Structural stability of the Brinkman–Forchheimer equations for flows in porous media with variable porosity

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Abstract

In this paper we are concerned with the structural stability (continuous dependence of solutions) of the Brinkman–Forchheimer model for flows in porous media. Payne and Straughan [1] established structural stability for the case of constant porosity. Herein we extend these results to porous media of variable porosity. The main differences when the porosity is no longer constant but a time-independent field variable are: i) the velocity field is not divergence-free, ii) the term related to the shear viscous stresses does not reduce to the Laplacian operator but involves the full deviatoric strain-rate tensor, and iii) there is an additional term due to normal viscous stresses. Herein we establish continuous dependence of solutions with respect to the physical parameters entering the equations, namely, the shear and bulk viscosities and the coefficients of the linear (Darcy) and quadratic (Forchheimer) terms for the interfacial drag. More specifically, we show continuous dependence in the weighted L^2 norm, with the porosity being the weight. Finally, we provide upper and lower bounds for the rate of decay of the kinetic energy. According to them, the kinetic energy decays exponentially with time but the decay rate depends on both the minimum and maximum porosities of the medium.

Keywords— Continuous dependence, structural stability, Brinkman–Forchheimer model, porous media, variable porosity, weighted norms

1 Introduction

The analysis of continuous dependence of solutions (known as structural stability) of partial differential equations is a particularly interesting topic that has been pursued intensively over the years. More specifically, structural stability examines the sensitivity of solutions to variations of the parameters entering the equations, whereas classical stability focuses on the sensitivity to changes in the initial data. Accordingly, structural stability can provide useful information about the properties of the solutions. Moreover, with regard to two-phase systems, such as porous media, structural stability represents an important test of the consistency and pertinence of relevant mathematical models. For a review of the scope and methods of structural stability, the interested reader is referred to Ames and Straughan [2].

The literature on structural stability of models for fluid flow and thermal convection in porous media is quite rich; see, for example, [1–13] as well as the monograph of Straughan [14]. Also, within the fields of elasticity and thermoelasticity, there has been a considerable amount of work on the structural stability of models for porous-media; see [15–20] and references therein. Beyond structural stability, recent analytical results for porous-media flows include magnetohydrodynamics [21–23] and traveling waves with non-linear diffusion [24].

A well-known model for flows in saturated porous media is the Brinkman–Forchheimer model [25]. Therein the interphasial drag is represented as the sum of a linear and a quadratic term, referred to as the Darcy and Forchheimer term, respectively. This model can be derived via different approaches, such as those presented in [26] and [27]. Typically, these approaches involve either a mixture-theoretic formalism or volume averaging. The continuous dependence of solutions of the Brinkman–Forchheimer equations in the L^2 norm has been demonstrated by Payne & Straughan [1] for the case of porous media with constant porosity. Subsequently, the authors of [28] have shown continuous dependence of solutions in the stronger H^1 norm.

In the present article we extend the results of [1] and derive continuous dependence of solutions in the (weighted) L^2 norm for flows in porous media with variable porosity. Relevant applications are flows through immobile sediments, flows through dense vegetation, filtration systems and others. The main complexities introduced by the variable porosity are the following. First, the velocity field is not divergence-free. Second, the term related to the shear viscous stresses no longer reduces to the Laplacian operator but involves the full deviatoric deformation tensor. Third, normal viscous stresses are now present, in addition to shear ones. Fourth, the Darcy and Forchheimer terms of the interphasial are functions of the porosity. Moreover, according to our formulation, the Forchheimer coefficient is also a function of the shear viscosity of the fluid. The main mathematical challenges associated to these complexities are the establishment of estimates for the viscous stress tensor and the handling of the Darcy-Forchheimer term which is a nonlinear sink term.

In our analysis the porosity field is introduced as the weight of the various norms. Then, we derive appropriate bounds for the term related to shear viscous stresses. Subsequently we establish continuous dependence of solutions on the Forchheimer and Darcy coefficients as well as the shear and bulk viscosity coefficients. Finally we provide results for the kinetic-energy bounds.

2 Problem statement and preliminaries

The Brinkman–Forchheimer equations for flow in porous media with variable porosity and constant fluid density are as follows,

$$\boldsymbol{\nabla}\cdot(\phi\boldsymbol{u}) \;=\; 0\,, \tag{1}$$

$$\phi\frac{\partial\boldsymbol{u}}{\partial t}+\phi\boldsymbol{\nabla}p \;=\; \boldsymbol{\nabla}\cdot(\phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{I})+\boldsymbol{\nabla}\cdot\big(\phi\nu\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})\big)-a^\star(\phi)\boldsymbol{u}-b^\star(\phi)|\boldsymbol{u}|\boldsymbol{u}+\phi\boldsymbol{g}\,. \tag{2}$$

The first of these equations represents the conservation of fluid mass and the second one the balance of the fluid momentum. Therein, ϕ is the porosity, $\boldsymbol{u}$ the fluid velocity in the porous medium, p the pressure divided by the constant density, ζ the kinematic bulk viscosity and ν the kinematic shear viscosity. Both ζ and ν are positive constants, $\zeta,\nu>0$. Also, $\boldsymbol{g}$ is the vector of the gravitational acceleration and $\boldsymbol{I}$ is the identity matrix. The Darcy term of the interphasial drag in (2) is given by $-a^\star(\phi)\boldsymbol{u}$ and the Forchheimer term by $-b^\star(\phi)|\boldsymbol{u}|\boldsymbol{u}$. The sum of these two terms represents the overall force that is exerted by the solid matrix to the interstitial fluid. The quantities $a^\star$ and $b^\star$ are functions of ϕ and are given by,

$$a^\star(\phi) \;=\; a\nu(1-\phi)\,, \qquad b^\star(\phi) \;=\; b(1-\phi)+d(1-\phi)^2\,. \tag{3}$$

In these expressions, a is the Darcy coefficient, while b and d are referred to herein as the first and second Forchheimer coefficients, respectively. The coefficients a, b and d are all constant and positive: $a, b, d > 0$.

Further, the term $\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})$ stands for twice the deviatoric component of the strain-rate tensor $\boldsymbol{V}$. In other words,

$$\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u}) \;=\; 2\left(\boldsymbol{V}(\boldsymbol{u}) - \frac{1}{3}\,(\boldsymbol{\nabla}\cdot\boldsymbol{u})\;\boldsymbol{I}\right)\,, \qquad \text{with} \quad \boldsymbol{V}(\boldsymbol{u}) \;=\; \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^\top\right) \tag{4}$$

The porosity ϕ is assumed to be a continuous function that is strictly bounded below by zero and above by unity. In other words,

$$0 \;<\; \phi_{\min} \;\le\; \phi \;\le\; \phi_{\max} \;<\; 1\,. \tag{5}$$

As a matter of fact, the first extreme case $\phi = 0$ (zero porosity) represents a solid material without voids and absence of fluid, whereas the second one $\phi = 1$ represents a pure-fluid domain and absence of a solid matrix.

We consider the associated initial-boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$. Homogeneous Dirichlet boundary conditions are imposed on the velocity field, that is, $\boldsymbol{u} = 0$ on $\partial\Omega$. We consider solutions $\boldsymbol{u} \in L^2(0,T;H_0^1(\Omega))$. Since ϕ is not constant, it is convenient to work with ϕ-weighted norms. Actually, weighted spaces are commonly employed in the mathematical analysis of equations arising in fluid mechanics. One well-known such example is the so-called lake equations; see, for example, [29–31]. Other examples are the shallow-water model for basins with varying bottom topography [32] and models for flows in mixed (porous-pure fluid) domains [33].

The weighted L^2 norm, with the porosity $\phi(\boldsymbol{x})$ being the weight, is defined by

$$\|\boldsymbol{u}\| \;:=\; \left(\int_\Omega \phi|\boldsymbol{u}|^2 d\boldsymbol{x}\right)^{1/2}\,. \tag{6}$$

It is straightforward to show that all standard inequalities (Poincaré, Hölder, Cauchy-Schwarz, Sobolev etc) are valid for the corresponding ϕ-weighted norms, possibly with different inequality constants that involve $\phi_{\max}$ and $\phi_{\min}$. For example, for smooth bounded domains, Poincaré's inequality for $\boldsymbol{u} \in H_0^1(\Omega)$ reads ([34], section 6.5)

$$\int_\Omega |u|^2 dx \;\le\; \frac{1}{\lambda_1}\int_\Omega |\boldsymbol{\nabla} u|^2 dx\,, \tag{7}$$

where λ_1 is the first eigenvalue of the Laplacian. From this inequality, we compute that

$$\lambda_1 \int_\Omega \phi|\boldsymbol{u}|^2\,d\boldsymbol{x} \;\le\; \lambda_1\phi_{\max}\int_\Omega |\boldsymbol{u}|^2\,d\boldsymbol{x} \;\le\; \phi_{\max}\int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{u}\,d\boldsymbol{x} \;\le\; \frac{\phi_{\max}}{\phi_{\min}}\int_\Omega \phi\boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{u}\,d\boldsymbol{x}\,. \tag{8}$$

In other words, we have that

$$\lambda_1 \|\boldsymbol{u}\|^2 \;\le\; \frac{\phi_{\max}}{\phi_{\min}} \|\boldsymbol{\nabla u}\|^2 \,, \tag{9}$$

which is Poincaré's inequality for weighted norms in $H_0^1(\Omega)$. In a similar manner one can establish the aforementioned inequalities in terms weighted norms. Therefore, those inequalities will be employed directly in our analysis.

For the purposes of showing continuous dependence of solutions, we need a lower bound of the weighted L^2 norm of the deviator tensor $\boldsymbol{V}^{\mathrm{d}}$. To this end, we observe that since $\boldsymbol{V}^{\mathrm{d}}$ is symmetric and traceless, the following relation holds

$$\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u}) : \boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u}) \;=\; 2\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u}) : \boldsymbol{\nabla u}\,, \tag{10}$$

where $:$ denotes the double-dot product between two second-order tensors, $\boldsymbol{A} : \boldsymbol{B} = \sum_i \sum_j A_{ij} B_{ij}$.

