Modelling of Tokamak plasmas as open GENERIC systems

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Abstract: In this paper, we investigate the formulation of open coupled electro–magnetic field with heat diffusion within the General Equilibrium Non–Equilibrium Reversible Irreversible Couplings, GENERIC, formulation. This two–potentials structured representation is developed to include the two fundamental laws of thermodynamics. The first potential is associated to the total energy of the system and the energy conservation law. The second one is a metric potential expressed the system evolution and is associated to the total entropy of the system and the entropy balance equation. In this contribution, we first define the to boundary formulation of GENERIC and then the two equivalent representations of a Tokamak plasma diffusion model in whose dynamics are governed by Maxwell's equations coupled to a heat diffusion equation. The main coupling is the plasma resistivity.

Keywords: Non–equilibrium thermodynamics, Plasmas, GENERIC, Boundary control systems, Multi–physics systems

1. INTRODUCTION

In this contribution, the double bracket geometric formulation of open infinite–dimensional thermodynamical systems is considered. This representation of dynamical systems (finite– and infinite–dimensional) was introduced in (Morrison, 1986), where metriplectic systems were proposed. The idea in this class of systems was to combine a Hamiltonian dynamics and a dissipative dynamics. Based on this idea, the General Equilibrium Non–Equilibrium Reversible Irreversible Couplings (GENERIC) formulation was proposed as a structured formalism built around the first and the second laws of thermodynamics (Grmela and Öttinger, 1997; Öttinger and Grmela, 1997; Öttinger, 2005a). This formulation is relevant for both microscopic and macroscopic levels of description. In the GENERIC formulation, the decomposition of the dynamics have a physical interpretation. The conservative part of the dynamics is associated with the conservation of energy (first law). The dissipative part of the dynamics is associated to an accumulation function that measures the advancement of losses and dissipation within the system (second law). As a dynamical system modelled using the GENERIC formalism satisfies the first and second laws of thermodynamics, it is a valuable tool to study and control irreversible thermodynamic systems (Favache et al., 2010).

An open formulation of GENERIC was proposed in (Öttinger, 2006), where boundary conditions are integrated within the Poisson and the dissipative brackets. The full brackets are decomposed into a bulk bracket and a boundary one. The state equations governing the bulk variable time evolution are only governed by the bulk brackets. Boundary brackets are involved in the time variation of total energy and entropy balance equations. A semi–group approach was considered in (Badlyan and Zimmer, 2018) to express the GENERIC framework as a boundary control system. In (Badlyan et al., 2018), a hydrodynamics example was considered and an entropy formulation was derived.

Tokamaks are magnetic fusion confinement experimental devices in which the fuel takes the form of the forth state of matter: the plasma state. The model–based control of Tokamak plasmas is a major challenge toward a viable future energy production device without nuclear waste (Pironti and Walker, 2005). Tokamak plasmas are multi–physics systems modelled by coupled Maxwell's and Navier–Stokes equations. For a review of plasma physics and Tokamak magnetic confinement devices, see the monograph (Wesson and Campbell, 2004).

In this contribution, we consider a 3–D plasma model representing the electro–magnetic and heat diffusion processes, as a simplified version of the model derived in (Vu et al., 2016; Vincent et al., 2017). The system is governed by the Maxwell equations and the internal energy diffusion equation, or alternatively the entropy balance equation. The main coupling in the model is Joule effect introduced with the plasma resistivity. The system possesses boundary and distributed inputs, defined both in the electro–

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magnetic and internal energy domains. As the GENERIC formulation is not unique we derive equivalent energy and entropy formulations of the plasma model.
The paper is structured as follows. In Section 2, we state the open GENERIC representation of infinite–dimensional systems. Boundary and distributed inputs are introduced to cope with a simplified 3–D plasma model, presented in Section 3. This model considers the Maxwell equations for the electro–magnetic part of the dynamics and the internal energy, alternatively, the entropy balance equations for the energy diffusion part of the dynamics. In Sections 4 and 5, the plasma model is re–formulated as a GENERIC open system with an entropy and an energy formulations, respectively. Areas for future investigations based on the proposed model formulations are discussed in Section 6.

