Two Results About Generic Non Cooperative Voting Games with Plurality Rule

Francesco De Sinopoli*

June, 1998

Abstract

In this paper we prove that for generic (non cooperative) voting games under plurality rule an equilibrium that induces a mixed distribution over the outcomes (i.e. with two or more candidates elected with positive probability) is isolated. From that we deduce also that the set of equilibrium distributions over outcomes is finite. Furthermore we offer an example (due to Govindan and McLennan) that shows the impossibility of extending such results to a general framework.

*CORE, Voie du Roman Pays 34, 1348 Louvain La Neuve, Belgium. The author thanks Professor Jean-Francois Mertens for his invaluable comments and suggestions in every step of the present research. The usual disclaimer applies.

In [2] it is proved that for generic normal form games the number of equilibrium points is finite and furthermore all equilibria are regular. In this paper we generalize the abstract model of a one stage voting procedure introduced in [6] in order to prove an "analogy" of Harsanyi's results for non cooperative voting games, in the case of plurality rule. More precisely given the set of voters (players), the set of candidates, the voting procedure and the payoffs that each voter gets in all the possible results of the election, we obtain a normal form game. Typically this game will be highly non generic, since the same outcome arises from many different pure strategy combinations. Hence in this class of games we cannot apply the results of [2], and it is clearly impossible to get the finiteness of the number of equilibria (i.e. for example, with plurality rule it is obvious that, independently of preferences, every strategy combination where the winner receives at least three votes more than any other candidate is an equilibrium). However we prove that, under plurality rule, for generic voting games the number of equilibrium distributions is finite. Moreover we find that, in this class of games, an equilibrium that induces a non completely degenerate distribution over outcomes is isolated. This result can be considered "analogous" to Harsanyi's regularity, since every regular equilibrium is isolated.

The finiteness of equilibrium distribution has also been proved, generically, in other classes of games. In [4] and [3] such a result is proved to hold for generic extensive form games and in [1] it is proved for games with two outcomes. Cf. also [1] for other related results.

In [1] an example is discussed of a game form for which there is an open set of payoffs, as function of the outcomes, such that each payoff in the set induces a game with a continuum of equilibrium distribution over outcomes. Since that game form could arise from a voting game, as we will show in the concluding section, it implies that our results cannot be extended to a very general class, even just of voting games.

1 The General Model

In this section we introduce and generalize the abstract model of a one stage voting procedure defined by Myerson and Weber in [6] .

Given the finite set of candidates $K = \{1, 2, ..., k\}$, each voter submits a ballot, which is a vector of k components, i.e. $v = (v_1, ...v_k)$, where v_1 votes are given to candidate 1, v_2 votes to candidate 2, and so on. An electoral system is then defined by the ballots that each voter can submit and by the election rule that, given the ballots cast, selects the winner from the set K. For example, if the election is done with plurality rule and there are three candidates, the strategy set of each voter is:

$$V_p = \{(1,0,0),(0,1,0),(0,0,1),(0,0,0)\}$$

where the last strategy represents the abstention. By contrast under approval voting the strategy set of each voter is given by:

$$V_a = \{(1,0,0),(0,1,0),(0,0,1),(0,0,0),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$$

while with Borda rule the strategy set is:

$$V_b = \{(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2),(0,0,0)\}$$

With the above three procedures the election rule selects the candidate that receives the largest total number of votes on all ballots cast. In this context we have to break ties. The most natural way to do this is to allow an equal probability lottery among the winners, or a given "statu quo", as under majority rule.

