Another approach to the issue as to how long diffusive processes can be ignored when tracking a water mass?

Eric Deleersnijder, 18 March 2018

We build on the previous working note by Deleersnijder (2018[1]), which is why the domain of interest and the notations are similar. Hence, they will not be defined explicitly again hereinafter.

The streamtube under study is defined by $0 \le x < \infty$ and $-H \le z \le H$. The objective is to calculate the amount of water originally present at the entrance (x=0) of the streamtube that is still in it at distance x to the entrance.

Let $C(x,z)$ represent the concentration (defined as a mass fraction) of the water mass under study. It is governed by the following partial differential problem

$$U\frac{\partial C}{\partial x} = K\frac{\partial^2 C}{\partial z^2} \quad , \tag{1}$$

$$C(0,z) = \begin{cases} 1 \ \text{ if } \ z \in]-H,+H[\\ 0 \ \text{ if } \ z \notin]-H,+H[\end{cases} \tag{2}$$

$$C(x,\pm\infty) < 1 \quad . \tag{3}$$

It is convenient to introduce the pseudo-time

$$\tau = \frac{x}{U} \quad , \tag{4}$$

which allows us to rewrite equation (1) as a diffusion equation

$$\frac{\partial C}{\partial \tau} = K\frac{\partial^2 C}{\partial z^2} \quad . \tag{5}$$

As was seen previously, this concentration is

$$C(\tau,z) = \frac{1}{2}\operatorname{erf}\left(\frac{H-z}{\sqrt{4K\tau}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{H+z}{\sqrt{4K\tau}}\right) \quad . \tag{6}$$

The fraction of this water that is present in the streamtube at $\tau = x/U$ is

$$r(\tau) = \frac{\int_{-H}^{H} C(\tau,z)\,dz}{\int_{-H}^{H} C(0,z)\,dz} = \frac{1}{2H}\int_{-H}^{H} C(\tau,z)\,dz = \operatorname{erf}\left(\frac{H}{\sqrt{K\tau}}\right) - \frac{1}{H}\sqrt{\frac{K\tau}{\pi}}\left(1 - e^{-H^2/(K\tau)}\right) \tag{7}$$

If a particle leaves the streamtube and comes back into at later stage under the impact of diffusive processes, it will be accounted for in (7). However, it may also be of interest to focus on the particles that have not yet left the streamtube, i.e. particles that have always been in it during the time interval $[0,\tau]$. Such particles are discarded at the moment they hit one of

[1] Deleersnijder E., 2018, Tracking a water mass: under which condition can diffusive processes be ignored?, Working Note, 9 pages

the boundaries of the streamtube, i.e. $z = \pm H$. The concentration $C'(\tau,z)$ of such water particles is defined only in the interval $-H \le z \le H$ and is the solution of the following partial differential problem

$$\frac{\partial C'}{\partial \tau} = K \frac{\partial^2 C'}{\partial z^2} \quad , \tag{8}$$

$$C'(0,z) = 1 \tag{9}$$

$$C'(\tau,\pm H) = 0 \quad . \qquad (10)^2$$

The concentration reads

$$C'(\tau,z) = \sum_{n=1}^{\infty} a_n \, e^{-\gamma_n \tau} \cos(k_n z) \quad , \tag{11}$$

with

$$k_n = \frac{\pi(2n-1)}{2H} \, , \quad a_n = \frac{2(-1)^n}{H\,k_n} \, , \quad \gamma_n = K k_n^2 \quad . \tag{12}$$

The fraction of this water that is present in the streamtube at $\tau = x/U$ is

$$r'(\tau) = \frac{\int_{-H}^{H} C'(\tau,z)\,dz}{\int_{-H}^{H} C'(0,z)\,dz} = \frac{1}{2H} \int_{-H}^{H} C'(\tau,z)\,dz = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{e^{-\gamma_n \tau}}{(2n-1)^2} \tag{13}$$

Clearly, the ratio $(r - r')/r$ represents the fraction of the water particles that were initially in the streamtube and have left it at least once.

The figures below show that particles very quickly leave the streamtube for the first time. The associated timescale is smaller than H^2/K. From (13), we may estimate that the relevant timescale is

$$\frac{1}{\gamma_1} = \frac{4H^2}{\pi^2 K} \approx 0.41 \frac{H^2}{K} \quad . \tag{14}$$

In other words, according to the present approach, diffusive processes can be neglected if the following criterion is satisfied

$$\text{simulation duration} \ll \frac{(\text{vertical size of the water mass patch})^2}{2.4 \times (\text{diapycnal diffusivity})} \tag{15}$$