2. OPEN GENERIC FORMULATION

Let us consider a fixed and bounded domain $\Omega \subset \mathbb{R}^3$ enclosed by the surface $\partial\Omega \subset \mathbb{R}^2$. The time variation of independent state variables $x \in \mathcal{X} = C^\infty(\Omega, \mathbb{R}^n)$, where $\mathcal{X}$ denotes the state space, is decomposed as:

$$\frac{\partial x}{\partial t} = L(x) \cdot \frac{\delta \mathbb{E}}{\delta x} + M(x) \cdot \frac{\delta \mathbb{S}}{\delta x} + g(x)u(t), \qquad (1)$$

where the real–valued functions $\mathbb{E} : \Lambda^3(\Omega) \to \mathbb{R} : x \mapsto \mathbb{E} = \int_\Omega e(x)$, and $\mathbb{S} : \Lambda^3(\Omega) \to \mathbb{R} : x \mapsto \int_\Omega s(x)$ denote the total system energy and entropy, respectively. The total energy density and the total entropy density variables are 3–forms defined as $e \in \Lambda^3(\Omega)$ and $s \in \Lambda^3(\Omega)$, respectively. The operators $L(x)(\cdot)$ and $M(x)(\cdot)$ are skew–symmetric and symmetric operators, both taking values in the state space $\mathcal{X}$, respectively. They can be function of the state variable x, as for example for hydrodynamical systems (Grmela and Öttinger, 1997). The skew–symmetric operator L defines a Poisson bracket and satisfy the Jacobi identities (see paragraph 2.1). The operator $\delta/\delta x$ is the variational derivative with respect to the state variable (Öttinger, 2005b). The first and second terms in the state equation (1) denote the conservative part and dissipative part of the dynamics, respectively. The system is subject to the following degeneracy conditions:

$$L(x) \cdot \frac{\delta \mathbb{S}}{\delta x} = M(x) \cdot \frac{\delta \mathbb{E}}{\delta x} = 0. \qquad (2)$$

A system described within the formalism of equations (1) and (2) were proposed originally in (Grmela and Öttinger, 1997) under the name of GENERIC. For open systems, *i.e.*, for systems with inputs and outputs, the system is subject to boundary conditions. These are set according to the definition of operators L and M. In the classical GENERIC formulation, these operators are defined with respect to Poisson and dissipative brackets when the system is closed. Here, keeping symmetry and skew–symmetry properties of these operators, extended brackets are defined through boundary operators L^∂ and M^∂. These operators are used to define an open representation of the system conserving the geometric structure. We shall recall in the following section the Poisson and dissipative brackets. Throughout the present note, functions are defined in the space of square integrable functions with the dual product $\langle f, g\rangle$ defined as

$$\langle f, g\rangle = \int_\Omega f \wedge g, \qquad (3)$$

where f and g are two dual k and $(n\text{-}k)$–forms with $f \in \Lambda^k(\Omega)$ and $g \in \Lambda^{(n-k)}(\Omega)$, respectively. Furthermore we shall use arbitrary state–dependent real–valued functions A , B, and C defined by their total density variables a, b, and c such that $A : \mathcal{X} \to \mathbb{R} : x \mapsto \int_\Omega a(x)$. The approach is the same for functions B and C defined later.

2.1 Poisson brackets

The reversible contribution of the dynamics (1) is governed by Poisson bracket defined as

$$\{A, B\} = \left\langle \frac{\delta A}{\delta x}, L(x) \cdot \frac{\delta B}{\delta x} \right\rangle, \qquad (4)$$

satisfying the skew-symmetry property

$$\{A, B\} = -\{B, A\}, \qquad (5)$$

the product or Leibniz rule

$$\{AB, C\} = A\{B, C\} + B\{A, C\}, \qquad (6)$$

and the Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0, \qquad (7)$$

where A, B, and C are arbitrary real–valued functions. Closed mechanical systems are defined by such structure and the port–Hamiltonian formulation takes its roots within this formalism (van der Schaft and Maschke, 2002). For an open system the Poisson bracket is split into an internal contribution and a boundary one (Öttinger, 2006) such that the bracket is decomposed as