By noting that $\boldsymbol{u}$ vanishes on $\partial\Omega$, and by using (10) and the porosity bound (5), we compute the following for the norm of $\boldsymbol{V}^{\mathrm{d}}$ in Ω,

$$\|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})\|^2 \;=\; \int_\Omega \phi \boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u}) : \boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})\, d\boldsymbol{x} \;\ge\; 2\phi_{\min} \int_\Omega \boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u}) : \boldsymbol{\nabla u}\, d\boldsymbol{x} \;=$$
$$2\phi_{\min} \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla u}\, d\boldsymbol{x} + 2\phi_{\min} \int_\Omega \boldsymbol{\nabla u} : (\boldsymbol{\nabla u})^\top\, d\boldsymbol{x} - \frac{4}{3}\phi_{\min} \int_\Omega (\mathrm{Tr}\, \boldsymbol{\nabla u})^2\, d\boldsymbol{x}\,. \tag{11}$$

Herein Tr denotes the trace of a second-order tensor. Accordingly, $\mathrm{Tr}\, \boldsymbol{V}(\boldsymbol{u}) = \mathrm{Tr}\, \boldsymbol{\nabla u} = \boldsymbol{\nabla} \cdot \boldsymbol{u}$. For the second term on the right-hand side of the second equality in (11) we employ integration by parts and use the fact that $\boldsymbol{u}$ vanishes on the boundary $\partial\Omega$. In this manner we get

$$\int_\Omega \boldsymbol{\nabla u} : (\boldsymbol{\nabla u})^\top\, d\boldsymbol{x} \;=\; -\int_\Omega (\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u})) \cdot \boldsymbol{u}\, d\boldsymbol{x}\,. \tag{12}$$

Applying integration by parts once again, and since $\boldsymbol{u} = 0$ on $\partial\Omega$, yields

$$\int_\Omega \boldsymbol{\nabla u} : (\boldsymbol{\nabla u})^\top\, d\boldsymbol{x} \;=\; \int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{u})^2\, d\boldsymbol{x} \;=\; \int_\Omega (\mathrm{Tr}\, \boldsymbol{\nabla u})^2\, d\boldsymbol{x}\,. \tag{13}$$

By inserting this result in (11) we find that

$$\|\boldsymbol{V}^{\mathrm{d}}\|^2 \;\ge\; 2\phi_{\min} \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla u}\, d\boldsymbol{x} + \frac{2}{3}\phi_{\min} \int_\Omega (\mathrm{Tr}\, \boldsymbol{\nabla u})^2\, d\boldsymbol{x} \;\ge\; 2\frac{\phi_{\min}}{\phi_{\max}} \int_\Omega \phi \boldsymbol{\nabla u} : \boldsymbol{\nabla u}\, d\boldsymbol{x}\,. \tag{14}$$

In summary, we have that

$$\|\boldsymbol{V}^{\mathrm{d}}\|^2 \;\ge\; 2\frac{\phi_{\min}}{\phi_{\max}} \|\boldsymbol{\nabla u}\|^2\,. \tag{15}$$

As a side note, we remark that the combination of (15) with Poincaré's inequality (9) yields a lower bound of $\|\boldsymbol{V}^{\mathrm{d}}\|$ in terms of the weighted L^2 norm $\|\boldsymbol{u}\|$ hence a lower bound in terms of the weighted H_0^1 norm of $\boldsymbol{u}$. Also, it is well known that

$$|\boldsymbol{\nabla} \cdot \boldsymbol{u}|^2 \le 3\, \nabla \boldsymbol{u} : \nabla \boldsymbol{u}\,, \tag{16}$$

from which we can derive that

$$\|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})\|^2 \;\leq\; 4\frac{\phi_{\max}}{\phi_{\min}}\|\boldsymbol{\nabla}\boldsymbol{u}\|^2\,. \tag{17}$$

In other words, the weighted L^2 norm of $\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})$ is equivalent to the weighted H_0^1 norm of $\boldsymbol{u}$,

$$C_1\|\boldsymbol{u}\|_{H_0^1} \;\leq\; \|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})\| \;\leq\; C_2\|\boldsymbol{u}\|_{H_0^1}\,, \tag{18}$$

with C_1 and C_2 constants that involve $\phi_{\min}$, $\phi_{\max}$ and the first eigenvalue of the Laplacian.

3 Continuous dependence on the Forchheimer coefficients

First we examine the continuous dependence on the first Forchheimer coefficient b. For this purpose, let us denote by $(\boldsymbol{u}, p)$ and $(\boldsymbol{v}, q)$ the solutions for the velocity and pressure of the following initial-boundary-value problems for different first Forchheimer coefficients b_1 and b_2 in a bounded domain $\Omega \in \mathbb{R}^3$ with boundary $\partial\Omega$,

$$\boldsymbol{\nabla}\cdot(\phi\boldsymbol{u}) \;=\; 0\,, \quad \text{in} \quad \Omega\times\{t>0\}\,, \tag{19}$$

$$\phi\frac{\partial\boldsymbol{u}}{\partial t} + \phi\boldsymbol{\nabla}p \;=\; \boldsymbol{\nabla}\cdot(\phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{I}) + \boldsymbol{\nabla}\cdot\big(\phi\nu\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})\big) - a^\star(\phi)\boldsymbol{u} - b_1^\star(\phi)|\boldsymbol{u}|\boldsymbol{u} + \phi\boldsymbol{g}\,, \tag{20}$$

$$\boldsymbol{u} \;=\; \boldsymbol{0}\,, \quad \text{on} \quad \partial\Omega\times\{t>0\}\,, \tag{21}$$

$$\boldsymbol{u}(\boldsymbol{x},0) \;=\; \boldsymbol{f}(\boldsymbol{x})\,, \quad \boldsymbol{x}\in\Omega\,. \tag{22}$$

and

$$\boldsymbol{\nabla}\cdot(\phi\boldsymbol{v}) \;=\; 0\,, \quad \text{in} \quad \Omega\times\{t>0\}\,, \tag{23}$$

$$\phi\frac{\partial\boldsymbol{v}}{\partial t} + \phi\boldsymbol{\nabla}q \;=\; \boldsymbol{\nabla}\cdot(\phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{v})I) + \boldsymbol{\nabla}\cdot\big(\phi\nu\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})\big) - a^\star(\phi)\boldsymbol{v} - b_2^\star(\phi)|\boldsymbol{v}|\boldsymbol{v} + \phi\boldsymbol{g}\,, \tag{24}$$

$$\boldsymbol{v} \;=\; \boldsymbol{0}\,, \quad \text{on} \quad \partial\Omega\times\{t>0\}\,, \tag{25}$$

$$\boldsymbol{v}(\boldsymbol{x},0) \;=\; \boldsymbol{f}(\boldsymbol{x})\,, \quad \boldsymbol{x}\in\Omega\,. \tag{26}$$

In the above problems, $\boldsymbol{f}$ represents the initial condition. Also, $b_1^\star(\phi) = b_1(1-\phi) + d(1-\phi)^2$ and $b_2^\star(\phi) = b_2(1-\phi) + d(1-\phi)^2$, in accordance with (3).

Next we introduce the difference variables $\boldsymbol{w}$, p and $\hat{b}^\star$,

$$\boldsymbol{w} \;=\; \boldsymbol{u}-\boldsymbol{v}\,, \quad \pi = p - q\,, \quad \hat{b}^\star(\phi) \;=\; b_1^\star - b_2^\star \;=\; \hat{b}(1-\phi)\,, \tag{27}$$

where $\hat{b}$ is the difference coefficient,

$$\hat{b} \;=\; b_1 - b_2\,. \tag{28}$$

Therefore $\boldsymbol{w}$ and π satisfy the following initial-boundary-value problem,

$$\boldsymbol{\nabla}\cdot(\phi\boldsymbol{w}) = 0\,, \quad \text{in} \quad \Omega\times\{t>0\}\,, \tag{29}$$

$$\begin{aligned}\phi\frac{\partial \boldsymbol{w}}{\partial t} &= \boldsymbol{\nabla}\cdot(\phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{w})\boldsymbol{I}) + \boldsymbol{\nabla}\cdot\big(\phi\nu\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{w})\big) - \\ & a^{\star}(\phi)\boldsymbol{w} - (b_1^{\star}(\phi)|\boldsymbol{u}|\boldsymbol{u} - b_2^{\star}(\phi)|\boldsymbol{v}|\boldsymbol{v}) - \phi\boldsymbol{\nabla}\pi\,, \quad \text{in} \quad \Omega\times\{t>0\}\,,\end{aligned} \tag{30}$$

$$\boldsymbol{w} = \boldsymbol{0}\,, \quad \text{on} \quad \partial\Omega\times\{t>0\}\,, \tag{31}$$

$$\boldsymbol{w}(\boldsymbol{x},0) = \boldsymbol{0}\,, \quad \boldsymbol{x}\in\Omega\,. \tag{32}$$

Upon multiplication of (30) with $\boldsymbol{w}$ and integration over Ω we have,

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi|\boldsymbol{w}|^2\,d\boldsymbol{x} &= -2\int_\Omega \phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{w})^2\,d\boldsymbol{x} - 2\int_\Omega \phi\nu\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{w}) : \boldsymbol{\nabla}\boldsymbol{w}\,d\boldsymbol{x} \\ -2\int_\Omega a^{*}(\phi)|\boldsymbol{w}|^2\,d\boldsymbol{x} &- 2\int_\Omega \Big(\hat{b}^{*}(\phi)|\boldsymbol{u}|\boldsymbol{u} + b_2^{*}(\phi)(|\boldsymbol{u}|\boldsymbol{u} - |\boldsymbol{v}|\boldsymbol{v})\Big)\cdot\boldsymbol{w}\,d\boldsymbol{x}\,,\end{aligned} \tag{33}$$

where we have used integration by parts and then rearranged the various terms.