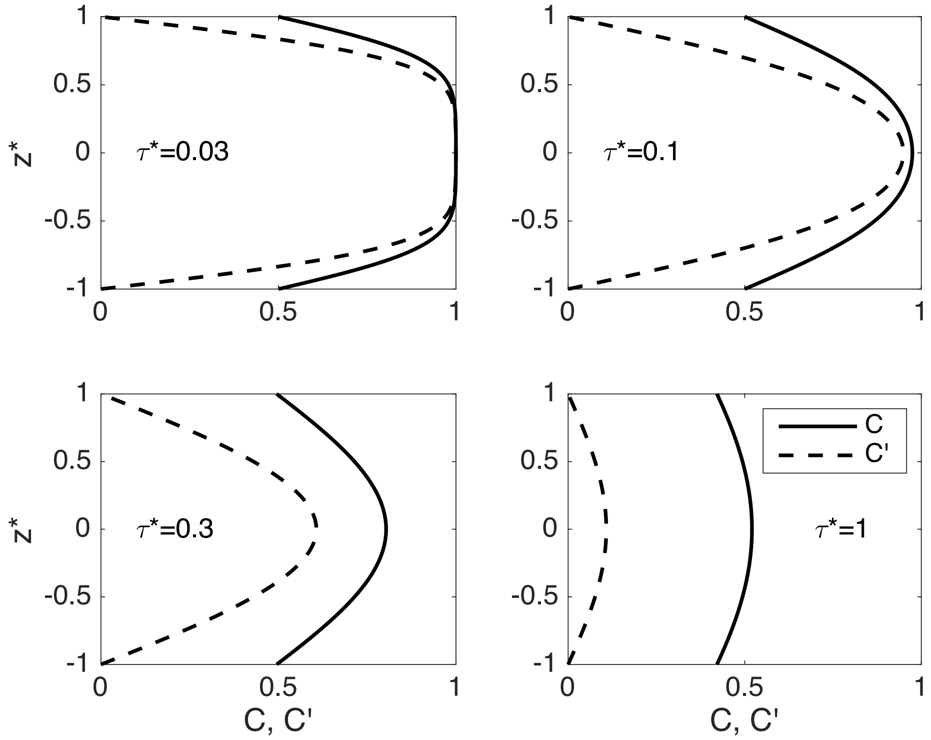
This condition is somewhat less constraining than that obtained in the previous study (but is not significantly different), which was

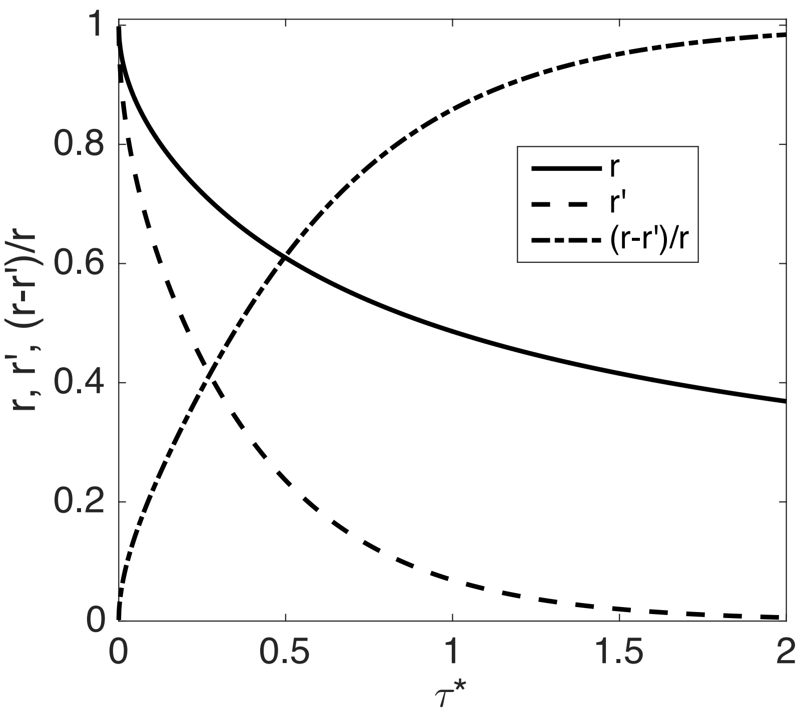
$$\text{simulation duration} \ll \frac{(\text{vertical size of the water mass patch})^2}{6 \times (\text{diapycnal diffusivity})} \tag{16}$$

