

CORRELATION IMPULSE RESPONSE FUNCTIONS

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REPRINT | 2023 / 30

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Correlation impulse response functions

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November 30, 2023

Abstract

Volatility impulse response functions are a widely used tool for analyzing the temporal impact of shocks on (co-)volatilities of financial time series. This paper proposes an extension to correlation impulse response functions (CIRF), based on a multivariate GARCH modelling framework. As we show, CIRF and corresponding covariance impulse response functions can react quite differently to a given shock and even move in opposite directions. Due to the inherent nonlinearity, no analytical form is available for CIRF, but we propose a straightforward algorithm to estimate the CIRF numerically. In an empirical application we focus on the change of the consensus protocol of Ethereum in 2022 and its effect on the correlation with Bitcoin.

Keywords: Dependence, causality, multivariate GARCH, conditional correlation, cryptocurrencies.

JEL Classification: C32, G15.

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1 Introduction

Volatility impulse response functions, introduced by Hafner and Herwartz (2006) in the framework of multivariate GARCH models, have become an important tool for analyzing the effects of shocks on volatilities and correlations in financial return time series. In the benchmark case of a so-called BEKK specification, introduced by Engle and Kroner (1995), analytical expressions are available to obtain the impulse response functions for conditional variances and covariances. For correlations such expressions are not available due to the nonlinearity of the transformation. However, correlations play a major role in understanding the linear dependencies between assets, and are important ingredients of, for example, portfolio allocation, starting with Markowitz (1952), and risk management decisions, see e.g. Engle (2009). Importantly, correlations depend on both covariances and variances, and hence, can move in opposite direction as the covariances. If, for example, a shock increases both volatilities and covariances, correlations may increase or decrease, depending on the relative size of these effects. Evaluation of the effects for the correlations is, however, difficult due to the inherent nonlinearity.

This paper defines correlation impulse response functions as the difference between the expected correlation conditional on a shock and past information, and the expected correlation conditioning only on the past information. The involved expectations are not available in analytical form for typical multivariate GARCH models, but we propose a suitable estimation algorithm based on numerical approximations, with inference based on a bootstrap approach as employed by Hafner and Herwartz (2023).

We provide an empirical illustration in an application to the correlation of the two largest cryptocurrencies (we will often use the term ‘cryptos’), Bitcoin (BTC) and Ethereum (ETH). Correlation of cryptos has become an important research area recently, both within cryptos and between cryptos and classical financial assets. For example, using data prior to January 2018, Katsiampa (2019) shows that ETH can be an effective hedge

against BTC, although correlations were at a much lower level than in the period following 2018. Dwita Mariana et al. (2021) show that BTC and ETH might serve as safe haven investments in times of crises, as they tend to correlate negatively with stocks, in particular during the recent Covid-19 pandemic. Nakagawa and Sakemoto (2022) analyze the dynamic behavior of the BTC-ETH correlation. They find that the correlation has increased with rising popularity of ETH, but that in states of high uncertainty the correlation is lower. Sifat et al. (2019) investigate price linkages and information flow transfer between BTC and ETH at intraday and daily levels, and confirm the close short run dependence between both cryptos. Aslanidis et al. (2019) confirm the status of cryptos as alternative assets, finding that correlations within cryptos are positive and time-varying, while the correlations of cryptos with classical assets are negligible. This could be explained by cryptocurrency-specific pricing factors, such as network factors, as shown by Liu and Tsyvinski (2020). Our contribution with respect to this literature is the analysis of correlations over time, conditional on certain shocks or events.

In our application we focus on the events around the so-called ‘Merge’ in September 2022, when the Ethereum blockchain changed its consensus protocol from Proof-of-Work to Proof-of-Stake, which modifies the way how transactions are validated on the blockchain. The inherent uncertainty of the eventual success of this event implied an ambiguous behavior of the BTC-ETH correlations: In the case of a success one might expect the return to high correlations, while in case of a failure, a decoupling of both cryptos with decreasing correlations. We analyze two consecutive days, where the CIRF on the first day would indicate a decoupling, and on the second day a return to high correlations. We show that these findings are genuine to CIRF and not visible in the conventional covariance IRF.

The following section introduces the model framework, defines CIRF and proposes an estimation algorithm. Section 3 presents the empirical application, and Section 4

concludes.

2 The methodology

In the following we first present the multivariate GARCH model framework, then define correlation impulse response functions, and propose an estimator.