By noting that the first term on the right-hand side of (33) is negative and by employing (10), relation (33) yields the following inequality

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi|\boldsymbol{w}|^2\,d\boldsymbol{x} \le & -\nu\|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{w})\|^2 - 2\int_\Omega a^{*}(\phi)|\boldsymbol{w}|^2\,d\boldsymbol{x} - \\ & 2\int_\Omega \Big(\hat{b}^{*}(\phi)|\boldsymbol{u}|\boldsymbol{u} + b_2^{*}(\phi)(|\boldsymbol{u}|\boldsymbol{u} - |\boldsymbol{v}|\boldsymbol{v})\Big)\cdot\boldsymbol{w}\,d\boldsymbol{x}\,.\end{aligned} \tag{34}$$

Next, we make use of the following inequality [1],

$$(|\boldsymbol{u}|\boldsymbol{u} - |\boldsymbol{v}|\boldsymbol{v})\cdot\boldsymbol{w} = \frac{1}{2}\left(|\boldsymbol{u}| + |\boldsymbol{v}|\right)|\boldsymbol{w}|^2 + \frac{1}{2}\big(|\boldsymbol{u}| - |\boldsymbol{v}|\big)^2(|\boldsymbol{u}| + |\boldsymbol{v}|\big) \ge 0\,. \tag{35}$$

Since $b_2^{*}(\phi)$ is strictly positive, we readily deduce that

$$\int_\Omega b_2^{*}(\phi)(|\boldsymbol{u}|\boldsymbol{u} - |\boldsymbol{v}|\boldsymbol{v})\cdot\boldsymbol{w}\,d\boldsymbol{x} \ge 0\,. \tag{36}$$

Now, by introducing (36) into (34) and by using the porosity bound (5), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi|\boldsymbol{w}|^2\,d\boldsymbol{x} \le & -\nu\|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{w})\|^2 - 2a\nu\left(\frac{1}{\phi_{\max}} - 1\right)\|\boldsymbol{w}\|^2 + \\ & 2\hat{b}\left(\frac{1}{\phi_{\min}} - 1\right)\int_\Omega \phi|\boldsymbol{u}|^2|\boldsymbol{w}|\,d\boldsymbol{x}\,.\end{aligned} \tag{37}$$

Next, by substituting (15) in (37) we arrive at

$$\frac{d}{dt}\|\boldsymbol{w}\|^2 \le -2\nu\frac{\phi_{\min}}{\phi_{\max}}\|\boldsymbol{\nabla}\boldsymbol{w}\|^2 - 2a\nu\left(\frac{1}{\phi_{\max}} - 1\right)\|\boldsymbol{w}\|^2 + 2\hat{b}\left(\frac{1}{\phi_{\min}} - 1\right)\int_\Omega \phi|\boldsymbol{u}|^2|\boldsymbol{w}|\,d\boldsymbol{x}\,. \tag{38}$$

In order to bound the term that involves the Forchheimer coefficient b in the above equation, we employ Hölder's inequality in the form

$$\int_\Omega \phi|\boldsymbol{u}|^2|\boldsymbol{w}|\,d\boldsymbol{x} \;\le\; \left(\int_\Omega \phi|\boldsymbol{u}|^3\,d\boldsymbol{x}\right)^{2/3}\left(\int_\Omega \phi|\boldsymbol{w}|^3\,d\boldsymbol{x}\right)^{1/3}. \tag{39}$$

For the last integral in (39) we employ successively the Cauchy-Schwarz inequality

$$\left(\int_\Omega \phi|\boldsymbol{w}|^3\,d\boldsymbol{x}\right)^{1/3} \;\le\; \left(\int_\Omega \phi|\boldsymbol{w}|^2\,d\boldsymbol{x}\right)^{1/6}\left(\int_\Omega \phi|\boldsymbol{w}|^4\,d\boldsymbol{x}\right)^{1/6} \tag{40}$$

and the Sobolev inequality

$$\int_\Omega \phi|\boldsymbol{w}|^4\,d\boldsymbol{x} \;\le\; c\,\frac{\phi_{\max}}{\phi_{\min}^2}\left(\int_\Omega \phi|\boldsymbol{w}|^2\,d\boldsymbol{x}\right)^{1/2}\left(\int_\Omega \phi\boldsymbol{\nabla w}:\boldsymbol{\nabla w}\,d\boldsymbol{x}\right)^{3/2}, \tag{41}$$

where c is the constant that appears in the corresponding Sobolev inequality for standard (non-weighted) L^q norms. By combining inequalities (40) and (41), we obtain the bound

$$\left(\int_\Omega \phi|\boldsymbol{w}|^3\,d\boldsymbol{x}\right)^{1/3} \;\le\; \left(c\,\frac{\phi_{\max}}{\phi_{\min}^2}\right)^{1/6}\left(\int_\Omega \phi|\boldsymbol{w}|^2\,d\boldsymbol{x}\right)^{1/4}\left(\int_\Omega \phi\boldsymbol{\nabla w}:\boldsymbol{\nabla w}\,d\boldsymbol{x}\right)^{1/4}. \tag{42}$$

In the same manner we can also derive that

$$\left(\int_\Omega \phi|\boldsymbol{u}|^3\,d\boldsymbol{x}\right)^{2/3} \;\le\; \left(c\,\frac{\phi_{\max}}{\phi_{\min}^2}\right)^{1/3}\left(\int_\Omega \phi|\boldsymbol{u}|^2\,d\boldsymbol{x}\right)^{1/2}\left(\int_\Omega \phi\boldsymbol{\nabla u}:\boldsymbol{\nabla u}\,d\boldsymbol{x}\right)^{1/2}. \tag{43}$$

We now substitute (39), (42) and (43) in (38) to get the following estimate

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{w}\|^2 \;\le\; & -2\nu\frac{\phi_{\min}}{\phi_{\max}}\|\boldsymbol{\nabla w}\|^2 - 2a\nu\left(\frac{1}{\phi_{\max}}-1\right)\|\boldsymbol{w}\|^2 + 2\hat{b}\left(\frac{1}{\phi_{\min}}-1\right)\left(c\,\frac{\phi_{\max}}{\phi_{\min}^2}\right)^{1/2}\\ \times & \left(\int_\Omega \phi|\boldsymbol{u}|^2\,d\boldsymbol{x}\right)^{1/2}\left(\int_\Omega \phi\boldsymbol{\nabla u}:\boldsymbol{\nabla u}\,d\boldsymbol{x}\right)^{1/2}\left(\int_\Omega \phi|\boldsymbol{w}|^2\,d\boldsymbol{x}\right)^{1/4}\left(\int_\Omega \phi\boldsymbol{\nabla w}:\boldsymbol{\nabla w}\,d\boldsymbol{x}\right)^{1/4}. \end{aligned}\tag{44}$$

For the last term of the above relation, we make use of Poincaré's inequality

$$\lambda_1\int_\Omega \phi|\boldsymbol{w}|^2\,d\boldsymbol{x} \;\le\; \frac{\phi_{\max}}{\phi_{\min}}\int_\Omega \phi\boldsymbol{\nabla w}:\boldsymbol{\nabla w}\,d\boldsymbol{x}, \tag{45}$$

Also, for the last term on the right-hand side of (44), we multiply (20) by $\boldsymbol{u}$ and then use integration by parts to derive

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \phi|\boldsymbol{u}|^2 d\boldsymbol{x} + \int_\Omega a^\star(\phi)|\boldsymbol{u}|^2 d\boldsymbol{x} + \int_\Omega b_1^\star(\phi)|\boldsymbol{u}|^3 d\boldsymbol{x} + \int_\Omega \phi\nu\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u}):\boldsymbol{\nabla u}\,d\boldsymbol{x} \;=\\ &-\int_\Omega \phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\,d\boldsymbol{x}.\end{aligned}\tag{46}$$

The above equation can be integrated to yield

$$2\int_0^t \mathrm{e}^{2a\nu\left(\frac{1}{\phi_{\max}}-1\right)(\eta-t)}\|\boldsymbol{\nabla u}\|^2\,d\eta \;\le\; \frac{\phi_{\max}}{\nu\phi_{\min}}\|\boldsymbol{f}\|^2\,\mathrm{e}^{-2a\nu\left(\frac{1}{\phi_{\max}}-1\right)t}. \tag{47}$$

Next we introduce (45) and the bound $\|\boldsymbol{u}\| \le \|\boldsymbol{f}\|$ (which holds by virtue of (46)) in (44). In this manner we arrive at

$$\frac{d}{dt}\|\boldsymbol{w}\|^2 \ \le \ -2\nu\frac{\phi_{\min}}{\phi_{\max}}\|\boldsymbol{\nabla w}\|^2 - 2a\nu\left(\frac{1}{\phi_{\max}}-1\right)\|\boldsymbol{w}\|^2 + \frac{2\hat{b}\ c^{1/2}}{\lambda_1^{1/4}}\left(\frac{1}{\phi_{\min}}-1\right)\frac{(\phi_{\max})^{3/4}}{(\phi_{\min})^{5/4}}\|\boldsymbol{f}\|\,\|\boldsymbol{\nabla u}\|\,\|\boldsymbol{\nabla w}\|\,. \tag{48}$$

For the last term on the right-hand side of (48), we can employ the arithmetic-geometric mean inequality

$$\|\boldsymbol{\nabla u}\|\,\|\boldsymbol{\nabla w}\| \ \le \ \frac{\|\boldsymbol{\nabla u}\|^2}{2\epsilon}+\frac{\epsilon\|\boldsymbol{\nabla w}\|^2}{2} \tag{49}$$

with the following choice for ϵ

$$\epsilon \ = \ \frac{2\nu{\lambda_1}^{1/4}}{\hat{b}\ c^{1/2}\|\boldsymbol{f}\|}\left(\frac{1}{\phi_{\min}}-1\right)^{-1}\frac{(\phi_{\min})^{9/4}}{(\phi_{\max})^{7/4}}\,. \tag{50}$$

In this manner, inequality (48) yields

$$\frac{d}{dt}\|\boldsymbol{w}\|^2 \ \le \ -2a\nu\left(\frac{1}{\phi_{\max}}-1\right)\|\boldsymbol{w}\|^2+\frac{\hat{b}^2\ c}{2\lambda_1^{1/2}\nu}\left(\frac{1}{\phi_{\min}}-1\right)^2\frac{(\phi_{\max})^{5/2}}{(\phi_{\min})^{7/2}}\|\boldsymbol{f}\|^2\|\boldsymbol{\nabla u}\|^2\,. \tag{51}$$

The above relation can be integrated to establish

$$\|\boldsymbol{w}\|^2 \ \le \ \frac{\hat{b}^2\ c}{2\lambda_1^{1/2}\nu}\left(\frac{1}{\phi_{\min}}-1\right)^2\frac{(\phi_{\max})^{5/2}}{(\phi_{\min})^{7/2}}\|\boldsymbol{f}\|^2\int_0^t\|\boldsymbol{\nabla u}\|^2\,\mathrm{e}^{2a\nu\left(\frac{1}{\phi_{\max}}-1\right)(\eta-t)}d\eta\,. \tag{52}$$

Finally, we introduce (47) in (52), which allows us to obtain the following estimate

$$\|\boldsymbol{w}\|^2 \ \le \ \frac{c}{4\lambda_1^{1/2}\nu^2}\left(\frac{1}{\phi_{\min}}-1\right)^2\frac{(\phi_{\max})^{7/2}}{(\phi_{\min})^{9/2}}\|\boldsymbol{f}\|^4\,\mathrm{e}^{-2a\nu\left(\frac{1}{\phi_{\max}}-1\right)t}\,\hat{b}^2\,. \tag{53}$$

Inequality (53) establishes the sought-after continuous dependence on the Forchheimer coefficient b in the weighted L^2 norm. The constant c that appears in this estimate is the one coming directly from the Sobolev inequality (41).

