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Inference in Dynamic, Nonparametric Models of Production for General Technologies

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Abstract. Nonparametric envelopment estimators are often used to estimate the attainable set and its efficient boundary, and to assess efficiency and changes in productivity. Kneip et al (2021) provide asymptotic results that can be used to make inference about expected changes in productivity measured by Malmquist indices and about the sources of productivity changes. These results require convexity of the attainable set, but in a number of situations this assumption is questionable. Recently, Kneip et al (2022) extend these results to allow for possibly non-convex technologies where the DEA estimators are known to be inconsistent. This paper summarizes these results, and explains how researchers should choose the appropriate method in a particular application.

Keywords: Nonparametric production frontiers, DEA, FDH, Malmquist Indices

1 Introduction

Nonparametric envelopment estimators are often used to estimate the attainable set and its efficient boundary, and to assess efficiency and changes in productivity. Kneip et al (2015) provide asymptotic theory enabling inference about expected efficiency, and Kneip et al (2016) provide asymptotic results that can be used to test convexity versus non-convexity of the production set as well as constant versus variable returns to scale (when the production set is convex) using these estimators. In addition, Kneip et al (2021) and Simar and Wilson (2019) provide asymptotic results that can be used to make inference about expected changes in productivity measured by Malmquist indices and about the sources of productivity changes, provided the underlying production sets are convex. But in a number of situations, convexity of the production set is questionable, and some studies (e.g., O'Loughlin and Wilson, 2021) have rejected convexity using the test of Kneip et al (2016). The results have recently been extended by Kneip et al (2022) to allow for possibly non-convex technologies, where the DEA estimators are known to be inconsistent. Kneip et al (2022) establish properties of the conical FDH estimator, a nonparametric envelopment estimator of distance to the boundary of the cone spanned by a possibly non-convex attainable set. These new results are then extended to make inference about Malmquist indices and their various components in non-convex settings. Recent results are summarized below, providing guidelines for empirical researchers that should be helpful for choosing appropriate estimators and for making inferences.

1.1 Basic Notation

A general nonparametric model of production is described as follows (e.g., see Shephard, 1970). Denote by $x \in \mathbb{R}_+^p$ the vector of input quantities that are used to produce a vector $y \in \mathbb{R}_+^q$ of output quantities. The set of feasible combinations of input and output quantities (i.e., the production set) is given by

$$\Psi = \{(x, y) \in \mathbb{R}_+^{p+q} \mid x \text{ can produce } y\}. \tag{1}$$

The efficient frontier of Ψ is given by

$$\Psi^\partial = \{(x, y) \mid (\gamma^{-1}x, \gamma y) \notin \Psi, \text{ for any } \gamma > 1\}. \tag{2}$$

The following assumptions on Ψ are typically made:

- Ψ is closed;
- no free lunches exist, i.e., $(x, y) \notin \Psi$ if $x = 0$ and $y \geq 0,\ y \neq 0$; and
- both inputs and outputs are freely disposable, i.e., for all $(x, y) \in \Psi$, if $x' \geq x,\ y' \leq y$ then $(x', y') \in \Psi$.

In addition, Ψ is sometimes (but not always) assumed to be convex (here, we do not assume Ψ is convex). Some studies assume constant returns to scale (CRS) so that Ψ is such that if $(x, y) \in \Psi$, then $(ax, ay) \in \Psi$ for all $a > 0$.

The efficiency of a firm represented by a point $(x, y) \in \Psi$ is measured by its distance to the efficient frontier Ψ^∂. Several such measures have appeared in the literature, including

- the radial, input-oriented measure $\theta(x, y \mid \Psi) = \inf\{\theta \mid (\theta x, y) \in \Psi\} \leq 1$;
- the radial, output-oriented measure $\lambda(x, y \mid \Psi) = \sup\{\lambda \mid (x, \lambda y) \in \Psi\} \geq 1$;
- the hyperbolic measure $\gamma(x, y \mid \Psi) = \inf\{\gamma \mid (\gamma x,\ \gamma^{-1} y) \in \Psi\} \leq 1$; and
- the directional distance measure $\beta(x, y; d_x, d_y \mid \Psi) = \sup\{\beta \mid (x - \beta d_x, y + \beta d_y) \in \Psi\} \geq 0$, where $d_x \geq 0$ and $d_y \geq 0$ are direction vectors.

