Performance of Compressive Sensing for Hadamard-Haar Systems

Amirafshar Moshtaghpour*, J. M. Bioucas-Dias†, and Laurent Jacques*
*ISPGroup, ICTEAM, UCLouvain, Belgium. †IT, IST, Universidade de Lisboa, Portugal

***Abstract*—We study the problem of image recovery from subsampled Hadamard measurements using Haar wavelet sparsity prior. This problem is of interest in, *e.g.*, computational imaging applications relying on optical multiplexing or single pixel imaging. While the high coherence between the Hadamard and Haar bases hinders the efficacy of such devices, the multilevel density sampling strategy solves this issue by adjusting the subsampling process to the local coherence between both bases; hence allowing for successful image recovery. In this work, we compute an explicit sample-complexity bound for Hadamard-Haar systems; a seemingly missing result in the related literature. The power of this approach is demonstrated on several experimental simulations.**

I. Introduction

Recovering an image from subsampled Hadamard measurements is now an emerging problem in several imaging applications of Compressive Sensing (CS), *e.g.*, single pixel camera [1], hyperspectral imaging [2–4], video imaging [5]. When the signal of interest is sparse in some wavelet basis, several theoretical and empirical evidences suggest variable density sampling [6] strategy, which densifies the subsampling of lower Hadamard frequencies, to obtain superior signal reconstruction quality.

The recovery of 1-D signals that are sparse in Haar or Daubechies wavelet basis (from subsampled Hadamard measurements) is well-studied in [7, 8]. In this work we focus on the recovery of 2-D images, that are sparse in the 2-D Discrete Haar Wavelet basis with Isotropic levels (IDHW) [9]. Note that our results also cover the case of DHW with anisotropic levels (not reported here). We build our analysis upon [6]. In particular, we compute the exact values of *multilevel coherence* (between Hadamard and IDHW basis), *i.e.*, of the main ingredient deriving the performance of CS systems relying on the partial sampling of orthonormal basis. The application of our approach to the compressive sensing of hyperspectral data with single pixel imaging [3, 4] has recently shown promising recovery performance.

II. Main Results

Let $\boldsymbol{X} \in \mathbb{R}^{N\times N}$ be the image to be recovered from the noisy subsampled Hadamard measurements

$$\boldsymbol{y} = \boldsymbol{P}_\Omega \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n} \in \mathbb{R}^M, \tag{1}$$

where $\boldsymbol{H} \in \{\pm 1/N\}^{N^2\times N^2}$ is the orthonormal Hadamard transform, $\boldsymbol{x} \in \mathbb{R}^{N^2}$ is the vectorization of $\boldsymbol{X}$, $\boldsymbol{n}$ models an additive noise, $\Omega := \{\omega_j\}_{j=1}^M \subset [\![N^2]\!] := \{1,\cdots,N^2\}$ stands for the subsampling index set, and $\boldsymbol{P}_\Omega \in \{0,1\}^{M\times N^2}$ is the restriction operator with $(\boldsymbol{P}_\Omega)_j = x_{\omega_j}$. In this context, our main question becomes: how to design an optimum sampling pattern Ω such that the solution $\hat{\boldsymbol{x}}$ to the program

$$\hat{\boldsymbol{x}} := \arg\min\nolimits_{\boldsymbol{u}\in\mathbb{R}^{N^2}} \|\boldsymbol{\Psi}^*\boldsymbol{u}\|_1 \text{ s.t. } \|\boldsymbol{y} - \boldsymbol{P}_\Omega\boldsymbol{H}\boldsymbol{u}\| \le \varepsilon, \tag{2}$$

gives a good approximation to $\boldsymbol{x}$. In (2), $\boldsymbol{\Psi} \in \mathbb{R}^{N^2\times N^2}$ is the IDHW basis (referred to as bivariate Haar wavelet basis in [10]) and $\varepsilon \ge \|\boldsymbol{n}\|$ bounds the noise power.