It is worth mentioning that, according to (53), the two different solutions $\boldsymbol{u}$ and $\boldsymbol{v}$ converge to one other as time passes. This in turn implies a stronger form of structural stability than would have been guaranteed if the difference between $\boldsymbol{u}$ and $\boldsymbol{v}$ has been merely bounded. Also, the lower $\phi_{\max}$ is, the faster the velocity difference $\boldsymbol{w}$ decays to zero. Analogously, the higher $\phi_{\min}$ is, the faster the solution difference $\boldsymbol{w}$ decays to zero. We may further observe that $\phi_{\max}$ enters the exponential term in the estimate (53) but $\phi_{\min}$ does not. This implies that, according to this estimate, the rate of the time decay of the velocity difference $\boldsymbol{w}$ is mainly controlled by $\phi_{\max}$ and to a lesser extent by $\phi_{\min}$.

We conclude this section by noting that the same procedure can be followed to derive continuous dependence of solutions on the second Forchheimer coefficient. The final result reads

$$\|\boldsymbol{w}\|^2 \ \le \ \frac{c}{4\lambda_1^{1/2}\nu^2}\left(\frac{1}{\phi_{\min}}-2+\phi_{\max}\right)^2\frac{(\phi_{\max})^{7/2}}{(\phi_{\min})^{9/2}}\|\boldsymbol{f}\|^4\,\mathrm{e}^{-2a\nu\left(\frac{1}{\phi_{\max}}-1\right)t}\,\hat{d}^2\,, \tag{54}$$

where $\hat{d}$ is the difference between the second Forchheimer coefficients d_1 and d_2. This estimate has the same properties as that for the first Forchheimer coefficient.

4 Continuous dependence on the Darcy coefficient

In our analysis of continuous dependence on the Darcy coefficient a, the staring point is again the Brinkman–Forchheimer equations for flow in porous media with variable porosity (1)-(2). We consider the solutions $(\boldsymbol{u}, p)$ and $(\boldsymbol{v}, q)$ of the following initial-boundary-value problems for different Darcy coefficients a_1 and a_2, respectively,

$$\boldsymbol{\nabla} \cdot (\phi \boldsymbol{u}) \;=\; 0\,, \quad \text{in} \quad \Omega \times \{t > 0\}\,, \tag{55}$$

$$\phi \frac{\partial \boldsymbol{u}}{\partial t} + \phi \boldsymbol{\nabla} p \;=\; \boldsymbol{\nabla} \cdot (\phi \zeta (\boldsymbol{\nabla} \cdot \boldsymbol{u}) \boldsymbol{I}) + \boldsymbol{\nabla} \cdot \left(\phi \nu \boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})\right) - a_1^\star(\phi) \boldsymbol{u} - b^\star(\phi) |\boldsymbol{u}| \boldsymbol{u} + \phi \boldsymbol{g}\,, \tag{56}$$

$$\boldsymbol{u} \;=\; \boldsymbol{0}\,, \quad \text{on} \quad \partial\Omega \times \{t > 0\}\,, \tag{57}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) \;=\; \boldsymbol{f}(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \Omega\,, \tag{58}$$

and

$$\boldsymbol{\nabla} \cdot (\phi \boldsymbol{v}) \;=\; 0\,, \quad \text{in} \quad \Omega \times \{t > 0\}\,, \tag{59}$$

$$\phi \frac{\partial \boldsymbol{v}}{\partial t} + \phi \boldsymbol{\nabla} q \;=\; \boldsymbol{\nabla} \cdot (\phi \zeta (\boldsymbol{\nabla} \cdot \boldsymbol{v}) \boldsymbol{I}) + \boldsymbol{\nabla} \cdot \left(\phi \nu \boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})\right) - a_2^\star(\phi) \boldsymbol{v} - b^\star(\phi) |\boldsymbol{v}| \boldsymbol{v} + \phi \boldsymbol{g}\,, \tag{60}$$

$$\boldsymbol{v} \;=\; \boldsymbol{0}\,, \quad \text{on} \quad \partial\Omega \times \{t > 0\}\,, \tag{61}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) \;=\; \boldsymbol{f}(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \Omega\,. \tag{62}$$

Also, $a_1^\star(\phi) = a_1 \nu (1 - \phi)$ and $a_2^\star(\phi) = a_2 \nu (1 - \phi)$.

Next we introduce the difference variables $\boldsymbol{w}$, π and $\hat{a}^\star$,

$$\boldsymbol{w} \;=\; \boldsymbol{u} - \boldsymbol{v}\,, \quad \pi \;=\; p - q\,, \quad \hat{a}^\star(\phi) \;=\; a_1^\star - a_2^\star \;=\; \hat{a} \nu (1 - \phi)\,, \tag{63}$$

where $\hat{a}$ is the difference coefficient,

$$\hat{a} \;=\; a_1 - a_2\,. \tag{64}$$

Therefore, the velocity and pressure differences, $\boldsymbol{w}$ and π respectively, satisfy the following boundary initial-value problem,

$$\boldsymbol{\nabla} \cdot (\phi \boldsymbol{w}) \;=\; 0\,, \quad \text{in} \quad \Omega \times \{t > 0\}\,, \tag{65}$$

$$\begin{aligned} \phi \frac{\partial \boldsymbol{w}}{\partial t} \;&=\; \boldsymbol{\nabla} \cdot (\phi \zeta (\boldsymbol{\nabla} \cdot \boldsymbol{w}) \boldsymbol{I}) + \boldsymbol{\nabla} \cdot \left(\phi \nu \boldsymbol{V}^{\mathrm{d}}(\boldsymbol{w})\right) - \\ &(a_1^\star(\phi) \boldsymbol{u} - a_2^\star(\phi) \boldsymbol{v}) - b^\star(\phi)(|\boldsymbol{u}|\boldsymbol{u} - |\boldsymbol{v}|\boldsymbol{v}) - \phi \boldsymbol{\nabla} \pi \quad \text{in} \quad \Omega \times \{t > 0\}\,, \end{aligned} \tag{66}$$

$$\boldsymbol{w} \;=\; \boldsymbol{0}\,, \quad \text{on} \quad \partial\Omega \times \{t > 0\}\,, \tag{67}$$

$$\boldsymbol{w}(\boldsymbol{x}, 0) \;=\; \boldsymbol{0}\,, \quad \boldsymbol{x} \in \Omega\,. \tag{68}$$

As a first step, we multiply the momentum equation (66) by $\boldsymbol{w}$ and integrate over Ω to obtain,

$$\frac{d}{dt}\int_\Omega \phi|\boldsymbol{w}|^2\,d\boldsymbol{x} \;=\; -2\int_\Omega \phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{w})^2\,d\boldsymbol{x} - 2\int_\Omega \phi\nu\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{w}) : \nabla\boldsymbol{w}\,d\boldsymbol{x}$$
$$-\,2\int_\Omega a_1^*(\phi)|\boldsymbol{w}|^2\,d\boldsymbol{x} - 2\int_\Omega \hat{a}^*(\phi)\boldsymbol{v}\cdot\boldsymbol{w}\,d\boldsymbol{x} - 2\int_\Omega b^*(\phi)(|\boldsymbol{u}|\boldsymbol{u} - |\boldsymbol{v}|\boldsymbol{v})\cdot\boldsymbol{w}\,d\boldsymbol{x}\,. \tag{69}$$

By noting that the first term on the right-hand side of (69) is negative and by invoking (36), we deduce from equation (69) that

$$\frac{d}{dt}\|\boldsymbol{w}\|^2 \;\le\; -2\int_\Omega \phi\nu\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{w}) : \nabla\boldsymbol{w}\,d\boldsymbol{x} - 2a_1\nu\left(\frac{1}{\phi_{\max}} - 1\right)\|\boldsymbol{w}\|^2 + 2\hat{a}\nu\left(\frac{1}{\phi_{\min}} - 1\right)\int_\Omega \phi|\boldsymbol{v}||\boldsymbol{w}|\,d\boldsymbol{x}\,. \tag{70}$$

For the first term on the right-hand side of this relation we combine (15) and Poincaré's inequality (45) to obtain

$$\frac{d}{dt}\|\boldsymbol{w}\|^2 \;\le\; -\left(2a_1\nu\left(\frac{1}{\phi_{\max}} - 1\right) + 2\lambda_1\nu\frac{\phi_{\min}^2}{\phi_{\max}^2}\right)\|\boldsymbol{w}\|^2 + 2\hat{a}\nu\left(\frac{1}{\phi_{\min}} - 1\right)\int_\Omega \phi|\boldsymbol{v}||\boldsymbol{w}|\,d\boldsymbol{x}\,. \tag{71}$$