See Simar and Wilson (2015) for a review and detailed references.

Of particular interest for purposes of the Malmquist Productivity Index (MPI) is the cone $\mathcal{C}(\Psi)$ spanned by Ψ, defined by

$$\mathcal{C}(\Psi) = \{(\tilde{x}, \tilde{y}) \mid \tilde{x} = ax, \tilde{y} = ay, \forall a \in \mathbb{R}_+ \text{ and } \forall (x, y) \in \Psi\}. \tag{3}$$

Since Ψ is not necessarily convex, $\mathcal{C}(\Psi)$ is not necessarily convex. In the particular case of CRS, we have $\mathcal{C}(\Psi) = \Psi$, for both convex and non-convex cases, but in general $\Psi \subseteq \mathcal{C}(\Psi)$.

In any realistic situation, Ψ is unknown, and hence $\mathcal{C}(\Psi)$ is also unknown. Consequently, both Ψ and $\mathcal{C}(\Psi)$, as well as the efficiency measures described above, must be estimated from a sample $\mathcal{X}_n = \{(X_i, Y_i)\}_{i=1}^n$ of observed input-output combinations. Typically, the observations in $\mathcal{X}_n$ are assumed to be identically and independently distributed (iid).

2 Nonparametric Estimators

2.1 A Data Generating Process (DGP)

The DGP is *the model*, i.e., the set of assumptions, that specify how the random sample $\mathcal{X}_n$ is generated. The DGP is determined by specifying the joint density $f_{XY}(x, y)$ of the random vectors (X, Y) with support on Ψ, or equivalently by specifying the probability

$$H_{XY}(x, y) = \Pr(X \le x, Y \ge y) \tag{4}$$

of a production plan (x, y) begin dominated. Under Free Disposability and regardless of whether Ψ is convex, Ψ is the support of H_{XY} and hence

$$\Psi = \{(x, y) \mid H_{XY}(x, y) > 0\}. \tag{5}$$

Then the efficiency measure for input-oriented case can be written as

$$\theta(x, y \mid \Psi) = \inf\{\theta \mid H_{XY}(\theta x, y) > 0\}. \tag{6}$$

In a nonparametric model, no particular structure is imposed on $H_{XY}(x, y)$ and a natural estimator is provided by its empirical analog $\widehat{H}_{XY}(x, y) = n^{-1} \sum_{i=1}^n \mathbb{1}(X_i \le x, Y_i \ge y)$, where $\mathbb{1}(\cdot)$ is the indicator function. This provides the basis of various envelopment estimators of Ψ.

2.2 Popular Envelopment Estimators

Under free disposability assumption of Ψ, without assuming convexity, Deprins et al (1984) propose the Free Disposal Hull (FDH) estimator obtained by applying the plug-in principle:

$$\begin{aligned}\widehat{\Psi}_{FDH}(\mathcal{X}_n) &= \{(x, y) \mid \widehat{H}_{XY}(x, y) > 0\} \\ &= \left\{(x, y) \in \mathbb{R}_+^{p+q} \mid y \le Y_i,\ x \ge X_i,\ (X_i, Y_i) \in \mathcal{X}_n\right\}.\end{aligned} \tag{7}$$

Imposing convexity of Ψ leads to the Data Envelopment Analysis (DEA) estimator

$$\begin{aligned}\widehat{\Psi}_{DEA}(\mathcal{X}_n) = \{(x, y) \in \mathbb{R}_+^{p+q} \mid y \le \sum_{i=1}^n \eta_i Y_i; x \ge \sum_{i=1}^n \eta_i X_i \text{ for } (\eta_1, \ldots, \eta_n) \\ \text{such that } \sum_{i=1}^n \eta_i = 1; \eta_i \ge 0, i = 1, \ldots, n\}\end{aligned} \tag{8}$$

which is the convex hull of $\widehat{\Psi}_{FDH}(\mathcal{X}_n)$ (see Farrell, 1957 and Charnes et al, 1978). Neither $\widehat{\Psi}_{FDH}(\mathcal{X}_n)$ nor $\widehat{\Psi}_{DEA}(\mathcal{X}_n)$ impose constant returns to scale.

