Subtleties in Reducing 3D Rotating Dynamics to a 2D Model

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Keywords: shallow flows, rotating flows, Ekman layer.

Abstract

In sufficiently shallow rotating flows where the Ekman number is on the order of unity, top and bottom Ekman layers merge without leaving a frictionless interior, and classical concepts such as Ekman pumping lose their significance. Different relations hold, and, in particular, it is found that in the case of a shallow layer of water subjected to both a surface wind stress and bottom friction, the vorticity of the depth-averaged horizontal flow is a function not only of the curl of the wind stress but also of its divergence.

The study further shows that the reduction of the 3D dynamics to a 2D model must be approached with great care. For example and contrary to expectations, a simple-minded representation of the bottom stress as a function of the depth-averaged velocity (a very common formulation in 2D modelling!) is incapable of preserving some required properties.

1. Introduction

Our questions concern (1) the relation between vertically averaged flow and forcing, and (2) the proper reduction of 3D rotating dynamics to a 2D model. The answers are not trivial. When Ekman layers are thin, the vertically averaged flow responds only to the wind stress curl, but there is no reason to believe that this continues to hold when the flow becomes shallow and the top and bottom Ekman layers merge. Pronounced veering throughout the water column also complicates the reduction of the three-dimensional dynamics to a two-dimensional model, and the question arises as to the form of terms and values of coefficients that ought to be used in the reduced model to remain faithful to the full dynamics.

2. Model and Key Result

We consider a shallow flow of water and solve the classical Ekman dynamics problem in the context of the following assumptions: f-plane, steady flow, low-Rossby number, hydrostatic balance ($\partial p/\partial z = -\rho g$), constant density ρ (no stratification), flat bottom and rigid lid ($-H \leq z \leq 0$, $H =$ constant), uniform eddy viscosity ν, and imposed surface wind stress (τ_x, τ_y functions of x and y). The neglect of the temporal derivatives and nonlinear advection terms is not justified in many circumstances, but we nonetheless ignore those terms here to place the focus on the combined role of Coriolis and friction forces in a mathematically tractable fashion, with the underlying assumption that the main findings of our study will continue to hold at least qualitatively in the broader dynamical setting that includes temporal derivatives and nonlinear advection terms.

We begin the analysis by expressing the horizontal pressure gradient force in terms of geostrophic flow components:

$$-fv_g = -\frac{1}{\rho}\,\frac{\partial p}{\partial x}\,, \qquad +fu_g = -\frac{1}{\rho}\,\frac{\partial p}{\partial y}\,, \tag{1}$$

from which follows that on an f-plane, the geostrophic flow is without divergence. It is also independent of the vertical coordinate z by virtue of the hydrostatic balance with constant density.

The equations governing the three velocity components u, v and w express momentum balance and mass conservation:

$$-fv = -fv_g + \nu \frac{\partial^2 u}{\partial z^2}, \qquad fu = fu_g + \nu \frac{\partial^2 v}{\partial z^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2}$$

At the bottom, we impose no flow on the frictional and impermeable bottom:

$$u = v = w = 0 \quad \text{at} \quad z = -H, \tag{3}$$

while at the top, we impose a surface wind stress and no vertical flow through a rigid lid:

$$\rho \nu \frac{\partial u}{\partial z} = \tau_x, \quad \rho \nu \frac{\partial v}{\partial z} = \tau_y, \quad w = 0 \quad \text{at} \quad z = 0. \tag{4}$$

The rigid-lid approximation is made for the same reason as stated above, namely to keep the focus on the combined role of Coriolis and friction forces with a minimum of other dynamics.

Equation set (2) contains a total of five z-derivatives while we have specified six boundary conditions in the z-direction. Thus, the system is over-specified by one, and we ought to expect a constraint on some of the variables that enter the equations, particularly a relationship between the geostrophic velocity (u_g, v_g) and the wind stress (τ_x, τ_y).

