

BECK-CHEVALLEY CONDITION AND GOURSAT CATEGORIES

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ABSTRACT. We characterise regular Goursat categories through a specific stability property of regular epimorphisms with respect to pullbacks. Under the assumption of the existence of some pushouts this property can be also expressed as a restricted Beck-Chevalley condition, with respect to the fibration of points, for a special class of commutative squares. In the case of varieties of universal algebras these results give, in particular, a structural explanation of the existence of the ternary operations characterising 3-permutable varieties of universal algebras.

A variety of universal algebras is called a Mal'tsev variety [25] when any pair of congruences R and S on the same algebra 2-permute, meaning that $RS = SR$. The celebrated Mal'tsev theorem asserts that the algebraic theory of such a variety is characterised by the existence of a ternary term $p(x, y, z)$ such that the identities $p(x, y, y) = x$ and $p(x, x, y) = y$ hold [22]. The weaker 3-permutability of congruences $RSR = SRS$, which defines 3-permutable varieties, is also equivalent to the existence of two ternary operations r and s such that the identities $r(x, y, y) = x$, $r(x, x, y) = s(x, y, y)$ and $s(x, x, y) = y$ hold [17]. A nice feature of 3-permutable varieties is the fact that they are congruence modular, a condition that plays a crucial role in the development of commutator theory [12, 16].

Many interesting results have been discovered in regular Mal'tsev categories [11] and in regular Goursat categories [10], which can be seen as the categorical extensions of Mal'tsev varieties and of 3-permutable varieties, respectively. The interested reader will find many properties

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of these categories in the references [2, 3, 5, 9, 10, 11, 13, 14, 19, 20, 21, 24], for instance.

For Mal'tsev categories, and many other algebraic categories, there are some elegant characterisations expressed in terms of the fibration of points [5, 3]. Recall that, given a category $\mathcal{C}$ with pullbacks, the fibration of points is the functor $\text{Pt}(\mathcal{C}) \rightarrow \mathcal{C}$ associating the codomain W to any “point” in $\mathcal{C}$, i.e. to any split epimorphism $g: U \rightarrow W$ with a given splitting $j: W \rightarrow U$. For any morphism $\beta: Y \rightarrow W$ in $\mathcal{C}$, the change-of-base functor, defined by pulling back along β , is denoted by $\beta^*: \text{Pt}_W(\mathcal{C}) \rightarrow \text{Pt}_Y(\mathcal{C})$. One of the goals of this paper is to give a characterisation of regular Goursat categories by using this fibration. It turns out that such a characterisation not only involves the change-of-base functors β^* with respect to the fibration of points, but also their left adjoints $\beta_!$, which exist as soon as the category admits pushouts of split monomorphisms [4]. It is precisely the so-called Beck-Chevalley condition (Theorem 1.4) for the commutative squares of the following type

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & U \\ f \downarrow & i & \downarrow g \\ Y & \xrightarrow{\beta} & W, \end{array}$$

where α and β are regular epimorphisms and f and g are split epimorphisms (i.e. the pair (α, β) is a regular epimorphism in the category $\text{Pt}(\mathcal{C})$). In fact, the Goursat property can be also expressed in terms of the functors $\beta_!$ alone: $\beta_!$ preserves binary products, for any regular epimorphism β .

The proof of these results also relies on the fact that the Shifting Lemma [16] holds in any regular Goursat category [6], since the lattice of equivalence relations on any object is modular [10]. A more general characterisation of Goursat categories among regular categories is also obtained without requiring the existence of pushouts along split monomorphisms and involves a stability property for regular epimorphisms (Theorem 1.3). In the varietal case, the existence of the ternary operations characterising 3-permutable varieties mentioned above can be deduced from this theorem by applying it to a suitable diagram involving free algebras (Remark 1.5).

As a consequence of the results in this paper, we obtain an extension to regular Goursat categories of a known result in the regular Mal'tsev context (Proposition 3.6 in [9]): the reflector to a (regular epi)-reflective subcategory of a regular Goursat category always preserves pullbacks of split epimorphisms along split epimorphisms (Proposition 2.1). It then

follows that the so-called internal Galois pregroupoid [18] associated to an extension is necessarily an internal groupoid (Corollary 2.3).

Let us finally mention that the possibility of formulating some exactness properties of Mal'tsev categories in terms of a suitable Beck-Chevalley condition has been first suggested to the authors by Zurab Janelidze during the *Workshop in Category Theory* at the University of Coimbra in 2012, where the work [14] was presented. We would like to warmly thank Zurab for this suggestion. The present paper shows that a suitable Beck-Chevalley condition characterises Goursat categories. Independently, Clemens Berger and Dominique Bourn have discovered a stronger condition that holds in the special case of exact Mal'tsev categories (Proposition 1.24 in [2]). We would like to thank Francis Borceux for some useful remarks concerning the proof of Theorem 1.3.

1. GOURSAT CATEGORIES

In this section we give a new characterisation for a regular category [1] to be a Goursat category through a stability property of regular epimorphisms, similar to the known characterisation for regular Mal'tsev categories given in Proposition 3.6 in [14].