Next, we apply the arithmetic-geometric mean inequality in the last term of (71)

$$|\boldsymbol{v}|\,|\boldsymbol{w}| \;\le\; \frac{|\boldsymbol{v}|^2}{2\epsilon} + \frac{\epsilon|\boldsymbol{w}|^2}{2}\,, \tag{72}$$

with the following choice for ϵ

$$\epsilon \;=\; \frac{a_1}{\hat{a}}\left(\frac{1}{\phi_{\max}} - 1\right)\left(\frac{1}{\phi_{\min}} - 1\right)^{-1}\,. \tag{73}$$

In this manner, (71) yields

$$\frac{d}{dt}\|\boldsymbol{w}\|^2 \;\le\; -\left(a_1\nu\left(\frac{1}{\phi_{\max}} - 1\right) + 2\lambda_1\nu\frac{\phi_{\min}^2}{\phi_{\max}^2}\right)\|\boldsymbol{w}\|^2 +$$
$$\frac{\hat{a}^2\nu}{a_1}\left(\frac{1}{\phi_{\min}} - 1\right)^2\left(\frac{1}{\phi_{\max}} - 1\right)^{-1}\|\boldsymbol{v}\|^2\,. \tag{74}$$

We can then multiply (60) by $\boldsymbol{v}$ and integrate over Ω to derive

$$\frac{d}{dt}\,\|\boldsymbol{v}\|^2 + 2a_2\nu\left(\frac{1}{\phi_{\max}} - 1\right)\|\boldsymbol{v}\|^2 \;\le\; 0\,. \tag{75}$$

The last relation can be readily integrated to yield

$$\|\boldsymbol{v}\|^2 \;\le\; \|\boldsymbol{f}\|^2\,\mathrm{e}^{-2a_2\nu\left(\frac{1}{\phi_{\max}} - 1\right)t} \tag{76}$$

By introducing the estimate (76) in (74), we get

$$\frac{d}{dt}\|\boldsymbol{w}\|^2 \;\leq\; -\left(a_1\nu\left(\frac{1}{\phi_{\max}}-1\right)+2\lambda_1\nu\frac{\phi_{\min}^2}{\phi_{\max}^2}\right)\|\boldsymbol{w}\|^2+$$
$$\frac{\hat{a}^2\nu}{a_1}\left(\frac{1}{\phi_{\min}}-1\right)^2\left(\frac{1}{\phi_{\max}}-1\right)^{-1}\|\boldsymbol{f}\|^2\,\mathrm{e}^{-2a_2\nu\left(\frac{1}{\phi_{\max}}-1\right)t}\,. \tag{77}$$

We now integrate the relation above to produce the following estimate

$$\|\boldsymbol{w}\|^2 \;\leq\; \left(\frac{\frac{1}{\phi_{\min}}-1}{\frac{1}{\phi_{\max}}-1}\right)^2\frac{\|\boldsymbol{f}\|^2}{a_1(a_1-2a_2)+2\lambda_1\left(\frac{\phi_{\min}}{\phi_{\max}}\right)^2}$$
$$\left(\mathrm{e}^{-2a_2\nu\left(\frac{1}{\phi_{\max}}-1\right)t}-\mathrm{e}^{-\left(a_1\nu\left(\frac{1}{\phi_{\max}}-1\right)+2\lambda_1\nu\frac{\phi_{\min}^2}{\phi_{\max}^2}\right)t}\right)\hat{a}^2\,. \tag{78}$$

Inequality (78) establishes the continuous dependence of solutions on the Darcy coefficient a.

According to this estimate, the solution difference $\boldsymbol{w}$ decays to zero as time passes, as is the case with the structural stability with respect to the Forchheimer coefficients. Now, however, both $\phi_{\max}$ and $\phi_{\min}$ enter the exponential term, which in turn implies that the decay rate of $\boldsymbol{w}$ is not necessarily controlled by $\phi_{\max}$ only.

Finally, we note that the factor 2 that multiples a_2 in the term $(a_1 - 2a_2)$ on the right-hand side of (78) emerges because of the specific way that we applied the arithmetic-geometric mean inequality to derive (74). If we had used a different choice of ϵ in this inequality, then a_2 would have been multiplied by a different factor. In this manner, one can avoid any potential singularity of the denominator in the above estimate. In other words, the particular term $a_1(a_1-2a_2)+2\lambda_1\left(\frac{\phi_{\min}}{\phi_{\max}}\right)^2$ does not pose any issue for the establishment of the structural stability of the model with respect to a.

5 Continuous dependence on the shear viscosity

We consider once again the Brinkman–Forchheimer equations for flow in porous media with variable porosity (1)-(2). Since we are now interested in continuous dependence on the shear viscosity, we consider the solutions $(\boldsymbol{u}, p)$ and $(\boldsymbol{v}, q)$ of the following initial-boundary-value problems for different shear viscosities, ν_1 and ν_2 respectively,

$$\boldsymbol{\nabla}\cdot(\phi\boldsymbol{u}) \;=\; 0\,, \quad \text{in} \quad \Omega\times\{t>0\}\,, \tag{79}$$

$$\phi\frac{\partial\boldsymbol{u}}{\partial t}+\phi\boldsymbol{\nabla}p \;=\; \boldsymbol{\nabla}\cdot(\phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{I})+\boldsymbol{\nabla}\cdot\left(\phi\nu_1\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})\right)-a_1^\star(\phi)\boldsymbol{u}-b^\star(\phi)|\boldsymbol{u}|\boldsymbol{u}+\phi\boldsymbol{g}\,, \tag{80}$$

$$\boldsymbol{u} \;=\; \boldsymbol{0}\,, \quad \text{on} \quad \partial\Omega\times\{t>0\}\,, \tag{81}$$

$$\boldsymbol{u}(\boldsymbol{x},0) \;=\; \boldsymbol{f}(\boldsymbol{x})\,, \quad \boldsymbol{x}\in\Omega\,, \tag{82}$$

and

$$\nabla\cdot(\phi\boldsymbol{v}) \;=\; 0\,,\quad \text{in}\quad \Omega\times\{t>0\}\,, \tag{83}$$

$$\phi\frac{\partial\boldsymbol{v}}{\partial t}+\phi\nabla q \;=\; \nabla\cdot(\phi\zeta(\nabla\cdot\boldsymbol{v})\boldsymbol{I})+\nabla\cdot\big(\phi\nu_2\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})\big)-a_2^{\star}(\phi)\boldsymbol{v}-b^{\star}(\phi)|\boldsymbol{v}|\boldsymbol{v}+\phi\boldsymbol{g}\,, \tag{84}$$

$$\boldsymbol{v} \;=\; \boldsymbol{0}\,,\quad \text{on}\quad \partial\Omega\times\{t>0\}\,, \tag{85}$$

$$\boldsymbol{v}(\boldsymbol{x},0) \;=\; \boldsymbol{f}(\boldsymbol{x})\,,\quad \boldsymbol{x}\in\Omega\,. \tag{86}$$

Moreover, $a_1^{\star}(\phi)=a\nu_1(1-\phi)$ and $a_2^{\star}(\phi)=a\nu_2(1-\phi)$.

Next we introduce the difference variables $\boldsymbol{w}$, π and $\hat{a}^{\star}$,

$$\boldsymbol{w} \;=\; \boldsymbol{u}-\boldsymbol{v}\,,\qquad \pi \;=\; p-q\,,\qquad \hat{a}^{\star}(\phi)=\; a_1^{\star}-a_2^{\star} \;=\; a\hat{\nu}(1-\phi)\,, \tag{87}$$

and the difference shear viscosity $\hat{\nu}$

$$\hat{\nu} \;=\; \nu_1-\nu_2\,. \tag{88}$$

Therefore, $\boldsymbol{w}$ and π satisfy the following initial-boundary-value problem

$$\nabla\cdot(\phi\boldsymbol{w}) \;=\; 0\,,\quad \text{in}\quad \Omega\times\{t>0\}\,, \tag{89}$$

$$\begin{aligned}\phi\frac{\partial\boldsymbol{w}}{\partial t} \;&=\; \nabla\cdot(\phi\zeta(\nabla\cdot\boldsymbol{w})\boldsymbol{I})+\nabla\cdot\big(\phi\nu_1\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{u})\big)-\nabla\cdot\big(\phi\nu_2\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})\big)\\ &-(a_1^{\star}(\phi)\boldsymbol{u}-a_2^{\star}(\phi)\boldsymbol{v})-b^{\star}(\phi)(|\boldsymbol{u}|\boldsymbol{u}-|\boldsymbol{v}|\boldsymbol{v})-\phi\nabla\pi\,,\quad \text{in}\quad \Omega\times\{t>0\}\,,\end{aligned} \tag{90}$$

$$\boldsymbol{w} \;=\; \boldsymbol{0}\,,\quad \text{on}\quad \partial\Omega\times\{t>0\}\,, \tag{91}$$

$$\boldsymbol{w}(\boldsymbol{x},0) \;=\; \boldsymbol{0}\,,\quad \boldsymbol{x}\in\Omega\,. \tag{92}$$

We multiply the momentum equation (90) by $\boldsymbol{w}$ and integrate over Ω to get

$$\begin{aligned}\frac{d}{dt}\int_\Omega\phi|\boldsymbol{w}|^2\,d\boldsymbol{x} \;&=\; -2\int_\Omega\phi\zeta(\nabla\cdot\boldsymbol{w})^2\,d\boldsymbol{x}-2\int_\Omega\phi\nu_1\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{w}):\nabla\boldsymbol{w}\,d\boldsymbol{x}-\int_\Omega\phi\hat{\nu}\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v}):\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{w})\,d\boldsymbol{x}\\ -2\int_\Omega a_1^{*}(\phi)|\boldsymbol{w}|^2\,d\boldsymbol{x}&-2\int_\Omega\hat{a}^{*}(\phi)\boldsymbol{v}\cdot\boldsymbol{w}\,d\boldsymbol{x}-2\int_\Omega b^{*}(\phi)(|\boldsymbol{u}|\boldsymbol{u}-|\boldsymbol{v}|\boldsymbol{v})\cdot\boldsymbol{w}\,d\boldsymbol{x}\,.\end{aligned} \tag{93}$$