The situation can be anticipated by considering the case of a deep water column compared to the Ekman layer thickness, defined as

$$d = \sqrt{\frac{2\nu}{f}}. \tag{5}$$

If $d \ll H$, friction is significant only in top and bottom boundary layers that occupy much less than the water depth, leaving a frictionless and geostrophic interior. In this case, surface Ekman dynamics generate a so-called Ekman pumping vertical velocity through the interior (Cushman-Roisin and Beckers, 2011, page 254) proportional to the wind stress curl, while bottom Ekman dynamics generate an interior vertical velocity proportional to the vorticity of the geostrophic flow (*op. cit.*, page 249):

$$w_{interior} = \frac{1}{\rho f}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right), \qquad w_{interior} = \frac{d}{2}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right). \tag{6}$$

Given that the vertical velocity in the interior must be vertically uniform in a flow that has no divergence, it follows that the two preceding expressions must be equal to each other:

$$\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{2}{\rho f d}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right). \tag{7}$$

In other words, in water deep compared to the Ekman layer thickness, the vorticity of the interior geostrophic flow must be proportional to the curl of the surface wind stress. The primary aim of this study is to determine what this relationship becomes when the water depth is comparable to the Ekman layer thickness ($d \sim H$, i.e. $\nu/fH^2 \sim 1$) and friction affects the entire water column.

Solving Equations (2) above, applying five of the six boundary conditions to determine all constants of integration, and then applying the sixth boundary condition as an additional constraint provides, after lengthy algebra, the following relation:

$$\begin{aligned}&\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right)\left(\cosh\frac{H}{d}\sinh\frac{H}{d} - \cos\frac{H}{d}\sin\frac{H}{d}\right)\\ &= \frac{2}{\rho f d}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)\left(\cosh^2\frac{H}{d} + \cos^2\frac{H}{d} - \cosh\frac{H}{d}\cos\frac{H}{d} - 1\right) + \frac{2}{\rho f d}\left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}\right)\sinh\frac{H}{d}\sin\frac{H}{d}.\end{aligned} \tag{8}$$

This last equation is the relation between the geostrophic flow (u_g, v_g), and the wind stress (τ_x, τ_y) that extends Equation (7) to the case of shallow water ($d \sim H$, i.e. $\nu/fH^2 \sim 1$). We note that the vorticity of the geostrophic flow is no longer simply proportional to the curl of the wind stress but is now a linear combination of both curl and divergence of the wind stress. In the limit $H \gg d$, Equation (8) reverts to (7) as it should.

A similar equation can be written for the vorticity of the depth-averaged flow (which is not the same as the geostrophic flow when Ekman dynamics are not relegated to thin boundary layers), but it is algebraically more cumbersome.

3. Discussion and Challenge

Numerical models of shallow seas, particularly tidal and storm surge models, have been two-dimensional models which purport to be depth-averaged models, but these rely on a parameterization of the bottom stress in terms of only the depth-averaged velocity.

In contrast, the previous calculations lead to formulas for the bottom stress components (τ_{bx}, τ_{by}) that depend not only on the geostrophic velocity (u_g, v_g), which is the barotropic pressure gradient in disguise [see Eqs. (1)], but also on the surface wind stress (τ_x, τ_y):

$$\tau_{bx} = \frac{1}{F_1^2+F_2^2}\left[\frac{\rho f d u_g}{2} F_3 - \frac{\rho f d v_g}{2} F_4 + \tau_x(F_1+F_2) + \tau_y(F_1-F_2)\right] \tag{9}$$

$$\tau_{by} = \frac{1}{F_1^2+F_2^2}\left[\frac{\rho f d u_g}{2} F_4 + \frac{\rho f d v_g}{2} F_3 - \tau_x(F_1-F_2) + \tau_y(F_1+F_2)\right] , \tag{10}$$

in which the coefficients F_1 to F_4 are combinations of trigonometric and hyperbolic sines and cosines taken at H/d, which are much too long to spell out here. With additional algebra, the previous formulas can be recast as (even more complicated) functions of the depth-averaged velocity ($\bar{u}$, $\bar{v}$) and surface wind stress (τ_x, τ_y).

In the limit of deep water ($H \gg d$), $u_g \approx \bar{u}$, $v_g \approx \bar{v}$, $F_3 \approx F_4 \approx F_1^2+F_2^2 \gg |F_1 \pm F_2|$, the dependency on the surface wind stress becomes negligible, and expressions (9)-(10) reduce to:

$$\tau_{bx} \approx \frac{\rho f d}{2}(\bar{u}-\bar{v}) \tag{11}$$

$$\tau_{by} \approx \frac{\rho f d}{2}(\bar{u}+\bar{v}), \tag{12}$$

which are classical results for a thin bottom Ekman layer. We note that the direct dependency on the surface wind stress disappears in the limit of deep water. In retropsective, it was to be expected that, in a shallow flow (one with the surface in close proximity to the bottom) the bottom stress would have some direct dependency on the surface wind stress.

