Optimal Growth Models with Bounded or Unbounded returns: A Unifying Approach

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October, 2000

Abstract

In this paper we propose a unifying approach to study optimal growth models with bounded or unbounded returns (above/below). We prove existence of optimal solutions. We prove also, without using contraction method, that the Value function is the unique solution to the Bellman equation in some classes of functions. The value function can be obtained by the usual algorithm defined by the operator provided by the Bellman equation. The well-known results, and those in Alvarez and Stokey (1998) can be obtained from this paper.

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1 Introduction

In infinite discrete time horizon optimal growth models, a limitation to the use of the tools of dynamic programming is the lack of a general approach for the cases where returns are unbounded. But when these ones are bounded below and bounded above by some function φ, the problem is overcome by introducing a Banach space of functions with a φ -sup-norm (see e.g. Boyd III, 1989, Duran, 2000). The value function will be the unique solution to the Bellman equation in this Banach space since it is a fixed point of a contraction mapping which is the operator defined by the Bellman equation.

When the returns are unbounded below, attempts to overcome the difficulty were briefly done in Boyd III (1990) by imposing an upper bound and a lower bound on the rate of growth of the state variables. Another very interesting approach is proposed by Alvarez and Stokey (1998) when the returns are homogeneous of degree θ or logarithm. But they impose that the technology is of constant returns to scale. Thanks to these assumptions, the value function V is either homogeneous of degree θ or of the type:

$$V(k) = V(\frac{k}{\|k\|}) + \frac{Ln(\|k\|)}{(1-\beta)}, \forall k \neq 0.$$

Here β is the discount factor which is less than one for the logarithm case.

They therefore consider the Bellman equation in the class of functions which are either homogeneous of degree θ, or of the previous type if the returns function is logarithm. When θ is positive, or when the returns function is logarithm, they use a contraction theorem in these kinds of Banach spaces.

The main limit of their approach is to exclude the very usual case with strictly decreasing returns, or the case with increasing returns.

In this paper we try to overcome this difficulty. But we would like also to propose a synthetic frame for optimal growth models with bounded or unbounded (below/above) returns. The returns could be of any type, i.e. decreasing, increasing or constant returns to scale. We prove existence of optimal solutions. The value function is upper-semi-continuous. It is the unique solution to the Bellman equation in some class of functions. This result is proved without contraction mapping technique. And as in the standard cases, the Bellman equation will provide an algorithm to find the value function. The well-known models in dynamic programming or the ones of Alvarez and Stokey (1998) will be particular cases of our setting.

Our paper is organized as follows:

In section 2 we present the model. Existence of optimal paths and properties of the value function are given in section 3. In section 4 we show that the value function is the unique upper-semi-continuous solution to the Bellman equation in a class of upper-semi-continuous functions which verify transversality conditions. In order to obtain that the value function is continuous, we add some more assumptions which are satisfied in the usual cases or in the models of Alvarez and Stokey (1998) (but they are also satisfied in non standard models) (section 5). In section 6, we show that the operator defined by the Bellman equation provides an algorithm to find the value function. In section 7, we apply our results to several models : bounded from below returns, Alvarez and Stokey models, homogeneous or logarithm criteria with strictly decreasing returns to scale technology, human capital model, learning by doing model.

2 The Model

In this section, we present the model and its assumptions. We make a few comments on the assumptions and introduce two lemmas about the set-up of the model.

Consider the following optimal growth model:

$$\left\{ \begin{array}{ll} \text{Maximize} \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) & \\ & \\ \text{s.t.} \quad x_{t+1} \in \Gamma(x_t), & \\ \qquad\ \ x_0 \in I\!R_+^n \text{ is given.} & \end{array} \right.$$

We assume that $\forall t, x_t \in I\!R_+^n$.

We make the following assumptions:

(H1) Γ is a continuous, non-empty compact-valued correspondence from $I\!R_+^n$ into $I\!R_+^n$.
$0 \in \Gamma(0)$.

Let $\Pi(x_0)$ denote the set of feasible sequences from x_0:

$$\Pi(x_0) = \left\{ \tilde{x} = (x_1, ..., x_t, ...) \in (I\!R_+^n)^{+\infty} / \forall t \geq 0, x_{t+1} \in \Gamma(x_t) \right\}$$

(H2) $\exists \gamma \geq 0, \gamma \neq 1, \gamma' \geq 0,$ such that $y \in \Gamma(x) \Rightarrow \|y\| \leq \gamma\|x\| + \gamma'$.
(Note that assuming that $\gamma \neq 1$ is not a restriction since if $\gamma = 1$ then take $\gamma = 1 + \varepsilon$ with $\varepsilon > 0$.)

(H3) $\forall x_0 \neq 0, \exists \tilde{x} \in \Pi(x_0)$ such that $\lim_{T \to +\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) > -\infty$.

(H4) The function $F : \text{graph}\Gamma \to I\!R \cup \{-\infty\}$ is continuous in any $(x, y) \in \text{graph}\Gamma$ such that $F(x, y) > -\infty$.
If $F(x, y) = -\infty$ and if $\lim_n (x^n, y^n) = (x, y)$ then $\lim_n F(x^n, y^n) = -\infty$.

(H5) $\beta > 0$.

(H6) $\forall x_0 \in I\!R^n_+, \forall \mathcal{V}(x_0)$ neighborhood of x_0 in $I\!R^n_+$, $\forall \varepsilon > 0, \exists T_0$ such that $\forall T \geq T_0, \forall x'_0 \in \mathcal{V}(x_0), \forall \tilde{x'} \in \Pi(x'_0)$, one has:
$\sum_{t=T}^{+\infty} \beta^t F^+(x'_t, x'_{t+1}) \leq \varepsilon$ where $F^+(x_t, x_{t+1}) = Max\,(0, F(x_t, x_{t+1}))$.

Lemma 1 *If* **(H2)** *holds , then* $\forall x_0 \in I\!R^n_+, \forall \tilde{x} \in \Pi(x_0)$, *one has:*

$$\forall t, \|x_t\| \leq \gamma^t \|x_0\| + \frac{1 - \gamma^t}{1 - \gamma}\gamma'$$

.

Proof: It can be easily proved by induction.

Q.E.D.

We will now introduce two assumptions which together with **(H2)** and **(H5)** induce **(H6)**:

(H7) $\forall (x, y) \in \text{graph}\Gamma, F(x, y) \leq A + B(\|x\| + \|y\|)$ with $A \geq 0, B \geq 0$.

(H8) $\beta < 1; \beta\gamma < 1$.

Comments on the assumptions

(H1) is quite standard. But we do not constraint that $\Gamma(0) = \{0\}$. In some models [1], this assumption may be false.

[1] see example 6 on the "Learning By Doing" model.

(H2) is usual. Note that we can replace **(H2)** by the following assumption: For any $x_0 \in I\!R^n_+$, there exists a neighborhood of x_0 in $I\!R^n_+$, $\mathcal{V}(x_0)$, there exists a sequence of compact sets of $I\!R^n_+$, $\{K_t(x_0)\}$, such that for any $x'_0 \in \mathcal{V}(x_0)$, for any $\tilde{x}' \in \Pi(x'_0)$, one has $x'_t \in K_t(x_0), \forall t \geq 0$.
This assumption may allow technologies with increasing returns. Observe that from Lemma 1, if **(H2)** holds, then the previous assumption also holds.

(H3) implies that $V(x_0) > -\infty$ for any $x_0 \neq 0$.
This assumption is satisfied in a one sector model where the technology is described by a concave function f which satisfies $f(0) = 0$ and $f'(0) > 1$.
Indeed, consider the model

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \\ \text{s.t.} \quad c_t + k_{t+1} \leq f(k_t), \\ \qquad\quad k_0 > 0 \text{ is given.} \end{cases}$$

with $\beta \in [0,1[$, and u concave.
For any $k_0 > 0$, let $\varepsilon > 0$ be such that $\varepsilon < k_0$ and $f'(\varepsilon) > 1$. Then we have $\varepsilon < f(\varepsilon)$. The sequence $\tilde{k} = (k_0, \varepsilon, \varepsilon, ..., \varepsilon, ...)$ is feasible and

$$\sum_{t=0}^{+\infty} \beta^t F(k_t, k_{t+1}) = \sum_{t=0}^{+\infty} \beta^t u(f(k_t) - k_{t+1}) = u(f(k_0) - \varepsilon) + \frac{\beta}{1-\beta} u(f(\varepsilon) - \varepsilon) > -\infty.$$

Hence **(H3)** is satisfied.
Becker, Boyd III and Foias (1991) impose in their Axiom of finitude (p.457) that $\sum_{t=0}^{+\infty} \beta^t u(f(\varepsilon) - \varepsilon f'(\varepsilon)) > -\infty$. One can see that $f(\varepsilon) - \varepsilon f'(\varepsilon) < f(\varepsilon) - \varepsilon$. Hence their condition implies **(H3)**. (But their paper deals with a more general utility function, the recursive preference).

(H4) and **(H5)** are usual.

(H6) is by far the most important assumption. It plays a crucial role in the proof of the u.s.c. of the value function. It is satisfied in the usual cases where **(H7)** and **(H8)** are fulfilled. It is also satisfied in the case[2] where the utility function is logarithm or homogeneous of degree θ, ($\theta < 0$).

(H7) and **(H8)** are quite usual. In particular, they ensure together with **(H2)** that $V(x_0) < +\infty$.

[2]see Section 7 about the examples.

Proposition 1 **(H2)**, **(H5)**, **(H7)** *and* **(H8)** *induce* **(H6)**.

Proof:
Consider $x_0 \in I\!R^n_+$. Under **(H2)**, Lemma 1 is true. Then, by **(H7)**:

$$F^+(x_t, x_{t+1}) \leq A + B\gamma' + B(1+\gamma)(\gamma^t \|x_0\| + \frac{1-\gamma^t}{1-\gamma}\gamma').$$

Then:

$$\sum_{t=T}^{+\infty} \beta^t F^+(x_t, x_{t+1}) \leq \frac{A\beta^T}{1-\beta} + \frac{2\beta\gamma'}{|1-\gamma|}\frac{\beta^T}{1-\beta} + \frac{\beta(1+\gamma)}{1-\gamma\beta}(\gamma\beta)^T \|x_0\|.$$

Let $\Phi(x) := \frac{A}{1-\beta} + \frac{2\beta\gamma'}{|1-\gamma|(1-\beta)} + \frac{\beta(1+\gamma)}{1-\gamma\beta}\|x\|$. Then:

$$\sum_{t=T}^{+\infty} \beta^t F^+(x_t, x_{t+1}) \leq max\{\beta^T, (\gamma\beta)^T\}\Phi(x_0).$$

Since Φ is continuous, if $\mathcal{V}(x_0)$ is a neighborhood of x_0, we define $C := sup\{\Phi(x); x \in \mathcal{V}(x_0)\}$. Given $\varepsilon > 0$, choose T_0 such that

$$max\{\beta^{T_0}, (\gamma\beta)^{T_0}\} < \varepsilon.$$

Q.E.D.