Recall that a regular category $\mathcal{C}$ is called a *Goursat* category [10] when the composition of (effective) equivalence relations R and S on a same object in $\mathcal{C}$ is 3-permutable: $RSR = SRS$. Given a relation $R = (R, r_1, r_2)$ from an object X to an object Y , we write R° for the opposite relation (R, r_2, r_1) from Y to X .

From [10] and [20] we have:

Theorem 1.1. *Let $\mathcal{C}$ be a regular category. The following conditions are equivalent:*

- (i) $\mathcal{C}$ is a Goursat category;
- (ii) $E^\circ \leq EE$, for any reflexive relation E ;
- (iii) $(1_X \wedge T)T^\circ(1_X \wedge T) \leq TT$, for any relation T on an object X ;
- (iv) $PP^\circ PP^\circ \leq PP^\circ$, for any relation P .

Proof. (i) $\Leftrightarrow$ (ii). By Theorem 1 in [20].

(i) $\Leftrightarrow$ (iv). By Theorem 3.5 of [10].

(i) $\Leftrightarrow$ (iii). This type of equivalence was mentioned at the end of [20]. Note that condition (iii) is a stronger version of condition (ii), so that it will suffice to prove that (iv) $\Rightarrow$ (iii). In a regular context, it suffices to give a proof in set-theoretical terms (see Metatheorem A.5.7 in [3], for instance). Suppose that $(x, x) \in T$, $(y, x) \in T$ and $(y, y) \in T$. We want to prove that $(x, y) \in TT$, i.e. that $(x, \alpha) \in T$ and $(\alpha, y) \in T$, for some

$\alpha \in X$. We define a relation P from $X \times X$ to X by: $((a, b), c) \in P$ if and only if $(a, c) \in T$ and $(c, b) \in T$. One then sees that

$$(x, x)Px, (y, x)Px, (y, x)Py, (y, y)Py$$

implying that

$$(x, x)PP^\circ PP^\circ(y, y).$$

By condition (iv) it follows that

$$(x, x)PP^\circ(y, y)$$

and there is an $\alpha \in X$ such that

$$(x, x)P\alpha, (y, y)P\alpha.$$

One concludes then that

$$xT\alpha Ty.$$

□

Before proving the main results of this section we need to recall a useful property of regular Goursat categories, namely the validity of the so-called *Shifting Lemma* [16]. In the context of varieties of universal algebras this property is equivalent to the modularity of the lattice of congruences on any of its algebras. The modularity of the lattice $(L_X, \vee, \wedge)$ of equivalence relations on any object X also holds in any regular Goursat category, as shown in [10]. More precisely, given equivalence relations R, S and T in L_X ,

$$R \leqslant T \Rightarrow (R \vee (S \wedge T) = (R \vee S) \wedge T).$$

In this context, the supremum $R \vee S$ of two equivalence relations in L_X is given by the triple relational composite $R \vee S = RSR$. By using generalized elements the validity of the Shifting Lemma can be expressed as follows:

Shifting Lemma

Given equivalence relations R, S and T on the same object X such that $R \wedge S \leqslant T$, whenever x, y, t, z are elements in X with $(x, y) \in R \wedge T$, $(x, t) \in S$, $(y, z) \in S$ and $(t, z) \in R$, it then follows that $(t, z) \in T$:

$$\begin{array}{ccc}
 & x & \xrightarrow{S} & t \\
 T \left(\begin{array}{c} \downarrow R \\ \downarrow R \end{array} \right. & & & \left. \downarrow R \right) T \\
 & y & \xrightarrow{S} & z
 \end{array}
 \tag{1}$$

The Shifting Lemma holds in any regular Goursat category [6], as we are now going to recall by using the internal logic of a regular category.

Given a diagram (1), the 3-permutability of the equivalence relations implies that

$$(t, z) \in S(R \wedge T)S = (R \wedge T)S(R \wedge T).$$

Accordingly, there exist a and b such that $(t, a) \in R \wedge T$, $(a, b) \in S$, and $(b, z) \in R \wedge T$. Then (a, b) is also in R , thus in T , since $R \wedge S \leq T$; it follows that $(t, z) \in T$, as desired.

The property expressed by the Shifting Lemma has been extended to a categorical context in [7], giving rise to the notion of Gumm category. Indeed, the Shifting Lemma can be equivalently reformulated in any finitely complete category $\mathcal{C}$ by asking that a specific class of internal functors are discrete fibrations (see [6] and [7] for more details).

One of the fundamental results in this paper is given in Theorem 1.3 below, where regular Goursat categories are characterised by a stability property of regular epimorphisms with respect to pullbacks. Such a stability condition is an extension of the following one where regular epimorphisms are stable with respect to kernel pairs:

Theorem 1.2. [13] *Let $\mathcal{C}$ be a regular category. The following conditions are equivalent:*

- (i) $\mathcal{C}$ is a Goursat category;
- (ii) any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & U \\ f \downarrow & i \uparrow & g \downarrow \\ Y & \xrightarrow{\beta} & W, \end{array} \quad j \uparrow$$

where f and g are split epimorphisms and α and β are regular epimorphisms (commuting also with the splittings), is a Goursat pushout: the comparison morphism $\lambda: \text{Eq}(f) \rightarrow \text{Eq}(g)$ induced by the universal property of the kernel pair $\text{Eq}(g)$ of g is also a regular epimorphism.