In general the strategy set can differ from player to player and, in addition, the election rule is not necessarily the one described above. Formally, given the set of voters, that in our case is finite, $N = \{1,2,...,n\}$ and the set of candidates, the voting procedure VP is defined by:

$$VP = \left(\left\{V^i\right\}_{i\in N}, D\right)$$

where V^i is the set of possible ballots that voter i can submit while D is the election rule. Denoting $V = \prod_{i=1}^{n} V^i$, if we want to allow the case that no candidate is elected:

$$D : V \rightarrow K \cup \{\emptyset\}$$

Given the set of voters N and the set of candidates K, the voting procedure VP determines the corresponding game form. To obtain the associated non cooperative game we need the collection of utility vectors $\{u^i\}_{i\in N}$, where $u^i = (u^i_1, ..., u^i_k)$ and each u^i_c represents the payoff that player i gets if candidate c is elected. In other words the triplet (N, K, VP) defines a family of games and each game in this family is identified with nk numbers. Typically the resulting normal form game is highly non generic since the same outcome arises from many different pure strategy combinations. For example with plurality rule each voter has $(k+1)$ pure strategies, therefore the corresponding normal form has $n(k+1)^n$ numbers. Hence we can not apply the results of Harsanyi [2] about the number of equilibria and their regularity for generic normal form games.

2 Two results under plurality rule

In this section we concentrate our attention on the model of plurality rule in order to prove that then, for generic voting games, an equilibrium which induces a mixed distribution over the elected candidates is isolated, and that hence the set of equilibrium distributions over outcomes is finite. The proof "basically" follows the same line as the one given by Harsanyi [2] to prove that for generic normal form games all equilibria are regular. It involves the definition of a map from the equilibrium graph to the space of games. It differs from Harsanyi's proof since where, for each unknown, he has one equation containing only this unknown, we have a linear system and furthermore, in our case, the construction has to be restricted to equilibria that induce a mixed distribution over outcomes. Moreover we exploit the fact that the map is semi-algebraic, instead of using Sard's theorem, to deduce that the results hold generically.

Given the set of candidates K and the set of voters N, the plurality rule determines the strategy space of each player. More precisely since each voter can either cast his vote for each candidate or abstain, we have that for each player i:

$$V^i = V = \{1, ..., k, 0\}$$

where each $c \in V - \{0\}$ is a vector of k components with all zeros except for a one in position c, and represents the vote for candidate c, while 0 is the zero vector representing the abstention. Denoting $K_0 = K \cup \{0\}$, the strategy space of each player is:

$$\Sigma = \Delta(K_0)$$

In the following we will use the superscript to indicate the voter and the subscript to denote the strategy, i.e. σ_c^i is the probability that player i votes for c while σ_0^i is the probability that he abstains. In order to determine the winner, we do not need to know the ballots cast by all the voters but it is enough to know their sum. Given a pure strategy vector $v \in V^n$, let $\omega = \sum_{i=1}^{n} v^i$. Clearly ω is a vector with k components and each coordinate represents the total number of votes obtained by the corresponding candidate; denoting by $p(c \mid v)$ the probability that candidate c is elected if v is played, we have:

$$p(c \mid v) = \begin{cases} 0 & if\ \exists m \in K\ s.t\ \omega_c < \omega_m \\ \frac{1}{q} & if\ \omega_c \geq \omega_m\ \forall m \in K\ and \\ & \#\{d \in K \mid \omega_d = \omega_c\} = q \end{cases} \tag{1}$$

Clearly we can extend (1) to mixed strategy vectors σ:

$$p(c \mid \sigma) = \sum p(c \mid v)\, \sigma(v) \tag{2}$$

where, as usual, $\sigma(v)$ denotes the probability of the strategy combination v under σ.

Denoting $p(\sigma) = (p(1 \mid \sigma), p(2 \mid \sigma), ..., p(k \mid \sigma))$, we have that $p(\sigma)$ is in Δ^K. Let's call a strategy combination σ *non-degenerate* if it leads to a non-degenerate distribution of candidates, i.e. if $p(\sigma)$ is not a vertex of Δ^K.

As mentioned in the previous section, given the set of players N, the set of candidates K and the plurality rule (PL), the corresponding game is defined by the utility vectors $\{u^i\}_{i \in N}$, where $u^i = (u_1^i, ..., u_k^i)$ and each u_c^i represents the payoff that player i gets if candidate c is elected.

