proof of [15, Proposition 9.2] and relies on the large s asymptotics of R. □

4.4 Integration of the differential identity in r

In this subsection, we compute the constant term C_r in the large gap asymptotics for the point process defined by $\mathbb{K}_r$. From [16, 15], these asymptotics are of the form

$$\det\big(1 - \mathbb{K}_r|_{[0,s]}\big) = C_r \exp\Big(- a_r s^{2\rho} + b_r s^\rho + c_r \ln s + \mathcal{O}\big(s^{-\rho}\big)\Big), \tag{4.53}$$

where the constants ρ, a_r, b_r and c_r are given by

$$\rho = \frac{1}{1+r}, \qquad a_r = \frac{r^{\frac{1-r}{1+r}}(r+1)^2}{4}, \qquad b_r = (1+r) r^{\frac{1}{1+r}} \nu, \qquad c_r = \frac{r-1}{12(r+1)} - \frac{r\nu^2}{2(r+1)}.$$

Proposition 4.6. *Let $\nu > -1$ and $r \geq 1$. Then*

$$\det\big(1 - \mathbb{K}_r|_{[0,s]}\big) = C_r \exp\Big(- a_r s^{2\rho} + b_r s^\rho + c_r \ln s + \mathcal{O}\big(s^{-\rho}\big)\Big),$$

where

$$\begin{aligned} C_r = {} & \frac{G(1+\nu)^r}{(2\pi)^{\frac{r\nu}{2}}} \exp\big\{ -(r-1)\zeta'(-1)\big\} \exp\left\{ \frac{-2 + r^2(r-1+12\nu^2)}{24(r+1)} \ln(r) \right\} \\ & \times \exp\left\{ - \frac{(r-1)^2 + 12 r\nu^2}{24} \ln(1+r) \right\}. \end{aligned}$$

Proof. It follows from the analysis of [16, 15] that the error term in (4.53) can be differentiated with respect to r and that its r-derivative is of order $\mathcal{O}(s^{-\rho}\ln(s^\rho))$ uniformly for r in compact subsets of $[1, +\infty)$. Therefore we have

$$\begin{aligned} \partial_r \ln \det\big(1 - \mathbb{K}_r|_{[0,s]}\big) = {} & -2\partial_r(\rho) a_r s^{2\rho} \ln s - \partial_r a_r s^{2\rho} + \partial_r(\rho) b_r s^\rho \ln s + \partial_r(b_r) s^\rho \\ & + \partial_r(c_r)\ln(s) + \partial_r(\ln C_r) + \mathcal{O}(s^{-\rho}\ln(s^\rho)) \end{aligned}$$

as $s \to +\infty$, uniformly for r in compact subsets of $[1, +\infty)$. Thus, from Lemma 3.1 and Propositions 4.1, 4.2, 4.3, 4.4, and 4.5, we infer that

$$\begin{aligned} \partial_r(\ln C_r) &= I_{1,r}^{(c)} + X_{1,r}^{(c)} + X_{2,r}^{(c)} + X_{3,r}^{(c)} + Z_r^{(c)} + I_{3,r}^{(c)} \\ &= -\frac{2 - 2r + 3r\nu^2}{24 r} - \frac{\nu \ln(2\pi)}{4} + \frac{1 + r(r - 1 + r^2 + 6(2+r)\nu^2)}{12(r+1)^2} \ln(r) \\ &\quad - \frac{1}{12}(r - 1 + 6\nu^2)\ln(1+r) + \frac{\nu-1}{2}\ln\Gamma\left(\frac{1+\nu}{2}\right) - \ln G\left(\frac{1+\nu}{2}\right) - \frac{\nu\ln(2\pi)}{4} \\ &\quad + \ln G(1+\nu) + \frac{3\nu^2 - 1}{24} - \zeta'\left(-1; \frac{1+\nu}{2} + 1\right) + \frac{\nu+1}{2}\ln\left(\frac{\nu+1}{2}\right). \end{aligned}$$

Applying the identity [1, Eq (18)]

$$\ln G(z+1) = \zeta'(-1) - \zeta'(-1, z+1) + z \ln\Gamma(z+1) \tag{4.54}$$

with $z = \frac{1+\nu}{2}$, we get

$$\partial_r(\ln C_r) = \frac{r-2}{24r} - \frac{\nu\ln(2\pi)}{2} + \frac{1+r(r-1+r^2+6(2+r)\nu^2)}{12(r+1)^2}\ln(r)$$
$$- \frac{1}{12}(r-1+6\nu^2)\ln(1+r) - \zeta'(-1) + \ln G(1+\nu).$$