2.1 MGARCH model framework

Consider a vector of N financial returns, $r_t = (r_{1t}, \dots, r_{Nt})'$, observed at time $t = 1, \dots, T$. Our general model can be cast into the following framework:

$$r_t = \mu_t + e_t, \quad (1)$$

$$e_t = Q_t \xi_t, \quad (2)$$

where $\mu_t = \mathbb{E}[r_t | \mathcal{F}_{t-1}]$ and $\mathcal{F}_t$ is the information set at time t , the shocks ξ_t are assumed i.i.d. with mean zero and identity covariance matrix, and Q_t denotes a decomposition of $H_t = \mathbb{V}\text{ar}(r_t | \mathcal{F}_{t-1})$ such that $H_t = Q_t Q_t'$. Examples are the spectral decomposition, the Cholesky decomposition, or orthogonal rotations that allow, e.g., to obtain independent components as in Hafner et al. (2022), or other identification schemes as in Fengler and Polivka (2022).

While the signal in Q_t is typically strong for high frequency financial returns due to pronounced time-variation of second order moments, the conditional mean μ_t is much more noisy and often statistically insignificant. For simplicity, we therefore set it to a constant, $\mu_t = \mu$, and for the following analysis consider de-meaned returns, $e_t = r_t - \mu$.

For the modelling of the conditional covariance matrix H_t , we use the BEKK(1,1) model proposed by Engle and Kroner (1995), i.e.,

$$H_t = CC' + Ae_{t-1}e_{t-1}'A' + GH_{t-1}G', \quad (3)$$

where C , A and G are $N \times N$ parameter matrices, and C is lower triangular.

2.2 Correlation impulse response functions

To recall, volatility impulse response functions (VIRF), as introduced by Hafner and Herwartz (2006), measure the impact of a shock ξ^* at time t on the subsequent variances and covariances of returns, conditional on the past, and relative to a benchmark that only conditions on the past. VIRFs at a given time horizon $\nu \geq 1$ are defined as

$$\mathcal{V}_{t+\nu}(\xi^*, \mathcal{F}_{t-1}) = \mathbb{E}[\text{vech}(H_{t+\nu}) | \mathcal{F}_{t-1}, \xi_t = \xi^*] - \mathbb{E}[\text{vech}(H_{t+\nu}) | \mathcal{F}_{t-1}], \quad (4)$$

where $\mathcal{F}_t$ denotes the information set generated by the process up to time t . The shock vector of interest, ξ^* , may be obtained empirically from the estimated model and for a given covariance decomposition Q_t , i.e. $\xi^* = \xi_t = Q_t^{-1}e_t$, in which case Fengler and Polivka (2022) speak of ‘historical VIRF’. They also propose a ‘scenario VIRF’ for the case where ξ_t can be interpreted as structural innovations, following an identification scheme for Q_t . In that case, specific hypothetical shocks of interest, e.g. tail events, may be considered. While Hafner and Herwartz (2006) derive analytical expressions of VIRF for the case of symmetric BEKK models, Hafner and Herwartz (2023) propose estimators of the VIRF for the asymmetric BEKK model.

The notion of volatility impulse response functions can now be extended to that of correlation impulse functions in the following way. Define the CIRF as

$$\mathcal{C}_{ij,t+\nu}(\xi^*, \mathcal{F}_{t-1}) = \mathbb{E} \left[\frac{H_{ij,t+\nu}}{\sqrt{H_{ii,t+\nu}H_{jj,t+\nu}}} | \mathcal{F}_{t-1}, \xi_t = \xi^* \right] - \mathbb{E} \left[\frac{H_{ij,t+\nu}}{\sqrt{H_{ii,t+\nu}H_{jj,t+\nu}}} | \mathcal{F}_{t-1} \right]. \quad (5)$$