The payoff that player i gets under σ is:

$$U^i(\sigma) = \sum_{c \in K} p(c \mid \sigma)\, u_c^i$$

Or in vector notation:

$$U^i(\sigma) = \langle p(\sigma), u^i \rangle \tag{3}$$

A strategy vector $\sigma \geq 0$ is a Nash equilibrium for the plurality game $\Gamma = (N, K, PL, u)$ if and only if it satisfies the following conditions:

$$\sigma_c^i \left[U^i(\sigma, c^i) - \max_{m \in K_0} U^i(\sigma, m^i) \right] = 0 \quad \forall i \in N,\ \forall c \in K_0 \tag{4}$$

$$\sum_{c=0}^{k} \sigma_c^i - 1 \quad = \quad 0 \quad \forall i \in N$$

where, as usual, (σ, c^i) denotes the strategy vector $(\sigma^{-i}, i$ votes for $c)$.

In the following we will also use the vector $\pi^c(\sigma^{-i})$ that expresses the difference in the probability distribution of the outcomes if player i votes for c or if he abstains, under the fixed strategy σ^{-i} of the others.

Definition 1 $\pi^c(\sigma^{-i}) = p(\sigma, c^i) - p(\sigma, 0^i)$.

The following results about the vector $\pi^c(\sigma^{-i})$ are quite obvious but since they will be extensively used, we prefer to put them in a separate lemma. Basically this just states that voting for a candidate, rather than abstaining, cannot increase the probability of the election of any other candidate.

Lemma 2 *The vector $\pi^c(\sigma^{-i})$ has k components (α) which sum up to zero and (β) the only component that can be positive is the c-th. (γ) Furthermore if $c^{\shortmid} \neq c$ and $p(c^{\shortmid} \mid \sigma, 0^i) = 0$ then $\pi^c_{c^{\shortmid}}(\sigma^{-i}) = 0$.*

Proof. (α) It is obvious for any vector which is a difference of two probability vectors. (β) For every fixed v^{-i}, if player i votes for c rather than abstaining, the total number of votes of any other candidate remains the same while that one for c increases, so the probability of electing any other candidate cannot increase. (γ) If a candidate has zero probability to be elected if player i abstains, what above implies that he cannot be elected even if i votes for every other candidate, hence the result. ■

Since we are interested in generic voting games (i.e. in generic point of $\Re^{nk}$) we can and will henceforth assume, without loss of generality, that no player is indifferent between two candidates. Formally it can be stated as:

Assumption 1: $\forall i \in N,\ u^i_c \neq u^i_{c^{\shortmid}}\ \ \forall c, c^{\shortmid} \in K,\ c \neq c^{\shortmid}$.

Under assumption 1 it is possible to prove that if at least two candidates are elected with positive probability under an equilibrium σ, then abstention is not a best reply and every player votes only for candidates for which his vote increases their probability of being elected as compared to the case when he abstain; and clearly he must then decrease the probability of election of some candidate that he prefers less, else abstention itself would be a better reply. This is the content of the next lemma and its corollary:

Lemma 3 *For every player i, if σ is a non-degenerate equilibrium*[1] *then abstention is not a best reply for i (hence $\sigma_0^i = 0$).*

Proof. Let's define:
$K_i^\sigma = \{c \in K \mid p(c \mid \sigma, 0^i) > 0\}$
$\tilde{K}_i^\sigma = \left\{ c \in K \mid \Pr(\omega_c \geq \max_{c' \in K} \omega_{c'} - 1 \mid (\sigma, 0^i)) > 0 \right\}$
$M = \arg\max_{c \in \tilde{K}_i^\sigma} r_c^i$

We will prove the lemma by contradiction, so let's suppose that the abstention is a best reply for player i.