Traditional CS for bounded orthonormal systems [11, 12] states that in order to reconstruct K-sparse signals via (2), one must draw $M \gtrsim N^2 \mu K \log(N)$ elements of Ω uniformly at random in $[\![N^2]\!]$, where $\mu = \mu(\boldsymbol{H\Psi}) := \max_{i,j} |(\boldsymbol{H\Psi})_{i,j}|^2$ is the coherence of $\boldsymbol{H\Psi}$. Unfortunately, in this framework, $\mu(\boldsymbol{H\Psi}) = 1$ and thus, $M \approx N^2$ samples are required for image reconstruction, which prevents any compression.

This limitation is related to the construction of Ω, *i.e.*, according to a Uniform Density Sampling (UDS). Adcock *et al.* [6] proposed a novel sampling framework that breaks the “coherence barrier” for such bases.

Following this approach, we partition the sampling (resp. signal) domain with $r \in \mathbb{N}$ disjoint sampling levels $\{\mathcal{W}_t\}_{t=1}^r$ (resp. sparsity levels $\{\mathcal{S}_l\}_{l=1}^r$). Roughly speaking, in order to reconstruct a signal that is k_l-sparse in the $l^{\rm th}$ sparsity level (*i.e.*, $\|\boldsymbol{P}_{\mathcal{S}_l}\boldsymbol{\Psi}^*\boldsymbol{x}\|_0 \le k_l$), for all $l \in [\![r]\!]$, we must draw [6]

$$m_t = O\Big(|\mathcal{W}_t|\,\big(\textstyle\sum_{l=1}^r \mu_{t,l}(\boldsymbol{H\Psi})\,k_l\big)\,\mathrm{polylog}(N)\Big), \tag{3}$$

measurements uniformly at random, in each $t^{\rm th}$ sampling level, where $\mu^2_{t,l}(\boldsymbol{H\Psi}) := \mu(\boldsymbol{P}_{\mathcal{W}_t}\boldsymbol{H\Psi})\,\mu(\boldsymbol{P}_{\mathcal{W}_t}\boldsymbol{H\Psi}\boldsymbol{P}_{\mathcal{S}_l}^\top)$ is the multilevel coherence between the Hadamard matrix $\boldsymbol{H}$ and the DHW basis $\boldsymbol{\Psi}$. However, the efficacy of this result relies on the proper partitioning of the sampling and sparsity domains, and on the ability to estimate multilevel coherence values.

To fix the ideas, set $r = \log_2(N)$ and let $\{\mathcal{T}_l\}_{l=1}^r$ denote a dyadic partitioning of the set $[\![N]\!]$, *i.e.*, $\mathcal{T}_1 := \{1,2\}$ and $\mathcal{T}_l := \{2^{l-1},\cdots,2^l\}$. The 2-D isotropic wavelet levels are then defined as $\mathcal{T}_l^{\rm iso} := \{\mathcal{T}_l \times \bar{\mathcal{T}}_l\} \cup \{\mathcal{T}_l \times \mathcal{T}_l\} \cup \{\bar{\mathcal{T}}_l \times \mathcal{T}_l\},\ l \in [\![r]\!]$, where $\bar{\mathcal{T}}_l := \bigcup_{j=1}^{l-1}\mathcal{T}_j$ for $l \ge 2$ and $\bar{\mathcal{T}}_1 := \{1\}$ (see the illustration in Fig. 1). These levels are used to partition both the sampling and sparsity domains. As illustrated in Fig. 2, such a partitioning permits to isolate the block-structure of the matrix $\boldsymbol{H\Psi}$, at each level.

With this structure in mind, we leverage the results of [6] to propose the following Multilevel Density Sampling (MDS) scheme. For each $t \in [\![r]\!]$, we draw a set $\Omega_t \subseteq \mathcal{T}_t^{\rm iso}$, of size $|\Omega_t| = m_t$ picked uniformly at random in the $t^{\rm th}$ sampling level. The subsampling set in (1) and (2) is then $\Omega = \bigcup_{t=1}^r \Omega_t$ of size $|\Omega| = M = m_1 + \cdots + m_r$. The next proposition is the main theoretical result of this work.