Hence, CIRFs quantify the correlation path expected to occur after a shock $\xi_t = \xi^*$ in comparison with a hypothetical benchmark ‘steady state’ correlation path. Note that the CIRF are not available in closed form, but can be estimated. For example, let $R_{ij,t+\nu|t}$ denote the first expectation on the right hand side of equation (5), and $R_{ij,t+\nu|t-1}$ the second

expectation, such that $\mathcal{C}_{ij,t+\nu}(\xi^*, \mathcal{F}_{t-1}) = R_{ij,t+\nu|t} - R_{ij,t+\nu|t-1}$. Draw a large number S of trajectories $\xi_{t+1}^{(s)}, \dots, \xi_{t+\nu_{max}}^{(s)}$, $s = 1, \dots, S$, from the empirical distribution of estimated ξ_t , where ν_{max} is the maximum desired time horizon. Then, generate $e_{t+1|t}^{(s)} := Q_{t+1} \xi_{t+1}^{(s)}$ and

$$H_{t+2|t}^{(s)} = CC' + Ae_{t+1|t}^{(s)} e_{t+1|t}^{(s)'} A' + GH_{t+1} G'$$

Then decompose $H_{t+2|t}^{(s)} = Q_{t+2|t}^{(s)} Q_{t+2|t}^{(s)'}$ and obtain $e_{t+2|t}^{(s)} := Q_{t+2|t}^{(s)} \xi_{t+2}^{(s)}$. Continue this procedure for $\nu = 3, 4, \dots, \nu_{max}$ until all trajectories $e_{t+\nu|t}^{(s)}$ are determined. Finally,

$$\hat{R}_{ij,t+\nu|t} := \frac{1}{S} \sum_{s=1}^S \frac{H_{ij,t+\nu|t}^{(s)}}{\sqrt{H_{ii,t+\nu|t}^{(s)} H_{jj,t+\nu|t}^{(s)}}}, \quad \nu \geq 1. \quad (6)$$

Likewise, $R_{ij,t+\nu|t-1}$ can be also be estimated by simulation, drawing additionally ξ_t , i.e., generating trajectories $\xi_t^{(s)}, \dots, \xi_{t+\nu_{max}}^{(s)}$, $s = 1, \dots, S$, and then recursively calculating $e_{t+\nu|t-1}^{(s)}$, $H_{t+1|t-1}^{(s)}$ and $Q_{t+1|t-1}^{(s)}$, as above.

Two points are worth stressing: First, neither VIRF nor CIRF depend on the ordering of the variables in the system, as it is the case for certain structural impulse response functions that rely on covariance matrix decompositions such as the Cholesky decomposition. Second, observing that the suggested CIRFs are highly nonlinear transformations of the QML estimators of the model parameters, we use bootstrap techniques to determine confidence intervals for estimated VIRF and CIRF. As in Hafner and Herwartz (2023), the bootstrap consists of an iid sampling from the model innovations $\{\hat{\xi}_t\}_{t=1}^T$, and the dynamic generation of a bootstrap imitation of the MGARCH process e_t .¹

¹After estimation of the BEKK model and implied model residuals in our empirical application (Section 3), we detect a few outliers of ξ_t . For the implementation of bootstrap inference, all those ξ_t that have at least one element larger than 6 in absolute value are skipped in order to robustify the procedure. The dynamic covariance profile for the b -th bootstrap replication is

$$H_t^{(b)} = \hat{C} \hat{C}' + \hat{A} e_{t-1}^{(b)} e_{t-1}^{(b)'} \hat{A}' + \hat{G} H_{t-1}^{(b)} \hat{G}', \quad t = 1, 2, \dots, T,$$

where $e_t^{(b)} = Q_t^{(b)} \xi_t^{(b)}$. Our empirical analysis relies on 1499 replications, i.e., $b = 1, 2, \dots, 1499$.

3 CIRF for the Bitcoin-Ethereum correlation

As an empirical application we consider the two largest cryptocurrencies by market capitalization, Bitcoin and Ethereum, for the period November 9, 2017 until December 31, 2022. The sample comprises 1878 daily returns.² The return data have been centered and standardized. Figure 1 shows the estimated volatilities and correlations. The model implied unconditional correlation is 0.867, which demonstrates the close linear dependence of Bitcoin and Ethereum, which has been much lower at the beginning of the sample.