From lemma 2 we know that:

$$U^i\left(\sigma, M^i\right) - U^i\left(\sigma, 0^i\right) = \left\langle \pi^M\left(\sigma^{-i}\right), u^i \right\rangle \geq 0 \tag{5}$$

where the inequality in (5) follows from the fact that $\pi^M\left(\sigma^{-i}\right)$ is a vector whose components sum up to zero and the only component that can be positive is the M-th which corresponds to the maximum of u^i (i.e. if $c' \notin \tilde{K}_i^\sigma$ then $\pi_{c'}^M\left(\sigma^{-i}\right) = 0$).

Under assumption 1, the inequality in (5) is strict if $\pi_M^M\left(\sigma^{-i}\right)$ is strictly greater than zero. Given the strategy of the others and the abstention of player i, this happens when, with positive probability, either candidate M has only one vote less than the winner(s) or when he gets the maximum number of votes, but he is not the only one. Then voting for M yields a strict improvement as compared to abstention except if, with probability one under $(\sigma, 0^i)$, either $\omega_M > \max_{c \neq M} \omega_c$ or $\omega_M < \max_{c \neq M} \omega_c - 1$. Using then that σ^{-i} is a product, so we can change one coordinate at a time, we deduce that only one of the two cases can have positive probability (if $0^i \in BR\left(\sigma^{-i}\right)$), hence probability one. This cannot be true for the second since $M \in \tilde{K}_i^\sigma$, hence it must be $K_i^\sigma = \{M\}$.

Then any σ^i of player i can only lead to a set of candidates having positive probability of election that consists of M and possibly some other candidates less preferred by i. Since $\{M\}$ itself is feasible for i, by abstaining, and any other such set is less preferred by i, it must be that under any best reply only M gets elected, which contradicts the hypothesis. ■

Corollary 4 *Under the same assumptions:*

[1] Or just if σ is a *non-degenerate* strategy combination and $\sigma^i \in BR\left(\sigma^{-i}\right)$.

(α) $\sigma_c^i > 0 \Longrightarrow \langle \pi^c(\sigma_{-i}), u^i \rangle > 0$
(β) $\sigma_c^i > 0 \Longrightarrow \exists m \prec_i c : \pi_m^c(\sigma_{-i}) < 0$
Proof. (α)It easy follows from the previous lemma, since if voting for candidate c does not increase, as compared to the abstention, the expected utility of player i, it cannot be an his best reply. (β) If voting for candidate c does not decrease the probability of election of any less preferred candidate, this strategy cannot give player i an higher expected payoff than abstention. ■

Let's fix a support $C \subseteq V^n$, a strategy vector $v_* \in C$ and let's denote $L_i = \{c \in K_0 \ s.t. \ c^i \in C^i - \{v_*^i\}\}$.

Take a game u that has a *non-degenerate* equilibrium σ with $C(\sigma) = C$. Then the equilibrium conditions (4) can be written as:

$$\begin{aligned} \sigma_c^i \left[U^i(\sigma, c^i) - U^i(\sigma, v_*^i) \right] &= 0 \quad \forall i \in N, \ \forall c^i \in K_0 - \{v_*^i\} \\ \sum_{c=0}^{k} \sigma_c^i - 1 &= 0 \quad \forall i \in N \end{aligned} \tag{6}$$

Denoting by $\pi_*^c(\sigma^{-i})$ the difference in the probability distribution of the outcomes if player i votes for c or if he plays v_*^i, under the fixed strategy σ^{-i} of the others, we have:

$$\pi_*^c(\sigma^{-i}) = p(\sigma, c^i) - p(\sigma, v_*^i) = \pi^c(\sigma^{-i}) - \pi^{v_*^i}(\sigma^{-i}) \tag{7}$$

Substituting (3) and (7) in (6) we get:

$$\sigma_c^i \langle \pi_*^c(\sigma^{-i}), u^i \rangle = 0 \quad \forall i \in N, \ \forall c^i \in K_0 - \{v_*^i\} \tag{8}$$