Integrating this identity with respect to r from $r = 1$ to a fixed $r \geq 1$, we obtain

$$\int_1^r \partial_{r'}(\ln C_{r'})dr' = \frac{\nu^2}{2}\ln(2) - \frac{\nu(r-1)\ln(2\pi)}{2} + (r-1)\Big(\ln G(1+\nu) - \zeta'(-1)\Big)$$
$$+ \frac{-2+r^2(r-1+12\nu^2)}{24(r+1)}\ln(r) - \frac{(r-1)^2+12r\nu^2}{24}\ln(1+r).$$

Since $\ln C_1 = \ln G(1+\nu) - \frac{\nu}{2}\ln(2\pi) - \frac{\nu^2}{2}\ln 2$ by (1.9), we arrive at

$$\ln C_r = r\ln G(1+\nu) - r\frac{\nu\ln(2\pi)}{2} - (r-1)\zeta'(-1)$$
$$+ \frac{-2+r^2(r-1+12\nu^2)}{24(r+1)}\ln(r) - \frac{(r-1)^2+12r\nu^2}{24}\ln(1+r)$$

and the proposition follows by exponentiating both sides. □

5 Asymptotics of the differential identities in ν_ℓ and μ_ℓ

In this section, we compute the large s asymptotics of the differential identities (3.16) and (3.17). For $\alpha \in \{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\}$, these identities express $\partial_\alpha \ln\det(1-\mathbb{K}|_{[0,s]})$ in terms of the quantities $I_{1,\alpha}$, $I_{2,\alpha}$, $I_{3,\alpha}(K)$, and $I_{4,\alpha}(K)$ defined in Lemma 3.1. By computing the asymptotics of these quantities and then integrating with respect to α, we can deduce the large s asymptotics of

$$\ln\det\big(1-\mathbb{K}|_{[0,s]}\big),$$

see also the outline in Section 1.3; in particular (1.31)-(1.32).

In the remainder of the paper, we let $r > q \geq 0$ be integers and let $\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q > -1$ be the parameters associated to the kernel $\mathbb{K}$ defined in (1.3). We set $\nu_{\min} := \min\{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\}$. The constants $c_1, \ldots, c_8$ and b_1, b_2 are defined in (2.5) and (2.8), respectively, and we will use the notation Ω introduced at the beginning of Section 4.

5.1 Asymptotics of $I_{1,\alpha}$

Proposition 5.1. *Let $I_{1,\alpha}$ be the function defined in (3.18). Then*

$$I_{1,\alpha} = \Omega\big(\ln(s^\rho)\big) + I_{1,\alpha}^{(c)} + \mathcal{O}\left(\frac{\ln(s^\rho)}{s^\rho}\right) \tag{5.1}$$

as $s \to +\infty$ uniformly for α in compact subsets of $(-1,+\infty)$, where

$$I_{1,\alpha}^{(c)} = -\frac{(c_1+c_2)}{2}\left(\frac{\ln(2\pi)}{2} - \ln\Gamma\Big(\frac{1+2\alpha-\nu_{\min}}{2}\Big)\right)\left(1-\frac{\operatorname{Im} b_2}{|b_2|}\right)$$
$$-(c_1+c_2)\frac{2\alpha-\nu_{\min}}{2}\left(\frac{\ln(|b_2|+\operatorname{Im} b_2)-\ln(2)}{2} - \frac{\operatorname{Im} b_2}{2|b_2|}\ln\left(\frac{2|b_2|^2}{|b_2|+\operatorname{Im} b_2}\right)\right). \tag{5.2}$$

Proof. Integrating $I_{1,\alpha}$ by parts as in the proof of Proposition 4.1, we find

$$I_{1,\alpha} = -i(c_1+c_2)\int_{\gamma_{b_2b_1}} \tilde{\Psi}(\zeta)\left(1-\frac{i\operatorname{Im} b_2}{\zeta}\right)\frac{1}{r(\zeta)}\frac{d\zeta}{2\pi i}, \tag{5.3}$$

where $\gamma_{b_2b_1}$ is defined as in (4.4) and

$$\tilde{\Psi}(\zeta) = -\,s^\rho \int_0^\zeta \psi\Big(\frac{1+2\alpha-\nu_{\min}}{2} - is^\rho\xi\Big)d\xi$$
$$= -\,i\ln\Gamma\Big(\frac{1+2\alpha-\nu_{\min}}{2} - is^\rho\zeta\Big) + i\ln\Gamma\Big(\frac{1+2\alpha-\nu_{\min}}{2}\Big).$$