Remark 1 *Assumptions* **(H5)** *and* **(H6)** *imply that* $\forall x_0 \in I\!R^n_+$, *the sum* $\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$ *is meaningful and equals the limit of* $\sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$ *when* $T \to +\infty$.
Indeed, one has:

$$\sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) = \sum_{t=0}^{T} \beta^t F^+(x_t, x_{t+1}) + \sum_{t=0}^{T} \beta^t F^-(x_t, x_{t+1})$$

By **(H6)** *the sum* $\sum_{t=0}^{+\infty} \beta^t F^+(x_t, x_{t+1})$ *is finite.*
By **(H5)**, *the sum* $\sum_{t=0}^{+\infty} \beta^t F^-(x_t, x_{t+1})$ *exists, hence* $\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$ *also exists.*

Lemma 2 *Assume* **(H1)** *and* **(H2)**. *Then:*

(i) $\forall x_0, \Pi(x_0)$ *is compact for the product topology.*

(ii) The correspondence Π *is continuous for the product topology.*

Proof:
(i) The proof is standard.
(ii)

- Let $(x_0^n)_n$ be a sequence that converges to x_0 and $\tilde{x}^n \in \Pi(x_0^n), \forall n$. By **(H2)**, for n big enough, $\tilde{x}^n$ belongs to a compact set of the product topology. Without loss of generality, $\tilde{x}^n$ can be assumed to converge to $\tilde{x}$. Since $x_1^n \in \Gamma(x_0^n)$, from **(H1)** one has $x_1 \in \Gamma(x_0)$. By induction, $x_{t+1} \in \Gamma(x_t), \forall t$. We have proved that Π is upper semi-continuous.

- Let us now prove that Π is lower semi-continuous.
Let $\tilde{x} \in \Pi(x_0)$ and let $(x_0^n)_n$ be a sequence that converges to x_0. Since Γ is l.s.c., there exists a sequence $(x_0^{n_1}, x_1^{n_1})$ where $x_1^{n_1} \in \Gamma(x_0^{n_1}), \forall n_1, (x_0^{n_1})$ is a subsequence of (x_0^n) and $x_1^{n_1}$ converges to x_1. Again by the semi-continuity of Γ, there exists a sequence $(x_0^{n_2}, x_1^{n_2}, x_2^{n_2})$ where $(x_0^{n_2}, x_1^{n_2})$ is a subsequence of $(x_0^{n_1}, x_1^{n_1})$, $x_2^{n_2} \in \Gamma(x_1^{n_2}), \forall n_2$, and $x_2^{n_2}$ converges to x_2. By induction, there exists a sequence $(x_0^{n_{t+1}}, x_1^{n_{t+1}}, ..., x_t^{n_{t+1}}, x_{t+1}^{n_{t+1}})$ where $(x_0^{n_{t+1}}, x_1^{n_{t+1}}, ..., x_t^{n_{t+1}}, x_{t+1}^{n_{t+1}}) \rightarrow (x_0, x_1, ..., x_t, x_{t+1})$ when $n_{t+1} \rightarrow +\infty$.
Now define

$$\tilde{z}^{n_1} = (x_1^{n_1}, \tilde{y}^{n_1}) \in \Pi(x_0^{n_1}), \text{ with } \tilde{y}^{n_1} \in \Pi(x_1^{n_1})$$

$$\tilde{z}^{n_2} = (x_1^{n_2}, x_2^{n_2}, \tilde{y}^{n_2}) \in \Pi(x_0^{n_2}), \text{ with } \tilde{y}^{n_2} \in \Pi(x_2^{n_2})$$

$$...$$

$$\tilde{z}^{n_t} = (x_1^{n_t}, ..., x_t^{n_t}, \tilde{y}^{n_t}) \in \Pi(x_0^{n_t}), \text{ with } \tilde{y}^{n_t} \in \Pi(x_t^{n_t})$$

Since $x_0^{n_t} \rightarrow x_0$, by **(H2)**, one can assume that $\tilde{z}^{n_t} \rightarrow \tilde{z} \in \Pi(x_0)$. Fix T. By construction, for t large enough one has $z_T^{n_t} = x_T^{n_t}$. Hence $\tilde{z} = \tilde{x}$.

Q.E.D.

3 Existence of an optimal solution - Value Function

From Remark 1, one can define:

$$u : \tilde{x} \in \Pi(x_0) \longrightarrow u(\tilde{x}) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}).$$

The problem then becomes:

$$\begin{cases} \text{Maximize } u(\tilde{x}) \\ \\ \tilde{x} \in \Pi(x_0). \end{cases}$$

Proposition 2 *Assume* **(H4)**, **(H5)** *and* **(H6)**. *Then u is upper semi-continuous on $\Pi(x_0)$ for the product topology.*

Proof:
Let $\tilde{x}^n$ be a sequence that converges to $\tilde{x}$, $\tilde{x}^n \in \Pi(x_0)$. Let $\varepsilon > 0$.

$$\begin{array}{rll} \textbf{(H6)} & \Rightarrow & \exists T_0 \text{ such that } \forall T \geq T_0 \\ & & u(\tilde{x}^n) \leq \sum_{t=0}^{T-1} \beta^t F(x_t^n, x_{t+1}^n) + \varepsilon \\ & \Rightarrow & \limsup_n u(\tilde{x}^n) \leq \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \varepsilon \\ & \Rightarrow & \limsup_n u(\tilde{x}^n) \leq \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) + \varepsilon \\ & \Rightarrow & \limsup_n u(\tilde{x}^n) \leq u(\tilde{x}) + \varepsilon \end{array}$$

Q.E.D.

We define the value function V as:

$$\forall x_0, V(x_0) = \begin{cases} Max \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) \\ \\ x_{t+1} \in \Gamma(x_t), \forall t \geq 0, \\ x_0 \in I\!R_+^n \text{ is given.} \end{cases}$$

Equivalently:

$$V(x_0) = \begin{cases} Max \;\; u(\tilde{x}) \\ \tilde{x} \in \Pi(x_0) \end{cases}$$

Theorem 1 *Assume* **(H1)**, **(H2)**,**(H4)**, **(H5)** *and* **(H6)**. *Then*

(i) *There exists an optimal solution.*

(ii) *The value function V is upper semi-continuous.*

(iii) *Assume moreover* **(H3)**. *Then $\forall x_0 \neq 0, V(x_0) > -\infty$.*

Proof:

(i) is obvious since u is upper semi-continuous and $\Pi(x_0)$ is a compact set.

(ii) We can not use the Maximum Theorem to prove the statement since $u(\tilde{x})$ may be equal to $-\infty$ for some $\tilde{x}$. We will give a direct proof in which **(H6)** has a crucial role.
Let x_0^n be a sequence that converges to x_0. Let $(x_0^{n_k})$ be a subsequence such that $\limsup_n V(x_0^n) = \lim_k V(x_0^{n_k})$. Let $\varepsilon > 0$. From **(H6)**, there exists k_0 and T_0 such that, $\forall k \geq k_0, \forall T \geq T_0$,

$$V(x_0^{n_k}) = \sum_{t=0}^{+\infty} \beta^t F(x_t^{n_k}, x_{t+1}^{n_k}) \leq \sum_{t=0}^{T} \beta^t F(x_t^{n_k}, x_{t+1}^{n_k}) + \varepsilon.$$

Fix $T \geq T_0$. By **(H2)**, the subsequence $(\tilde{x}^{n_k})$ which belongs to $\Pi(x_0^{n_k})$ may be assumed to converge to some $\tilde{x} \in \Pi(x_0)$.
Let $k \to +\infty$:

$$\limsup_n V(x_0^n) \leq \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \varepsilon.$$

Let $T \to +\infty$:

$$\limsup_n V(x_0^n) \leq u(\tilde{x}) + \varepsilon \leq V(x_0) + \varepsilon.$$

(iii) is obvious.

Q.E.D.

Proposition 3 *Assume* **(H1)**, **(H2)**,**(H4)**, **(H5)** *and* **(H6)**.
A sequence $\tilde{x}^*$ is a solution if and only if:

$$\forall t, V(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*)$$

Proof: It is quite standard.

Q.E.D.

4 The Bellman Equation

Let us consider the following set:

$$\Pi'(x_0) = \{\tilde{x} \in \Pi(x_0) / u(\tilde{x}) > -\infty\}$$

Assumption **(H3)** guarantees that $\forall x_0 \neq 0, \Pi'(x_0) \neq \emptyset$.

Proposition 4 *Assume* **(H1)**, **(H2)**,**(H3)**, **(H4)**, **(H5)** *and* **(H6)**. *Then the value function* V *verifies:*

(i) $\forall x_0, \forall \mathcal{V}(x_0)$ *neighborhood of* x_0 *in* $\mathbb{R}^n_+, \forall \varepsilon > 0, \exists T_0$ *such that* $\forall T > T_0$, $\forall x'_0 \in \mathcal{V}(x_0), \forall \tilde{x'} \in \Pi(x'_0)$, *one has:* $\beta^T V(x'_T) \leq \varepsilon$.
And hence:

(ibis) $\forall x_0, \forall \tilde{x} \in \Pi(x_0), \limsup \beta^t V(x_t) \leq 0$.

(ii) $\forall x_0$ *such that* $\Pi'(x_0) \neq \emptyset, \forall \tilde{x} \in \Pi'(x_0), \lim_{T\to+\infty} \beta^T V(x_T) = 0$.