2.3 Cone Spanned by Ψ

In general, $\Psi \subseteq \mathcal{C}(\Psi)$, and $\Psi = \mathcal{C}(\Psi)$ if and only if CRS holds; otherwise, $\Psi \subset \mathcal{C}(\Psi)$. This is true regardless of whether Ψ is convex. If Ψ is convex, then $\mathcal{C}(\Psi)$ is the conical hull of Ψ and $\mathcal{C}(\Psi)$ is convex. Alternatively, if Ψ is *not* convex, then $\mathcal{C}(\Psi)$ is not necessarily convex. Note that CRS may (or may not) hold regardless of whether Ψ is convex.

Under free disposability, but without assuming convexity of Ψ, $\mathcal{C}(\Psi)$ can be estimated by the *conical* FDH (CFDH) estimator

$$\widehat{\Psi}_{CFDH}(\mathcal{X}_n) := \mathcal{C}(\widehat{\Psi}_{FDH}(\mathcal{X}_n)) = \big\{(x,y) \in \mathbb{R}_+^{p+q} \mid (x,y) = (a\widetilde{x}, a\widetilde{y}) \text{ for some } a \geq 0 \text{ and } (\widetilde{x},\widetilde{y}) \in \widehat{\Psi}_{FDH}(\mathcal{X}_n)\big\}. \tag{9}$$

If in addition CRS holds, then $\widehat{\Psi}_{CFDH}$ also estimates Ψ. When both free disposability and convexity of Ψ are assumed, either $\widehat{\Psi}_{CFDH}(\mathcal{X}_n)$ or the *conical* DEA (CDEA) estimator

$$\widehat{\Psi}_{CDEA}(\mathcal{X}_n) := \mathcal{C}(\widehat{\Psi}_{DEA}(\mathcal{X}_n)) := \Big\{(x,y) \in \mathbb{R}_+^{p+q} \mid y \leq \sum_{i=1}^{n} \eta_i Y_i ; x \geq \sum_{i=1}^{n} \eta_i X_i \text{ with } \eta_i \geq 0 \ \forall\ i = 1, \ \ldots,\ n\Big\} \tag{10}$$

can be used to estimate $\mathcal{C}(\Psi)$. If in addition CRS holds as well as free disposability and convexity, then both $\widehat{\Psi}_{CFDH}(\mathcal{X}_n)$ and $\widehat{\Psi}_{CDEA}(\mathcal{X}_n)$ estimate Ψ. But if CRS does not hold, then $\widehat{\Psi}_{CDEA}(\mathcal{X}_n)$ estimates $\mathcal{C}(\Psi) \neq \Psi$ provided Ψ is convex, and $\widehat{\Psi}_{CFDH}(\mathcal{X}_n)$ estimates $\mathcal{C}(\Psi)$ regardless of whether Ψ is convex.

2.4 Efficiency Estimators

Results for the input-oriented case are given below, but similar results hold for the output orientation as well as both hyperbolic and directional distances. Efficiency estimates are obtained by "plugging in" the the appropriate estimator of Ψ, depending on the assumptions one makes about Ψ.

If only free disposability (but not convexity) is assumed, then the appropriate estimator of Ψ is $\widehat{\Psi}_{FDH}(\mathcal{X}_n)$, and hence an estimate distance to Ψ^{∂} in the input direction is given by

$$\widehat{\theta}_{FDH}(x,y) = \min_{i \in D(x,y)} \left[\max_{j=1,\ldots,p} \left(\frac{X_{ij}}{x_j} \right) \right], \tag{11}$$

where $D(x,y) = \{i \mid X_i \leq x, Y_i \geq y\}$ is the set of observations dominating (x,y). For estimating Malmquist indices, estimates of distance to the boundary of $\mathcal{C}(\Psi)$ are needed, and if only free disposability is assumed these distances are estimated by distance to the boundary of $\mathcal{C}(\Psi_{FDH}(\mathcal{X}_n)$ given by

$$\widehat{\theta}_{CFDH}(x,y) = \min_{i=1,\ldots,n} \frac{\max_{j=1,\ldots,p} \frac{X_{ij}}{x_j}}{\min_{j=1,\ldots,q} \frac{Y_{ij}}{y_j}} \tag{12}$$

as shown by Kneip et al (2022). Note that if CRS holds, then $\widehat{\theta}_{CFDH}(x,y)$ also estimates efficiency.