A direct computation using the asymptotics (4.7) yields

$$\tilde{\Psi}(\zeta) = \Omega\big(\ln(s^\rho)\big) - i\frac{2\alpha-\nu_{\min}}{2}\ln(-i\zeta) - i\frac{\ln(2\pi)}{2} + i\ln\Gamma\Big(\frac{1+2\alpha-\nu_{\min}}{2}\Big) + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big) \tag{5.4}$$

as $s\to+\infty$ uniformly for α in compact subsets of $(-1,+\infty)$, and uniformly for $\zeta\in\gamma_{b_2b_1}$. We obtain (5.1)-(5.2) after substituting (5.4) into (5.3), using (4.9) and then simplifying. □

5.2 Asymptotics of $I_{2,\alpha}$

Let $I_{2,\alpha}$ be the function defined in (3.19). Using (4.10), we can write

$$I_{2,\alpha} = \int_\sigma \mathcal{F}(\zeta)p'(\zeta)\frac{d\zeta}{2\pi i},$$

where $p(\zeta)$ is defined by (2.12) and $\mathcal{F}(\zeta)$ is defined by

$$\mathcal{F}(\zeta) = -\psi\Big(\frac{1+2\alpha-\nu_{\min}}{2} - is^\rho\zeta\Big). \tag{5.5}$$

We integrate by parts and then deform the contour and use the jumps for p as in the beginning of Section 4.2. This yields

$$I_{2,\alpha} = Z_\alpha + X_\alpha,$$

where

$$Z_\alpha = -2\int_{\gamma_{b_2b_1}} \mathcal{F}'(\zeta)p(\zeta)\frac{d\zeta}{2\pi i}, \qquad X_\alpha = \int_{\Sigma_5}\mathcal{F}'(\zeta)\ln\mathcal{G}(\zeta)\frac{d\zeta}{2\pi i}. \tag{5.6}$$

Proposition 5.2. *Let X_α be defined by (5.6). Then*

$$X_\alpha = \Omega\big(\ln(s^\rho)\big) + X_{1,\alpha}^{(c)} + X_{2,\alpha}^{(c)} + X_{3,\alpha}^{(c)} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big) \qquad \text{as } s\to+\infty, \tag{5.7}$$

uniformly for α in compact subsets of $(-1,+\infty)$, where

$$X_{1,\alpha}^{(c)} = \frac{1}{2\pi i}\Big\{c_5\big(\ln(-ib_1)\ln(ib_1) - \ln(-ib_2)\ln(ib_2)\big)$$
$$+ c_6\big(\ln^2(-ib_1) - \ln^2(-ib_2)\big) + c_7\big(\ln(-ib_1) - \ln(-ib_2)\big)\Big\}, \tag{5.8}$$

$$X_{2,\alpha}^{(c)} = \sum_{k=0}^{\infty}\left\{\ln\left(k + \frac{1+2\alpha-\nu_{\min}}{2}\right) - \psi\big(k+1+\alpha\big) + \frac{\nu_{\min}}{2}\ln\left(1+\frac{1}{k+1}\right)\right\}, \tag{5.9}$$

$$X_{3,\alpha}^{(c)} = \left\{-\frac{\nu_{\min}}{2} + \sum_{j=1}^{r}\frac{2\nu_j-\nu_{\min}}{2} - \sum_{k=1}^{q}\frac{2\mu_k-\nu_{\min}}{2}\right\}\frac{\ln(b_1/b_2)\ln(|b_2|)}{2\pi i}. \tag{5.10}$$

Proof. Recalling the definition (2.6) of $\mathcal{G}(\zeta)$ and using the identity $\rho^{-1} = c_1 + c_2$, we see that

$$\ln\mathcal{G}(\zeta) = \ln\Gamma\Big(\frac{1+\nu_{\min}}{2} + is^\rho\zeta\Big) + \sum_{k=1}^{q}\ln\Gamma\Big(\frac{1+2\mu_k-\nu_{\min}}{2} - is^\rho\zeta\Big) - \sum_{j=1}^{r}\ln\Gamma\Big(\frac{1+2\nu_j-\nu_{\min}}{2} - is^\rho\zeta\Big)$$
$$- is^\rho\zeta\big(c_1\ln(is^\rho\zeta) + c_2\ln(-is^\rho\zeta) + c_3\big).$$