Proof:
(i) By **(H6)**, $\exists T_0, \forall T > T_0, \forall x'_0 \in \mathcal{V}(x_0), \varepsilon > 0, \forall \tilde{x'} \in \Pi(x'_0)$,

$$\sum_{t=T}^{+\infty} \beta^t F(x'_t, x'_{t+1}) \leq \sum_{t=T}^{+\infty} \beta^t F^+(x'_t, x'_{t+1}) \leq \varepsilon$$

Let $\tilde{x}'$ be in $\Pi(x'_0), T \geq T_0, \tilde{\hat{x}} \in \Pi(x'_T)$
One has $(x'_1, ..., x'_T, \hat{x}_{T+1}, ...) \in \Pi(x'_0)$.
Then $\beta^T F(x'_T, \hat{x}_{T+1}) + \beta^{T+1} F(\hat{x}_{T+1}, \hat{x}_{T+2}) + ... \leq \varepsilon$.
Finally $\beta^T V(x'_T) \leq \varepsilon$.
$(ibis)$ follows from (i).
(ii) $\forall \tilde{x} \in \Pi'(x_0)$, one has:

$$-\infty < u(\tilde{x}) \leq \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \beta^{T+1} V(x_{T+1}).$$

Then:

$$0 = \lim_T \left[u(\tilde{x}) - \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) \right] \leq \liminf_T \beta^{T+1} V(x_{T+1}).$$

From $(ibis)$ one gets $\lim_T \beta^{T+1} V(x_{T+1}) = 0$.

Q.E.D.

Comparison with some other results

When the return function is bounded below and bounded above by some continuous function φ, one can use the contraction mapping technique with a $\varphi - norm$. It is the case in Boyd III (1990) and Duran (2000). The contraction technique is also used by Dana and Le Van (1991) to study the Pareto- optimality with recursive preferences. They transform the Pareto problem in an optimal growth model.

When the returns are not bounded below, Boyd III (1990) imposes the rate of growth of the state variables to lie between two bounds, in order to use the contraction method. Stokey and Lucas (1989, p.73) show that the value function is the unique solution to the Bellman equation under a stronger condition than ours,which is: $\forall x_0, \forall \tilde{x} \in \Pi(x_0), \lim_n \beta^n V(x_n) = 0$. They do not use the contraction technique. Alvarez and Stokey (1998) consider returns which are homogeneous of degree θ or Logarithm. They impose the technology to be a cone.

Our assumption **(H6)** allows to get the transversality conditions (*ibis*) and (*ii*).

Streufert (1990, 1992), in an abstract setting, introduces the notion of biconvergence, a limiting condition ensuring that returns of any feasible path are, on the one hand, sufficiently discounted from above (upper convergence) andfrom below (lower convergence). However, in the case where one-stage return of $-\infty$ is admissible, the return function fails to be lower convergent. Notice that Streufert does not use the contraction method to prove that the value function is the unique solution to the Bellman equation.

In the next theorem, we prove the uniqueness of the solution to the Bellman equation in a certain class of functions, without using any contraction mapping argument. Let us introduce such a class of functions:

Let S be the set of functions f which are upper semi-continuous, from $\mathbb{R}^n_+$ into $\mathbb{R} \cup \{-\infty\}$ and which verify:

(i) $\forall x_0, \forall \mathcal{V}(x_0)$ neighborhood of x_0 in $\mathbb{R}^n_+, \forall \varepsilon > 0, \exists T_0$ such that $\forall T > T_0$, $\forall x'_0 \in \mathcal{V}(x_0), \forall \tilde{x'} \in \Pi(x'_0)$, one has:

$$\beta^T f(x'_T) \leq \varepsilon$$

(ii) $\forall x_0$ such that $\Pi'(x_0) \neq \emptyset, \forall \tilde{x} \in \Pi'(x_0), \lim_{T\to+\infty} \beta^T f(x_T) = 0$.

Let S' be the set of functions f which are upper semi-continuous, from $\mathbb{R}^n_+$ into $\mathbb{R} \cup \{-\infty\}$ and which verify:

(ibis) $\forall x_0, \forall \tilde{x} \in \Pi(x_0), \limsup_t \beta^t f(x_t) \leq 0$.

(ii) $\forall x_0$ such that $\Pi'(x_0) \neq \emptyset, \forall \tilde{x} \in \Pi'(x_0), \lim_{T\to+\infty} \beta^T f(x_T) = 0$.

One has $S \subset S'$.

Theorem 2 *Assume* **(H1)**, **(H2)**, **(H4)**, **(H5)** *and* **(H6)**.

(i) *V verifies the Bellman equation:*
$(\mathcal{B}) \qquad \forall x_0, V(x_0) = Max\{F(x_0, y) + \beta V(y); y \in \Gamma(x_0)\}.$

(ii) *V is the unique solution in S' to the Bellman equation. Hence it is the unique solution in S.*

Proof:
(i) is standard.

(ii) Let $W \in S$ be another upper semi-continuous solution to the Bellman equation. Since W is u.s.c., from **(H4)**, the function $F(x_0, .) + \beta W(.)$ is u.s.c. $\Gamma(x_0)$ is compact. There thus exists $x_1 \in \Gamma(x_0)$ such that:

$$W(x_0) = F(x_0, x_1) + \beta W(x_1).$$

By induction, one has:

$$\begin{array}{rcl} W(x_0) & = & \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \beta^{T+1} W(x_{T+1}) \\ & \leq & \lim_{T \to +\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \limsup_T \beta^{T+1} W(x_{T+1}) \\ & \leq & u(\tilde{x}) \\ & \leq & V(x_0). \end{array}$$

We will show that $W(x_0) \geq V(x_0), \forall x_0$. We first assume that $x_0 \neq 0$ and take $\tilde{x} \in \Pi'(x_0)$. One has:

$$W(x_0) \geq F(x_0, x_1) + \beta W(x_1).$$

By induction, one has:

$$\begin{array}{rcl} W(x_0) & \geq & \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \beta^{T+1} W(x_{T+1}) \\ & \geq & \lim_{T \to +\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \lim_T \beta^{T+1} W(x_{T+1}) \\ & \geq & u(\tilde{x}) \end{array}$$

Hence $W(x_0) \geq V(x_0)$.
So we have proved $W(x_0) = V(x_0)$ if $x_0 \neq 0$.
Assume now that $x_0 = 0$. If $\Pi'(0) = \emptyset$, then $V(0) = -\infty \leq W(0)$. Since $W(x_0) \leq V(x_0), \forall x_0$, one has $W(0) = V(0)$. If $\Pi'(0) \neq \emptyset$, take $\tilde{x} \in \Pi'(0)$. By the same proof as before, one obtains that $W(0) \geq V(0)$, hence $W(0) = V(0)$.

Q.E.D.

5 On the Continuity of the Value Function

Let us introduce the following assumptions:

(H9) $\forall x_0 \in I\!R^n_+, \exists \tilde{x} \in \Pi(x_0)$ such that $\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) \geq 0$.

(H10) Γ, F and β verify: $\forall (x,y) \in \text{graph}\Gamma$, $\forall \alpha \in]0,1]$

(a) $(\alpha x, \alpha y) \in \text{graph}\Gamma$.

(b) $F(\alpha x, \alpha y) \geq \Phi_1(\alpha) F(x,y) + \Phi_2(\alpha)$
where Φ_1 and Φ_2 are continuous on $]0,1]$, $\Phi_1(1) = 1, \Phi_2(1) = 0$
and if $\beta \geq 1$ then $\Phi_2(\alpha) = 0$.

(c) Let $\hat{x}$ be in $I\!R^n_+$. $\forall y \in \Gamma(\hat{x})$,
y verifies $\forall \varepsilon > 0$ sufficiently small, there exists a neighborhood of $\hat{x}$ in $I\!R^n_+, \mathcal{V}(\hat{x})$, such that $(1-\varepsilon)y \in \Gamma(x), \forall x \in \mathcal{V}(\hat{x})$.

(H11) $\Gamma(0) = \{0\}, F(0,0) = -\infty$.
There exists a continuous function Φ that verifies:
$\forall x_0$ such that $\Pi'(x_0) \neq \emptyset$, $\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) \geq \Phi(x_0)$
and $\forall \tilde{x} \in \Pi'(x_0), \lim_t \beta^t \Phi(x_t) = 0$.

Comments on the assumptions

(H9) ensures that $V(x_0) \geq 0, \forall x_0$. It is less restrictive than the assumption that F is positive. If $\beta < 1$, then one can replace it by: $\exists a \in I\!R$ such that $\forall x_0 \in I\!R^n_+, \exists \tilde{x} \in \Pi(x_0), \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) \geq a$. Obviously, if $\beta < 1$, this assumption is satisfied if F is bounded below. It can be weakened in: $\forall x_0 \in I\!R^n_+, V(x_0) \geq 0$.

(H10) If graphΓ is convex or is a cone then obviously **(H10a)** holds. If F is concave and $F(0,0) > -\infty$ then **(H10b)** holds with $\Phi_1(\alpha) = \alpha$, and
$\Phi_2(\alpha) = (1-\alpha)F(0,0)$.
We give examples in which F is concave and $F(0,0) = -\infty$, and **(H10b)** holds.
(i) $\lambda > 0, F(\lambda x, \lambda y) = \lambda^\theta F(x,y)$ with $\theta < 0$. One can check that $\Phi_1(\alpha) = \alpha^\theta$ and $\Phi_2(\alpha) = 0$.
(ii) $F(x,y) = F_1(f(x) - y)$ where the function F_1 is homogeneous of degree $\theta < 0$

and f is concave with $f(0) = 0$. We have

$$\begin{array}{rcccl} F(\alpha x, \alpha y) & = & F_1(f(\alpha x) - \alpha y) & & \\ & \geq & F_1(\alpha(f(x) - y)) & = & \alpha^\theta F_1(f(x) - y) \\ & & & = & \alpha^\theta F(x, y). \end{array}$$

(iii) $F(x, y) = ln(f(x) - y)$ where the function f is concave with $f(0) = 0$. We have

$$\begin{array}{rcl} F(\alpha x, \alpha y) & = & ln(f(\alpha x) - \alpha y) \\ & \geq & ln(\alpha(f(x) - y)) \\ & \geq & ln(\alpha) + F(x, y). \end{array}$$

Here, $\Phi_1(\alpha) = 1, \Phi_2(\alpha) = ln\alpha$.
(H10c) is verified in the following case: $\Gamma(x) = \{y \in I\!R, y \leq f(x)\}$ where f is strictly increasing and continuous, and such that $f(0) \geq 0$.
Let $x > 0$ and $y \leq f(x)$. If $y > 0$, then $(1 - \varepsilon)y < f(x)$. By the continuity of f, $(1 - \varepsilon)y < f(x'), \forall x' \in \mathcal{V}(x)$. If $y = 0$ then $0 < f(x)$ and $(1 - \varepsilon)0 = 0 < f(x'), \forall x' \in \mathcal{V}(x)$.
Let $x = 0$ and $y \leq f(x)$. If $f(0) > 0$ the previous argument holds. If $f(0) = 0$, then $y = 0$ and $(1 - \varepsilon)0 = 0 < f(x'), \forall x' > 0$.