Using relations (10), (15) and (36) we arrive at

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{w}\|^2 \;\le\; &-2a\nu_1\left(\frac{1}{\phi_{\max}}-1\right)\|\boldsymbol{w}\|^2+2a\hat{\nu}\left(\frac{1}{\phi_{\min}}-1\right)\int_\Omega\phi|\boldsymbol{v}||\boldsymbol{w}|\,d\boldsymbol{x}\\ &-2\nu_1\frac{\phi_{\min}}{\phi_{\max}}\|\nabla\boldsymbol{w}\|^2+2\hat{\nu}\int_\Omega\phi|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v}):\nabla\boldsymbol{w}|\,d\boldsymbol{x}\,.\end{aligned} \tag{94}$$

For the positive terms inside the integrals of the above relation, we apply the arithmetic-geometric mean inequality. More specifically, we have that

$$|\boldsymbol{v}|\,|\boldsymbol{w}| \;\le\; \frac{|\boldsymbol{v}|^2}{2\epsilon}+\frac{\epsilon|\boldsymbol{w}|^2}{2}\,, \tag{95}$$

with the following choice for ϵ

$$\epsilon \;=\; \frac{\nu_1}{\hat{\nu}}\left(\frac{1}{\phi_{\max}}-1\right)\left(\frac{1}{\phi_{\min}}-1\right)^{-1}\,. \tag{96}$$

Also, by applying successively the Cauchy-Schwarz and arithmetic-geometric mean inequalities, we get

$$|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v}) : \boldsymbol{\nabla w}| \;\leq\; |\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})|\,|\boldsymbol{\nabla w}| \;\leq\; \frac{|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})|^2}{2\epsilon'}+\frac{\epsilon'|\boldsymbol{\nabla w}|^2}{2}\,, \tag{97}$$

with the following choice for ϵ

$$\epsilon' \;=\; \frac{2\nu_1\phi_{\min}}{\hat{\nu}\phi_{\max}}\,. \tag{98}$$

In this manner (94) yields,

$$\frac{d}{dt}\|\boldsymbol{w}\|^2 \;\leq\; -\,a\nu_1\left(\frac{1}{\phi_{\max}}-1\right)\|\boldsymbol{w}\|^2+\frac{a\hat{\nu}^2}{\nu_1}\left(\frac{1}{\phi_{\min}}-1\right)^2\left(\frac{1}{\phi_{\max}}-1\right)^{-1}\|\boldsymbol{v}\|^2+$$
$$\frac{\hat{\nu}^2\phi_{\max}}{2\nu_1\phi_{\min}}\|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})\|^2\,. \tag{99}$$

We now multiply (84) by $\boldsymbol{v}$ and integrate over Ω to get

$$\frac{d}{dt}\|\boldsymbol{v}\|^2+2a\nu_2\left(\frac{1}{\phi_{\max}}-1\right)\|\boldsymbol{v}\|^2+\nu_2\|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})\|^2 \;\leq\; 0\,. \tag{100}$$

This inequality is then integrated to yield the relations

$$\|\boldsymbol{v}\|^2 \;\leq\; \|\boldsymbol{f}\|^2\,\mathrm{e}^{-2a\nu_2\left(\frac{1}{\phi_{\max}}-1\right)t}\,, \tag{101}$$

and

$$\int_0^t \|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})\|^2\,\mathrm{e}^{a\nu_2\left(\frac{1}{\phi_{\max}}-1\right)(\eta-t)}d\eta \;\leq\; \frac{1}{\nu_2}\|\boldsymbol{f}\|^2\;\mathrm{e}^{-a\nu_2\left(\frac{1}{\phi_{\max}}-1\right)t}\,. \tag{102}$$

By substituting (101) in (99) we get

$$\frac{d}{dt}\left(\|\boldsymbol{w}\|^2\,\mathrm{e}^{a\nu_1\left(\frac{1}{\phi_{\max}}-1\right)t}\right) \;\leq\; \frac{a\hat{\nu}^2}{\nu_1}\left(\frac{1}{\phi_{\min}}-1\right)^2\left(\frac{1}{\phi_{\max}}-1\right)^{-1}\|\boldsymbol{f}\|^2\,\mathrm{e}^{a\left(\frac{1}{\phi_{\max}}-1\right)(\nu_1-2\nu_2)t}+$$
$$\frac{\hat{\nu}^2\phi_{\max}}{2\nu_1\phi_{\min}}\|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})\|^2\;\mathrm{e}^{a\nu_1\left(\frac{1}{\phi_{\max}}-1\right)t}\,. \tag{103}$$

Integration of the above inequality allows us to obtain

$$\|\boldsymbol{w}\|^2 \;\leq\; \left(\frac{1}{\phi_{\min}}-1\right)^2\frac{\hat{\nu}^2}{\nu_1\left(\frac{1}{\phi_{\max}}-1\right)^2(\nu_1-2\nu_2)}\|\boldsymbol{f}\|^2\,(\mathrm{e}^{-2a\nu_2\left(\frac{1}{\phi_{\max}}-1\right)t}-\mathrm{e}^{-a\nu_1\left(\frac{1}{\phi_{\max}}-1\right)t})+$$
$$\frac{\hat{\nu}^2\phi_{\max}}{2\nu_1\phi_{\min}}\int_0^t\|\boldsymbol{V}^{\mathrm{d}}(\boldsymbol{v})\|^2\;\mathrm{e}^{a\nu_1\left(\frac{1}{\phi_{\max}}-1\right)(\eta-t)}d\eta\,. \tag{104}$$

Then, by introducing (102) in the above relation we obtain the estimate

$$\|\boldsymbol{w}\|^2 \;\leq\; \left(\frac{\frac{1}{\phi_{\min}}-1}{\frac{1}{\phi_{\max}}-1}\right)^2 \frac{1}{\nu_1(\nu_1-2\nu_2)}\|\boldsymbol{f}\|^2 \,\left(\mathrm{e}^{-2a\nu_2\left(\frac{1}{\phi_{\max}}-1\right)t}-\mathrm{e}^{-a\nu_1\left(\frac{1}{\phi_{\max}}-1\right)t}\right)\,\hat{\nu}^2+$$

$$\frac{\phi_{\max}}{\phi_{\min}}\frac{1}{2\nu_1\nu_2}\|\boldsymbol{f}\|^2\,\mathrm{e}^{-a\nu_2\left(\frac{1}{\phi_{\max}}-1\right)t}\,\hat{\nu}^2\,. \tag{105}$$

Inequality (105) establishes continuous dependence of the solution on the shear viscosity ν. Once again, we can observe that the solution difference decays to zero with time. Moreover, according to this estimate, only $\phi_{\max}$ enters the exponential term in the estimate. Whereas $\phi_{\min}$ enters only the pre-exponential factor. In other words, the decay rate of the velocity difference $\boldsymbol{w}$ is mainly controlled by $\phi_{\max}$ and to a lesser extent by $\phi_{\min}$.

Once again, we clarify that the factor 2 that multiples ν_2 in the term $(\nu_1-2\nu_2)$ on the right-hand side of (105) emerges because of the specific form of the arithmetic-geometric mean inequality that we used to derive (99). If we had used a different choice of ϵ for this inequality, then ν_2 would have been multiplied by a different factor. In this manner, one avoids any potential singularity of the denominator in the above estimate.

6 Continuous dependence on the bulk viscosity

We consider the Brinkman–Forchheimer equations for flow in porous media with variable porosity (1)-(2). Since we are now interested in continuous dependence on the bulk viscosity, we consider the solutions $(\boldsymbol{u},p)$ and $(\boldsymbol{v},q)$ of the initial-boundary-value problems for different shear viscosities ζ_1 and ζ_2 respectively. Next we introduce the difference variables $\boldsymbol{w}$ and π, and the difference bulk viscosity ζ,

$$\boldsymbol{w} \;=\; \boldsymbol{u}-\boldsymbol{v}\,,\qquad \pi \;=\; p-q\,,\qquad \hat{\zeta} \;=\; \zeta_1-\zeta_2\,, \tag{106}$$

so that $\boldsymbol{w}$ and π satisfy the following initial-boundary-value problem

$$\boldsymbol{\nabla}\cdot(\phi\boldsymbol{w}) \;=\; 0\,,\quad \text{in}\quad \Omega\times\{t>0\}\,, \tag{107}$$

$$\phi\frac{\partial\boldsymbol{w}}{\partial t} = \boldsymbol{\nabla}\cdot(\phi\zeta_1(\boldsymbol{\nabla}\cdot\boldsymbol{u})I)-\boldsymbol{\nabla}\cdot(\phi\zeta_2(\boldsymbol{\nabla}\cdot\boldsymbol{v})I)+\boldsymbol{\nabla}\cdot\left(\phi\nu\boldsymbol{V}^{\mathrm{d}}\right),$$
$$-\,a^\star(\phi)\mathbf{w}-b^\star(\phi)(|\boldsymbol{u}|\boldsymbol{u}-|\boldsymbol{v}|\boldsymbol{v})-\phi\boldsymbol{\nabla}\pi\,,\quad \text{in}\quad \Omega\times\{t>0\}\,, \tag{108}$$

$$\boldsymbol{w} \;=\; \boldsymbol{0}\,,\quad \text{on}\quad \partial\Omega\times\{t>0\}\,, \tag{109}$$

$$\boldsymbol{w}(\boldsymbol{x},0) \;=\; \boldsymbol{0}\,,\quad \boldsymbol{x}\in\Omega\,. \tag{110}$$

Continuous dependence on ζ is established by repeating the same steps as in the previous section. The final estimate reads

$$\|\boldsymbol{w}\|^2 \leq \frac{1}{4\zeta_1\zeta_2}\|\boldsymbol{f}\|^2\,\mathrm{e}^{-2a\nu(\frac{1}{\phi_{\max}}-1)t}\,\hat{\zeta}^2\,. \tag{111}$$

Once again, the solution difference decays to zero as time passes, which is indicative of a stronger form of structural stability. It is also interesting to observe that, according to this estimate, the decay rate of the solution difference is controlled by $\phi_{\max}$. Moreover, it is independent of $\phi_{\min}$, in contrast to the estimates for the structural stability on the other physical parameters.