Denoting $l_i = \#L_i = \#C^i - 1$ and $l = \sum_{i=1}^{n} l_i$, define the following subvectors of $u \in \Re^{nk}$: $u_* \in \Re^n$, that specifies for each player the utility that he gets if the candidate that he votes for in v_* is elected (i.e. lemma 3 assures us that C contains no abstention), $u_{**} \in \Re^{nk-l-n}$, that specifies for each voter his utility for the candidates for whom he does not vote, and $u^\circ \in \Re^l$, that specifies for each player i his utility for the candidates in L_i. To simplify the notation let $u^* = (u_*, u_{**})$. Clearly $u = (u^*, u^\circ)$. The equilibrium conditions in (8) imply:

$$\langle \pi_*^c(\sigma^{-i}), u^i \rangle = 0 \quad \forall i, \ \forall c \in L_i \tag{9}$$

Given σ, v_* and u^*, (9) is a linear system of l equations in l unknowns, the u°. The system can be written as:

$$\left\langle \pi_*^{\circ c}\left(\sigma^{-i}\right), u^{\circ i}\right\rangle = \left\langle -\pi_*^{*c}\left(\sigma^{-i}\right), u^{*i}\right\rangle \quad \forall i,\ \forall c \in L_i$$

where $\pi_*^{\circ c}\left(\sigma^{-i}\right)$ and $u^{\circ i}$ are vectors of dimension l_i, while $-\pi_*^{*c}\left(\sigma^{-i}\right)$ and u^{*i} are vectors of dimension $K - l_i$. Since for every player i we have l_i equations the system can be written as:

$$\Pi_*^\circ\left(\sigma^{-i}\right) u^{\circ i} = -\Pi_*^*\left(\sigma^{-i}\right) u^{*i} \quad \forall i$$

where $\Pi_*^\circ\left(\sigma^{-i}\right)$ and $u^{\circ i}$ are, respectively, a square matrix and a column vector of dimension l_i, while $-\Pi_*^*\left(\sigma^{-i}\right) u^{*i}$ is a column vector. The whole system has the following matrix representation:

$$\Pi_*^\circ(\sigma)\, u^\circ = -\Pi_*^*(\sigma)\, u^* \tag{10}$$

The system in (10) has an unique solution if the square matrix $\Pi_*^\circ(\sigma)$ of dimension l is not singular. The following lemma will prove it.

Lemma 5 *The matrix $\Pi_*^\circ(\sigma)$ has a strictly positive determinant, hence it is not singular.*

Proof. The matrix $\Pi_*^\circ(\sigma)$ is block diagonal, where each block coincides with a matrix $\Pi_*^\circ\left(\sigma^{-i}\right)$, hence if each $\Pi_*^\circ\left(\sigma^{-i}\right)$ has a strictly positive determinant $\Pi_*^\circ(\sigma)$ also has. As mentioned $\Pi_*^\circ\left(\sigma^{-i}\right)$ is a square matrix of dimension l_i. The (c, m) entry of the square matrix $\Pi_*^\circ\left(\sigma^{-i}\right)$ is given by the probability that candidate m is elected if player i votes for c minus the probability that m is elected if i plays v_*^i, with $c, m \in L_i$. By definition:

$$\pi_*^c\left(\sigma^{-i}\right) = p\left(\sigma, c^i\right) - p\left(\sigma, v_*^i\right) = \pi^c\left(\sigma^{-i}\right) - \pi^{v_*^i}\left(\sigma^{-i}\right)$$

then the (c, m) element of the matrix $\Pi_*^\circ\left(\sigma^{-i}\right)$ is:

$$\Pi_*^\circ\left(\sigma^{-i}\right)_{cm} = \left(a_{cm} + b_m\right) \qquad c, m \in L_i$$

where:

$$a_{cm} = \pi_m^c\left(\sigma^{-i}\right) \text{ and } b_m = -\pi_m^{v_*^i}\left(\sigma^{-i}\right)$$