We give now an example where graphΓ is not convex and **(H10a)**, **(H10b)** hold.
Let $\Gamma(x) = \{y \in I\!R^n_+, y \leq f(x)\}$
where

$$\begin{array}{rcll} f(x) & = & a & \text{for } 0 \leq x \leq a \\ & = & x & \text{for } x \geq a. \end{array}$$

Let $F(x, y) = \frac{(f(x) - y)^\theta}{\theta}, \theta < 0$. One can check that **(H10b)** holds and $0 < \alpha < 1$, and

$$F(\alpha x, \alpha y) = \frac{(f(\alpha x) - \alpha y)^\theta}{\theta} \geq \alpha^\theta F(x, y) + (1 - \alpha)\frac{a^\theta}{\theta}.$$

We give now an example where **(H10)** is not satisfied and the value function is not continuous.
Let Γ be defined as follows:

$$\begin{array}{rcll} \Gamma(x) & = & \{0\} & \text{for } x \in [0, \frac{1}{2}] \\ \Gamma(x) & = & \{2x - 1\} & \text{for } x \in [\frac{1}{2}, 1] \\ \Gamma(x) & = & \{1\} & \text{for } x \geq 1. \end{array}$$

and let $F(x, y) = ln(2x - y)$.
Then, if $(x, y) \in$ graphΓ, one has:

$$\begin{array}{rcll} F(x, y) & = & ln(2x) & \text{if } x \in [0, \frac{1}{2}] \\ & = & 0 & \text{if } x \in [\frac{1}{2}, 1] \\ & = & ln(2x - 1) & \text{if } x \geq 1. \end{array}$$

One has $V(1) = 0$. Let $0 < \varepsilon < 1$. Let us compute $V(1-\varepsilon)$. Posit $x_0 = 1-\varepsilon$. Then $x_1 = 2(1-\varepsilon) - 1 = 1 - 2\varepsilon, x_2 = 1 - 4\varepsilon, ..., x_t = 1 - 2^t\varepsilon$. There is a T such that for $t \geq T, 1 - 2^t\varepsilon \leq \frac{1}{2}$. Hence, for $t \geq T, x_t = 0$. One can then easily check that $V(1-\varepsilon) = -\infty$.
In this example, **(H10)** is not satisfied.

(H11) If $\Phi = 0$, then it coincides with **(H9)** except for $x_0 = 0$. We think that it is interesting to make a distinction between the two cases because of the "singular" point which is the origin

Theorem 3 *Assume* **(H1)**, **(H2)**, **(H4)**, **(H5)**, **(H6)**.

(i)If we have in addition **(H9)**, *then V is continuous and is the unique solution in S to the Bellman equation.*

(ii) If **(H10)** *holds then the same claim is true.*

(iii) If **(H11)** *and* **(H3)** *holds then we have the same conclusion.*

Proof:
(i) It suffices to prove that V is l.s.c.
Let x_0^n be such that $\lim_n x_0^n = x_0$ and $\tilde{x}$ verify $V(x_0) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$. Since Π is l.s.c., there exists $\tilde{x}^n$ such that $\lim_n \tilde{x}^n = \tilde{x}$ and $\tilde{x}^n \in \Pi(x_0^n), \forall n$. Let us fix N. One has:

$$V(x_0^n) \geq \sum_{t=0}^{N} \beta^t F(x_t^n, x_{t+1}^n) + \beta^N V(x_{N+1}^n).$$

(H9) implies that $V(x) \geq 0, \forall x \in I\!\!R_+^n$. Hence, $V(x_0^n) \geq \sum_{t=0}^{N} \beta^t F(x_t^n, x_{t+1}^n)$. Let $\liminf_n V(x_0^n) = \lim_k V(x_0^{n_k})$. Hence, $\liminf_n V(x_0^n) \geq \sum_{t=0}^{N} \beta^t F(x_t, x_{t+1})$. Let $N \to +\infty$: $\liminf_n V(x_0^n) \geq \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) = V(x_0)$

(ii) As before, we will show that V is l.s.c.
Let $x_1 \in \Gamma(x_0)$ verify $V(x_0) = F(x_0, x_1) + \beta V(x_1)$. Let $x_0^n \to x_0$. Let $\varepsilon > 0$ be sufficiently small. From **(H10c)**, for any n large enough, $(1-\varepsilon)x_1 \in \Gamma(x_0^n)$. Hence $V(x_0^n) \geq F(x_0^n, (1-\varepsilon)x_1) + \beta V((1-\varepsilon)x_1)$. From **(H10a)**, **(H10b)**, one has:

$$V((1-\varepsilon)x_1) \geq \Phi_1(1-\varepsilon)V(x_1) + \frac{\Phi_2(1-\varepsilon)}{1-\beta} \text{ if } \beta < 1,$$

and

$$V((1-\varepsilon)x_1) \geq \Phi_1(1-\varepsilon)V(x_1) \text{ if } \beta \geq 1.$$

Hence

$$\liminf_n V(x_0^n) \geq F(x_0, x_1) + \beta\Phi_1(1-\varepsilon)V(x_1) + \frac{\beta\Phi_2(1-\varepsilon)}{1-\beta} \text{ if } \beta < 1,$$

and

$$\liminf_n V(x_0^n) \geq F(x_0, x_1) + \beta\Phi_1(1-\varepsilon)V(x_1) \text{ if } \beta \geq 1.$$

Let $\varepsilon \to 0$: $\liminf_n V(x_0^n) \geq F(x_0, x_1) + \beta V(x_1) = V(x_0)$.

(iii) Let $\tilde{x} \in \Pi(x_0)$ verify $V(x_0) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$. Let x_0^n be such that $\lim_n x_0^n = x_0$. Since Π is l.s.c., there exists $\tilde{x}^n \in \Pi(x_0^n), \forall n$ and $\lim_n \tilde{x}^n = \tilde{x}$. Let N be fixed. One has:

$$V(x_0^n) \geq \sum_{t=0}^{N} \beta^t F(x_t^n, x_{t+1}^n) + \beta^{N+1} V(x_{N+1}^n).$$

We have, for any t, $\Pi'(x_t) \neq \emptyset$. Since $\Gamma(0) = \{0\}$ and $F(0,0) = -\infty$, we have $x_t \neq 0, \forall t$. We have $\lim_n x_{N+1}^n = x_{N+1}$, hence for any n large enough, $x_{N+1}^n \neq 0$. From **(H3)**, $\Pi'(x_{N+1}^n) \neq \emptyset$ for n large enough, $V(x_{N+1}^n) \geq \Phi(x_{N+1}^n)$. Thus:

$$V(x_0^n) \geq \sum_{t=0}^{N} \beta^t F(x_t^n, x_{t+1}^n) + \beta^{N+1} \Phi(x_{N+1}^n),$$

and

$$\liminf_n V(x_0^n) \geq \sum_{t=0}^{N} \beta^t F(x_t, x_{t+1}) + \beta^{N+1} \Phi(x_{N+1})$$

Let $N \to +\infty$:

$$\liminf_n V(x_0^n) \geq V(x_0) + \lim_N \beta^{N+1} \Phi(x_{N+1}) = V(x_0).$$

Obviously, V is the unique solution in S to the Bellman equation.

Q.E.D.

REMARK 2 *The continuity of V must be understood in a generalized sense as follows:*
If $V(x_0) = -\infty$ and if x_n converges to x_0 then $V(x_n)$ converges to $-\infty$.

Proposition 5 *Let G be the optimal policy, i.e.*

$$G(x) = Argmax\{F(x,y) + \beta V(y); y \in \Gamma(x)\}.$$

Under the assumptions of Theorem 3, G is an upper semi-continuous correspondence.

Proof: We can not directly use the Berge Theorem because V may be equal to $-\infty$ at some point.We give here an elementary proof.
Let x_0^n be a sequence that converges to x_0 and let $y^n \in G(x_0^n), \forall n$. We have $y^n \in \Gamma(x_0^n), \forall n$. Since Γ is u.s.c., one can assume that y^n converges to $y \in \Gamma(x_0)$. Let $z \in \Gamma(x_0)$. Γ being l.s.c., there exists $\{z^\nu\} \to z$ and $z^\nu \in \Gamma(x_0^\nu), \forall \nu$. We have:

$$\begin{aligned} \forall \nu, V(x_0^\nu) &= F(x_0^\nu, y^\nu) + \beta V(y^\nu) \\ &\geq F(x_0^\nu, z^\nu) + \beta V(z^\nu) \end{aligned}$$

Let $\nu \to +\infty$:

$$\begin{aligned} V(x_0) &= F(x_0, y) + \beta V(y) \\ &\geq F(x_0, z) + \beta V(z) \end{aligned}$$

since V is continuous (in a generalized sense) and F is continuous (in a generalized sense). Thus $y \in G(x_0)$.

Q.E.D.

6 Algorithm to find the Value Function

Let T be the mapping which associates with any u.s.c. function f from $\mathbb{R}_+^n$ into $\mathbb{R} \cup \{-\infty\}$ the u.s.c. function Tf defined as follows:

$$Tf(x) = max\{F(x, y) + \beta f(y)\}; y \in \Gamma(x)\}.$$

Lemma 3 *Assume* **(H1)**, **(H2)**, **(H4)**, **(H5)**, **(H6)**. *T maps S into S and maps S' into S'.*