Proposition 1. *In the sampling context specified above,*

$$\mu_{t,l}(\boldsymbol{H\Psi}) = 2^{-2(t-1)}\delta_{t,l},$$

where $\delta_{t,l} = 1$ if $t = l$ (and zero otherwise).

From (3), this exact estimation yields the minimum number of measurements $m_t = O(k_t\,\mathrm{polylog(N)})$ at each level and $M = O(K\,\mathrm{polylog(N)})$, with $K := \sum_t k_t$.

In a nutshell, our results implies that the number of samples in every sampling level should be proportional to the sparsity value at the corresponding sparsity level. This statement remains true when the sparsity basis is set as the 2D DHW with anisotropic levels, though it is not reported here. We provide in Fig. 3 and 4 a few experiments demonstrating the proposed MDS sampling for Hadamard-Haar systems.

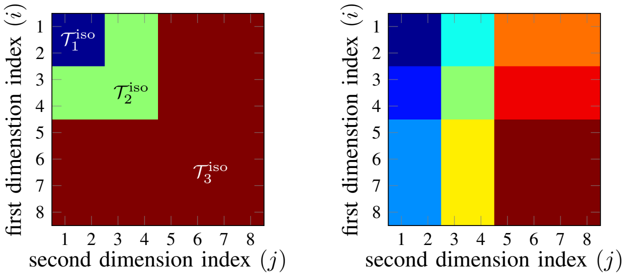

Fig. 1: Small dimensional example of 2D isotropic (left) versus 2D anisotropic wavelet levels (right) with $N = 8$. Each color represents the subset of pair of indices (i, j) that belong to a level. Note that the number of levels r equals $\log_2(N)$ for isotropic levels, while it is $\log_2^2(N)$ for anisotropic levels. The case of 2D DHW basis with anisotropic levels is not reported here, for the sake of simplicity.

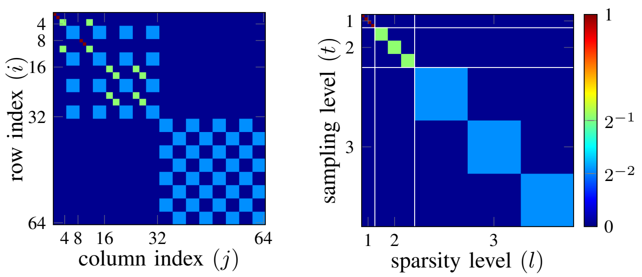

Fig. 2: Structure of the matrix $\boldsymbol{H\Psi}$ with $N = 8$. (left) Values of $|(\boldsymbol{H\Psi})_{i,j}|$; (right) rearrangement of rows and columns according to the 2D isotropic partitioning. Each white rectangle centered at (t, l) corresponds to the $\boldsymbol{P}_{\mathcal{T}_t^{\text{iso}}} \boldsymbol{H\Psi P}_{\mathcal{T}_l^{\text{iso}}}^\top$ block.

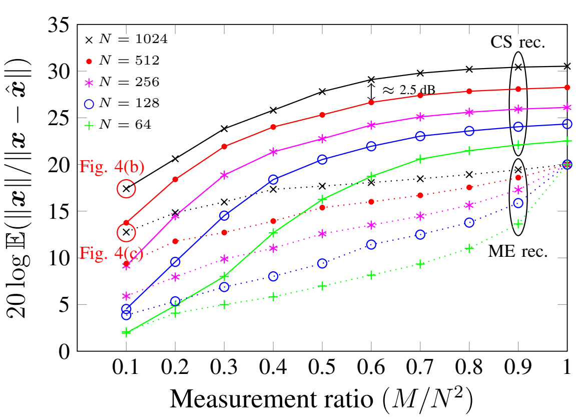

Fig. 3: Signal to Reconstruction Error (SRE) in dB as a function of the measurement ratio (M/N^2) for different sizes of the problem. We compare the recovery performance of CS framework (2) (in solid curves) with Minimal Energy (ME) solution (in dotted curves), *i.e.*, applying the pseudo-inverse of $\boldsymbol{P}_\Omega \boldsymbol{H}$ on the measurement vector. A synthetic phantom image of size $N \times N$ is generated as the ground truth. The variance of the additive i.i.d. Gaussian noise is set such that $20 \log(\|\boldsymbol{x}\|/(\sigma N)) \approx 20$ dB. The results are averaged over 10 trials (*i.e.*, over random generation of noise and random selection of Ω according to the proposed MDS scheme). The effectiveness of both CS and ME frameworks relies on the size of the problem. The recovery of CS always outperforms the one of ME, as ME does not take into account the sparsity of the signal.