7 Bounds for the kinetic energy

In this section we derive lower and upper bounds for the kinetic energy of the fluid. In order to fix ideas, let $\boldsymbol{u}$ be a solution to the initial-boundary-value problem associated with the Brinkman-Forchheimer equations for flow in porous media with variable porosity (1)-(2). First we derive a lower bound for $\|\boldsymbol{u}\|$ based on the procedure presented in [1] for the case of a medium with constant porosity. To this end, we introduce $K(t)$ as twice the total amount of kinetic energy,

$$K(t) \;=\; \|\boldsymbol{u}\|^2 \,. \tag{112}$$

Then by computation and by using (10) and (20), in which we have replaced b_1^* by b^*, we get

$$\begin{aligned}\frac{dK}{dt} &= 2\int_\Omega \phi\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t}\,d\boldsymbol{x} \;=\\ &\quad -2\int_\Omega \phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\,d\boldsymbol{x} - \int_\Omega \phi\nu\boldsymbol{V}^{\mathrm{d}} : \boldsymbol{V}^{\mathrm{d}}\,d\boldsymbol{x} - 2\int_\Omega a^*(\phi)|\boldsymbol{u}|^2\,d\boldsymbol{x} - 2\int_\Omega b^*(\phi)|\boldsymbol{u}|^3\,d\boldsymbol{x}\\ &\le \xi(t)\,.\end{aligned} \tag{113}$$

The quantity $\xi(t)$ is defined as

$$\xi(t) \;=\; -\int_\Omega \phi\nu\boldsymbol{V}^{\mathrm{d}} : \boldsymbol{V}^{\mathrm{d}}\,d\boldsymbol{x} - 2\int_\Omega \phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\,d\boldsymbol{x} - 2\int_\Omega a^*(\phi)|\boldsymbol{u}|^2\,d\boldsymbol{x} - \frac{4}{3}\int_\Omega b^*(\phi)|\boldsymbol{u}|^3\,d\boldsymbol{x}\,. \tag{114}$$

To further analyze the behavior of the kinetic energy, we compute the time derivative of $\xi(t)$. Using (10), we have

$$\begin{aligned}\frac{d\xi}{dt} \;&=\; -\int_\Omega \phi\nu\boldsymbol{V}^{\mathrm{d}} : \frac{\partial\boldsymbol{V}^{\mathbf{d}}}{\partial t}d\boldsymbol{x} - 2\int_\Omega \phi\nu\boldsymbol{\nabla}\left(\frac{\partial\boldsymbol{u}}{\partial t}\right) : \boldsymbol{V}^{\mathrm{d}}d\boldsymbol{x} - 4\int_\Omega \phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{u})(\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{u}}{\partial t})d\boldsymbol{x}\\ &- 4\int_\Omega a^*(\phi)\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t}d\boldsymbol{x} - 4\int_\Omega b^*(\phi)|\boldsymbol{u}|\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t}d\boldsymbol{x}\,.\end{aligned} \tag{115}$$

By applying integration by parts and by inserting (20) (again with b_1^* replaced by b^*), we obtain

$$\begin{aligned}\frac{d\xi}{dt} \;&=\; -\int_\Omega \phi\nu\boldsymbol{V}^{\mathrm{d}} : \frac{\partial\boldsymbol{V}^{\mathbf{d}}}{\partial t}d\boldsymbol{x} + 2\int_\Omega \phi|\frac{\partial\boldsymbol{u}}{\partial t}|^2 d\boldsymbol{x} + 2\int_\Omega \boldsymbol{\nabla}(\phi\zeta(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\frac{\partial\boldsymbol{u}}{\partial t}d\boldsymbol{x}\\ &- 2\int_\Omega a^*(\phi)\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t}d\boldsymbol{x} - 2\int_\Omega b^*(\phi)|\boldsymbol{u}|\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t}d\boldsymbol{x}\,.\end{aligned} \tag{116}$$

From the last equation, by integrating by parts the term involving ζ and by combining the result with (115), we get

$$\frac{d\xi}{dt} \;=\; = \frac{1}{2}\frac{d\xi}{dt} + 2\int_\Omega \phi|\frac{\partial\boldsymbol{u}}{\partial t}|^2 d\boldsymbol{x}\,, \tag{117}$$

or, in other words,

$$\frac{d\xi}{dt} \;=\; 4\left\|\frac{\partial\boldsymbol{u}}{\partial t}\right\|^2\,. \tag{118}$$

Also, from (113) we have that

$$\left(-\frac{dK}{dt}\right)(-\xi) \;\leq\; \left(\frac{dK}{dt}\right)^2 \;=\; 4\left(\int_\Omega \phi \boldsymbol{u}\cdot\frac{\partial \boldsymbol{u}}{\partial t}\,d\boldsymbol{x}\right)^2 . \tag{119}$$

By applying successively the Cauchy-Schwarz inequality and (118) we get

$$4\left(\int_\Omega \phi \boldsymbol{u}\cdot\frac{\partial \boldsymbol{u}}{\partial t}\,d\boldsymbol{x}\right)^2 \;\leq\; 4\|\boldsymbol{u}\|^2\left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2 \;=\; K\frac{d\xi}{dt} . \tag{120}$$

Introducing the last relation in (119) gives

$$\frac{dK}{dt}\xi \;\leq\; K\frac{d\xi}{dt} . \tag{121}$$

By integrating the above relation and exponentiating the result, we obtain

$$-\xi(t) \;\leq\; -\frac{K(t)}{\|\boldsymbol{f}\|^2}\xi(0) . \tag{122}$$

Now, since ξ is negative the following relation holds

$$\frac{3}{2}\xi \;\leq\; \frac{dK}{dt} \;\leq\; \xi . \tag{123}$$

By combining this relation with (122) we readily deduce that

$$K(t)\frac{\xi(0)}{\|\boldsymbol{f}\|^2} \;\leq\; \xi(t) \;\leq\; \frac{2}{3}\frac{dK}{dt} . \tag{124}$$

By considering the inequality between the first and last parts of (124), we arrive, upon integration at,

$$K(t) \;\geq\; \|\boldsymbol{f}\|^2\,\mathrm{e}^{\frac{3\xi(0)}{2\|\boldsymbol{f}\|^2}t} . \tag{125}$$

This estimate serves as a lower bound for the kinetic energy.

Next we consider the initial-boundary value problem (19)-(22) associated with $\boldsymbol{u}$ in which b_1^* is replaced by b^*. We multiply (20) with $\boldsymbol{u}$ and make use of (45) and the porosity bound (5) to derive

$$\frac{d}{dt}\|\boldsymbol{u}\|^2 + \left(2a\nu\left(\frac{1}{\phi_{\max}}-1\right) + \frac{2\lambda_1\nu\phi_{\min}^2}{\phi_{\max}^2}\right)\|\boldsymbol{u}\|^2 + 2b\left(\frac{1}{\phi_{\max}}-1\right)\int_\Omega \phi|\boldsymbol{u}|^3\,d\boldsymbol{x} \;\leq\; 0 . \tag{126}$$

This relation is then integrated to obtain

$$\|\boldsymbol{u}\|^2 \;\leq\; \|\boldsymbol{f}\|^2\,\mathrm{e}^{-\left(2a\nu\left(\frac{1}{\phi_{\max}}-1\right)+\frac{2\lambda_1\nu\phi_{\min}^2}{\phi_{\max}^2}\right)t} . \tag{127}$$

This estimate serves as a first upper bound for $K(t)$ hence for the kinetic energy.

This estimate can be improved by noting that, by virtue of (125), $\|\boldsymbol{u}\|$ is strictly positive. To this end, we employ the interpolation inequality for standard (non-weighted) L^q norms ([34],

Appendix B) for the function $\phi^{1/3}\boldsymbol{u}$ to get

$$\left(\int_\Omega \phi^{2/3}|\boldsymbol{u}|^2 d\boldsymbol{x}\right)^2 \;\leq\; \int_\Omega \phi^{1/3}|\boldsymbol{u}| d\boldsymbol{x} \int_\Omega \phi|\boldsymbol{u}|^3\, d\boldsymbol{x}\,. \tag{128}$$

We also employ the integral form of the Cauchy-Schwarz inequality

$$\int_\Omega \phi^{1/3}|\boldsymbol{u}| d\boldsymbol{x} \;\leq\; \left(\int_\Omega 1 d\boldsymbol{x}\right)^{1/2} \left(\int_\Omega \phi^{2/3}|\boldsymbol{u}|^2 d\boldsymbol{x}\right)^{1/2}\,, \tag{129}$$

By combining (128) and (129) we deduce that

$$\int_\Omega \phi|\boldsymbol{u}|^3 d\boldsymbol{x} \;\geq\; \frac{||\boldsymbol{u}||^3}{(m\phi_{\max})^{1/2}}\,. \tag{130}$$

where $m = m(\Omega)$ is the measure (volume) of the domain Ω. Upon substitution of the last inequality in (126), we arrive at

$$\frac{d}{dt}\|\boldsymbol{u}\|^2 + \left(2a\nu\left(\frac{1}{\phi_{\max}} - 1\right) + \frac{2\lambda_1\nu\phi_{\min}^2}{\phi_{\max}^2}\right)\|\boldsymbol{u}\|^2 + \frac{2b}{(m\phi_{\max})^{1/2}}\left(\frac{1}{\phi_{\max}} - 1\right)\|\boldsymbol{u}\|^3 \;\leq\; 0\,. \tag{131}$$

Next we divide the above relation with $-2\|\boldsymbol{u}\|^3$ we get

$$\frac{d}{dt}\frac{1}{\|\boldsymbol{u}\|} - \left(a\nu\left(\frac{1}{\phi_{\max}} - 1\right) + \frac{\lambda_1\nu\phi_{\min}^2}{\phi_{\max}^2}\right)\frac{1}{\|\boldsymbol{u}\|} - \frac{b}{(m\phi_{\max})^{1/2}}\left(\frac{1}{\phi_{\max}} - 1\right) \;\geq\; 0\,, \tag{132}$$