Hence, recalling the values (2.5) of c_1, c_2 and c_3,

$$\frac{\mathcal{G}'(\zeta)}{\mathcal{G}(\zeta)} = is^\rho\Bigg(\psi\Big(\frac{1+\nu_{\min}}{2}+is^\rho\zeta\Big) - \sum_{k=1}^{q}\psi\Big(\frac{1+2\mu_k-\nu_{\min}}{2}-is^\rho\zeta\Big) + \sum_{j=1}^{r}\psi\Big(\frac{1+2\nu_j-\nu_{\min}}{2}-is^\rho\zeta\Big) - \ln(is^\rho\zeta) - (r-q)\ln(-is^\rho\zeta)\Bigg).$$

Therefore, after integrating by parts, we can write

$$X_\alpha = X_{1,\alpha} + X_{2,\alpha} + X_{3,\alpha},$$

where

$$\begin{aligned}
X_{1,\alpha} &= \frac{\mathcal{F}(\zeta)\ln\mathcal{G}(\zeta)}{2\pi i}\Bigg|_{\zeta=b_1}^{b_2},\\
X_{2,\alpha} &= \int_{is^\rho\Sigma_5}\psi\Big(\frac{1+2\alpha-\nu_{\min}}{2}-w\Big)\Big\{\psi\Big(\frac{1+\nu_{\min}}{2}+w\Big) - \ln(w) - \frac{\nu_{\min}}{2(w-m)}\Big\}\frac{dw}{2\pi i},\\
X_{3,\alpha} &= \int_{is^\rho\Sigma_5}\psi\Big(\frac{1+2\alpha-\nu_{\min}}{2}-w\Big)\Big\{-\sum_{k=1}^{q}\psi\Big(\frac{1+2\mu_k-\nu_{\min}}{2}-w\Big)\\
&\qquad + \sum_{j=1}^{r}\psi\Big(\frac{1+2\nu_j-\nu_{\min}}{2}-w\Big) - (r-q)\ln(-w) + \frac{\nu_{\min}}{2(w-m)}\Big\}\frac{dw}{2\pi i}.
\end{aligned}$$

Here we have used the change of variables $is^\rho\zeta = w$ in the expressions for $X_{2,\alpha}$ and $X_{3,\alpha}$, and m is an arbitrary constant which lies in $\mathbb{C}\setminus is^\rho\Sigma_5$; it will be convenient to henceforth choose $m = \frac{1+2\alpha-\nu_{\min}}{2}$. As in the proof of Proposition 4.2, we have added and substracted one term in order to ensure that $X_{2,\alpha}$ has a limit as $s\to+\infty$. It remains to compute the large s asymptotics of $X_{1,\alpha}$, $X_{2,\alpha}$, and $X_{3,\alpha}$.

Asymptotics of $X_{1,\alpha}$. By combining (4.23) and (4.38), we obtain directly that

$$X_{1,\alpha} = \Omega(\ln(s^\rho)) + X_{1,\alpha}^{(c)} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big), \qquad s\to+\infty,$$

where $X_{1,\alpha}^{(c)}$ is given by (5.8).

Asymptotics of $X_{2,\alpha}$. The expansion (4.21) implies that

$$X_{2,\alpha} = \int_{-i\infty}^{i\infty}\psi\Big(\frac{1+2\alpha-\nu_{\min}}{2}-w\Big)\Big\{\psi\Big(\frac{1+\nu_{\min}}{2}+w\Big) - \ln(w) - \frac{\nu_{\min}}{2(w-m)}\Big\}\frac{dw}{2\pi i} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big)$$

as $s\to+\infty$. We deform the contour of integration to infinity in the right half-plane and pick up infinitely many residue contributions at the points $m+k$, $k=1,2,\ldots$, of the form

$$-\Big\{\psi(k+1+\alpha) - \ln\Big(k+\frac{1+2\alpha-\nu_{\min}}{2}\Big) - \frac{\nu_{\min}}{2k}\Big\}$$

and one residue contribution $-\psi(1+\alpha)+\ln(m)-\frac{\gamma_{\mathrm{E}}\nu_{\min}}{2}$ at the point m. Using the identity (4.25), it follows that

$$\begin{aligned}
X_{2,\alpha} &= -\sum_{k=1}^{\infty}\Big\{\psi\Big(k+1+\alpha\Big) - \ln\Big(k+\frac{1+2\alpha-\nu_{\min}}{2}\Big) - \frac{\nu_{\min}}{2k}\Big\}\\
&\quad - \psi(1+\alpha) + \ln\Big(\frac{1+2\alpha-\nu_{\min}}{2}\Big) - \frac{\gamma_{\mathrm{E}}\nu_{\min}}{2} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big)\\
&= X_{2,\alpha}^{(c)} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big), \qquad s\to+\infty.
\end{aligned}$$