Under both free disposability and convexity of Ψ, efficiency is estimated by $\widehat{\theta}_{DEA}(x,y) = \min\{\theta \mid (\theta x, y) \in \widehat{\Psi}_{DEA}(\mathcal{X}_n)\}$. For estimating Malmquist indices, estimates of distance to the boundary of $\mathcal{C}(\widehat{\Psi}_{DEA})$ are needed, and these are provided by $\widehat{\theta}_{CDEA}(x,y) = \min\{\theta \mid (\theta x, y) \in \mathcal{C}(\widehat{\Psi}_{DEA}(\mathcal{X}_n))\}$. Once again, if CRS holds, then $\widehat{\theta}_{CDEA}(x,y)$ also estimates efficiency. Both the DEA and CDEA estimators $\widehat{\theta}_{DEA}(x,y)$ and $\widehat{\theta}_{CDEA}(x,y)$ can be computed by solving familiar linear programs (e.g., see Kneip et al, 2021).

2.5 Statistical properties

The properties of the estimators discussed above depend crucially on assumptions made about the DGP. In all cases, the asymptotic properties rest on mild regularity assumptions, e.g., smoothness of boundaries and of f_{XY} near the relevant boundary. The results assume that $f_{XY}(x,y) > 0$ on the boundary of Ψ. See Park et al (2000), Park et al (2010), Kneip et al (2008, 2021, 2022) and Simar and Wilson (2015) for additional details.

Under appropriate assumptions, each of the distance estimators discussed above is statistically consistent and converges to a stable, non-degenerate limiting distribution at rate n^{κ} as $n \to \infty$. For the FDH and CFDH estimators, the limiting distribution is Weibull, and for the DEA and CDEA cases (under convexity) the limiting distribution is a regular distribution depending on unknown parameters, but no closed-form representations are available when p or $q > 1$. If Ψ is non-convex, the results for the DEA and CDEA estimators do not hold; neither DEA nor CDEA estimators are consistent without convexity of Ψ. The rates of convergence n^{κ} depend on the particular case, and on the values of $p+q$. The error of estimation of each efficiency measure for a fixed point (x,y) is of order $O_p\left(n^{-\kappa}\right)$, and so the *curse of dimensionality* that is common among nonparametric estimators is present here, too. Table 1 summarizes the results, with free disposability holding in all cases.

It is important to note that with free disposability, the FDH and CFDH estimators always provides consistent estimates, regardless of whether Ψ is convex. This is not the case with the DEA and CDEA estimators, which requires convexity to be consistent. Note also that the CFDH, DEA and CDEA estimators converge at faster rates under CRS than when CRS does not hold.

Inference (i.e., confidence intervals) for measures of distance from a given point (x,y) to the boundary of Ψ or $\mathcal{C}(\Psi)$ can be obtained by bootstrap methods (see Simar and Wilson, 2015 for a survey and references). Ordinary Central Limit Theorems (CLTs) (e.g., the Lindeberg-Feller CLT) do not hold in general for mean efficiency or distances, but recent CLT results are available making possible inference about expected efficiency or mean efficiency over groups of firms. See Kneip et al (2015, 2021 and 2022) for details.

As a practical matter, inference requires information about convergence rates, and Table 1 makes clear that convergence rates of the various estimators depend

Table 1. Values of κ for the various distance estimators

Estimator	CRS, CONV	$\overline{\text{CRS}}$, CONV	CRS, $\overline{\text{CONV}}$	$\overline{\text{CRS}}$, $\overline{\text{CONV}}$
FDH	$1/(p+q)$	$1/(p+q)$	$1/(p+q)$	$1/(p+q)$
CFDH	$1/(p+q-1)$	$1/(p+q-1/2)$	$1/(p+q-1)$	$1/(p+q-1/2)$
DEA	$2/(p+q)$	$2/(p+q+1)$	—	—
CDEA	$2/(p+q)$	$2/(p+q+1)$	—	—

"CRS" and "$\overline{\text{CRS}}$" denote that CRS holds or does not hold, respectively. "CONV" and "$\overline{\text{CONV}}$" denote that convexity of Ψ holds or does not hold, respectively.

on whether CRS or convexity hold. Whether CRS or convexity hold are ultimately empirical questions requiring inference via hypothesis testing. Kneip et al (2016) develop a test of convexity and a test for CRS versus non-CRS for convex production sets, and Kneip et al (2022) provide a test of CRS versus non-CRS that does not require convexity of production sets.