$$\{A, B\} = \{A, B\}_\Omega + \{A, B\}_{\partial\Omega}. \qquad (8)$$

The first and the second term on the right hand side of the full bracket definition (8) denote the bulk and boundary contributions, respectively. The bulk bracket is identical to the Poisson bracket formulated for closed systems as (Öttinger, 2006):

$$\{A, B\}_\Omega = \left\langle \frac{\delta A}{\delta x}, L(x) \cdot \frac{\delta B}{\delta x} \right\rangle, \qquad (9)$$

where $L(x)$ is the differential operator used in the state equation (1), with the scalar product (3). The bulk bracket is defined as follows (Öttinger, 2006):

$$\{A, B\}_\partial = \int_{\partial\Omega} \frac{\delta A}{\delta x} \wedge L^\partial(x) \cdot \frac{\delta B}{\delta x}, \qquad (10)$$

where $L^\partial(x)$ is the boundary operator associated to the conservative part of the dynamics. The integration is carried out over the boundary surface $\partial\Omega$. The operator $L^\partial(x)$ can be derived by verifying the full Poisson bracket skew–symmetry property (5). This is achieved for

$$\{A, B\}_\Omega + \{B, A\}_\Omega = -\int_{\partial\Omega} \frac{\delta B}{\delta x} \wedge \left(L^{\partial *} + L^\partial\right) \cdot \frac{\delta A}{\delta x}, \qquad (11)$$

where $L^{\partial *}$ is the adjoint boundary operator defined as (Öttinger, 2006):

$$\int_{\partial\Omega} \frac{\delta A}{\delta x} \wedge L^\partial(x) \cdot \frac{\delta B}{\delta x} = \int_{\partial\Omega} \frac{\delta B}{\delta x} \wedge L^{\partial *}(x) \cdot \frac{\delta A}{\delta x}. \qquad (12)$$

Usually, the Poisson operator is a first order differential operator hence the boundary Poisson operator is constituted of scalar values. By definition the bulk and boundary Poisson operators (9) and (10) verify the Leibniz rule. The degeneracy condition remains a bulk property of the system as

$$L(x) \cdot \frac{\delta \mathbb{S}}{\delta x} = 0, \qquad (13)$$

holds. The full Poisson bracket is defined by both the bulk and the boundary Poisson operators $\big(L(x), L^{\partial}(x)\big)$.

2.2 Dissipative brackets

The irreversible contribution of the dynamics is modelled by the dissipative elements of the system. The dynamics is driven by the entropy gradient and defines a dissipative bracket:

$$[A,\, B] = \left\langle \frac{\delta A}{\delta x},\, M(x) \cdot \frac{\delta B}{\delta x} \right\rangle, \tag{14}$$

satisfying the symmetry property

$$[A,\, B] = [B,\, A], \tag{15}$$

the product of Leibniz rule

$$[AB,\, C] = A[B,\, C] + B[A,\, C], \tag{16}$$

and the non-negativity condition

$$[A,\, A] \geq 0, \tag{17}$$

where A, B, and C are arbitrary real–valued functions. The full dissipative bracket is decoupled into a bulk contribution and a boundary one (Öttinger, 2006) such that:

$$[A,\, B] = [A,\, B]_{\Omega} + [A,\, B]_{\partial\Omega}. \tag{18}$$

The bulk dissipative bracket is equivalent to the closed dissipative bracket (14) as:

$$[A,\, B]_{\Omega} = \left\langle \frac{\delta A}{\delta x},\, M(x) \cdot \frac{\delta B}{\delta x} \right\rangle, \tag{19}$$

while the boundary dissipative bracket is defined as (Öttinger, 2006):

$$[A,\, B]_{\partial\Omega} = \int_{\partial\Omega} \frac{\delta A}{\delta x} \wedge M^{\partial}(x) \cdot \frac{\delta B}{\delta x}, \tag{20}$$