Proof:
It is clear that Tf is u.s.c. Let us show that T maps S into S:
Let us show that Tf verifies **(i)**.
Let $x_0 \in \mathbb{R}_+^n, \mathcal{V}(x_0)$ a neighborhood of x_0 in $\mathbb{R}_+^n, x_0' \in \mathcal{V}(x_0), \tilde{x}' \in \Pi(x_0'), \varepsilon > 0$.
(H6) implies that $\exists T_0', \forall T \geq T_0', \sum_{t=T}^{+\infty} \beta^t F^+(x_t', x_{t+1}') \leq \frac{\varepsilon}{2}$.
$f \in S$ implies by **(i)** that $\exists T_0'', \forall T \geq T_0'', \beta^T f(x_T') \leq \frac{\varepsilon}{2}$.
Let $\tilde{\hat{x}} = (x_T', \hat{x}_{T+1}, \hat{x}_{T+2}, ...) \in \Pi(x_T')$. Then $(x_1', ..., x_T', \hat{x}_{T+1}, ...) \in \Pi(x_0')$. Let $\hat{x}_T := x_T'$. One has:

$$\sum_{t=T}^{+\infty} \beta^t F^+(\hat{x}_t, \hat{x}_{t+1}) \leq \frac{\varepsilon}{2},$$

and

$$\beta^{T+1} f(\hat{x}_{T+1}) \leq \frac{\varepsilon}{2}$$

Then

$$\forall \tilde{\hat{x}} \in \Pi(x'_T), \sum_{t=T}^{+\infty} \beta^t F^+(\hat{x}_t, \hat{x}_{t+1}) + \beta^{T+1} f(\hat{x}_{T+1}) \leq \varepsilon,$$

and thus one has $\forall y \in \Gamma(x'_T), \beta^T F^+(x'_t, y) + \beta^{T+1} f(y) \leq \varepsilon$. But one has $\beta^T F(x'_t, y) \leq \beta^T F^+(x'_t, y)$, so one finally has:

$$\forall y \in \Gamma(x'_T), \beta^T F(x'_t, y) + \beta^{T+1} f(y) \leq \varepsilon$$

and

$$\beta^T T f(x'_T) \leq \varepsilon.$$

Let us show that Tf verifies **(ii)**.
Let x_0 be such that $\Pi'(x_0) \neq \emptyset$ and let $x' \in \Pi'(x_0)$. Let us show that $\lim_T \beta^T T f(x_T) = 0$. Let $y \in \Gamma(x_T)$:

$$Tf(x_T) \geq F(x_T, y) + \beta f(y).$$

One has

$$\forall \tilde{x} \in \Pi'(x_0), -\infty < \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) < +\infty.$$

Then:

$$(*) \lim_t \beta^t F(x_t, x_{t+1}) = 0.$$

Then one has, by $(*)$ and $f \in S$:

$$\lim_T \beta^T T f(x_T) \geq \lim_T \beta^T F(x_T, x_{T+1}) + \lim_T \beta^{T+1} f(x_{T+1}) \geq 0$$

Let us show that T maps S' into S':
Let us show that Tf verifies **(ibis)**. For any $\tilde{x} \in \Pi(x_0)$ one has:

$$\sum_{t=T}^{+\infty} \beta^t F^+(x_t, x_{t+1}) \leq \varepsilon,$$

and

$$\limsup_t \beta^t f(x_t) \leq 0.$$

Then one has:

$$\forall \tilde{\hat{x}} \in \Pi(x'_T), \beta^T F(\hat{x}_T, \hat{x}_{T+1}) + \beta^{T+1} f(\hat{x}_{T+1}) \leq \varepsilon + \beta^{T+1} f(\hat{x}_{T+1}),$$

$$\forall y \in \Gamma(x'_T), \beta^T F(x'_T, y) + \beta^{T+1} f(y) \leq \varepsilon + \beta^{T+1} f(y)$$

and finally

$$\limsup_T \beta^T Tf(x_T) \leq \varepsilon.$$

Q.E.D.

Theorem 4 *Assume* **(H1)**, **(H2)**, **(H4)**, **(H5)**, **(H6)**. *Then*

(i) $\forall x \in I\!\!R^n_+, V(x) = \lim_n T^n f(x)$, *where f is any function in S.*
In particular $V(x) = \lim_n T^n 0(x)$.

(ii) $\forall x \in I\!\!R^n_+, V(x) = \lim_n T^n f(x)$, *where f is in S' verifying $Tf \leq f$.*

(iii) Assume that F is positive. Then V is the uniform limit on any compact set of $I\!\!R^n_+$ of $T^n f$ where $f \in S$ and is positive. In particular, $T^n 0$ converges to V uniformly on any compact set of $I\!\!R^n_+$.

(iv) Assume moreover **(H8)**. *Let f be an u.s.c. function from $I\!\!R^n_+$ into $I\!\!R \cup \{-\infty\}$ which verifies:*
(a) $\forall x \in I\!\!R^n_+, f(x) \leq A'\|x\| + B'$, *with* $A' > 0, B' > 0$.
(b) $\forall \tilde{x} \in \Pi'(x_0), \lim_T \beta^T f(x_T) = 0$.
Then $\forall x_0 \in I\!\!R^n_+, \lim_n T^n f(x_0) = V(x_0)$.
In particular, for any constant a, $\lim_n T^n a(x_0) = V(x_0)$

Proof:
(i) Let $x_0 \in I\!\!R^n_+$. Let l be a cluster point of $\{T^n f(x_0)\}$: $l = \lim_\nu T^\nu f(x_0)$.
Let us show that $l \geq V(x_0)$.
If $\Pi'(x_0) = \emptyset : V(x_0) = -\infty$ and $l \geq V(x_0)$.
If $\Pi'(x_0) \neq \emptyset :$
Let $\tilde{x} \in \Pi'(x_0)$.
$T^\nu f(x_0) \geq F(x_0, x_1) + \beta F(x_1, x_2) + ... + \beta^{\nu-1} F(x_{\nu-1}, x_{\nu-1}) + \beta^\nu F(x_\nu)$
$\Rightarrow l \geq \lim_\nu \sum_{t=0}^{\nu-1} \beta^t F(x_t, x_{t+1}) + \lim_\nu \beta^\nu F(x_\nu)$
$\Rightarrow l \geq u(\tilde{x})$ (f verifies **(ii)** and $\Pi'(x_0) \neq \emptyset$).
$\Rightarrow l \geq V(x_0)$.
Let us show that $l \leq V(x_0)$.
Let us associate with any ν the sequence $\tilde{y}^\nu = (x_1^\nu, x_2^\nu, ..., x_N^\nu, y_{N+1}^\nu, ...) \in \Pi(x_0)$ such that

$$\begin{aligned} T^\nu f(x_0) &= F(x_0, x_1^\nu) + \beta F(x_1^\nu, x_2^\nu) + ... + \beta^{N-1} F(x_{N-1}^\nu, x_N^\nu) + \beta^N f(x_N^\nu) \\ &\leq F(x_0, x_1^\nu) + \beta F(x_1^\nu, x_2^\nu) + ... + \beta^{N-1} F(x_{N-1}^\nu, x_N^\nu) + \varepsilon \end{aligned}$$

Without loss of generality, we can assume that $\tilde{y}^\nu$ converges to
$\tilde{y} = (x_1, x_2, ..., x_N, y_{N+1}, ...)$.

Let ν converge to $+\infty$, then

$$l \le F(x_0, x_1) + \beta F(x_1, x_2) + ... + \beta^{N-1} F(x_{N-1}, x_N) + \varepsilon$$

Let ε converge to 0, and N to $+\infty$, then

$$l \le V(x_0)$$

(ii) The proof that $l \ge V(x_0)$ is the same as in the case (i). Let us show now

that $l \le V(x_0)$.
As previously let us associate with any ν the sequence
$\tilde{y}^\nu = (x_1^\nu, x_2^\nu, ..., x_N^\nu, y_{N+1}^\nu, ...) \in \Pi(x_0)$ such that

$$\begin{array}{rcl} T^\nu f(x_0) & = & F(x_0, x_1^\nu) + \beta F(x_1^\nu, x_2^\nu) + ... + \beta^{N-1} F(x_{N-1}^\nu, x_N^\nu) + \beta^N T^{\nu-N} f(x_N^\nu) \\ & \le & F(x_0, x_1^\nu) + \beta F(x_1^\nu, x_2^\nu) + ... + \beta^{N-1} F(x_{N-1}^\nu, x_N^\nu) + \beta^N f(x_N^\nu) \text{ since } Tf \le f. \end{array}$$

Let ν converge to $+\infty$ and N to $+\infty$, and then one has $l \le V(x_0)$.

(iii) Let X be a compact set of $\mathbb{R}_+^n$. Suppose the assertion is false:

$$\exists \varepsilon > 0, \exists \{x^n\} \subset X \text{ such that } \forall n,\ |T^n f(x^n) - V(x^n)| > \varepsilon.$$

Since X is compact, x^n can be assumed to converge to $x_0 \in X$. Let l be a cluster point of $\{T^n f(x^n)\}$: $l = \lim_\nu T^\nu f(x^\nu)$. There exists T_0, there exists N such that for any $\nu \ge T_0$, $\nu \ge N$ one has:

$$\beta^\nu f(x_\nu^\nu) \le \varepsilon.$$

One has

$$(1) T^\nu f(x^\nu) = F(x^\nu, x_1^\nu) + \beta F(x_1^\nu, x_2^\nu) + ... + \beta^{N-1} F(x_{N-1}^\nu, x_N^\nu) + \beta^N f(x_N^\nu)$$

Let us associate with any ν the sequence $\tilde{y}^\nu = (x_1^\nu, x_2^\nu, ..., x_N^\nu, y_{N+1}^\nu, ...) \in \Pi(x^\nu)$. As x_ν converges to x_0, and without loss of generality $\tilde{y}^\nu$ can be assumed to converge to $\tilde{x}' \in \Pi(x_0)$. Then by $\nu \to +\infty$ and since f is u.s.c.:

$$(1) \Rightarrow l \le F(x_0, x_1) + \beta F(x_1, x_2) + ... + \beta^{N-1} F(x_{N-1}, x_N) + \varepsilon$$

Since F is positive, one has then $l \le u(\tilde{x}') + \varepsilon \le V(x_0) + \varepsilon$.
Now, let $\tilde{x} \in \Pi(x_0)$. As x^ν converges to x_0 and Π is u.s.c.,
$\exists \tilde{x'}^\nu \in \Pi(x^\nu), \forall \nu, \tilde{x'}^\nu$ converges to $\tilde{x}$. Let N be fixed, $\nu \ge N$. Since F is positive, F verifies **(H9)** and:

$$\begin{array}{rcl} T^\nu f(x^\nu) & \ge & F(x^\nu, {x'}_1^\nu) + \beta F({x'}_1^\nu, {x'}_2^\nu) + ... + \beta^N F({x'}_N^\nu, {x'}_{N+1}^\nu) \\ & & + ... + \beta^{\nu-1} F({x'}_{\nu-1}^\nu, {x'}_\nu^\nu) + \beta^\nu f({x'}_\nu^\nu) \\ & \ge & F(x^\nu, {x'}_1^\nu) + \beta F({x'}_1^\nu, {x'}_2^\nu) + ... + \beta^N F({x'}_N^\nu, {x'}_{N+1}^\nu) \end{array}$$

Let $\nu \to +\infty$:

$$l \geq F(x_0, x_1) + \beta F(x_1, x_2) + ... + \beta^N F(x_N, x_{N+1}),$$

and then let $N \to +\infty$:

$$l \geq u(\tilde{x})$$

and finally

$$l \geq V(x_0).$$

Since V is continuous, $V(x_n)$ converges to $V(x_0)$, we have a contradiction.