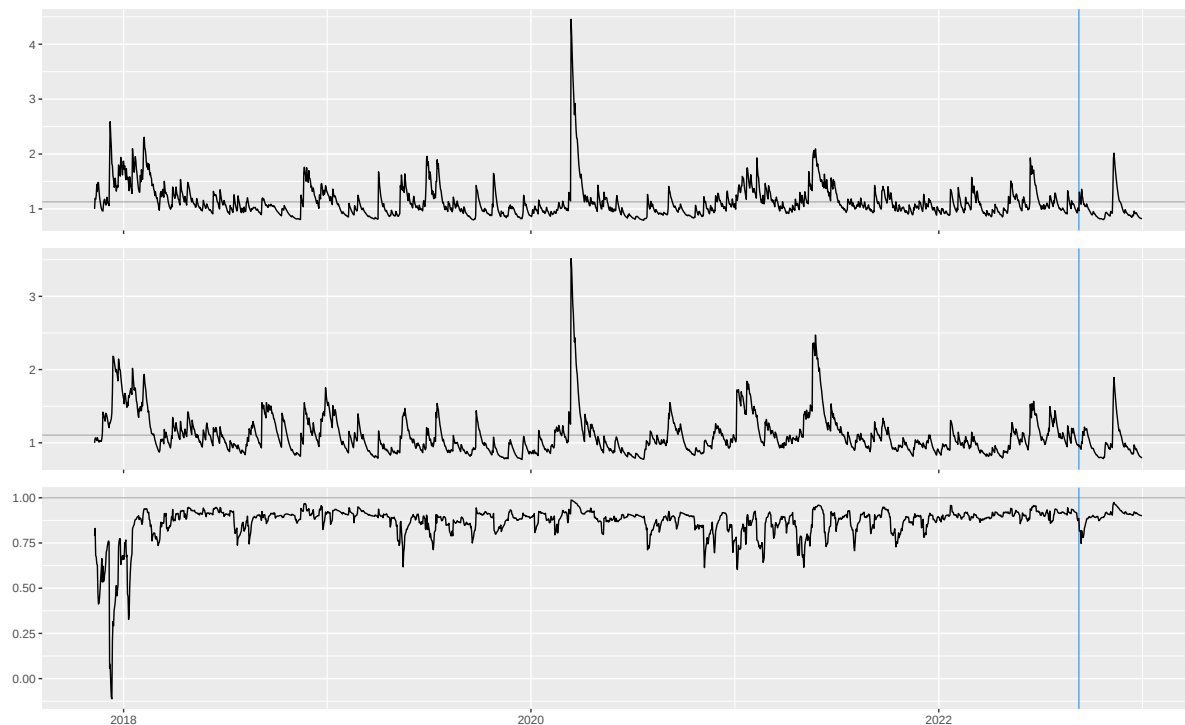


Figure 1: Estimated volatilities for Bitcoin (top) and Ethereum (middle), and conditional correlations (bottom). The vertical line indicates September 9, 2022.

²The daily prices in US Dollar have been downloaded from Yahoo.finance (yahoo codes: BTC-USD and ETH-USD).

We obtain the following QML parameter estimates (joint with bootstrap t -ratios):³

$$\hat{C} = \begin{pmatrix} 0.325 & 0 \\ (6.51) & \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0.360 & -0.004 \\ (6.39) & (-0.84) \\ 0.016 & 0.312 \\ (0.26) & (5.72) \end{pmatrix}, \quad \hat{G} = \begin{pmatrix} 0.844 & -0.051 \\ (25.6) & (-1.83) \\ 0.073 & 0.978 \\ (1.95) & (30.6) \end{pmatrix}$$

For the analysis of the reaction of correlations to events in periods of uncertainty, we consider the period around the change of the consensus protocol of Ethereum from Proof-of-Work (PoW) to Proof-of-Stake (PoS) in September 2022. This event, known as “The Merge” as it essentially merged the Ethereum blockchain with the PoS Beacon blockchain, took place in two steps: The first, on September 6, by an upgrade of the consensus layer (Bellatrix), and the second, on September 15, by an upgrade of the executive layer (Paris). We have investigated the correlations in this ten-day period, marked by uncertainty about the eventual success of The Merge, and in particular focus on the two days of September 9 and 10.

Figure 2 shows the estimated volatility IRF, covariance IRF and correlation IRF for September 9, 2022. The current state of the volatility matrix H_t on this day was close to the unconditional level: $\text{vech}(H_t) = (0.93, 0.81, 0.89)'$, implying a current correlation of 0.89, compared with an unconditional correlation of 0.867. The observed returns on this day were 2.49 for Bitcoin and 0.95 for Ethereum, i.e., a strong positive shock for Bitcoin, and a moderately positive one for Ethereum.⁴ This leads to a strong positive effect of the VIRF for Bitcoin, and a slightly negative one for Ethereum. We see a strong positive effect in the covariance IRF for small lags, triggered by the large Bitcoin return, while the correlation IRF is much smaller and insignificant, becoming significantly negative after a few lags. Intuitively, this can be explained by the fact that the CIRF standardizes the

³Unreported diagnostics such as portmanteau statistics of squared standardized returns confirm the chosen model specification.