Theorem 1.3. *Let $\mathcal{C}$ be a regular category. The following conditions are equivalent:*

- (i) $\mathcal{C}$ is a Goursat category;

(ii) for any commutative cube

$$\begin{array}{ccccc}
 X \times_Y Z & \xrightarrow{\lambda} & U \times_W V & & \\
 \uparrow & \swarrow & \uparrow & \swarrow & \\
 X & \xrightarrow{\gamma} & Z & \xrightarrow{\gamma} & V \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 X & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & W \\
 \uparrow & \swarrow & \uparrow & \swarrow & \uparrow \\
 Y & \xrightarrow{\beta} & U & \xrightarrow{\beta} & W \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 Y & \xrightarrow{\beta} & U & \xrightarrow{\beta} & W
 \end{array}
 \quad (2)$$

where the left and right faces are pullbacks of split epimorphisms and α, β and γ are regular epimorphisms (commuting also with the splittings), then the comparison morphism $\lambda: X \times_Y Z \rightarrow U \times_W V$ is also a regular epimorphism;

(iii) for any commutative cube

$$\begin{array}{ccccc}
 X \times_Y Z & \xrightarrow{\delta} & A & & \\
 \uparrow & \swarrow & \uparrow & \swarrow & \\
 X & \xrightarrow{\gamma} & Z & \xrightarrow{\gamma} & V \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 X & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & W \\
 \uparrow & \swarrow & \uparrow & \swarrow & \uparrow \\
 Y & \xrightarrow{\beta} & U & \xrightarrow{\beta} & W \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 Y & \xrightarrow{\beta} & U & \xrightarrow{\beta} & W
 \end{array}
 \quad (3)$$

where the left face is a pullback of split epimorphisms, the right face is a commutative diagram of split epimorphisms and the horizontal arrows $\alpha, \beta, \gamma, \delta$ are regular epimorphisms (commuting also with the splittings), then the right face is a pullback.

Proof. (i) $\Rightarrow$ (ii). Again, it suffices to give a proof in set-theoretical terms. Let $(u, v) \in U \times_W V$. Then there exist $x \in X$ and $z \in Z$ such that $\alpha(x) = u$ and $\gamma(z) = v$; thus $g\alpha(x) = h\gamma(z)$. We define a binary relation R on $Y \times U \times V$ by:

$$((y_1, u_1, v_1), (y_2, u_2, v_2)) \in R$$

if $y_1 = l(\bar{z}), u_1 = \alpha(\bar{x}), y_2 = f(\bar{x})$ and $v_2 = \gamma(\bar{z})$, for some $\bar{x}$ and $\bar{z}$. Categorically, R is defined as follows. Let

$$A = (Y \times U \times V) \times (Y \times U \times V) \times (X \times Z)$$

and consider the following four equalisers, where π_i denotes the i -th projection:

$$\begin{aligned} E_1 &\rightrightarrows A \begin{array}{c} \xrightarrow{l\pi_8} \\ \xrightarrow{\pi_1} \end{array} Y \\ E_2 &\rightrightarrows A \begin{array}{c} \xrightarrow{\alpha\pi_7} \\ \xrightarrow{\pi_2} \end{array} U \\ E_3 &\rightrightarrows A \begin{array}{c} \xrightarrow{f\pi_7} \\ \xrightarrow{\pi_4} \end{array} Y \\ E_4 &\rightrightarrows A \begin{array}{c} \xrightarrow{\gamma\pi_8} \\ \xrightarrow{\pi_6} \end{array} Y \end{aligned}$$

Let $E = E_1 \wedge E_2 \wedge E_3 \wedge E_4$ be their intersection (as subobjects of A), and define R as the regular image given by

$$\begin{array}{c} E \rightrightarrows A \longrightarrow (Y \times U \times V) \times (Y \times U \times V) \\ \searrow \quad \nearrow \\ R \end{array}$$

along the obvious projection $A \rightarrow (Y \times U \times V) \times (Y \times U \times V)$.

We have the following relations:

$$\cdot (f(x), \alpha(x), \gamma kf(x)) R (f(x), \alpha(x), \gamma kf(x)), \text{ for } \bar{x} = x \text{ and } \bar{z} = kf(x);$$

$$\cdot (l(z), \alpha if(x), \gamma(z)) R (f(x), \alpha(x), \gamma kl(z)), \text{ for } \bar{x} = if(x) \text{ and } \bar{z} = kl(z);$$

$$\cdot (l(z), \alpha il(z), \gamma(z)) R (l(z), \alpha il(z), \gamma(z)), \text{ for } \bar{x} = il(z) \text{ and } \bar{z} = z.$$

From $g\alpha(x) = h\gamma(z)$, we get that $\gamma kf(x) = \gamma kl(z)$ and $\alpha if(x) = \alpha il(z)$. So:

$$(f(x), \alpha(x), \gamma kf(x))(1_{Y \times U \times V} \wedge R)R^\circ(1_{Y \times U \times V} \wedge R)(l(z), \alpha il(z), \gamma(z)).$$