Asymptotics of $X_{3,\alpha}$. For $X_{3,\alpha}$, we first deform the contour $is^\rho\Sigma_5$ to $is^\rho\gamma_{b_2b_1}$, then we apply the change of variables $w = is^\rho\zeta$ and the large s asymptotics (4.38) of ψ. This gives

$$X_{3,\alpha} = \Omega\big(\ln(s^\rho)\big) + \bigg\{ -\frac{\nu_{\min}}{2} + \sum_{j=1}^{r} \frac{2\nu_j - \nu_{\min}}{2} - \sum_{k=1}^{q} \frac{2\mu_k - \nu_{\min}}{2} \bigg\} \int_{\gamma_{b_2b_1}} \frac{\ln(-i\zeta)}{\zeta} \frac{d\zeta}{2\pi i} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big)$$

as $s \to +\infty$. Using the second integral in (4.33), we obtain

$$X_{3,\alpha} = \Omega\big(\ln(s^\rho)\big) + X_{3,\alpha}^{(c)} + \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big), \qquad \text{as } s \to +\infty.$$

□

Proposition 5.3. *Let Z_α be defined by (5.6). Then*

$$Z_\alpha = \Omega\big(\ln(s^\rho)\big) + Z_\alpha^{(c)} + \mathcal{O}\Big(\frac{\ln s^\rho}{s^\rho}\Big) \tag{5.11}$$

as $s \to +\infty$ uniformly for α in compact subsets of $[0, +\infty)$, where

$$\begin{aligned} Z_\alpha^{(c)} = \frac{1}{4\pi i}\bigg\{ &-4\pi i \ln(|b_2|)(c_5 - c_6) + 4i\pi c_6 \ln\Big(\frac{2}{|b_2| + \operatorname{Im} b_2}\Big) \\ &+ \ln(b_1)(-2c_7 + i\pi(3c_5 + c_6)) - (c_5 + c_6)(\ln b_1)^2 \\ &+ \ln(b_2)(2c_7 + i\pi(c_5 - c_6)) + (c_5 + c_6)(\ln b_2)^2 + 2\pi^2 c_5 \bigg\}. \end{aligned} \tag{5.12}$$

Proof. A formula for the large s asymptotics of the function $\mathcal{F}(\zeta)$ defined in (5.5) can be deduced from (4.38) (with ν replaced by $2\alpha - \nu_{\min}$). This formula is uniform for $\zeta \in \gamma_{b_2b_1}$ and can be differentiated with respect to ζ. Hence

$$\mathcal{F}'(\zeta) = -\frac{1}{\zeta} + O(s^{-\rho}) \qquad \text{as} \qquad s \to +\infty, \tag{5.13}$$

uniformly for $\zeta \in \gamma_{b_2b_1}$. Substituting (5.13) and the asymptotic formula (4.39) for $p(\zeta)$ into the definition (5.6) of Z_α, we infer that

$$Z_\alpha = -\frac{c_4}{\rho}\ln(s^\rho) \int_{\gamma_{b_2b_1}} \frac{1}{\zeta}\frac{d\zeta}{2\pi i} + \int_{\gamma_{b_2b_1}} \frac{\mathcal{B}(\zeta)}{\zeta}\frac{d\zeta}{2\pi i} + \mathcal{O}\Big(\frac{\ln s^\rho}{s^\rho}\Big)$$

as $s \to +\infty$ uniformly for α in compact subsets of $(-1, +\infty)$, where $\mathcal{B}(\zeta)$ is defined by (4.40). Recalling (4.45), the proposition follows. □

5.3 Asymptotics of $I_{3,\alpha}(K)$ and $I_{4,\alpha}(K)$

We next show that the quantities $I_{3,\alpha}(K)$ and $I_{4,\alpha}(K)$ defined in (3.20) and (3.21) vanish as s tends to $+\infty$ for $K = s^\rho$.

Proposition 5.4. *Let $\nu > -1$, $K = s^\rho$, and let $N \geq 1$ be an integer. Then*

$$I_{3,\alpha}(K) = \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big) \quad \text{and} \quad I_{4,\alpha}(K) = \mathcal{O}\big(s^{-N\rho}\big) \tag{5.14}$$

as $s \to +\infty$ uniformly for α in compact subsets of $(-1, +\infty)$.