(iv) Let us show that such a function f belongs to S.
Let us show that f verifies **(i)**.
Let $x_0 \in I\!R^n_+$, $\mathcal{V}(x_0)$ be a neighborhood of x_0 in $I\!R^n_+$, and $\varepsilon > 0$. By **(H2)**, one has:

$$\begin{array}{rcl} f(x_T) & \leq & A'(\gamma^T \|x_0\| + \gamma' \frac{1-\gamma^T}{1-\gamma}) + B' \\ & \leq & \gamma^T A'(\|x_0\| + \frac{\gamma'}{|1-\gamma|}) + \frac{A'\gamma'}{|1-\gamma|} + B' \\ \beta^T f(x_T) & \leq & (\beta\gamma)^T (A'\|x_0\| + \frac{A'\gamma'}{|1-\gamma|}) + \beta^T (\frac{A'\gamma'}{|1-\gamma|} + B') \\ & \leq & \delta^T (A'\|x_0\| + \frac{2A'\gamma'}{|1-\gamma|} B'). \end{array}$$

where $\delta^T = max\{(\beta\gamma)^T, \beta^T\}$.
Let $\Psi(x_0) = A'\|x_0\| + \frac{2A'\gamma'}{|1-\gamma|} B'$. Then

$$\beta^T f(x_T) \leq \delta^T \Psi(x_0).$$

Since Ψ is continuous, define $D := sup\{\Psi(x); x \in \mathcal{V}(x_0)\}$. Then, by **(H8)**, there exists T_0 such that for any $x'_0 \in \mathcal{V}(x_0)$, for any $\tilde{x'} \in \Pi(x'_0)$, for any $T \geq T_0$, one has $\beta^T f(x_T) \leq \varepsilon$.

f verifies **(ii)** ((a)). Then f belongs to S.

Q.E.D.

Corollary 1 *Assume* **(H1)**, **(H2)**, **(H4)**, **(H5)** *and* **(H6)**. *Let $\hat{V}$ be an u.s.c. function from $I\!R^n_+$ into $I\!R \cup \{-\infty\}$ such that*

(i) $T\hat{V} \leq \hat{V}$.

(ii) $\forall x_0 \in I\!R^n_+, \forall \tilde{x} \in \Pi(x_0), \limsup_t \beta^t \hat{V}(x_t) \leq 0$.

(iii) $u(\tilde{x}) \leq \hat{V}(x_0), \forall x_0 \in I\!R^n_+, \forall \tilde{x} \in \Pi(x_0)$.

Then $V = \lim_n T^n \hat{V}$.

Proof:
It suffices to prove that $\hat{V} \in S'$. The conclusion follows Theorem 4, statement

(ii). It remains to prove that $\forall x_0 \in I\!R^n_+, \forall \tilde{x} \in \Pi'(x_0), \lim_t \beta^t \hat{V}(x_t) = 0$. But (ii) implies that $V(x_0) \leq \hat{V}(x_0), \forall x_0$. Hence $\forall \tilde{x} \in \Pi'(x_0)$,
$0 = \lim_t \beta^t V(x_t) \leq \liminf_t \beta^t \hat{V}(x_t)$. Since $\limsup_t \beta^t \hat{V}(x_t) \leq 0$, we have $\lim_t \beta^t \hat{V}(x_t) = 0$.

Q.E.D.

REMARK 3 *Stokey and Lucas (1989, p.93) obtain the same result with a redundant assumption which is* $\lim_n T^n \hat{V}$ *is a fixed point of* T.

7 Examples and Applications

7.1 Example 1: the case with bounded from below returns

Consider the benchmark model in section 1. Assume **(H1)**, **(H2)**, **(H4)**, **(H7)**, **(H8)** and that F is positive. In this case we have:

Proposition 6 *There exists an optimal solution. The value function* V *is positive, continuous and verifies:*

$$\forall x, V(x) \leq a\|x\| + b$$

for some $a \geq 0, b \geq 0$.

V *is the uniform limit on any compact set of* $I\!R^n_+$ *of* $T^n f$ *where* f *is continuous, positive, bounded by a linear function* g*, i.e.* $g(x) \leq a'\|x\| + b'$ *for some non negative numbers* a', b'.

Proof: It follows Theorem3, statement (i), and Theorem 4, statement (iii).

Q.E.D.

REMARK 4 *In this case, one can use, as in Dana and Le Van (1990) or Duran (2000), the contraction mapping* T *provided by the Bellman equation, on the space* E *of continuous functions endowed with the norm* $\|f\| = \sup_{x \in I\!R^n_+} \frac{|f(x)|}{(1+\|x\|)}$. V *is the limit in this norm of the sequence* $\{T^m f\}$ *for any* f *in* E. *Hence it is the uniform limit on any compact set of this sequence. Our result is in conformity with this well-known result.*

7.2 Example 2: the utility function is homogeneous of degree $\theta < 0$, the technology exhibits strictly decreasing returns

Consider the model

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t \frac{c_t^\theta}{\theta} \\ \\ \text{s.t.} \quad \forall t, c_t + k_{t+1} \leq f(k_t), c_t \geq 0, k_t \geq 0, \\ \qquad\quad k_0 \geq 0, \text{ is given.} \end{cases}$$

where $\theta < 0, \beta \in [0, 1[$.

Let $F(x, y) = \frac{(f(x)-y)^\theta}{\theta}, \Gamma(x) = [0, f(x)]$.

The problem becomes:

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t \frac{(f(x_t)-x_{t+1})^\theta}{\theta} \\ \\ \text{s.t.} \quad \forall t, x_{t+1} \in \Gamma(x_t), \\ \qquad\quad x_0 \geq 0 \text{ is given .} \end{cases}$$

Proposition 7 *Assume that f is concave, continuous, strictly increasing, differentiable, $f(0) = 0, f'(0) > 1, f'(\infty) < 1$. There exists an optimal solution. The value function V is continuous. $V(0) = -\infty, V(x) > -\infty, \forall x > 0$. $V = \lim T^n 0$, where T is the operator defined by the Bellman equation (see section 5).*

Proof: **(H1)**, **(H4)**, **(H5)** are trivial. Let us prove that **(H2)** is true with $\gamma < 1$. Indeed, let $\bar{x}$ be such that $f(\bar{x}) = \bar{x}$. Then $f'(\bar{x}) < 1$. From the concavity of f, one has $f(x) \leq f'(\bar{x})(x - \bar{x})$.

Now since f is concave and the utility function is concave and increasing, **(H7)** is true. Obviously, **(H8)** is also true. **(H10)** is verified (see comments on **(H10)** in section 6). Let us prove **(H3)**.

If $x_0 < \bar{x}$ then $f(x_0) > x_0$ and the sequence $(x_0, x_0, \ldots)$ is feasible from x_0 and $\sum_t \beta^t F(x_t, x_{t+1}) = \frac{F(x_0,x_0)}{(1-\beta)} > -\infty$.

If $x_0 \geq \bar{x}$, then choose $x_1 > 0$ such that $0 < x_1 < \bar{x}$.
The sequence $(x_0, x_1, x_1, \ldots)$ is feasible and
$\sum_t \beta^t F(x_t, x_{t+1}) = F(x_0, x_1) + \frac{\beta}{(1-\beta)} F(x_1, x_1) > -\infty$.

The conclusion now follows Theorem 1, Theorem 3 statement (ii), and Theorem 4.

Q.E.D.

7.3 Example 3: the criterion function F is homogeneous of degree θ and the technology is a cone (Alvarez, Stokey, 1998)

Consider the model in section 1. We assume as in Alvarez and Stokey (1998):

(A1): **(H1)** of section 1; graph Γ is a cone.

(A2): $\beta > 0$; there exists a continuous and homogeneous of degree one selection g of Γ and a strictly positive number ξ with $\mu = \beta\xi^{\theta} < 1$ such that

$$\|g(x)\| \geq \xi\|x\|, \forall x.$$

(A3): $F : graph\Gamma \setminus \{0\} \rightarrow \mathbb{R}_{-}$ is continuous, homogeneous of degree $\theta < 0$ and for $b > 0$

$$F(x, y) \leq -b\|x\|^{\theta}, \forall (x, y) \in graph\Gamma$$
$$F(0, 0) = -\infty.$$

Proposition 8 *There exists an optimal solution. The value function V is continuous and homogeneous of degree θ.*

$V = \lim T^m f$, where f is negative, continuous, homogeneous of degree θ. In particular, $V = lim T^m 0$.

Proof: Obviously, **(H1)**, **(H4)**, **(H5)** are satisfied. Let us check **(H2)**. One has:

If $x \neq 0, y \in \Gamma(x)$ implies $y \in \|x\|\Gamma(\frac{x}{\|x\|})$.

Hence if $y \in \Gamma(x)$ then $\|y\| \leq \alpha\|x\|$, where $\alpha = \max\{\|z\| \mid z \in \Gamma(u), \|u\| = 1\}$, since Γ is continuous. Let us check **(H3)**: **(A2)** and **(A3)** imply that (see Alvarez and Stokey, 1998, for the proof) $\forall x_0$ the sequence $\tilde{x} = (x_0, g(x_0), g^2(x_0), \ldots, g_t(x_0), \ldots)$ verifies:

$$\sum_t \beta^t F(x_t, x_{t+1}) \geq -B\|x_0\|^{\theta}\frac{1 + \xi^{\theta}}{1 - \mu}.$$

(H6) is true because F is negative.

We claim that **(H11)** is true. Indeed, for any $x_0 \neq 0$ one has:

$$-\infty < \sum_t \beta^t F(x_t, x_{t+1}) \leq -b \sum \beta^t \|x_t\|^\theta, \forall \tilde{x} \in \Pi'(x_0).$$

Hence $\lim_t \beta^t \|x_t\|^\theta = 0, \forall \tilde{x} \in \Pi'(x_0)$.

Posit $\phi(x_0) = -B\|x_0\|^\theta \frac{1+\xi^\theta}{1-\mu}$.

One has $\forall x_0 \neq 0, \forall \tilde{x} \in \Pi'(x_0), \lim_t \beta^t \phi(x_t) = 0$.