In the figures below, the following dimensionless variables are used

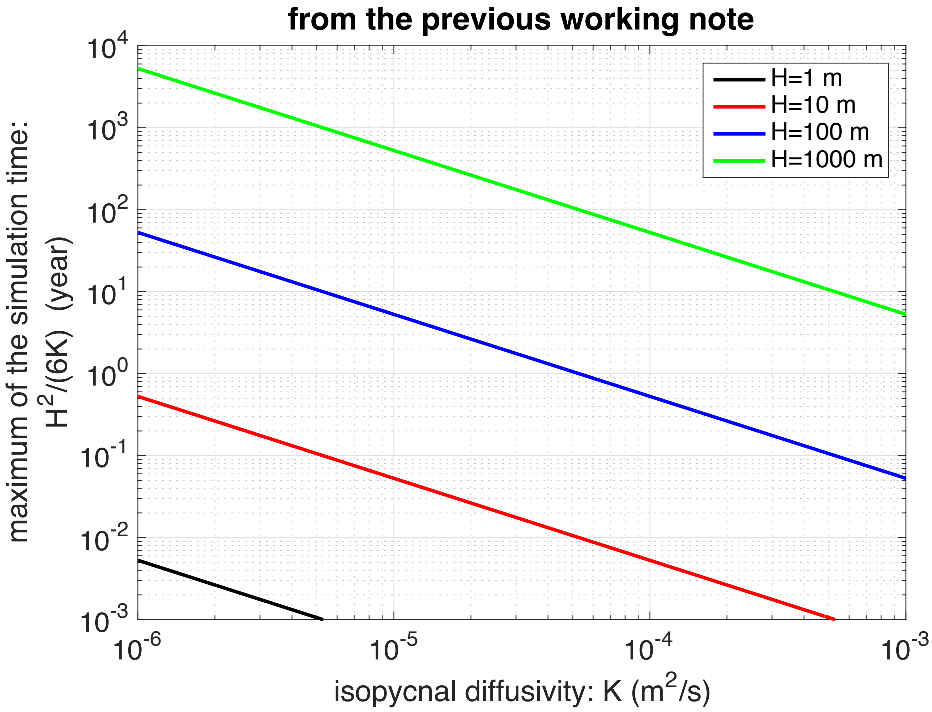
[2] At first glance, this boundary condition may seem to be somewhat surprising. However, a justification of it may be found in the following article: Delhez E.J.M. and E. Deleersnijder, 2006, The boundary layer of the residence time field, *Ocean Dynamics*, 56, 139-150

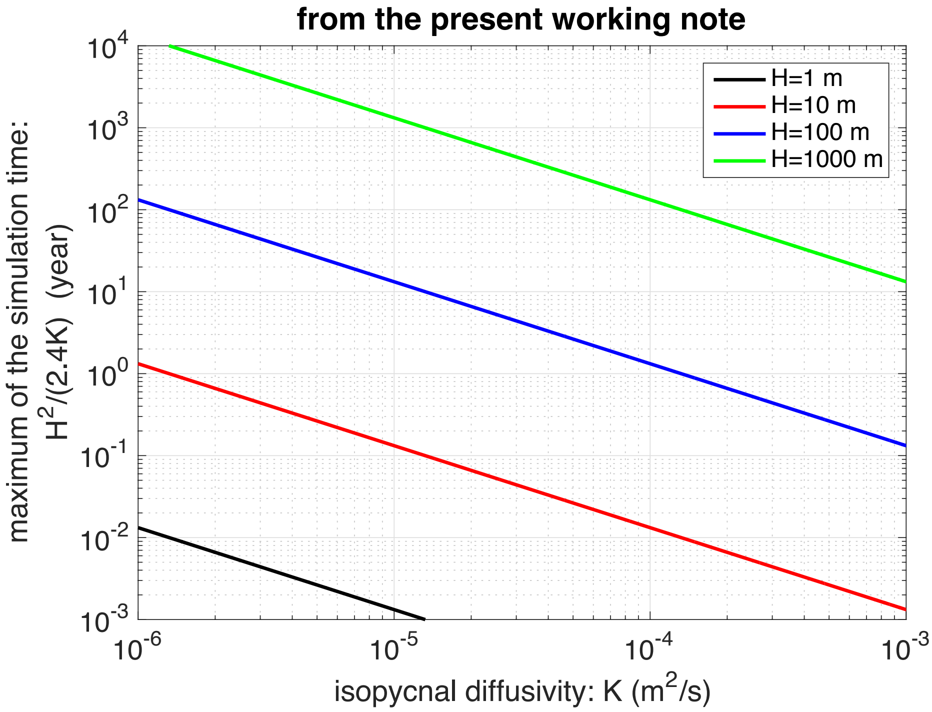
$$\tau^* = \frac{\tau}{H^2/K} = \frac{x/U}{H^2/K} \ , \quad z^* = \frac{z}{H} \quad . \tag{17}$$





The maximum duration of the simulation (i.e. the simulation time during which diffusion can be ignored to a certain degree) is represented graphically in the figures below.





The previous working note and the present one rely on significantly different approaches to determine the simulation time during which diffusive processes can be ignored to a certain degree. However, they yield rather similar values of the maximum simulation time.