Proof. The proof for $I_{3,\alpha}(K)$ is similar to (but easier than) the proof of Proposition 4.4. In fact, it follows from (4.38) that

$$\psi\Big(\frac{1 + 2\alpha - \nu_{\min}}{2} - is^\rho\zeta\Big) = \mathcal{O}(\ln(s^\rho))$$

as $s \to +\infty$ uniformly for $\zeta \in \tilde{\sigma}$ (see the proof of Proposition 4.4 for the definition of $\tilde{\sigma}$), from which we immediately deduce that

$$I_{3,\alpha}(K) = \mathcal{O}\Big(\frac{\ln(s^\rho)}{s^\rho}\Big), \qquad \text{as } s \to +\infty.$$

The proof for $I_{4,\alpha}(K)$ is analogous to the proof of Proposition 4.5 and we omit the details. □

5.4 Integration of the differential identities in ν_ℓ and μ_ℓ

By using the results of the previous subsections, we can compute the large s asymptotics of $\partial_\alpha \ln C$ for any $\alpha \in \{\nu_1,\ldots,\nu_r,\mu_1,\ldots,\mu_q\}$. Integration with respect to α then yields the following proposition.

Proposition 5.5. *Let $r > q \geq 0$ be integers and suppose that $\nu_1,\ldots,\nu_r,\mu_1,\ldots,\mu_q > -1$. Let $\nu_{\min} = \min\{\nu_1,\ldots,\nu_r,\mu_1,\ldots,\mu_q\}$. If $\ell \in \{1,\ldots,r\}$, then*

$$\begin{aligned}\int_{\nu_{\min}}^{\nu_\ell} \partial_{\nu_\ell'}(\ln C) d\nu_\ell' &= (\nu_\ell - \nu_{\min})\Bigg(\sum_{\substack{j=1\\ j\neq \ell}}^{r} \nu_j - \sum_{k=1}^{q}\mu_k\Bigg)\big(\ln(r-q) - \ln(1+r-q)\big) \\ &\quad + \frac{\nu_\ell^2-\nu_{\min}^2}{2}\frac{1+r-q-(r-q)^2}{1+r-q}\ln(r-q) - (2+q-r)\frac{\nu_\ell^2-\nu_{\min}^2}{2}\ln(1+r-q) \\ &\quad + \ln G(1+\nu_\ell) - \ln G(1+\nu_{\min}) + (\nu_{\min}-\nu_\ell)\frac{\ln(2\pi)}{2}. \end{aligned} \tag{5.15}$$

If $\ell \in \{1,\ldots,q\}$, then

$$\begin{aligned}\int_{\nu_{\min}}^{\mu_\ell} \partial_{\mu_\ell'}(\ln C) d\mu_\ell' &= (\nu_{\min} - \mu_\ell)\Bigg(\sum_{j=1}^{r} \nu_j - \sum_{\substack{k=1\\ k\neq\ell}}^{q}\mu_k\Bigg)\big(\ln(r-q) - \ln(1+r-q)\big) \\ &\quad + \frac{\mu_\ell^2-\nu_{\min}^2}{2}\frac{1+r-q+(r-q)^2}{1+r-q}\ln(r-q) - (r-q)\frac{\mu_\ell^2-\nu_{\min}^2}{2}\ln(1+r-q) \\ &\quad - \ln G(1+\mu_\ell) + \ln G(1+\nu_{\min}) - (\nu_{\min}-\mu_\ell)\frac{\ln(2\pi)}{2}. \end{aligned} \tag{5.16}$$

Proof. We only consider the case $\alpha = \nu_\ell$, $\ell \in \{1,\ldots,r\}$. The case $\alpha = \mu_\ell$, $\ell \in \{1,\ldots,q\}$, is analogous. We start by integrating the term $X_{2,\nu_\ell}^{(c)}$ defined in (5.9). By Fubini's theorem, we can interchange the order of integration and summation, which implies

$$\begin{aligned}\int_{\nu_{\min}}^{\nu_\ell} X_{2,\nu_\ell'}^{(c)} d\nu_\ell' = \sum_{k=0}^{\infty}\Bigg\{ &-\ln\Gamma\Big(k+1+\nu_\ell\Big) - \nu_\ell + \Big(k + \frac{1+2\nu_\ell-\nu_{\min}}{2}\Big)\ln\Big(k+\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big) \\ &+ \frac{\nu_{\min}}{2}(\nu_\ell-\nu_{\min})\ln\Big(1+\frac{1}{k+1}\Big) + \ln\Gamma\Big(k+1+\nu_{\min}\Big) + \nu_{\min} \\ &- \Big(k+\frac{1+\nu_{\min}}{2}\Big)\ln\Big(k+\frac{1+\nu_{\min}}{2}\Big)\Bigg\}. \end{aligned}$$