It follows from Theorem 1 that an optimal solution exists and that $V(x_0) > -\infty, \forall x_0 \neq 0$. Obviously $V(0) = -\infty$. It can be easily checked that V is homogeneous of degree θ. To end the proof, we have just to prove that any function f, negative, continuous, homogeneous of degree θ, belongs to the class of functions S defined in section 3. Since f is negative, condition **(i)** in the definition of S is obviously verified. Since we have proved that $\lim_t \beta^t \|x_t\|^\theta = 0, \forall \tilde{x} \in \Pi'(x_0)$, and since $|f|$ is bounded on the unit-sphere, condition **(ii)** in the definition of S is also true. The conclusion follows the statement (i) of Theorem 3.

Q.E.D.

Remark 5 *In fact assumption A2(e) in Alvarez and Stokey, 1998, page 174, is not necessary for their results. Our conclusion is stronger than theirs which states that $V = \lim T^m 0$ or $V = \lim T^m h$, where h (negative, homogeneous of degree θ) is the total discounted return from following a stationary policy (see their Theorem 4).*

7.4 Example 4: the utility function is logarithm and the technology is of strictly decreasing returns

The model is

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t Ln(c_t) \\ \\ \text{s.t.} \quad \forall t, c_t + k_{t+1} \leq f(k_t), c_t \geq 0, k_t \geq 0, \\ \qquad\quad k_0 \geq 0, \text{ is given.} \end{cases}$$

where $\beta \in [0, 1]$.
Let $F(x, y) = Ln(f(x) - y), \Gamma(x) = [0, f(x)]$.

The problem becomes:

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t Ln(f(x_t) - x_{t+1}) \\ \\ \text{s.t.} \quad \forall t, x_{t+1} \in \Gamma(x_t), \\ \qquad\quad x_0 \geq 0 \text{ is given }. \end{cases}$$

Proposition 9 *Assume that f is concave, continuous, strictly increasing, differentiable, $f(0) = 0, f'(0) > 1, f'(\infty) < 1$.*

There exists an optimal solution for any $x_0 \geq 0$. The value function V is continuous. $V(0) = -\infty; V(x_0) > -\infty, \forall x_0 \neq 0$. $V = \lim_m T^m 0$.

Proof: As in Example 2, **(H1)**, **(H2)**, **(H3)**, **(H4)**, **(H7)**, **(H8)** and **(H10)** are satisfied. The conclusion follows Theorem 1, statement (ii) of Theorem 3, and Theorem 4.

Q.E.D.

7.5 Example 5: the criterion function F is logarithm and the technology is a cone (Alvarez and Stokey, 1998, p.183)

Consider the model of section 1. The following assumptions come from Alvarez and Stokey, 1998, p.183:

(B1): **(H1)** of section 1; graph Γ is a cone.

(B2): $\beta \in]0, 1[$. There exists a continuous and homogeneous of degree one selection g of Γ and a strictly positive number ξ such that

$$\|g(x)\| \geq \xi \|x\|, \forall x.$$

(B3): $F : graph\Gamma \setminus \{0\} \to I\!R$ verifies:

$F(x, y) = Ln(\psi(x, y))$ where $\psi : graph\Gamma \setminus \{0\} \to I\!R_+$ is continuous and homogeneous of degree one, and $\psi(0, 0) = 0$.

There exists $0 < b < B$ such that:

$$b\|x\| \leq \psi(x, g(x)), \forall x.$$

$$\psi(x, y) \leq B(\|x\| + \|y\|), \forall y \in \Gamma(x), \forall x.$$

Proposition 10 *There exists an optimal solution for any x_0 in $\mathbb{R}^n_+$. The value function V is continuous. $V(x_0) > -\infty, \forall x_0 \neq 0. V(0) = -\infty$.*

$\forall x \neq 0, V(x) = V(\frac{x}{\|x\|}) + \frac{Ln(\|x\|)}{1-\beta}$.

$V = \lim_m T^m 0$. *More generally, $V = \lim_m T^m f$ where f is continuous, verifies $f(0) = -\infty$ and*

$\forall x \neq 0, f(x) = f(\frac{x}{\|x\|}) + \frac{Ln(\|x\|)}{1-\beta}$.

Proof: Let us check that **(H1)**, **(H2)**, **(H3)**, **(H4)**, **(H5)**, **(H6)**, **(H11)** are satisfied.

(H1) is obvious. Since Γ is continuous and $graph\Gamma$ is a cone, **(H2)** is satisfied as in example 3. More precisely, one has: $y \in \Gamma(x) \Rightarrow \|y\| \leq \gamma\|x\|$.

Now let $x_0 \neq 0$. Let $\tilde{x}$ satisfy $x_{t+1} = g(x_t), \forall t \geq 0$. One has:

$\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = \sum_{t=0}^{\infty} \beta^t Ln(\psi(x_t, x_{t+1})) \geq \sum_{t=0}^{\infty} \beta^t Ln(b\|x_t\|) \geq \frac{Ln(b)}{1-\beta} + \sum_{t=0}^{\infty} \beta^t Ln(\xi^t\|x_0\|) = \frac{Ln(b)}{1-\beta} + Ln(\xi)\sum_{t=0}^{\infty} t\beta^t + \frac{Ln(\|x_0\|)}{1-\beta} > -\infty$. Hence **(H3)** is satisfied.

(H4), **(H5)** are obvious.

Let us check **(H6)**. One has:

$F(x_t, x_{t+1}) = Ln(\psi(x_t, x_{t+1})) \leq Ln(B(\|x_t\| + \|x_{t+1}\|)) \leq Ln(B(1+\gamma)\gamma^t\|x_0\|)$.

Hence $F^+(x_t, x_{t+1}) \leq |Ln(B(1+\gamma))| + \max\{0, Ln(\|x_0\|)\} + t|Ln(\gamma)|$. **(H6)** is therefore satisfied.

Let us check **(H11)**. Obiously $\Gamma(0) = \{0\}$ and $F(0,0) = -\infty$. We have seen that $\forall x_0 \neq 0$, the following inequality holds:

$\sum_{t=0}^{\infty} \beta^t F(x_t, g(x_t)) \geq \frac{Ln(b)}{1-\beta} + Ln(\xi)\sum_{t=0}^{\infty} t\beta^t + \frac{Ln(\|x_0\|)}{1-\beta}$.

Define

$\phi(x_0) = \frac{Ln(b)}{1-\beta} + Ln(\xi)\sum_{t=0}^{\infty} t\beta^t + \frac{Ln(\|x_0\|)}{1-\beta}$.

Let $\tilde{x} \in \Pi'(x_0)$. One has:

$$\begin{array}{rcl} -\infty < \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) & \leq & \sum_{t=0}^{\infty} \beta^t Ln(B(1+\gamma)\|x_t\|) \\ & \leq & \sum_{t=0}^{\infty} \beta^t Ln(B(1+\gamma)) + \sum_{t=0}^{\infty} \beta^t Ln(\|x_t\|) \\ & \leq & \frac{Ln(B(1+\gamma))}{1-\beta} + \sum_{t=0}^{\infty} \beta^t Ln(\gamma^t\|x_0\|) \\ & < & \infty. \end{array}$$

Hence the sum $\sum_{t=0}^{\infty} \beta^t Ln(\|x_t\|)$ exists, and $\lim_t \beta^t Ln(\|x_t\|) = 0$.
Thus $\lim_t \beta^t \phi(x_t) = 0$ for any $\tilde{x} \in \Pi'(x_0)$: **(H11)** is satisfied.

It follows from Theorem 1, statement (iii) of Theorem 3, Theorem 4, that V is continuous, $V(x_0) > -\infty$, $\forall x_0 \neq 0. V(0) = -\infty$, and $V = \lim_m T^m 0$.

It is easy to check that $\forall x \neq 0, V(x) = V(\frac{x}{\|x\|}) + \frac{Ln(\|x\|)}{1-\beta}$.

Now, let f be continuous and satisfy:

$\forall x \neq 0, f(x) = f(\frac{x}{\|x\|}) + \frac{Ln(\|x\|)}{1-\beta}$. $f(0) = -\infty$.

We claim that $f \in S$. Indeed let $\tilde{x} \in \Pi(x_0)$. One has, if $x_t \neq 0$,

$f(x_t) \leq \max_{\|x\|=1} |f(x)| + \frac{Ln(\gamma^t \|x_0\|)}{1-\beta} = A + \frac{tLn(\gamma)}{1-\beta} + \frac{Ln(\|x_0\|)}{1-\beta}$.

Since $f(x_t) = -\infty$ if $x_t = 0$, we have $\limsup \beta^t f(x_t) \leq 0$ for any $\tilde{x} \in \Pi(x_0)$, for any x_0. Condition **(i)** in the definition of S is fulfilled.

Now let $\tilde{x} \in \Pi'(x_0)$. Then $\forall t, x_t \neq 0$ and

$$\beta^t f(x_t) = \beta^t f(\frac{x_t}{\|x_t\|}) + \beta^t \frac{Ln(\|x_t\|)}{1-\beta}.$$

We have seen that $\beta^t Ln(\|x_t\|) \to 0, \forall \tilde{x} \in \Pi'(x_0)$.
Since, obviously $\beta^t f(\frac{x_t}{\|x_t\|}) \to 0$ because f is continuous, finite valued on the unit-sphere, condition **(ii)** in the definition of S is satisfied. Use statement (i) of Theorem 4 to conclude.

Q.E.D.

7.5.1 Application 1: The AK Model

Consider the model:

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \\ \text{ s.t.} \quad \begin{array}{l} \forall t, c_t + I_t \leq Ak_t, \text{ with } A > 0 \\ I_t = k_{t+1} - (1-\delta)k_t \\ c_t \geq 0, I_t \geq 0, k_t \geq 0 \\ k_0 \geq 0 \text{ is given.} \end{array} \end{cases}$$

where $\beta \in]0,1[, \delta \in \in]0,1]$

Here, $u(c) = Ln(c)$ or $\frac{c^\theta}{\theta}$ with $\theta \neq 0$.

The problem is equivalent to

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t u((A+1-\delta)k_t - k_{t+1}) \\ \\ \text{s.t.} \quad (1-\delta)k_t \leq k_{t+1} \leq (A+1-\delta)k_t, \forall t \\ \qquad\quad k_0 \geq 0 \text{ is given.} \end{cases}$$

If $\theta > 0$, it is a problem with returns bounded from below (example 1). If $\theta < 0$ or if $u(c) = Ln(c)$ then it is an application of example 3 or example 5. The selection g may be defined as $g(x) = rx$ with $1 - \delta < r < A + 1 - \delta$. One can check than all the other assumptions of example 3 or example 5 are fulfilled.

7.5.2 Application 2: A Human Capital Model (Stokey and Lucas, 1989, p.111)

We here present a simplified version of the Lucas model which is given in Stokey and Lucas, 1989, p.111.