From Theorem 1.1(iii), we can conclude that

$$(f(x), \alpha(x), \gamma kf(x)) R R (l(z), \alpha il(z), \gamma(z)).$$

Then there exists (y', u', v') such that

$$(f(x), \alpha(x), \gamma kf(x)) R (y', u', v') R (l(z), \alpha il(z), \gamma(z)).$$

From the definition of R we can conclude that, for some $\bar{x}, \bar{z}, \bar{\bar{x}}, \bar{\bar{z}}, l(\bar{z}) = f(x)$, $\alpha(\bar{x}) = \alpha(x)$, $f(\bar{x}) = y'$, $\gamma(\bar{z}) = v'$ and $l(\bar{\bar{z}}) = y'$, $\alpha(\bar{\bar{x}}) = u'$, $f(\bar{\bar{x}}) = l(z)$, $\gamma(\bar{\bar{z}}) = \gamma(z)$. Then, there exists $(\bar{x}, \bar{\bar{z}}) \in X \times_Y Z$, since $f(\bar{x}) = y' = l(\bar{\bar{z}})$, such that $\lambda(\bar{x}, \bar{\bar{z}}) = (\alpha(\bar{x}), \gamma(\bar{\bar{z}})) = (\alpha(x), \gamma(z)) = (u, v)$.

(ii) $\Rightarrow$ (i). We can consider the left and right faces in the cube (2) to be kernel pairs of split epimorphisms. Then condition (ii) translates into the statement of Theorem 1.2(ii).

(ii) $\Rightarrow$ (iii). By assumption we know that the induced arrow

$$\lambda: X \times_Y Z \rightarrow U \times_W V$$

in diagram (2) is a regular epimorphism, and this implies that the unique induced arrow $c: A \rightarrow U \times_W V$ such that $c\delta = \lambda$ (where δ is defined as in diagram (3)) is a regular epimorphism as well. In order to show that c is also a monomorphism it will suffice to show that $\text{Eq}(\lambda) \leq \text{Eq}(\delta)$, since one always has $\text{Eq}(\delta) \leq \text{Eq}(\lambda)$.

As a preliminary step, we first show that $\text{Eq}(\pi_X) \wedge \text{Eq}(\lambda) \leq \text{Eq}(\delta)$. For that, we define a new relation P as follows: if

$$\text{Eq}(\pi_X) \wedge \text{Eq}(\delta) \xrightarrow{\langle p_1, p_2 \rangle} (X \times_Y Z) \times (X \times_Y Z)$$

is the intersection of the relations $\text{Eq}(\pi_X)$ and $\text{Eq}(\delta)$ on $X \times_Y Z$, the relation P is given by the monomorphism

$$P = \text{Eq}(\pi_X) \wedge \text{Eq}(\delta) \xrightarrow{\langle p_1, \pi_Z p_2 \rangle} (X \times_Y Z) \times Z.$$

In set-theoretical terms P is the relation from $X \times_Y Z$ to Z defined as follows:

$((a, b), c) \in P$ if and only if $((a, b), (a, c)) \in \text{Eq}(\delta)$, (remark that $(a, c) \in X \times_Y Z$ by definition). As we have observed above, under the assumption (ii) $\mathcal{C}$ is necessarily a Goursat category, and then by applying Theorem 1.1 to the relation P we get the equality $PP^\circ PP^\circ = PP^\circ$.

Now, let $((x, z), (x, w))$ be an element in $\text{Eq}(\pi_X) \wedge \text{Eq}(\lambda)$. This implies that $f(x) = l(z) = l(w)$, and $(z, w) \in \text{Eq}(\gamma)$; accordingly, $((il(z), z), (il(w), w)) \in \text{Eq}(\delta)$. Then:

- $((x, z), z) \in P$;
- $((il(z), w), z) \in P$, since $l(z) = l(w)$ and $((il(z), z), (il(w), w)) \in \text{Eq}(\delta)$, as observed above;
- $((il(w), w), w) \in P$;
- $((x, w), w) \in P$.

It follows that $((x, z), (x, w)) \in PP^\circ PP^\circ = PP^\circ$. There is then a θ such that $((x, z), \theta) \in P$ and $((x, w), \theta) \in P$. The fact that $\text{Eq}(\delta)$ is an equivalence relation implies that $((x, z), (x, w)) \in \text{Eq}(\delta)$, since $((x, z), (x, \theta)) \in \text{Eq}(\delta)$ and $((x, w), (x, \theta)) \in \text{Eq}(\delta)$. It follows that $\text{Eq}(\pi_X) \wedge \text{Eq}(\lambda) \leq \text{Eq}(\delta)$.