Simplification gives (see (4.27), (4.28), and (4.30))

$$\begin{aligned}\int_{\nu_{\min}}^{\nu_\ell} X_{2,\nu_\ell'}^{(c)} d\nu_\ell' &= \frac{\ln(2\pi)}{2}(\nu_{\min}-\nu_\ell) + \ln G(1+\nu_\ell) - \ln G(1+\nu_{\min}) \\ &\quad + \frac{1+2\nu_\ell-\nu_{\min}}{2}\ln\Big(\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big) - \frac{1+\nu_{\min}}{2}\ln\Big(\frac{1+\nu_{\min}}{2}\Big) \\ &\quad - \zeta'\Big(-1;\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big) + \zeta'\Big(-1;\frac{1+\nu_{\min}}{2}\Big) + \frac{\nu_\ell}{2}(\nu_\ell-\nu_{\min}). \end{aligned} \tag{5.17}$$

On the other hand, part (b) of Lemma 3.1 and Propositions 5.1, 5.2, 5.3, and 5.4 imply that

$$\begin{aligned}\partial_{\nu_\ell}(\ln C) - X_{2,\nu_\ell}^{(c)} &= \ln\Gamma\Big(\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big) - \frac{\ln(2\pi)}{2} \\ &\quad + \Bigg(\sum_{\substack{j=1\\ j\neq\ell}}^{r}\nu_j - \sum_{k=1}^{q}\mu_k\Bigg)\ln(r-q) + \nu_\ell\frac{1+r-q-(r-q)^2}{1+r-q}\ln(r-q) \\ &\quad - \Bigg(\sum_{\substack{j=1\\ j\neq\ell}}^{r}\nu_j - \sum_{k=1}^{q}\mu_k\Bigg)\ln(1+r-q) - (2+q-r)\nu_\ell\ln(1+r-q). \end{aligned} \tag{5.18}$$

Using the identity (4.5), we can integrate the term in (5.18) involving the Gamma function:

$$\begin{aligned}\int_{\nu_{\min}}^{\nu_\ell} \ln\Gamma\Big(\frac{1+2\nu_\ell'-\nu_{\min}}{2}\Big)d\nu_\ell' = \frac{\ln(2\pi)}{2}(\nu_\ell-\nu_{\min}) - \frac{\nu_\ell}{2}(\nu_\ell-\nu_{\min})\\ - \frac{1-2\nu_\ell+\nu_{\min}}{2}\ln\Gamma\Big(\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big) + \frac{1-\nu_{\min}}{2}\ln\Gamma\Big(\frac{1+\nu_{\min}}{2}\Big)\\ - \ln G\Big(\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big) + \ln G\Big(\frac{1+\nu_{\min}}{2}\Big).\end{aligned}$$

Integrating also the other terms in (5.18) and utilizing (5.17), we arrive at

$$\begin{aligned}&\int_{\nu_{\min}}^{\nu_\ell} \partial_{\nu_\ell'}(\ln C)d\nu_\ell' = \frac{\ln(2\pi)}{2}(\nu_\ell-\nu_{\min}) - \frac{\nu_\ell}{2}(\nu_\ell-\nu_{\min}) - \frac{1-2\nu_\ell+\nu_{\min}}{2}\ln\Gamma\Big(\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big)\\ &+ \frac{1-\nu_{\min}}{2}\ln\Gamma\Big(\frac{1+\nu_{\min}}{2}\Big) - \ln G\Big(\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big) + \ln G\Big(\frac{1+\nu_{\min}}{2}\Big) - \frac{\ln(2\pi)}{2}(\nu_\ell-\nu_{\min})\\ &+ (\nu_\ell-\nu_{\min})\Big(\sum_{\substack{j=1\\ j\neq\ell}}^{r}\nu_j - \sum_{k=1}^{q}\mu_k\Big)\big(\ln(r-q)-\ln(1+r-q)\big) + \frac{\ln(2\pi)}{2}(\nu_{\min}-\nu_\ell)\\ &+ \frac{\nu_\ell^2-\nu_{\min}^2}{2}\frac{1+r-q-(r-q)^2}{1+r-q}\ln(r-q) - (2+q-r)\frac{\nu_\ell^2-\nu_{\min}^2}{2}\ln(1+r-q)\\ &+ \ln G(1+\nu_\ell) - \ln G(1+\nu_{\min}) + \frac{1+2\nu_\ell-\nu_{\min}}{2}\ln\Big(\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big) - \frac{1+\nu_{\min}}{2}\ln\Big(\frac{1+\nu_{\min}}{2}\Big)\\ &- \zeta'\Big(-1;\frac{1+2\nu_\ell-\nu_{\min}}{2}\Big) + \zeta'\Big(-1;\frac{1+\nu_{\min}}{2}\Big) + \frac{\nu_\ell}{2}(\nu_\ell-\nu_{\min}).\end{aligned}$$