Lemma 2 implies that the matrix $A = (a_{cm})$ is an improper Minkowsky-matrix (i.e. a matrix such that all non diagonal elements are non positive and the sum of each row is non negative) and corollary 4(α) implies that all the diagonal elements of A are strictly positive. The next step is to show that the matrix A is dominant diagonal (in the definition of McKenzie [5]p.47). To this end let's define the following functions:

$$s_c(d) = a_{cc}d_c - \sum_{m \in L_i/c} |a_{cm}| \, d_m = \sum_{m \in L_i} a_{cm}d_m \quad c \in L_i$$

We have to show that a $d^* \gg 0$ exists such that $s_c(d^*) > 0 \ \forall \ c \in L_i$. By lemma 2($\alpha$) we know that $\sum_{m \in K} a_{cm}$ is equal to zero and by corollary 4(α) we know that $\sum_{m \in K} a_{cm}u^i_m > 0$ hence for every $\varepsilon > 0$, $\sum_{m \in K} a_{cm}\varepsilon u^i_m > 0$. For ε sufficiently close to zero $1 + \varepsilon u^i_m > 0 \ \ \forall m \in K$. Choose an $\tilde{\varepsilon}$ that satisfies these conditions: hence $d^* = \vec{1} + \tilde{\varepsilon}u^i \gg 0$ and $s_c(d^*) > 0 \ \ \forall c \in L_i$. In fact

$$s_c(d^*) \geq \sum_{m \in K} a_{cm}(1 + \tilde{\varepsilon}u^i_m) > 0 \quad \forall c \in L_i$$

where the first inequality follows from lemma 2(β) and the positivity of each $1 + \tilde{\varepsilon}u^i_m$, and the second from lemma 2(α) and the positivity of each $\sum_{m \in K} a_{cm}\tilde{\varepsilon}u^i_m$. This proves that the matrix A has a (positive) dominant diagonal and implies that all the principal minors of A are strictly positive (cf. [7] Th.21.1 p.386). Then the matrix A is an M-matrix, in the definition of Ostrowsky ([8] p.95).

In [8] (p.97) the following result is contained:

If Q is an $n \times n$ M-matrix then:

$$|q_{ij} + d_j| \geq |q_{ij}| > 0 \quad for \ d_j \geq 0 \ \ (j = 1, \ldots, n)$$

This result with the fact that $b_m \geq 0$ (lemma 2(β)) implies the claim of the lemma. ■

Now we can prove the following theorem:

Theorem 6 *For generic $u \in \Re^{nk}$, a non-degenerate equilibrium is isolated.*

Proof. To prove the theorem it is enough to prove that, for every possible support C, except for a closed set of measure zero, there is a finite number of *non-degenerate* equilibria with support C (i.e. since the number of players and candidates is finite also the set of possible supports is finite and moreover the continuity of the distribution over candidates implies that if there is a finite number of *non-degenerate* equilibria each one is isolated). For every C, let's fix a corresponding pure strategy combination $v_* \in C$. Then let's consider a support C. Lemma 5 implies that given a *non-degenerate* equilibrium σ, with support C, and the corresponding subvector u^* we can reconstruct uniquely the entire vector u.

Let's define the following sets:

$I = \left\{ u \in \Re^{nk} \mid \exists i, c, c^{\text{l}} \; s.t. \; u^i_c = u^i_{c^{\text{l}}} \right\}$

$\bar{\Re}^{nk} = \Re^{nk} - I$

$E^C = \left\{ (u, \sigma) \mid u \in \bar{\Re}^{nk}, \; C(\sigma) = C, \; \sigma \text{ is a } \textit{non-degerate} \text{ eq. of } u \right\}$

(i.e. $\bar{\Re}^{nk}$ is the set of games that satisfy assumption 1 and the set I is a lower dimensional set, while E^C is the graph of the correspondence that associates to each game in $\bar{\Re}^{nk}$ its *non-degenerate* equilibria with support equal to C). Denoting $E^C_* = proj_{\left(\Sigma^n \times \bar{\Re}^{nk-l}\right)} E^c$, the above construction implies that there is a function $F^{C_*} : E^C_* \to \bar{\Re}^{nk}$ that maps (u^*, σ) into $u = (u^*, u^\circ)$.