⁴Keep in mind that returns are standardized to have unit standard deviation.

covariance IRF by the volatilities, so that the magnitude of the Bitcoin return does not directly matter for the correlation.

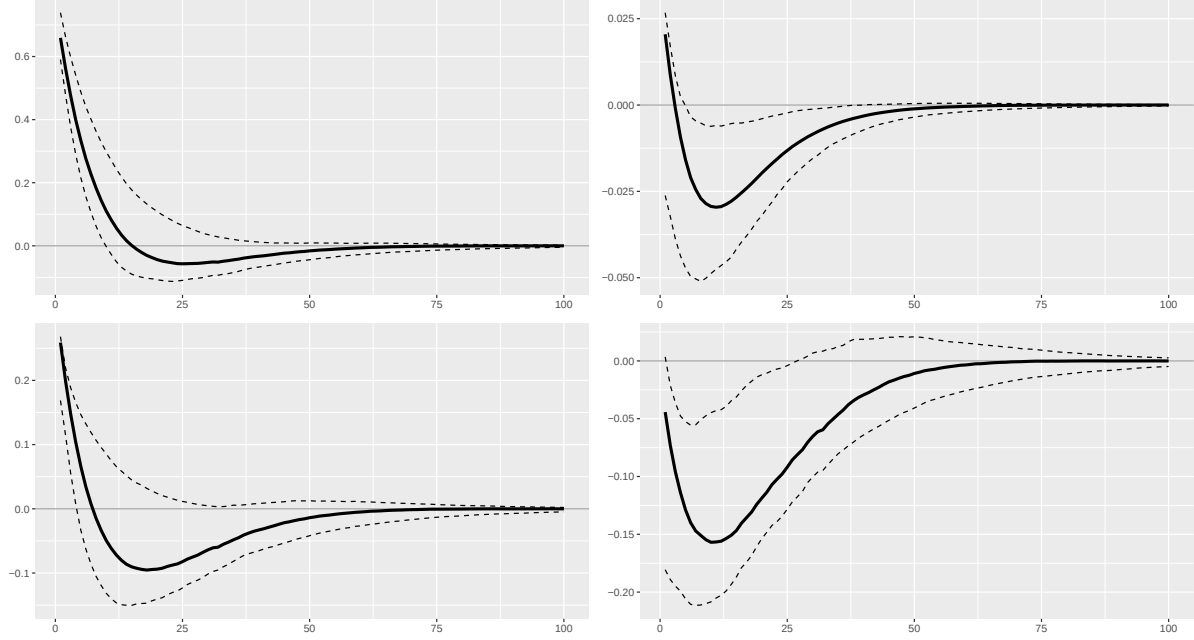


Figure 2: IRFs for 9/9/22: VIRF for Bitcoin (upper left), Ethereum (lower right), covariance (lower left), and correlation (upper right). Graphs show estimates (thick lines) joint with 95% pointwise bootstrap based confidence intervals. All impulse response estimates are based on $S = 10000$ trajectories of model innovations as described in Section 2.2.

On September 10, 2022 (Figure 3), current volatility has changed to $\text{vech}(H_t) = (1.71, 1.03, 0.94)'$, with a current correlation of 0.81. Note that current Bitcoin volatility has substantially increased due to the large shock on the previous day. However, Ethereum volatility and the covariance have only slightly increased, and importantly, the correlation has even decreased. This drop in correlations is also visible in the time series of conditional correlations in Figure 1. The returns on this day were much smaller in magnitude: 0.334 for Bitcoin and 0.623 for Ethereum, resulting in a significantly negative VIRF for both cryptos. Again, the covariance IRF is affected by the negative VIRF and is pulled down

to negative values for lags larger than 4, although insignificant. On the other hand, given the fact that both cryptos move in the same direction of about similar size, and hitting a state of correlation below the unconditional level, the correlation IRF shows a strong positive effect that remains significant until lag 12. This clearly highlights the difference between covariance and correlation IRF: An interpretation based on covariances only could erroneously conclude a negative effect of the shock on the dependencies of both cryptos, while in fact the effect is positive when adjusting for the scale.