Finally, to prove that $\text{Eq}(\lambda) \leq \text{Eq}(\delta)$, we consider then an element $((x_1, z_1), (x_2, z_2)) \in \text{Eq}(\lambda)$: this means that $f(x_1) = l(z_1)$, $f(x_2) = l(z_2)$ and $\lambda(x_1, z_1) = \lambda(x_2, z_2)$ (thus, in particular, $\alpha(x_1) = \alpha(x_2)$). We are going to show that $\delta(x_1, z_1) = \delta(x_2, z_2)$. To do so, we apply the Shifting

Lemma to the following situation

$$\begin{array}{ccc}
 (x_1, kf(x_1)) & \xrightarrow{\text{Eq}(\pi_X)} & (x_1, z_1) \\
 \text{Eq}(\delta) \left(\begin{array}{c} \left| \text{Eq}(\lambda) \right. \\ \left| \text{Eq}(\lambda) \right. \end{array} \right. & & \left. \begin{array}{c} \left| \text{Eq}(\lambda) \right. \\ \left| \text{Eq}(\lambda) \right. \end{array} \right) \text{Eq}(\delta) \\
 (x_2, kf(x_2)) & \xrightarrow{\text{Eq}(\pi_X)} & (x_2, z_2),
 \end{array} \tag{4}$$

where the solid lines represent relations holding by assumption. Note that all the elements $(x_1, z_1), (x_2, z_2), (x_1, kf(x_1)), (x_2, kf(x_2))$ are in $X \times_Y Z$. Since $\alpha(x_1) = \alpha(x_2)$, we have

$$((x_1, kf(x_1)), (x_2, kf(x_2))) \in \text{Eq}(\delta)$$

by the commutativity of the upward back face of (3); it follows that $((x_1, kf(x_1)), (x_2, kf(x_2))) \in \text{Eq}(\lambda)$ since $\text{Eq}(\delta) \leq \text{Eq}(\lambda)$. The inequality $\text{Eq}(\pi_X) \wedge \text{Eq}(\lambda) \leq \text{Eq}(\delta)$ that we have already checked above allows us to apply the Shifting Lemma in the situation described by the bold lines in (4), so that the relation in the dashed line holds, i.e. $((x_1, z_1), (x_2, z_2)) \in \text{Eq}(\delta)$, as desired.

(iii) $\Rightarrow$ (ii). This implication easily follows by taking the (regular epimorphism, monomorphism) factorisation of the comparison morphism λ given in diagram (2), say $\lambda = m\delta$. One then obtains a cube of the type (3) which, by assumption, is such that the right face is a pullback. Consequently, λ is isomorphic to the regular epimorphism δ . $\square$

In the last part of this section we give a characterisation of regular Goursat categories through the fibration of points. Thus, we add Goursat categories to the list of (many) algebraic categories characterised in these terms (see [5, 3]).

A *point* in a category $\mathcal{C}$ is a split epimorphism $f: X \rightarrow Y$ together with a fixed splitting $i: Y \rightarrow X$, usually depicted as

$$X \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{f} \end{array} Y.$$

The category of points in $\mathcal{C}$ is denoted by $\text{Pt}(\mathcal{C})$. When $\mathcal{C}$ has pullbacks of split epimorphisms, the functor sending a point to its codomain

$$\begin{array}{ccc}
 \text{Pt}(\mathcal{C}) & \rightarrow & \mathcal{C} \\
 U \begin{array}{c} \xleftarrow{j} \\ \xrightarrow{g} \end{array} W & \mapsto & W
 \end{array}$$

is a fibration, called the *fibration of points* [4]. Given a morphism $\beta: Y \rightarrow W$, the change-of-base functor with respect to this fibration is

denoted by $\beta^*: \text{Pt}_W(\mathcal{C}) \rightarrow \text{Pt}_Y(\mathcal{C})$. If $\mathcal{C}$ has, moreover, pushouts along split monomorphisms, then any pullback functor β^* has a left adjoint

$$\begin{aligned} \beta_! : \quad \text{Pt}_Y(\mathcal{C}) &\rightarrow \text{Pt}_W(\mathcal{C}), \\ X \begin{smallmatrix} \xleftarrow{i} \\ \xrightarrow{f} \end{smallmatrix} Y &\mapsto \beta_!(X) \begin{smallmatrix} \xleftarrow{\beta_!(i)} \\ \xrightarrow{\beta_!(f)} \end{smallmatrix} W \end{aligned}$$

where $(\beta_!(X), \beta_!(f), \beta_!(i)) \in \text{Pt}_W(\mathcal{C})$ is determined by the right hand part of the following pushout:

$$\begin{array}{ccc} X & \xrightarrow{\bar{\beta}} & \beta_!(X) \\ i \uparrow & & \uparrow \beta_!(i) \\ Y & \xrightarrow{\beta} & W. \end{array}$$

Observe that the arrow $\bar{\beta}$ in the diagram above is a regular epimorphism whenever β is a regular epimorphism.

Theorem 1.4. *Let $\mathcal{C}$ be a regular category with pushouts along split mono-morphisms. Then the following conditions are equivalent:*

- (i) $\mathcal{C}$ is a Goursat category;
- (ii) for any regular epimorphism $\beta: Y \rightarrow W$ in $\mathcal{C}$ the functor $\beta_!: \text{Pt}_Y(\mathcal{C}) \rightarrow \text{Pt}_W(\mathcal{C})$ preserves binary products;
- (iii) for any commutative square

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & U \\ f \downarrow \uparrow i & & g \downarrow \uparrow j \\ Y & \xrightarrow{\beta} & W \end{array}$$

where f and g are split epimorphisms and α and β are regular epimorphisms (commuting also with the splittings), the Beck-Chevalley condition holds: there is a functor isomorphism $\alpha_! f^* \cong g^* \beta_!$.