$$F^{C_*} : \begin{array}{l} u^* = u^* \\ u^\circ = \left(\Pi^\circ_* (\sigma)\right)^{-1} \left(-\Pi^*_* (\sigma) \, u^*\right) \end{array}$$

The map F^{C_*} is continuous since it involves additions, subtractions, multiplications and only the division by the determinant of $\Pi^\circ_* (\sigma)$, greater than zero by lemma 5. Furthermore the sets E^C_* and $\bar{\Re}^{nk}$ are semi-algebraic as well as the map F^{C_*} and the dimension of E^C_* is at most nk (i.e. since we have fixed C, $\dim (\Sigma^n \mid C) = l$). We can apply the General Local Triviality theorem (cf. [9]p.20) to deduce that the set of games such that their inverse image has dimension greater than zero in E^C_* has dimension lower than nk. If the inverse image of a game u has dimension zero then it implies that u has a finite number of *non-degenerate* equilibria (i.e. a 0-dimensional semi-algebraic set is finite). So we have to prove that

$$\dim \left\{ u \in \bar{\Re}^{nk} \mid \dim F^{C_*^{-1}} (u) > 0 \right\} < nk \tag{11}$$

The General Local Triviality theorem assures us that there is a semi-algebraic set $B \subset \bar{\Re}^{nk}$ with $\dim B < nk$ such that for each of the (finite number of) connected components D_j of $\bar{\Re}^{nk} \backslash B$ there is a semi-algebraic set Q_j (the "fiber") and a semi-algebraic homeomorphism $h_j : D_j \times Q_j \to F^{C_*^{-1}} (D_j)$ such that $F^{C_*} \circ h_j (u, q) = u$ for each $u \in D_j$, $q \in Q_j$.

Clearly if D_j and Q_j are as above we have

$$\dim F^{C_*^{-1}}(D_j) = \dim D_j + \dim Q_j$$

and for $u \in D_j$

$$\dim F^{C_*^{-1}}(u) = \dim h_j^{-1}\left(F^{C_*^{-1}}(u)\right) = \dim(\{u\} \times Q_j) = \dim Q_j$$

hence

$$\dim F^{C_*^{-1}}(u) = \dim F^{C_*^{-1}}(D_j) - \dim D_j$$

Since the dimension of E_*^C is at most nk, if $\dim D_j = nk$ the inverse image of a point $u \in D_j$ is not positive. Hence the set of games that have an infinite number of *non-degenerate* equilibria with support C is contained in the union (T) of I and B with the components D_j of dimension strictly less than nk. Since each member of this (finite) union is semi-algebraic with dimension less than nk, also T is a semi-algebraic set and $\dim T < nk$. Then the closure of T has dimension lower than nk, hence measure zero in $\Re^{nk}$. Since each game $u \in \Re^{nk} - \bar{T}$ has a finite number (possibly zero) of *non-degenerate* equilibria with support C, the result follows. ■

Remark 1 *Since the map F^{C_*} is smooth and E_*^C is semi-algebraic, Sard's theorem offers an alternative way to prove the above result.*

With the theorem above it is easy to prove:

Theorem 7 *For generic $u \in \Re^{nk}$, the set of equilibrium distributions over outcomes is finite.*

Proof. This immediately follows from theorem 6 and the fact that the number of connected components of equilibria is finite ([3] Proposition 1). In fact theorem 6 implies that, for generic games every equilibrium in the same connected component induces the same distribution over the outcomes, hence the result. ■

3 Conclusion

The results proved in the previous section imply that, in spite of the typically infiniteness of the set of Nash equilibria, for almost all plurality games we have a finite number of equilibrium distributions over outcomes.