3 Dynamic Settings

An objective in dynamic, multivariate ($p,\ q \geq 1$) settings is to measure any gains or losses of productivity by firms over time, and perhaps to analyze the source of these changes. This is achieved by defining a Malmquist Productivity Index (MPI); e.g., see Malmquist (1953) and Caves et al (1982). Various studies have decomposed the MPI into indices of changes in efficiency, technology, scale efficiency, etc.

In the literature, DEA and CDEA estimators are invariably used to estimate MPIs and component indices, but as discussed above, these estimators require Ψ to be convex. Kneip et al (2021, 2022) provide the tools for providing consistent estimators of the index under various assumptions (i.e., convexity or non-convexity, CRS or non-CRS), as described in the preceding section. They also derive appropriate CLTs for making inference on averages of the MPI over the units in a entire sample.

3.1 Malmquist Productivity Indices

Consider two periods t_1 and t_2 and suppose that $p = q = 1$. The relative productivity changes (using the output orientation) is easily defined as

$$\mathcal{M} = \frac{y^{t_2}/x^{t_2}}{y^{t_1}/x^{t_1}}.$$

If productivity increases between periods t_1 and t_2, then $\mathcal{M} > 1$, and if productivity decreases, then $\mathcal{M} < 1$. If there is no change, then $\mathcal{M} = 1$. For $p,\ q \geq 1$, the idea of MPI is to use distances to the boundary of $\mathcal{C}(\Psi)$ to measure these productivity changes. It turns out that the MPI defined below is equivalent to the ratio $\mathcal{M}$ when $p = q = 1$.

In order to handle dynamic settings, denote the attainable set at time t by

$$\Psi^t = \{(x, y) \mid x \text{ can produce } y \text{ at time } t\} \tag{13}$$

and its efficient boundary by

$$\Psi^{t,\partial} = \{(x, y) \mid (x, y) \in \Psi^t, (\gamma x, \gamma^{-1} y) \notin \Psi^t\ \forall \gamma \in (0, 1)\}. \tag{14}$$

Denote the cone spanned by Ψ^t by $\mathcal{C}(\Psi^t)$.

Defining the MPI in terms of hyperbolic distances

$$\gamma(x, y \mid \mathcal{C}(\Psi^t)) = \inf \left\{\gamma \mid (\gamma x,\ \gamma^{-1} y) \in \mathcal{C}(\Psi^t)\right\} \tag{15}$$

to the boundary of $\mathcal{C}(\Psi^t)$ ensures that the MPI is well-defined for all of the observations in a given sample. Recall that under CRS, $\mathcal{C}(\Psi^t) = \Psi^t$, but in general, $\mathcal{C}(\Psi^t) \subseteq \Psi^t$. Then the MPI for a firm moving from (x^1, y^1) in period 1 to (x^2, y^2) in period 2 is defined as

$$\mathcal{M} = \left(\frac{\gamma(x^2, y^2 \mid \mathcal{C}(\Psi^1))}{\gamma(x^1, y^1 \mid \mathcal{C}(\Psi^1))} \times \frac{\gamma(x^2, y^2 \mid \mathcal{C}(\Psi^2))}{\gamma(x^1, y^1 \mid \mathcal{C}(\Psi^2))}\right)^{1/2}. \tag{16}$$

The boundaries of $\mathcal{C}(\Psi^1)$ and of $\mathcal{C}(\Psi^2)$ serve as benchmarks against which any change in productivity is measured. The MPI in (16) is the geometric mean of the changes (the ratios) of the distances of the firm to the boundary of the cone spanned by the possibility sets at the two periods. The geometric mean is natural when considering means of ratios. All of this is valid under convexity or non-convexity of the attainable sets, and with or without the CRS assumption.