Assume that at date t, the growth of rate of the human capital, is given by the formula

$$b_t = \phi(\frac{h_{t+1}}{h_t})$$

where h_t is the human capital at date t and b_t is the number of working hours.

We assume that ϕ is continuous, decreasing, $\phi(1+\lambda) = 0, \phi(1-\delta) = 1$, where $\lambda > 0$ and $0 \leq \delta < 1$.

The model is

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t \frac{c_t^\theta}{\theta} \\ \\ \text{s.t.} \quad \forall t, 0 \leq c_t \leq (b_t h_t)^\alpha, \alpha \in]0,1[. \end{cases}$$

where $\theta \neq 0, \theta < 1$.

Proposition 11 *Assume* $\beta(1+\lambda) < 1$.

The value function V *is continuous. Moreover,* $\forall h, V(h) = Ah^{\alpha\theta}$.

The optimal path $\tilde{h}^*$ *verifies:* $\forall t, h_{t+1}^* = u^* h_t^*$ *for some* $u^* \in [1-\delta, 1+\lambda]$.

Proof: The model becomes

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t \frac{[h_t \phi(\frac{h_{t+1}}{h_t})]^{\alpha\theta}}{\theta} \\ \\ \text{s.t.} \quad \forall t, (1-\delta)h_t \leq h_{t+1} \leq (1+\lambda)h_t \\ \qquad\quad h_0 \geq 0 \text{ is given.} \end{cases}$$

If $\theta > 0$, the model is of the class of models in Example 1 and V is continuous.

Let us consider the case where $\theta < 0$. We have a model of Example 3 where the degree is $\alpha\theta$. More precisely:

$$F(x,y) = \frac{1}{\theta}(x\phi(\frac{y}{x}))^{\alpha\theta}, \Gamma(x) = [(1-\delta)x, (1+\lambda)x].$$

Assumption **(A2)** in example 3 is satisfied with $g(x) = x, \forall x$ while **(A3)** is satisfied with $b = -\frac{(1+\lambda)^{\alpha\theta}}{\theta} > 0$.

We have $V = \lim T^m 0$. Let us successively compute $T^m 0$. Let $x \neq 0$.

One has:

$$T0(x) = \max_{y \in [(1-\delta)x, (1+\lambda)x]} \{\frac{1}{\theta}(x\phi(\frac{y}{x}))^{\alpha\theta}\} = x^{\alpha\theta} \max_{u \in [(1-\delta),(1+\lambda)]} \{\frac{1}{\theta}\phi(u)^{\alpha\theta}\} = A_1 x^{\alpha\theta}.$$

and

$$T^2 0(x) = \max_{y \in [(1-\delta)x, (1+\lambda)x]} \{\frac{1}{\theta}(x\phi(\frac{y}{x}))^{\alpha\theta} + \beta A_1 y^{\alpha\theta}\} = x^{\alpha\theta} \max_{u \in [(1-\delta),(1+\lambda)]} \{\frac{1}{\theta}\phi(u)^{\alpha\theta} + \beta A_1 u^{\alpha\theta}\} = A_2 x^{\alpha\theta}.$$

By induction: $T^n 0(x) = A_n x^{\alpha\theta}$.

Hence, $A_n \to A$ and $V(x) = Ax^{\alpha\theta}$. Obviously, $V(0) = -\infty$.

We have the Bellman equation:

$$\forall x \neq 0, V(x) = \max_{y \in [(1-\delta)x, (1+\lambda)x]} \{\frac{1}{\theta}(x\phi(\frac{y}{x}))^{\alpha\theta} + \beta A y^{\alpha\theta}\} = x^{\alpha\theta} \max_{u \in [(1-\delta),(1+\lambda)]} \{\frac{1}{\theta}\phi(u)^{\alpha\theta} + \beta A u^{\alpha\theta}\}.$$

Let u^* be an element of the Argmax. Then if $\tilde{h}^*$ is an optimal path then it verifies: $\forall t, h^*_{t+1} = u^* h^*_t$.

Q.E.D.

7.6 Example 6: Learning By Doing Model (Stokey and Lucas, 1989, p.108)

In this model, we have a case where $\Gamma(0) \neq \{0\}$, the criterion function F is not bounded below but the value function is continuous and finite valued.

Let C denote the cost function of a monopoly. C depends on the production at date t, q_t, and on the cumulated production Q_t, where $Q_t = Q_{t-1} + q_{t-1}$.

More precisely, $C_t = C(q_{t-1}, Q_t)$. We assume that C is convex, continuously differentiable and verifies: $\forall Q, C(0,Q) = 0, 0 < \underline{c} \leq \frac{\partial C}{\partial q}(0,Q) < \overline{c}$.

The price is given by an inverse demand function ψ which is continuously differentiable, strictly decreasing and such that the income function $q\psi(q)$ is strictly concave. We assume that $\psi(0) > \overline{c}$ and $\psi(\infty) < \underline{c}$.

We maximize the intertemporal profit of the monopoly:

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t [(Q_{t+1} - Q_t)\psi(Q_{t+1} - Q_t) - C(Q_{t+1} - Q_t, Q_t)] \\ \\ \text{s.t.} \quad Q_{t+1} \geq Q_t, \forall t \\ \qquad\quad Q_0 \geq 0 \text{ is given .} \end{cases}$$

The criterion function F is: $F(x,y) = (y-x)\psi(y-x) - C(y-x, x)$.

We will show that given $x \geq 0$, $F(x,y) \to -\infty$ when $y \to \infty$.

Since C is convex, we have $C(x,y) = C(x,y) - C(0,y) \geq \frac{\partial C}{\partial q}(0,y)x \geq \underline{c}x$.

Consider the function f defined by $f(x) = x\psi(x) - \underline{c}x$. We have $\frac{f(x)}{x}$ converges to $\psi(\infty) - \underline{c} < 0$, when $x \to \infty$. That means that $f(x) \to -\infty$ when $x \to \infty$. Since $F(x,y) \leq (y-x)\psi(y-x) - \underline{c}(y-x) = f(y-x)$, then $F(x,y) \to -\infty$ when y converges to ∞.

Proposition 12 *Assume $\beta \in [0,1[$. There exists an optimal solution. The value function is positive and continuous.*

Proof: For any $Q_0 \geq 0$, the sequence $(Q_0, Q_0, \ldots, Q_0, \ldots)$ is feasible. Thus $V(Q_0) \geq 0$.

Consider the function f defined above: $f(x) = x\psi(x) - \underline{c}x$. f is strictly concave, $f(0) = 0$. We have $\frac{f(x)}{x}$ converges to $\psi(0) - \underline{c} > \overline{c} - \underline{c} > 0$ when x converges to 0. In other words, $f'(0) > 0$. We have seen that $f'(\infty) < 0$. There exists a unique maximum point $\overline{x}$ with $f'(\overline{x}) = 0$. Hence $f(x) \leq M, \forall x \geq 0$.

But $F(x,y) \leq f(x-y)$. We then have $0 \leq V(Q_0) \leq \frac{M}{1-\beta}, \forall Q_0 \geq 0$.

We also have: $\sum_{t=0}^{\infty} \beta^t [(Q_{t+1} - Q_t)\psi(Q_{t+1} - Q_t) - C(Q_{t+1} - Q_t, Q_t)] \leq (Q_1 - Q_0)\psi(Q_1 - Q_0) - C(Q_1 - Q_0), Q_0) + \beta\frac{M}{1-\beta} \leq (Q_1 - Q_0)\psi(Q_1 - Q_0) - \underline{c}(Q_1 - Q_0) + \beta\frac{M}{1-\beta} = f(Q_1 - Q_0) + \beta\frac{M}{1-\beta}$.

The function $g(x) = f(x) + \beta\frac{M}{1-\beta}$ is strictly concave, $g(0) = \beta\frac{M}{1-\beta} > 0$, $g'(0) > 0$, $g'(\infty) < 0$. Therefore, there exists a unique point $\hat{x}$ such that
$g(x) \geq 0 \Leftrightarrow x \leq \hat{x}$.

Since $V(Q_0) \geq 0$, we must choose Q_1 such that $Q_1 - Q_0 \leq \hat{x}$.

The problem now becomes:

$$\begin{cases} \text{Maximize} \sum_{t=0}^{\infty} \beta^t F(Q_t, Q_{t+1}) \\ \\ \text{ s.t.} \quad \forall t, Q_t \leq Q_{t+1} \leq Q_t + \hat{x} \\ \qquad\quad Q_0 \geq 0 \text{ is given .} \end{cases}$$

Here we have $\Gamma(x) = [x, x + \hat{x}]$.

Since $x\psi(x)$ is concave,**(H7)** is verified because there exists $a, b \geq 0$ such that $F(x, y) \leq a(y - x) + b \leq a(y + x) + b$.

Here **(H2)** is satisfied with $\gamma = 1 + \varepsilon, \forall \varepsilon > 0$. Since $\beta < 1$, one chooses ε small enough such that $\beta(1 + \varepsilon) < 1$ and **(H8)** is satisfied. Thus **(H6)** is true.

(H9) is also true since if $\tilde{x} = (Q_0, Q_0, \ldots, Q_0, \ldots)$ one has $\sum_{t=0}^{\infty} \beta^t F(Q_t, Q_{t+1}) = 0$.

The conclusion now follows statement (i) of Theorem 3.

Q.E.D.

References

[1] F. Alvarez and N. Stokey, Dynamic programming with homogeneous functions. *Journal of Economic Theory* **82**(1998),167-189.

[2] R.A.Becker, J.H. Boyd III and C.Foias, The existence of Ramsey equilibrium. *Econometrica* **50**,N2(1991), 441-460.

[3] J.H. Boyd III, Recursive utility and the Ramsey problem, *Journal of Economic Theory* **50** (1990), 326-345.

[4] R.A. Dana and C. Le Van, Optimal growth and pareto optimality, *Journal of Mathematical Economics* **20** (1991), 155-180.

[5] J. Duran, On dynamic programming with unbounded returns, *Economic Theory* (2000).

[6] N.L. Stokey, R.E. Lucas, Jr, and E. C. Prescott, "Recursive Methods in Economic Dynamics", Harvard University Press, Cambridge, MA, 1989.

[7] P.A. Streufert, Stationary recursive utility and dynamic programming under the assumption of biconvergence, *Review of Economic Studies* **57** (1990), 79-97.

[8] P.A. Streufert, An abstract topological approach to dynamic programming, *Journal of Mathematical Economics* **21** (1992), 59-88