This casts some serious doubts on the application of classical two-dimensional models such as those of Heaps (1969) for the North Sea, Dube *et al.* (2009) for the Bay of Bengal and Arabian Sea, and Davies *et al.* (2011) for Celtic-Irish Seas, as well as others, in regions where the water depth H is not much greater than the Ekman layer depth d.

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What happens to Ekman dynamics when the flow is shallow.

Subtleties in reducing 3D rotating dynamics to a 2D model

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4th International Symposium on Shallow Flows
Eindhoven, The Netherlands
27 June 2017

Outline

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- The easy part – When the flow is not shallow
- Complication – When the flow is shallow
- Surprise – A different relationship
- Subtlety – Caution with reduction from 3D to 2D
- Recap

1. Introduction

Ekman's paper of 1905 is well known.

Almost all analytical studies are using boundary layer theory
based on the smallness of the Ekman number

$$Ek = \frac{\upsilon}{fH^2} << 1$$

Few papers exist that do the algebra without the boundary layer simplification.
These present just more algebra without exploring the possibility of richer dynamics.
The richer dynamics remained buried in the weeds.

Here, we ask: When the flow is shallow ($Ek \geq 1$)

- Are there new dynamics?
- How do you reduce 3D to 2D faithfully?

Governing Equations

(steady, low Rossby number, f-plane, no horizontal friction to isolate Ekman dynamics)

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u}{\partial z^2} \quad\rightarrow\quad \boxed{-fv = -fv_g + \upsilon\frac{\partial^2 u}{\partial z^2}}$$

$$+fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \upsilon\frac{\partial^2 v}{\partial z^2} \quad\rightarrow\quad \boxed{+fu = +fu_g + \upsilon\frac{\partial^2 v}{\partial z^2}}$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

with

$$v_g = +\frac{1}{\rho f}\frac{\partial p}{\partial x}$$

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y}$$

$$\downarrow$$

$$\boxed{\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0}$$

Boundary conditions

At top $z = 0:$ $\quad \rho\upsilon\frac{\partial u}{\partial z} = \tau_x\,, \quad \rho\upsilon\frac{\partial v}{\partial z} = \tau_y\,, \quad w = 0$

Wind stress, rigid lid

At bottom $z = -H:$ $\quad u = 0\,, \quad v = 0\,, \quad w = 0$

No slip, impermeable

Count

z-derivatives: 2+2+1=5

boundary conditions in z: 3+3=6

→ System is overdetermined by 1.

→ There exists a constraint.

For convenience, define the Ekman layer thickness as

$$d = \sqrt{\frac{2\upsilon}{f}} \quad \Leftrightarrow \quad H$$

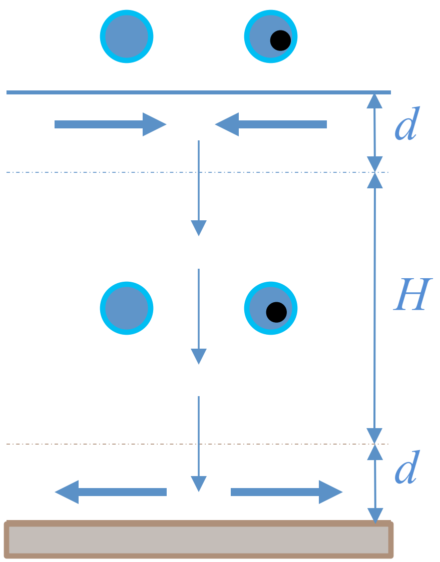
Cases to be explored: $H >> d$ Deep, $Ek << 1$, boundary layers + interior

$H \approx d$ Shallow, $Ek = O(1)$, friction throughout

$H << d$ Very shallow, $Ek >> 1$, strong friction throughout

2. When the flow is not shallow

Large depth $H >> d$ → Small Ekman number $Ek << 1$
→ Thin boundary layers separated by an interior



Ekman pumping velocity coming out of top boundary layer:

$$w_{\text{interior}} = \frac{1}{\rho f}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

Ekman pumping velocity into bottom boundary layer:

$$w_{\text{interior}} = \frac{d}{2}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right)$$

Must be same vertical velocity w_{interior} because $\dfrac{\partial w}{\partial z} = 0$ in the interior.