3.2 Sources of Productivity Changes

Several decompositions of MPI have been proposed in the literature; see Simar and Wilson (2019) for discussion and references. To facilitate notation, let $w^t = (x^t, y^t)$, $t \in \{1, 2\}$. We focus below on the decomposition

$$\mathcal{M} = \mathcal{E} \times \mathcal{T} \times \mathcal{S}_1 \times \mathcal{S}_2, \tag{17}$$

where

$$\begin{aligned}
\mathcal{E} &= \frac{\gamma(w^2 \mid \Psi^2)}{\gamma(w^1 \mid \Psi^1)} && \text{(change in efficiency)},\\
\mathcal{T} &= \left[\frac{\gamma(w^2 \mid \Psi^1)}{\gamma(w^2 \mid \Psi^2)} \times \frac{\gamma(w^1 \mid \Psi^1)}{\gamma(w^1 \mid \Psi^2)}\right]^{1/2} && \text{(change in technology)},\\
\mathcal{S}_1 &= \frac{\gamma(w^2 \mid \mathcal{C}(\Psi^2))/\gamma(w^2 \mid \Psi^2)}{\gamma(w^1 \mid \mathcal{C}(\Psi^1))/\gamma(w^1 \mid \Psi^1)} && \text{(change in scale efficiency)}
\end{aligned} \tag{18}$$

and $\mathcal{S}_2$ may be viewed as a residual term. Simar and Wilson (2019) discuss how $\mathcal{S}_2$ can be interpreted as a change in the scale or shape of the technology. Note that the true production sets Ψ^t serve as the reference sets for defining $\mathcal{E}$ and $\mathcal{T}$. The scale efficiency term $\mathcal{S}_1$ measures the differences between Ψ^t and $\mathcal{C}(\Psi^t)$. Again, note that this does not require convexity of Ψ^t nor CRS, but clearly under CRS in both periods, $\mathcal{S}_1 = \mathcal{S}_2 = 1$.

3.3 Nonparametric Estimation

Estimation requires random samples of firms for two different time periods $t \in \{1, 2\}$, i.e. n iid pairs $\{(W_i^1, W_i^2)\}_{i=1}^n$, where $W_i^t = (X_i^t, Y_i^t)$. Since there may be additional observations in each period, with no counterpart in the other period, we might consider $\mathcal{X}_{n_1}^1$ and $\mathcal{X}_{n_2}^2$, with $n_t \geq n$. The Malmquist indices for each firm $i = 1, \ldots, n$ can be written as

$$\mathcal{M}_i = \left(\frac{\gamma(X_i^2, Y_i^2 \mid \mathcal{C}(\Psi^1))}{\gamma(X_i^1, Y_i^1 \mid \mathcal{C}(\Psi^1))} \times \frac{\gamma(X_i^2, Y_i^2 \mid \mathcal{C}(\Psi^2))}{\gamma(X_i^1, Y_i^1 \mid \mathcal{C}(\Psi^2))} \right)^{1/2}. \tag{19}$$

For estimation of the Malmquist index at fixed point, for a single firm $w^t = (x^t, y^t)$, $t \in \{1, 2\}$, first simplify the notation by defining $\widehat{\gamma}_C^s(w^t) = \gamma(w^t \mid \mathcal{C}(\widehat{\Psi}^s))$ for $s, t = 1, 2$, and letting this denote either the CFDH hyperbolic distance for general (possibly non-convex) technologies, or the CDEA estimator for convex technologies, of w^t to the boundary of the cone spanned by the FDH or DEA estimator of Ψ^s (respectively) estimated from the sample $\mathcal{X}_{n_s}^s$, for $s, t = 1, 2$. To be explicit,

$$\widehat{\mathcal{M}}(w^1, w^2) = \left(\frac{\widehat{\gamma}_C^1(w^2)}{\widehat{\gamma}_C^1(w^1)} \times \frac{\widehat{\gamma}_C^2(w^2)}{\widehat{\gamma}_C^2(w^1)} \right)^{1/2}. \tag{20}$$

The properties of this estimator are based on some mild regularity assumptions (see Kneip et al, 2021, 2022 for details). These regularity assumptions include (i) for $t = 1, 2$, we have iid observations (X_i^t, Y_i^t), $i = 1, \ldots, n_t$; (ii) for $n = \min\{n_1, n_2\}$, $((X_i^1, Y_i^1), (X_i^2, Y_i^2))$ are iid with joint density f_{12}, with (X_i^s, Y_i^s) independent of (X_j^t, Y_j^t) for $i \neq j$; and (iii) some regularity assumptions on the densities of (X^t, Y^t), f_t, $t = 1, 2$ and on $f_{12}(X^1, Y^1, X^2, Y^2)$ ensuring that $\mathcal{M}(w^1, w^2)$ and $\log \mathcal{M}(w^1, w^2)$ are well defined. Then

$$n^\kappa \big(\widehat{\mathcal{M}}(w^1, w^2) - \mathcal{M}(w^1, w^2) \big) \xrightarrow{\mathcal{L}} F_{w^1, w^2}, \tag{21}$$

as $n = \min\{n_1, n_2\} \to \infty$, where F_{w^1, w^2} is a regular distribution depending on the particular estimators used and on various features of the DGP. The rate κ depends on the assumption made on Ψ^1 and Ψ^2 (see Table 1). For instance, without the CRS assumption we have $\kappa = 1/(p + q - 0.5)$ when CFDH is used, and $\kappa = 2/(p + q + 1)$ for CDEA if the convexity assumption is valid.