Proof. (i) $\Rightarrow$ (iii). Given a point (Z, l, k) over Y consider the pullback defining $f^*(Z, l, k)$:

$$\begin{array}{ccc} X \times_Y Z & \begin{smallmatrix} \xleftarrow{\pi_Z} \\ \xrightarrow{\pi_X} \end{smallmatrix} & Z \\ \pi_X \downarrow \uparrow & & l \downarrow \uparrow k \\ X & \begin{smallmatrix} \xleftarrow{i} \\ \xrightarrow{f} \end{smallmatrix} & Y. \end{array} \tag{5}$$

The following commutative cube is obtained by applying the functor $\alpha_!$ and $\beta_!$ to the points $f^*(Z, l, k)$ (over X) and (Z, l, k) (over Y), respectively:

$$\begin{array}{ccccc}
 X \times_Y Z & \xrightarrow{\bar{\alpha}} & \alpha_!(X \times_Y Z) & & \\
 \uparrow \pi_Z & \searrow \pi_Z & \uparrow \text{!} & \searrow \text{!} & \\
 X & \xrightarrow{\pi_X} & Z & \xrightarrow{\bar{\beta}} & \beta_!(Z) \\
 \downarrow \pi_X & \downarrow \pi_X & \downarrow \pi_X & \downarrow \text{!} & \downarrow \text{!} \\
 X & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & W \\
 \uparrow i & \uparrow l & \uparrow k & \uparrow j & \uparrow j \\
 Y & \xrightarrow{f} & Y & \xrightarrow{\beta} & W
 \end{array}$$

This diagram is of the form (3). Consequently, the right face is a pullback by Theorem 1.3(iii), so that $g^*\beta_! \cong \alpha_!f^*$.

(iii) $\Rightarrow$ (ii). Let (X, f, i) and (Z, l, k) be points over Y and consider their product in $\text{Pt}_Y(\mathcal{C})$, which is given by the pullback (5). We take its image through the functor $\beta_!: \text{Pt}_Y(\mathcal{C}) \rightarrow \text{Pt}_W(\mathcal{C})$:

$$\begin{array}{ccccc}
 X \times_Y Z & \xrightarrow{\bar{\beta}} & \beta_!(X \times_Y Z) & & \\
 \uparrow \pi_Z & \searrow \pi_Z & \uparrow \text{!} & \searrow \text{!} & \\
 X & \xrightarrow{\pi_X} & Z & \xrightarrow{\bar{\beta}} & \beta_!(Z) \\
 \downarrow \pi_X & \downarrow \pi_X & \downarrow \pi_X & \downarrow \text{!} & \downarrow \text{!} \\
 X & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & W \\
 \uparrow i & \uparrow l & \uparrow k & \uparrow j & \uparrow j \\
 Y & \xrightarrow{f} & Y & \xrightarrow{\beta} & W
 \end{array} \tag{6}$$

By applying the assumption to the bottom commutative face, we conclude that the right face is a pullback, i.e. $\beta_!$ preserves binary products.

(ii) $\Rightarrow$ (i). Consider the diagram (2), where α, β, γ are assumed to be regular epimorphisms, and let us show that the induced arrow λ is also a regular epimorphism. The image of the points over Y of the left face of (2) by the functor $\beta_!$ determines the commutative diagram (6).

One then obtains the following commutative diagram

$$\begin{array}{ccccc}
 \beta_!(X \times_Y Z) & \xrightarrow{\quad \phi \quad} & U \times_W V & & \\
 \updownarrow & \swarrow & \updownarrow & \swarrow & \\
 & \beta_!(Z) & \xrightarrow{\quad \tau \quad} & V & \\
 \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
 \beta_!(X) & \xrightarrow{\quad \sigma \quad} & U & \xrightarrow{\quad \lambda \quad} & V \\
 \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
 & W & \xrightarrow{\quad 1_W \quad} & W &
 \end{array} \quad (7)$$

where the arrows σ, τ and ϕ are induced by the universal properties of the pushouts defining $\beta_!(X), \beta_!(Z)$ and $\beta_!(X \times_Y Z)$, respectively, and $\phi\bar{\beta} = \lambda$. The fact that α and γ are regular epimorphisms implies that σ and τ are regular epimorphisms, while the assumption guarantees that the left face in the diagram (7) is a pullback in $\mathcal{C}$. It then follows that the induced arrow ϕ is a regular epimorphism as well (as a product of the regular epimorphisms σ and τ in $\text{Pt}_W(\mathcal{C})$), and so is then the arrow $\lambda: X \times_Y Z \rightarrow U \times_W V$. This shows that condition (ii) in Theorem 1.3 is satisfied, and $\mathcal{C}$ is then a Goursat category, as desired. $\square$

Remark 1.5. A variety of universal algebras is 3-permutable when its algebraic theory has two ternary operations r and s such that the identities $r(x, y, y) = x$, $r(x, x, y) = s(x, y, y)$ and $s(x, x, y) = y$ hold [17]. We can prove the existence of such ternary operations by applying the property stated in Theorem 1.3(ii) to the following commutative cube

$$\begin{array}{ccccc}
 P & \xrightarrow{\quad \lambda \quad} & \text{Eq}(\nabla) & & \\
 \updownarrow & \swarrow & \updownarrow & \swarrow & \\
 & 3X & \xrightarrow{\quad \nabla+1 \quad} & 2X & \\
 \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
 3X & \xrightarrow{\quad 1+\nabla \quad} & 2X & \xrightarrow{\quad \nabla \quad} & X \\
 \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
 & 2X & \xrightarrow{\quad \nabla+1 \quad} & X &
 \end{array}$$