In view of (4.54) and the identities $\Gamma(z+1) = z\Gamma(z)$ and $G(z+1) = \Gamma(z)G(z)$, this expression simplifies to (5.15). The proof of (5.16) is analogous. □

6 Proof of Theorem 1.1

We use the strategy described in Section 1.3 to prove Theorem 1.1. Let $C^{(\ell)}$ be the multiplicative constant arising in the large gap asymptotics for the point process induced by $\mathbb{K}^{(\ell)}$, $\ell \in \{0,\dots,r+q\}$, where $\mathbb{K}^{(\ell)}$ is given by (1.33). The final constant is given by

$$\ln C = \ln C_{r-q} + \sum_{\ell=1}^{r}\int_{\nu_{\min}}^{\nu_\ell}\partial_{\nu_\ell'}(\ln C^{(\ell)})d\nu_\ell' + \sum_{\ell=1}^{q}\int_{\nu_{\min}}^{\mu_k}\partial_{\mu_\ell'}(\ln C^{(r+\ell)})d\mu_\ell', \tag{6.1}$$

where C_{r-q} is the constant in (4.53) associated with $\mathbb{K}_{r-q}$. Proposition 4.6 with r replaced by $r-q$ gives

$$\begin{aligned}\ln C_{r-q} = (r-q)\ln G(1+\nu_{\min}) - \frac{\ln(2\pi)}{2}\nu_{\min}(r-q) - (r-q-1)\zeta'(-1)\\ + \frac{-2+(r-q)^2(r-q-1+12\nu_{\min}^2)}{24(r-q+1)}\ln(r-q) - \frac{(r-q-1)^2+12(r-q)\nu_{\min}^2}{24}\ln(1+r-q).\end{aligned} \tag{6.2}$$

On the other hand, Proposition 5.5 shows that, for $\ell = 1,\dots,r$,

$$\begin{aligned}\int_{\nu_{\min}}^{\nu_\ell}\partial_{\nu_\ell'}(\ln C^{(\ell)})d\nu_\ell' = (\nu_\ell-\nu_{\min})\Big(\sum_{j=1}^{\ell-1}\nu_j + (r-q-\ell)\nu_{\min}\Big)\big(\ln(r-q)-\ln(1+r-q)\big)\\ + \frac{\nu_\ell^2-\nu_{\min}^2}{2}\frac{1+r-q-(r-q)^2}{1+r-q}\ln(r-q) - (2+q-r)\frac{\nu_\ell^2-\nu_{\min}^2}{2}\ln(1+r-q)\\ + \ln G(1+\nu_\ell) - \ln G(1+\nu_{\min}) + (\nu_{\min}-\nu_\ell)\frac{\ln(2\pi)}{2}\end{aligned} \tag{6.3}$$

and, for $\ell = 1, \dots, q$,

$$\int_{\nu_{\min}}^{\mu_\ell} \partial_{\mu'_\ell}(\ln C^{(r+\ell)}) d\mu'_\ell = (\nu_{\min} - \mu_\ell)\bigg(\sum_{j=1}^{r} \nu_j - \sum_{k=1}^{\ell-1} \mu_k - (q-\ell)\nu_{\min}\bigg)\big(\ln(r-q) - \ln(1+r-q)\big)$$
$$+ \frac{\mu_\ell^2 - \nu_{\min}^2}{2} \frac{1+r-q+(r-q)^2}{1+r-q} \ln(r-q) - (r-q)\frac{\mu_\ell^2 - \nu_{\min}^2}{2} \ln(1+r-q)$$
$$- \ln G(1+\mu_\ell) + \ln G(1+\nu_{\min}) - (\nu_{\min} - \mu_\ell)\frac{\ln(2\pi)}{2}. \tag{6.4}$$

By substituting (6.2), (6.3), and (6.4) into (6.1), we find the expression (1.15). This completes the proof of Theorem 1.1. □

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