Inference, e.g., confidence intervals for $\mathcal{M}(w^1, w^2)$, can then conducted by using the bootstrap. One can perform the bootstrap on pairs $\{(W_i^1, W_i^2)\}_{i=1}^n$ using the subsampling techniques described by Simar and Wilson (2011). The same applies for all the components appearing in the decomposition (17), as discussed by Simar and Wilson (2019).

3.4 Inference for Geometric Means

In empirical studies, researchers often compute the geometric mean of the MPI over all the units belonging to a particular group. We can also provide inference (confidence intervals or tests of hypotheses) about the productivity changes of the entire group.

To derive the results, first consider the mean $\mu_{\mathcal{M}} = E(\log \mathcal{M}_i)$ where

$$\log \mathcal{M}_i = \frac{1}{2}\left[\log \gamma_C^1(W_i^2) + \log \gamma_C^2(W_i^2) - \log \gamma_C^1(W_i^1) - \log \gamma_C^2(W_i^1)\right]. \tag{22}$$

Then define $\widehat{\mu}_{\mathcal{M},n} = n^{-1}\sum_{i=1}^{n} \log \widehat{\mathcal{M}}_i$ where

$$\log \widehat{\mathcal{M}}_i = \frac{1}{2}\left[\log \widehat{\gamma}_C^1(W_i^2) + \log \widehat{\gamma}_C^2(W_i^2) - \log \widehat{\gamma}_C^1(W_i^1) - \log \widehat{\gamma}_C^2(W_i^1)\right]. \tag{23}$$

Kneip et al (2021, 2022) derive CLTs for $\mu_{\mathcal{M}}$ by establishing that

$$\sqrt{n}\,(\widehat{\mu}_{\mathcal{M},n} - \mu_{\mathcal{M}} - \mathcal{R}_n) \xrightarrow{\mathcal{L}} N(0, \sigma_{\mathcal{M}}^2) \tag{24}$$

as $n \leq \min\{n_1, n_2\} \to \infty$, where $\mathcal{R}_n := C_{\mathcal{M}} n^{-\kappa} + \xi_{n,\kappa}$ is the leading bias term, with $\xi_{n,\kappa} = O\big(n^{-2\kappa}\big)$ when CFDH estimators are used and $\xi_{n,\kappa} = O\big(n^{-3\kappa/2}\big)$ if CDEA estimators are used. This property ensures that $\widehat{\mu}_{\mathcal{M},n} \xrightarrow{p} \mu_{\mathcal{M}}$. In addition, a consistent estimator of the variance is given by

$$\widehat{\sigma}_{\mathcal{M}}^2 = n^{-1}\sum_{i=1}^{n}\big(\log \widehat{\mathcal{M}}_i - \widehat{\mu}_{\mathcal{M},n}\big)^2 \xrightarrow{p} \sigma_{\mathcal{M}}^2. \tag{25}$$

Using the Delta Method, it is easy to obtain the desired result for the geometric mean $M_n = \left(\prod_{i=1}^{n} \mathcal{M}_i\right)^{1/n} = \exp(\mu_{\mathcal{M}})$, which is consistently estimated by $\widehat{M}_n = \left(\prod_{i=1}^{n} \widehat{\mathcal{M}}_i\right)^{1/n} = \exp(\widehat{\mu}_{\mathcal{M},n})$. Kneip et al (2021, 2022) prove that

$$\sqrt{n}\left(\widehat{M}_n - \widetilde{M}\right) \xrightarrow{\mathcal{L}} N(0, \exp(2\mu_{\mathcal{M}})\sigma_{\mathcal{M}}^2) \tag{26}$$