$\begin{matrix} (i_1 \ i_2) \\ (i_2 \ i_3) \end{matrix}$

where X is the free algebra on one element, kX denotes a k -indexed copower of X and i_j the canonical j -th injection, P denotes the object part of the pullback defining the left face and $\nabla = \begin{pmatrix} 1_X & 1_X \end{pmatrix} : 2X \rightarrow X$ is the codiagonal. By Theorem 1.3(ii), the comparison morphism λ is a surjective homomorphism. The terms $p_1(x, y) = x$ and $p_2(x, y) = y$ are such that $(p_1, p_2) \in \text{Eq}(\nabla)$. Since λ is surjective, there exist ternary terms, say r and s , such that:

$(r, s) \in P$, from which we deduce that $r(x, x, y) = s(x, y, y)$, and $\lambda(r, s) = (p_1, p_2)$, which gives $r(x, y, y) = x$ and $s(x, x, y) = y$.

2. REFLECTIONS FOR GOURSAT CATEGORIES

In this section we show that any (regular epi)-reflective subcategory $\mathcal{X}$ of a regular Goursat category $\mathcal{C}$ has the property that the reflector $I : \mathcal{C} \rightarrow \mathcal{X}$ preserves pullbacks of split epimorphisms along split epimorphisms, extending the same result which is known for regular Ma'tsev categories (Proposition 3.6 in [9]). Consequently, every internal Galois pregroupoid of an extension is an internal groupoid.

Proposition 2.1. *Consider a (regular epi)-reflective subcategory $\mathcal{X}$ of a regular Goursat category $\mathcal{C}$*

$$\mathcal{C} \begin{array}{c} \xrightarrow{I} \\ \xleftarrow{\perp} \\ \xleftarrow{U} \end{array} \mathcal{X},$$

where $U : \mathcal{X} \rightarrow \mathcal{C}$ is a full inclusion. Then the reflector $I : \mathcal{C} \rightarrow \mathcal{X}$ preserves pullbacks of pairs of split epimorphisms.

Proof. Consider the following commutative diagram where the left face is a pullback of split epimorphisms, the right face is its image through the functor $UI : \mathcal{C} \rightarrow \mathcal{C}$ and η denotes the unit of the adjunction:

$$\begin{array}{ccccc}
 X \times_Y Z & \xrightarrow{\eta_{X \times_Y Z}} & UI(X \times_Y Z) & & \\
 \uparrow \pi_X & \swarrow \pi_Z & \uparrow \eta_Z & \swarrow & \\
 X & \xrightarrow{\eta_X} & UI(X) & & \\
 \downarrow f & \swarrow i & \downarrow k & \swarrow & \\
 Y & \xrightarrow{\eta_Y} & UI(Y) & &
 \end{array}$$

$\begin{array}{c} \text{Left face: } X \times_Y Z \text{ is a pullback of } \pi_X : X \times_Y Z \rightarrow X \text{ and } \pi_Z : X \times_Y Z \rightarrow Z. \\ \text{Right face: } UI(X \times_Y Z) \text{ is a pullback of } UI(\pi_X) : UI(X \times_Y Z) \rightarrow UI(X) \text{ and } UI(\pi_Z) : UI(X \times_Y Z) \rightarrow UI(Z). \end{array}$

This commutative diagram verifies the conditions of the commutative diagram (3) in Theorem 1.3. It follows that the right face is a pullback,

since $\mathcal{C}$ is a Goursat category. But $U: \mathcal{X} \rightarrow \mathcal{C}$ is a full inclusion that preserves and reflects pullbacks, and this completes the proof. $\square$

We follow [19] in calling a (regular epi)-reflective subcategory $\mathcal{X}$ of an exact category $\mathcal{C}$ a *Birkhoff* subcategory if, moreover, $\mathcal{X}$ is closed in $\mathcal{C}$ under regular quotients.

Corollary 2.2. *Consider a Birkhoff subcategory $\mathcal{X}$ of an exact Goursat category $\mathcal{C}$*

$$\mathcal{C} \begin{array}{c} \xrightarrow{I} \\ \xleftarrow{\perp} \\ \xleftarrow{U} \end{array} \mathcal{X},$$

where $U: \mathcal{X} \rightarrow \mathcal{C}$ is a full inclusion. Then the reflector $I: \mathcal{C} \rightarrow \mathcal{X}$ preserves pullbacks of split epimorphisms along regular epimorphisms.