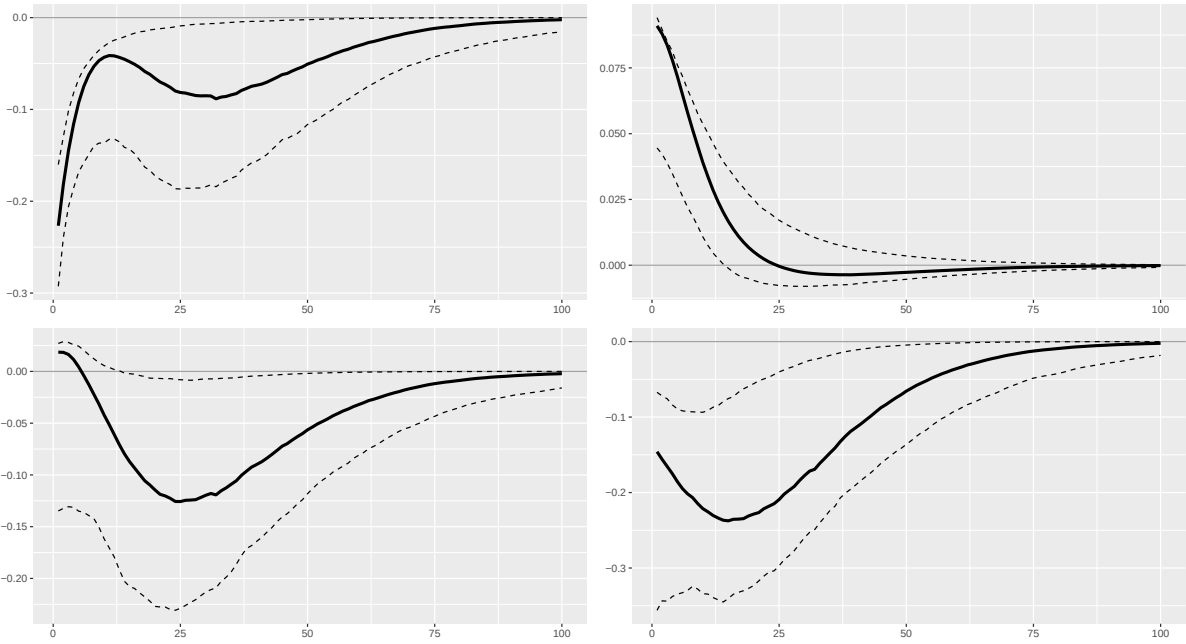


Figure 3: IRFs for 9/10/22: VIRF for Bitcoin (upper left), Ethereum (lower right), covariance (lower left), and correlation (upper right). Graphs show estimates (thick lines) joint with 95% pointwise bootstrap based confidence intervals. All impulse response estimates are based on $S = 10000$ trajectories of model innovations as described in Section 2.2.

Altogether, these results suggest that the period around The Merge was characterized by high uncertainty. We picked two consecutive days in 2022 for which the market’s perception of the Bitcoin-Ethereum correlation, and implicitly the evaluation of the success

of The Merge, was quite different: September 9, where the CIRF was predominantly negative, possibly indicating a decoupling of the two major cryptos, and September 10, for which the CIRF was mostly positive, implying a return to a high correlation state. We emphasize that these results are genuine to CIRF and could not be obtained using the conventional covariance IRF analysis.

4 Conclusions

We have proposed a new tool for analyzing the effects of shocks in multivariate time series models, which allows to trace the effects of the shocks on the correlations over time. The analytical treatment is difficult due to the inherent nonlinearity, but we propose ways to estimate the impulse response functions, and do inference. In an empirical application to the two largest cryptocurrencies, we have demonstrated the substantially different implications of covariance and correlation impulse response functions. Our methodology could be extended to other multivariate volatility models, such as the asymmetric model of Hafner and Herwartz (2023). Furthermore, analogous definitions of impulse response functions may be obtained for rank correlations, such as Kendall’s tau and Spearman’s rho, see e.g. McNeil et al. (2015). We leave this for future research.

Acknowledgements

We would like to thank three anonymous reviewers for their helpful comments. The first author is grateful for financial support of the research grant ARC 18-23/089 of the Belgian Federal Science Policy Office. The second author gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft (HE2188/14-1).

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