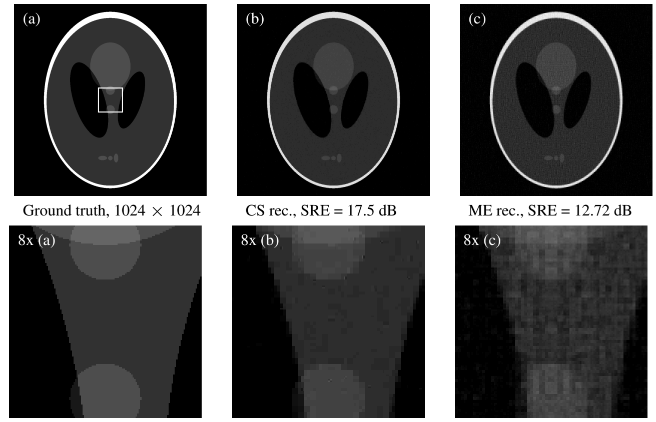

Fig. 4: An example of the reconstructed images from 10% of the Hadamard measurements. Superior quality of the CS reconstruction is obvious both visually and quantitatively. We notice that as the resolution increase the wavelet coefficients of the image become (asymptotically) sparser, and the multilevel coherence between Hadamard and Haar bases decreases asymptotically (see Prop. 1).

REFERENCES

[1] M. F. Duarte, M. A. Davenport, D. Takhar, J. N. Laska, T. Sun, K. E. Kelly, and R. G. Baraniuk, "Single-pixel imaging via compressive sampling," *IEEE signal processing magazine*, vol. 25, no. 2, p. 83, 2008.

[2] B. Roman, A. C. Hansen, and B. Adcock, "On asymptotic structure in compressed sensing," *arXiv preprint arXiv:1406.4178*, 2014.

[3] A. Moshtaghpour, J. M. Bioucas-Dias, and L. Jacques, "Compressive hyperspectral imaging: Fourier transform interferometry meets single pixel camera," in *international Traveling Workshop on Interactions between low-complexity data models and Sensing Techniques (iTWIST)*, 2018.

[4] ——, "Compressive single-pixel Fourier transform imaging using structured illumination," *arXiv preprint arXiv:1810.13200*, 2018.

[5] Y. Zhang, M. P. Edgar, B. Sun, N. Radwell, G. M. Gibson, and M. J. Padgett, "3D single-pixel video," *Journal of Optics*, vol. 18, no. 3, p. 035203, 2016.

[6] B. Adcock, A. C. Hansen, C. Poon, and B. Roman, "Breaking the coherence barrier: A new theory for compressed sensing," in *Forum of Mathematics, Sigma*, vol. 5. Cambridge University Press, 2017.

[7] V. Antun, "Coherence estimates between Hadamard matrices and Daubechies wavelets," Master's thesis, 2016.

[8] V. Antun, B. Adcock, A. Hansen, and Ø. Ryan, "Uniform recovery guarantees for Hadamard sampling and wavelet reconstruction," in *Signal Processing with Adaptive Sparse Structured Representations workshop (SPARS)*, 2017.

[9] S. Mallat, *A wavelet tour of signal processing: the sparse way*. Academic press, 2008.

[10] F. Krahmer and R. Ward, "Stable and robust sampling strategies for compressive imaging," *IEEE transactions on image processing*, vol. 23, no. 2, pp. 612–622, 2014.

[11] E. Candès and J. Romberg, "Sparsity and incoherence in compressive sampling," *Inverse problems*, vol. 23, no. 3, p. 969, 2007.

[12] S. Foucart and H. Rauhut, *A mathematical introduction to compressive sensing*. Birkhäuser Basel, 2013.