Proof. Let

$$\begin{array}{ccc} P & \xrightarrow{p_2} & C \\ p_1 \updownarrow & & \updownarrow g \\ A & \xrightarrow{p} & B \end{array} \quad (8)$$

be a pullback of a split epimorphism g along a regular epimorphism p . By taking the kernel pairs $\text{Eq}(p_2)$ of p_2 and $\text{Eq}(p)$ of p and then applying UI one gets the commutative diagram

$$\begin{array}{ccccc} UI(\text{Eq}(p_2)) & \rightrightarrows & UI(P) & \xrightarrow{UI(p_2)} & UI(C) \\ \updownarrow & & \updownarrow UI(p_1) & & \updownarrow UI(g) \\ UI(\text{Eq}(p)) & \rightrightarrows & UI(A) & \xrightarrow{UI(p)} & UI(B). \end{array} \quad (9)$$

The two (downward oriented) commutative squares on the left are pullbacks, since UI preserves pullbacks of split epimorphisms along split epimorphisms. It suffices to prove that (A) is a pullback, since U reflects pullbacks. Now, by taking the regular image of $\text{Eq}(p_2)$ along the regular epimorphism η_P one gets an effective equivalence relation $\eta_P(\text{Eq}(p_2))$ on $UI(P)$, since $\mathcal{C}$ is an exact Goursat category. Moreover, $UI(p_2)$ is its coequaliser in $\mathcal{C}$, since $\mathcal{X}$ is stable in $\mathcal{C}$ under regular quotients. Accordingly, $\eta_P(\text{Eq}(p_2)) \cong \text{Eq}(UI(p_2))$. Similarly,

$\eta_A(\text{Eq}(p)) \cong \text{Eq}(UI(p))$ yielding the following commutative diagram:

$$\begin{array}{ccccc}
 UI(\text{Eq}(p_2)) & \cdots\cdots\cdots\rightarrow & \text{Eq}(UI(p_2)) & \xlongequal{\quad} & UI(P) \\
 \updownarrow & & \updownarrow & & \updownarrow \\
 & & & (B) & UI(p_1) \\
 UI(\text{Eq}(p)) & \cdots\cdots\cdots\rightarrow & \text{Eq}(UI(p)) & \xlongequal{\quad} & UI(A).
 \end{array}$$

The exterior rectangles are the left pullbacks in diagram (9) and the dotted arrows are regular epimorphisms by construction. By applying Proposition 4.1 in [15] we conclude that the squares (B) are pullbacks. By the usually called Barr-Kock theorem [1] we conclude that (A) is a pullback, as desired. $\square$

We finally observe that the so-called internal Galois pregroupoids associated to an extension are always internal groupoids in the Goursat context.

First recall from [18] that an *internal precategory* in a category $\mathcal{C}$ is a diagram of the form

$$\begin{array}{ccccc}
 & \xrightarrow{p_1} & & \xrightarrow{d_1} & \\
 P_2 & \xrightarrow[m]{p_2} & P_1 & \xleftarrow[s]{d_2} & P_0 \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} &
 \end{array}$$

with

- (1) $d_1 s = 1_{P_0} = d_2 s$;
- (2) $d_2 p_1 = d_1 p_2$;
- (3) $d_1 p_1 = d_1 m$, $d_2 p_2 = d_2 m$.

In other words, an internal precategory in $\mathcal{C}$ can be seen as what remains of the definition of an internal category in $\mathcal{C}$ when one cancels all references to pullbacks.

Every extension (= regular epimorphism) $f: A \rightarrow B$ in $\mathcal{C}$ induces the internal groupoid given by the equivalence relation determined by the kernel pair of f :

$$\begin{array}{ccccc}
 \text{Eq}(f) \times_A \text{Eq}(f) & \xrightarrow{p_1} & & \xrightarrow{d_1} & \\
 & \xrightarrow[m]{p_2} & \text{Eq}(f) & \xleftarrow[s]{d_2} & A. \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} &
 \end{array}$$

By applying the left adjoint $I: \mathcal{C} \rightarrow \mathcal{X}$ of the inclusion functor $U: \mathcal{X} \rightarrow \mathcal{C}$ to this equivalence relation one always obtains an internal precategory in $\mathcal{X}$:

$$\begin{array}{ccccc}
 & \xrightarrow{I(p_1)} & & \xrightarrow{I(d_1)} & \\
 I(\text{Eq}(f) \times_A \text{Eq}(f)) & \xrightarrow[I(m)]{I(p_2)} & I(\text{Eq}(f)) & \xleftarrow[I(d_2)]{I(s)} & I(A). \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} &
 \end{array}$$

This special kind of internal precategory is called the *internal Galois pregroupoid of f* (see [18] for more details), and is denoted by $\mathbf{Gal}(f)$. It turns out to always be an internal groupoid in the Goursat context:

Corollary 2.3. *Consider a (regular epi)-reflective subcategory $\mathcal{X}$ of a regular Goursat category $\mathcal{C}$*

$$\mathcal{C} \begin{array}{c} \xrightarrow{I} \\ \perp \\ \xleftarrow{U} \end{array} \mathcal{X},$$

where $U: \mathcal{X} \rightarrow \mathcal{C}$ is a full inclusion. Given any extension $f: A \rightarrow B$ in $\mathcal{C}$, then the internal Galois pregroupoid $\mathbf{Gal}(f)$ is an internal groupoid.

Proof. An internal precategory is an internal groupoid when certain commutative squares of split epimorphisms are pullbacks (see [3], for instance). The result follows immediately from the fact that those pullbacks are preserved by I (Proposition 2.1). $\square$

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