as $n \leq \min\{n_1, n_2\} \to \infty$, where

$$\begin{aligned} \widetilde{M} := \exp(\mu_{\mathcal{M}} + \mathcal{R}_n) &= \exp(\mu_{\mathcal{M}})(1 + \mathcal{R}_n) + O\left(n^{-2\kappa}\right) \\ &= M_n + \exp(\mu_{\mathcal{M}}) C_{\mathcal{M}} n^{-\kappa} + \xi_{n,\kappa}, \end{aligned} \tag{27}$$

where $C_{\mathcal{M}}$ is a constant. The variance $\exp(2\mu_{\mathcal{M}})\sigma_{\mathcal{M}}^2$ is estimated consistently by $\exp(2\widehat{\mu}_{\mathcal{M}})\widehat{\sigma}_{\mathcal{M}}^2$. Note that if κ is too small, i.e., less than or equal to $1/2$, then the leading term of the bias, $\exp(\mu_{\mathcal{M}})C_{\mathcal{M}} n^{-\kappa} = O\big(n^{-\kappa}\big)$, does not vanish when multiplied by $\sqrt{n}$ and $n \to \infty$.

Kneip et al (2021, Section 4) describe in detail a procedure for estimating the leading bias term by using a generalized jackknife method on the sample of common pairs $\mathcal{W}_n = \big\{(W_i^1, W_i^2)\big\}_{i=1}^{n}$. This provides a consistent estimate $\widehat{B}_{n,\kappa}$

of $\exp(\mu_{\mathcal{M}})C_{\mathcal{M}}n^{-\kappa}$, with error of the same order as $\xi_{n,\kappa} = o(n^{-\kappa})$. So, when $\kappa \geq 1/2$, which occurs only in limited cases (see Table 1), the CLT result

$$\sqrt{n}\left(\widehat{M}_n - \widehat{B}_{n,\kappa} - M_n + \xi_{n,\kappa}\right) \xrightarrow{\mathcal{L}} N(0, \exp(2\mu_{\mathcal{M}})\sigma^2_{\mathcal{M}}) \tag{28}$$

as $n \to \infty$ can be used to make inference about M_n. However, if $\kappa < 1/2$, the bias correction is not enough; from (27), it is evident that the bias explodes as $n \to \infty$ when $\kappa < 1/2$. This problem requires defining the geometric mean over a smaller number $n_\kappa = \lfloor n^{2\kappa} \rfloor < n$ of randomly selected units to reduce the rate of convergence. Then as $n \to \infty$,

$$\sqrt{n_\kappa}\left(\widehat{M}_{n_\kappa} - \widehat{B}_{n,\kappa} - M_n + \xi_{n,\kappa}\right) \xrightarrow{\mathcal{L}} N(0, \exp(2\mu_{\mathcal{M}})\sigma^2_{\mathcal{M}}), \tag{29}$$

where the geometric mean over the subsample, i.e. $\widehat{M}_{n_\kappa} = \left(\prod_{i=1}^{n_\kappa} \widehat{\mathcal{M}}_i\right)^{1/n_\kappa}$ replaces $\widehat{M}_n$ in (28). Then confidence intervals for M_n can be estimated using normal quantiles and $\exp(2\widehat{\mu}_{\mathcal{M}})\widehat{\sigma}^2_{\mathcal{M}}$ as a consistent estimator of the variance.

4 Malmquist Indices in Action

The recent results of Kneip et al (2021, 2022) enable inference on Malmquist indices and their component indices under quite general technologies, i.e., with our without convexity of Ψ and with or without CRS. Moreover, results are provided for testing whether Ψ is convex or whether CRS holds. The computations are easy, and can be done using commands in the *FEAR* package described by Wilson (2008) for the *R* programming language. Results from intensive Monte-Carlo experiments are provided by Kneip et al (2021, 2022), indicating that even with small sample sizes, the CLTs provide good approximations that are usable in practice.

The methods described here have already been used in several fields of applications. For example, Simar and Wilson (2019) revisit the analysis of productivity in Swedish pharmacies by Färe et al (1992) to provide inference. Simar and Wilson (2023b) re-examine the analysis of productivity changes among U.S. hospitals by Burgess and Wilson (1995), while Simar and Wilson (2023a) update the analysis of productivity changes among OECD countries by Färe et al (1994), providing inference in both cases using the methods described above. The results and methods described above will likely prove useful in many other applications in the near future.

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