Theorem 7 assures us that, for generic plurality games, every equilibrium in the same connected component is outcome equivalent. In a way we are in a situation analogous to an extensive form game where the behavior out of the equilibrium path has some "degree of freedom" while here it is the behavior of a not decisive voter that has such a "degree of freedom".

It would be interesting if the above results could be extended to other voting systems, but all the proofs given here depend crucially upon the plurality. Furthermore the following example, due to Govindan and McLennan shows that the above results cannot be extended to the general framework.

$$\begin{array}{cc} & U \\ & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} N \\ E \\ S \\ W \end{array} & \begin{pmatrix} a & a \\ b & b \\ a & b \\ e & f \end{pmatrix} \end{array} \qquad \begin{array}{cc} & D \\ & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} N \\ E \\ S \\ W \end{array} & \begin{pmatrix} c & c \\ d & d \\ c & d \\ e & f \end{pmatrix} \end{array} \tag{12}$$

In [1] it is shown that for the following game form there is an open set of games such that everyone in this set has a continuum of completely mixed equilibria inducing a one dimensional set of distributions over outcomes. For a complete discussion of this game we refer to [1] (pp.6-8). In this context it is enough to notice that, as pointed out by Govindan and McLennan, the crucial aspect is that "the lottery over outcomes resulting from S is a convex combination of the lottery resulting from N and E, so that whenever agent 1 is indifferent between N and E, she is also indifferent between either of these strategies and S." [1](p.7).

It is not difficult to rephrase this game in terms of a voting game. Since there are six "candidates" each strategy is a vector with 6 components. Using the same election rule as in the plurality case it is easy to find out a set of eight vectors associated to the pure strategies of (12) that gives rise to the same normal form game. For example, the following will do the trick.

$U = (3, 3, 0, 0, 0, 0)$
$D = (0, 0, 3, 3, 0, 0)$
$L = (1, 0, 1, 0, 1, 0)$
$R = (0, 1, 0, 1, 0, 1)$
$N = (3, 0, 3, 0, 0, 0)$
$E = (0, 3, 0, 3, 0, 0)$
$S = (3, 3, 3, 3, 0, 0)$
$W = (0, 0, 0, 0, 5, 5)$

If we try to apply the same kind of proof made for the plurality rule to this "voting procedure" we get that it falls in lemma 5, for the above mentioned equivalence between the strategy S and a convex combination of N and E.

This example shows the impossibility of extending our results to a general framework but still leaves open the problem for other voting procedure such as Borda or approval.

References

[1] Govindan S. and A. McLennan (1997): "On the Generic Finiteness of Equilibrium Outcome Distributions in Game Forms" mimeo

[2] Harsanyi J.C. (1973): "Oddness of the Number of Equilibrium points: A New Proof" *Internat. J. Game Theory* vol.2, pp.235-250.

[3] Kohlberg E. and J.F. Mertens (1986): "On the Strategic Stability of Equilibria" *Econometrica* vol.54, pp.1003-1037.

[4] Kreps D. and R. Wilson (1982): "Sequential Equilibria" *Econometrica* vol.50, pp. 863-894.

[5] McKenzie L.W. (1960): "Matrices with Dominant Diagonal and Economic Theory" in *Mathematical Methods in Social Sciences*, ed. by Arrow, Karlin and Suppes, Stanford , Calif., Stanford University Press.

[6] Myerson R.B. and R.J. Weber (1993): "A Theory of Voting Equilibria" *Amer. Pol. Sc. Rev.* vol.87, pp.102-114.

[7] Nikaido H. (1968): Convex Structure and Economic Theory. New York, USA: Academec Press.

[8] Ostrowsky A.M. (1995): "Note on Bounds for Some Determinants" *Duke Math. J.* vol.22, pp.95-102.

[9] Schanuel S.H., L.K. Simon and W.R. Zame (1991): "The Algebraic Geometry of Games and the Tracing Procedure" in *Game Equilibrium Models, Vol. II: Methods, Morals and Markets*, ed. by Selten . Berlin, Springer-Verlag.