The integration of this inequality reads

$$\frac{1}{\|\boldsymbol{u}\|} + \frac{b(\phi_{\max})^{1/2}(1-\phi_{\max})}{m^{1/2}\nu(a\phi_{\max}(1-\phi_{\max}) + \lambda_1\phi_{\min}^2)}\left(1 - \mathrm{e}^{\left(a\nu\left(\frac{1}{\phi_{\max}}-1\right)+\frac{\lambda_1\nu\phi_{\min}^2}{\phi_{\max}^2}\right)t}\right) \;\geq$$
$$\frac{\mathrm{e}^{\left(a\nu\left(\frac{1}{\phi_{\max}}-1\right)+\frac{\lambda_1\nu\phi_{\min}^2}{\phi_{\max}^2}\right)t}}{\|\boldsymbol{f}\|}\,. \tag{133}$$

We can solve inequality (133) to derive an upper bound that is sharper than (127). In summary, the following upper and lower bounds are established,

$$\|\boldsymbol{f}\|\, e^{\frac{3\xi(0)}{4\|\boldsymbol{f}\|^2}t} \;\leq\; \|\boldsymbol{u}\| \;\leq$$
$$\frac{\|\boldsymbol{f}\|(a\phi_{\max}(1-\phi_{\max}) + \lambda_1\phi_{\min}^2)}{a\phi_{\max}(1-\phi_{\max}) + \lambda_1\phi_{\min}^2 + \|\boldsymbol{f}\|b\nu^{-1}\,(m\phi_{\max})^{1/2}\,(1-\phi_{\max})(1-A(t))}A(t)\,, \tag{134}$$

where the quantity $A(t)$ is given by,

$$A(t) \;=\; \mathrm{e}^{-\left(a\nu\left(\frac{1}{\phi_{\max}}-1\right)+\frac{\lambda_1\nu\phi_{\min}^2}{\phi_{\max}^2}\right)t}\,. \tag{135}$$

We observe that the kinetic energy decays exponentially with time, just as in the Navier-Stokes equations. In fact, in the limiting case of a pure-fluid domain, $\phi_{\min} = \phi_{\max} = 1$, the above bounds reduce to those for the kinetic energy of a viscous incompressible fluid described by the Navier-Stokes equations [35, 36]. Further, since $\phi_{\min}$ and $\phi_{\max}$ are less than unity, the decay rate of $\|\boldsymbol{u}\|$ is faster than in the pure-fluid case due to interphasial drag, as expected. It is also interesting to observe that, according to (134), the lower bound of the decay rate of $\|\boldsymbol{u}\|$ does not depend explicitly on either $\phi_{\min}$ or $\phi_{\max}$; nonetheless, since $\|\boldsymbol{f}\|$ and $\xi(0)$ depend implicitly on them, so does the lower bound.

. On the other hand, both $\phi_{\min}$ and $\phi_{\max}$ enter explicitly the expression of the time exponential of the upper bound. In other words, the upper bound of the energy-decay rate is controlled by both porosity extrema.

8 Conclusions

In this paper we considered the Brinkman-Forchheimer model for fluid flow in porous media of variable porosity and established continuous dependence of solutions on the parameters of the problem, namely, the Darcy and Forchheimer coefficients of interphasial drag and the shear and bulk fluid viscosities of the fluid. According to our analysis, the solution difference decays always with time. Also, in the estimates for the Forchheimer coefficients and shear viscosity, the decay rate of the velocity difference is principally controlled by the maximum porosity. More specifically the smaller the maximum porosity is, the fastest the solution difference decays to zero. Further, in the estimate for the bulk viscosity, the decay rate is independent of the minimum porosity. Finally, we derived lower and upper bounds for the kinetic energy of the fluid. Our analysis shows that the kinetic energy decays exponentially but faster than in pure-fluid domains because of the interphasial drag.

Author contributions

Evangelos Petridis: Conceptualization; investigation; methodology; writing-original draft. **Miltiadis V. Papalexandris**: Conceptualization; investigation; methodology; writing-original draft.

Acknowledgments

The authors gratefully acknowledge the financial support provided by European Union's (EU) Horizon Europe Framework Programme (HORIZON) via SEDIMARE (Grant Agreement No. 101072443), an MSCA Doctoral Network (HORIZON-MSCA-2021-DN-01).

Conflict of interest

The authors declare no potential conflict of interest.

References

[1] L. E. Payne and B. Straughan. Convergence and continuous dependence for the Brinkman-Forchheimer equations. *Stud. Appl. Math.*, 102:419–439, 1999.

[2] K. A. Ames and B. Straughan. *Non-standard and improperly posed problems.* Academic Press, 1997.

[3] K. A. Ames and L. E. Payne. On stabilizing against modeling errors in a penetrative convection problem for a porous medium. *Math. Models Methods Appl. Sci.*, 4:733–740, 1994.

[4] Y. Qin and P. N. Kaloni. Convective instabilities in anisotropic porous media. *Stud. Appl. Math.*, 91:189–204, 1994.

[5] L. E. Payne and B. Straughan. Stability in the initial-time geometry problem for the Brinkman and Darcy equations of flow in a porous media. *J. Math. Pures Appl.*, 75: 225–271, 1996.

[6] P. N Kaloni and J. Guo. Doubly diffusive convection in a porous medium, nonlinear stability, and the Brinkman effect. *Stud. Appl. Math.*, 94:341–358, 1995.

[7] J. Chadam and Y. Qin. Nonlinear convective stability in a porous medium with temperature dependent viscosity and inertial drag. *Stud. Appl. Math.*, 96:273–288, 1996.

[8] J. Guo and P. N. Kaloni. Steady nonlinear double-diffusive convection in a porous medium based upon the Brinkman-Forchheimer model. *J. Math. Anal. Appl.*, 204:138–155, 1996.

[9] P. N. Kaloni and Y. Qin. Spatial decay estimates for plane flow in the Brinkman-Forchheimer model. *Quart. Appl. Math.*, 56:71–87, 1998.

[10] L. E. Payne and B. Straughan. Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modeling questions. *J. Math. Pures Appl.*, 77:317–354, 1998.

[11] L. E. Payne and B. Straughan. Structural stability for the darcy equations of flow in porous media. *Proc. R. Soc. Lond. A*, 454:1691–1698, 1998.

[12] L. E. Payne, J. C. Song, and B. Straughan. Continuous dependence and convergence results for Brinkman and Forchheimer models with variable viscosity. *Proc. R. Soc. Lond. A*, 455: 2173–2190, 1999.

[13] B. Straughan and K. Hutter. A priori bounds and structural stability for double-diffusive convection incorporating the Soret effect. *Proc. R. Soc. Lond. A*, 455:766–777, 1999.

[14] B. Straughan. *Stability and Wave Motion in Porous Media.* Springer, 2008.

[15] R. J. Knops and L. E. Payne. Continuous data dependence for the equations of classical elastodynamics. *Math. Proc. Cambridge Phil. Soc.*, 66:481–491, 1969.

[16] R. J. Knops and L. E. Payne. Improved estimates for continuous data dependence in linear elastodynamics. *Math. Proc. Cambridge Phil. Soc.*, 103:535–559, 1988.

[17] S. Chiriţă, M. Ciarletta, and B. Straughan. Structural stability in porous elasticity. *Proc. R. Soc. A*, 462:2593–2605, 2006.

[18] S. Chiriţă and M. Ciarletta. On the structural stability of thermoelastic model of porous media. *Math. Methods Appl. Sci.*, 31:19–34, 2008.

[19] Y. Liu. Continuous dependence for a thermal convection model with temperature-dependent solubility. *Appl. Math. Comput.*, 308:18–30, 2017.

[20] J. de Castro Motta, V. Zampoli, S. Chiriţă, and M. Ciarletta. On the structural stability for a model of mixture of porous solids. *Math. Methods Appl. Sci.*, pages 1–17, 2023.

[21] J. L. Díaz Palencia. A mathematical analysis of an extended MHD Darcy–Forchheimer type fluid. *Sci. Rep.*, 12:5228, 2022.

[22] S. u. Rahman and J. L. Díaz Palencia. Regularity and analysis of solutions for a MHD flow with a p-Laplacian operator and a generalized Darcy–Forchheimer term. *Eur. Phys. J. Plus*, 137:1328, 2022.

[23] S. u. Rahman and J. L. Díaz Palencia. Regularity and profiles of solutions to a higher order eyrin-powell fluid with darcy-forchheimer porosity term. *Math. Methods Appl. Sci.*, 48:7166–7187, 2022.

[24] S. u. Rahman, J. L. Díaz Palencia, and J. Roa González. Analysis and profiles of travelling wave solutions to a Darcy-Forchheimer fluid formulated with a non-linear diffusion. *AIMS Math.*, 7:6898–6914, 2022.

[25] D. A. Nield and A. Bejan. *Convection in Porous Media.* Springer, 2017.

[26] T. Giorgi. Derivation of the Forchheimer law via matched asymptotic expansions. *Transp. Porous Media*, 29:191–206, 1997.

[27] S. Whitaker. The Forchheimer equation: A theoretical development. *Transp. Porous Media*, 25:27–62, 1996.

[28] A. O. Çelebi, V. K. Kalantarov, and D. Uğurlu. Continuous dependence for the convective Brinkman-Forchheimer equations. *Appl. Anal.*, 31:117–130, 2008.

[29] C. D. Levermore, M. Oliver, and E. S. Titi. Global well-posedness for the lake equations. *Phys. D*, 98:492–509, 1996.

[30] D. Bresch and G. Métivier. Global existence and uniqueness for the lake equations with vanishing topography: elliptic estimates for degenerate equations. *Nonlinearity*, 19:591, 2006.

[31] Q. Jiu and D. Niu. Vanishing viscous limits for the 2D lake equations with Navier boundary conditions. *J. Math. Anal. Appl.*, 338:1070–1080, 2008.

[32] C. D. Levermore and M. Sammartino. A shallow water model with eddy viscosity for basins with varying bottom topography. *Nonlinearity*, 14:1403–1515, 2001.

[33] C. Varsakelis and M. V. Papalexandris. On the well-posedness of the Darcy-Brinkman-Forchheimer equations for coupled porous media-clear fluid flow. *Nonlinearity*, 30:1449–1464, 2017.

[34] L. C. Evans. *Partial Differential Equations.* American Mathematical Society, 1998.

[35] O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous, Incompressible Flow.* Gordon and Breach, 1969.

[36] J. C. Robinson, J. L. Rodrigo, and W. Sadowski. *The Three-Dimensional Navier-Stokes equations.* Cambridge University Press, 2016.