where $M^{\partial}(x)$ is the dissipative boundary operator. The dissipative boundary bracket is obtained using the symmetry property for the full bracket which gives:

$$[A,\, B]_{\Omega} - [B,\, A]_{\Omega} = \int_{\partial\Omega} \frac{\delta B}{\delta x} \cdot \big(M^{\partial} - M^{\partial *}\big) \cdot \frac{\delta A}{\delta x}, \tag{21}$$

where the operator $M^{\partial *}(x)$ is the adjoint boundary dissipative operator defined as (Öttinger, 2006):

$$\int_{\partial\Omega} \frac{\delta A}{\delta x} \wedge M^{\partial}(x) \cdot \frac{\delta B}{\delta x} = \int_{\partial\Omega} \frac{\delta B}{\delta x} \wedge M^{\partial *}(x) \cdot \frac{\delta A}{\delta x}. \tag{22}$$

Usually the dissipative bracket is a second–order differential operator, and the boundary operator resulting from integration by part is a first–order one. The full dissipative bracket is defined by the pair of the bulk and the boundary operators $\big(M(x), M^{\partial}(x)\big)$. By definition the bulk and boundary dissipative operators (19) and (20) verify the Leibniz rule. The degeneracy condition remains a bulk property of the system as

$$L(x) \cdot \frac{\delta \mathbb{S}}{\delta x} = 0, \tag{23}$$

holds. The full Poisson bracket is defined by both the bulk and the boundary Poisson operators $\big(L(x), L^{\partial}(x)\big)$.

Remark 1. Only the bulk operators $M(x)$ and $L(x)$ are involved in the state equation (1). However the boundary operators M^{∂} and L^{∂} contribute to the total energy and entropy time variation.

2.3 Balance equations

The time derivative of an arbitrary real–valued function A is given by (Öttinger, 2006):

$$\frac{dA}{dt} = \{A,\, \mathbb{E}\} + [A,\, \mathbb{S}], \tag{24}$$

where the full brackets can be decomposed into bulk and boundary bracket by applying definitions (8) and (18). By re–arranging the terms, one obtains (Öttinger, 2006) the following formulation:

$$\begin{aligned} \frac{dA}{dt} = &\int_{\Omega} \left(-\frac{\delta \mathbb{E}}{\delta x} \cdot L(x) + \frac{\delta \mathbb{S}}{\delta x} \cdot M(x) \right) \wedge \frac{\delta A}{\delta x} \\ &- \int_{\Omega} \frac{\delta \mathbb{E}}{\delta x} \cdot \big(L^{\partial} + L^{\partial *}\big) \wedge \frac{\delta A}{\delta x} \\ &+ \int_{\Omega} \frac{\delta \mathbb{S}}{\delta x} \cdot \big(M^{\partial} - M^{\partial *}\big) \wedge \frac{\delta A}{\delta x} \\ &+ \int_{\Omega} \frac{\delta A}{\delta x} B u_d(t). \end{aligned} \tag{25}$$

One note that the time variation of a total density variable (25) is associated to the variation of bulk variables, symmetric part of the conservative boundary inputs, skew–symmetric part of the dissipative boundary sources terms, and the distributed sources. Replacing the arbitrary real–valued function A by the total energy $\mathbb{E}$, its balance equation is given as:

$$\begin{aligned} \frac{d\mathbb{E}}{dt} = &\int_{\partial\Omega} \left(-\frac{\delta \mathbb{E}}{\delta x} \cdot L^{\partial} + \frac{\delta \mathbb{S}}{\delta x} \cdot \big(M^{\partial} - M^{\partial *}\big) \right) \cdot \frac{\delta \mathbb{E}}{\delta x} \\ &+ \int_{\Omega} \frac{\delta \mathbb{E}}{\delta x} B u_d, \end{aligned} \tag{26}$$

where we have used the full bracket definition (8), the degeneracy conditions (13) and (23). We verify that the energy is conserved when there is no energy exchange with the surrounding. Similarly the total entropy balance equation reads:

$$\begin{aligned} \frac{d\mathbb{S}}{dt} = &[\mathbb{S}, \mathbb{S}] + \int_{\Omega} \frac{\delta \mathbb{S}}{\delta x} B u_d \\ &- \int_{\partial\Omega} \left(\frac{\delta \mathbb{E}}{\delta x} \cdot \big(L^{\partial} + L^{\partial *}\big) + \frac{\delta \mathbb{S}}{\delta x} \cdot M^{\partial *} \right) \wedge \frac{\delta \mathbb{S}}{\delta x}. \end{aligned} \tag{27}$$