Thus,

$$\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{2}{\rho f d}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

Vorticity of geostrophic velocity flow is proportional to wind-stress curl.

Since the boundary layers are very thin,
the interior velocity is also the average velocity:

$$\bar{u} = u_g \qquad \bar{v} = v_g$$

$$\Rightarrow \boxed{\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} = \frac{2}{\rho f d}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)}$$

Vorticity of vertically averaged flow
is proportional to wind-stress curl.

The average flow is in 2D in (x,y).

The preceding equation establishes a **local** constraint.

The remaining variation is **non-local**.
(It depends on non-local wind, depths elsewhere and lateral boundary conditions.)

3. When the flow is shallow

Depth H not large → Ekman number ~ 1 → Boundary layers merge.

Heavy algebra with lots of trigonometric and hyperbolic sines/cosines.

Approach:

1. Define scaled variable $\zeta = \frac{z + H}{d} \quad (0 \le \zeta \le 1)$

2. Solve not for $u,\ v$ but rather for divergence and vorticity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \cosh\zeta\,(D\cos\zeta - C\sin\zeta) + \sinh\zeta\,(B\cos\zeta - A\sin\zeta)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \cosh\zeta\,(A\cos\zeta + B\sin\zeta) + \sinh\zeta\,(C\cos\zeta + D\sin\zeta)$$

At this stage, there are four constants of integration: A, B, C and D.

3. Apply the 2 bottom boundary conditions on u and v.
   Apply the 2 top boundary conditions on u and v.
   The 4 constants of integration are now known
   in terms of the surface stress and geostrophic flow.

4. Obtain the vertical velocity from

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \rightarrow \quad w=-\int_{-H}^{z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz$$

$w=0$ at bottom

which yields
$$w=\frac{d}{2}(B+C)(1-\cosh\zeta\cos\zeta)-\frac{d}{2}\sinh\zeta\sin\zeta$$
$$+\frac{d}{2}A\,(\cosh\zeta\sin\zeta-\sinh\zeta\cos\zeta)$$

5. Finally, enforce $w=0$ at the top.
   From this results the sought-after constraint.

4. The Surprise

The sought-after constraint resulting from the application of $w = 0$ at the top is:

New!

$$\boxed{\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right)} \left(\cosh\frac{H}{d}\sinh\frac{H}{d} - \cos\frac{H}{d}\sin\frac{H}{d} \right)$$

$$= \frac{2}{\rho f d} \boxed{\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right)} \left(\cosh^2\frac{H}{d} + \cos^2\frac{H}{d} - \cosh\frac{H}{d}\cos\frac{H}{d} - 1 \right)$$

$$+ \frac{2}{\rho f d} \boxed{\left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} \right)} \sinh\frac{H}{d}\sin\frac{H}{d}$$

Vorticity of geostrophic flow

Curl of wind stress

Divergence of wind stress

We find that now, for shallow depth, the vorticity of the geostrophic flow is not only a function of the curl of the wind stress but also of its divergence.

- For deep water ($H >> d$), the above expression reduces to previously known constraint.

- For very shallow water ($H << d$), the above expression reduces to:

$$\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{3}{2\rho f H} \left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} \right)$$

No more curl, all divergence.

Dynamics when $H \sim d$

$$\left(\frac{\partial v_g}{\partial x}-\frac{\partial u_g}{\partial y}\right)=\frac{\cdots}{\rho f d}\left(\frac{\partial \tau_y}{\partial x}-\frac{\partial \tau_x}{\partial y}\right)+\frac{\cdots}{\rho f d}\left(\frac{\partial \tau_x}{\partial x}+\frac{\partial \tau_y}{\partial y}\right)$$

When the interior does not exist, and the entire column is “boundary layer,” the depth-averaged flow is no longer the geostrophic flow:

$$\bar{u} = u_g + \frac{\tau_y}{\rho f H} - \frac{d}{H} D$$

$$\bar{v} = v_g - \frac{\tau_x}{\rho f H} + \frac{d}{H} C$$

The previous relation was in terms of the geostrophic flow.
In terms of the depth-averaged flow, the relation becomes:

$$\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} = \left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) - \frac{1}{\rho f H} \left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} \right) + \frac{d}{H} \left(\frac{\partial C}{\partial x} + \frac{\partial D}{\partial y} \right)$$

Lots of algebra later, relation is found to retain the same structure:

$$\boxed{\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} = \frac{1}{\rho f d} F\left(\frac{H}{d} \right) \left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right) + \frac{1}{\rho f d} G\left(\frac{H}{d} \right) \left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} \right)}$$

Complicated!

5. Reduction from 3D to 2D

A often used depth-averaged 2D model with rotation and friction is the following
(stripped as before: steady, low Rossby number, f-plane, no horizontal friction to isolate Ekman dynamics)

$$-f\bar{v} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\tau_x}{\rho H} - \cancel{\frac{\tau_{bx}}{\rho H}} \quad - r\,\bar{u}$$

$$+f\bar{u} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\tau_y}{\rho H} - \cancel{\frac{\tau_{by}}{\rho H}} \quad - r\,\bar{v}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial x} = 0$$

with r serving as
a bottom friction coefficient

Elimination of pressure p by cross-differentiation of the momentum equations and subsequent use of the continuity equation leads to:

$$\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} = \frac{1}{\rho H r}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right) + \text{ no term in } \left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}\right)$$

The classical 2D model is not a legitimate reduction of the 3D dynamics!
Ironically, the shallow the water, the worse the 2D model.

When trying better by including an angle between bottom stress and averaged velocity,

$$-f\bar{v} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\tau_x}{\rho H} - r(\bar{u}\cos\alpha - \bar{v}\sin\alpha) \qquad \leftarrow \quad -\frac{\partial}{\partial y}$$

and sum

$$+f\bar{u} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\tau_y}{\rho H} - r(\bar{u}\sin\alpha + \bar{v}\cos\alpha) \qquad \leftarrow \quad +\frac{\partial}{\partial x}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial x} = 0$$

to obtain:

$$\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} = \frac{1}{\rho H r\cos\alpha}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right) + \text{ no term in } \left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}\right)$$

We find that the problem persists. It still does not work!

What we find is that it is not possible to express the bottom stress as a function of depth-average velocity only.

For a 2D model to be faithful to 3D dynamics when $H \sim d$ or smaller, the bottom stress must be expressed in terms of BOTH depth-averaged velocity AND surface stress.

$$\frac{\tau_{bx}}{\rho H} = r\,(\bar{u}\cos\alpha - \bar{v}\sin\alpha) + \frac{s}{\rho H}(\tau_x \cos\beta + \tau_y \sin\beta)$$

$$\frac{\tau_{by}}{\rho H} = r\,(\bar{u}\sin\alpha + \bar{v}\cos\alpha) + \frac{s}{\rho H}(-\tau_x \sin\beta + \tau_y \cos\beta)$$

Two parameters: r and s
Two angles: *a* and *b*

which yields:

$$\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} = \frac{1 - s\cos\beta}{\rho H r\cos\alpha}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right) + \frac{s\sin\beta}{\rho H r\cos\alpha}\left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}\right)$$

6. Conclusions

- We considered Ekman dynamics when the influence of friction extends over the full water column (as opposed to being relegated to thin upper and lower boundary layers).

- We found that the curl of the depth-averaged velocity then depends not only on the curl of the surface stress but also on its divergence.

- The shallower the water layer, the more pronounced the effect.

- Reduction of 3D dynamics to a 2D shallow-water model must be approached with great care.

- It appears that conventional 2D shallow-water models become faulty when the depth of the water is shallow enough that friction acts from top to bottom.

- Ironically, the shallower the system (*i.e*. the more two-dimensional the geometry), the worse the two-dimensional model!

Extras

In an attempt to find the appropriate value for the friction coefficient r, we compare the 3D and 2D relations:

3D

$$\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right)\left(\cosh\frac{H}{d}\sinh\frac{H}{d} - \cos\frac{H}{d}\sin\frac{H}{d}\right)$$

$$= \frac{2}{\rho f d}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)\left(\cosh^2\frac{H}{d} + \cos^2\frac{H}{d} - \cosh\frac{H}{d}\cos\frac{H}{d} - 1\right)$$

$$+ \frac{2}{\rho f d}\left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}\right)\sinh\frac{H}{d}\sin\frac{H}{d}$$

curl

divergence

2D

$$\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{\rho r H}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right) + \frac{1}{\rho f H}\left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}\right)$$

We have two coefficients to match and only one parameter to adjust.
It does not work! (Except in limit $\mathrm{H} >> d$ when the divergence term does not matter.)

The classical 2D model is not a legitimate reduction of the 3D dynamics!
Ironically, the shallow the water, the worse the 2D model.