The total entropy rate of change (27) is due to the symmetric contribution of the Poisson boundary operator L^{∂}.

3. PLASMA MODEL

To illustrate the GENERIC formulation of open systems we consider a simplified plasma model whose dynamics are governed by Maxwell's electro–magnetic equations (Frankel, 2011) and a heat diffusion equation (Vu et al., 2016; Vincent et al., 2017).

3.1 Electro–magnetic dynamics

Let $\Omega \subset \mathbb{R}^3$ be a fixed domain closed by the boundary surface $\partial\Omega \subset \mathbb{R}^2$. The Maxwell–Faraday's equation maps the magnetic flux $B \in \Lambda^2(\Omega)$, a 2–form, and the electric field $E \in \Lambda^1(\Omega)$, a 1–form, written as

$$\frac{\partial B}{\partial t} = -dE. \tag{28}$$

Ampere's formula is given by

$$\frac{\partial D}{\partial t} = dH - j, \tag{29}$$

where the distributed current $j \in \Lambda^2(\Omega)$ is a 2–form, the electric flux is the 2–form $D \in \Lambda^2(\Omega)$, and the magnetic field is the 1–form $H \in \Lambda^1(\Omega)$. The operator d denotes for the exterior derivative and maps a k–form into a $(k+1)$–form. The current density is composed of an ohmic current, source of internal dissipations due to the resistivity $\eta \in \Lambda^0(\Omega)$, and of an external input current density such that the total current reads

$$j = j_{\mathrm{ohm}} + j_{\mathrm{ext}} = \star_{1/\eta} E + j_{\mathrm{ext}}. \tag{30}$$

The Hodge star operator $\star$ maps a k–form into a $(n$-$k)$–form. Maxwell's relations are subject to the following phenomenological relations $B = \star_{\mu_0} E$, and $D = \star_{\varepsilon} H$, where $\mu_0 \in \Lambda^0(\Omega)$ and $\varepsilon \in \Lambda^0(\Omega)$ denote the medium permittivity and permeability, respectively. Casimir functions of equations (29) and (28) are defined by the following degeneracy conditions $dB = 0$ and $dD = 0$. The electro–magnetic energy density function is the sum of the electric and the magnetic energy densities:

$$\mathcal{H}_{em} = \frac{1}{2} D \wedge E + \frac{1}{2} B \wedge H. \tag{31}$$

The time derivative of the electro–magnetic energy is

$$\frac{d\mathcal{H}_{em}}{dt} = -d\,(E \wedge H) + j \wedge E, \tag{32}$$

where the flux $(E \wedge H)$ denotes the Poynting vector and represents the conservative part of the electro–magnetic energy. The last term in the right–hand side of equation (32) is the source of electro–magnetic energy due to the current density resulting from a resistive and a source term (30).

The Maxwell equations (28) and (29) are subject to the following boundary conditions $tr(H)$ and $tr(E)$, respectively. $tr(\cdot)$ denotes the trace operator on the boundary $\partial\Omega$. Theses represent the electric field and the magnetic field at the plasma boundary $\partial\Omega$.

3.2 Heat diffusion equation

The internal energy $u \in \Lambda^3(\Omega)$ diffusion equation reads

$$\frac{\partial u}{\partial t} = -d\left(j_q\right) + E \wedge j_{\mathrm{ohm}} + \sigma_u, \tag{33}$$

where the internal energy source term $\sigma_u \in \Lambda^3(\Omega)$ is a 3–form, and the 2–form $j_q \in \Lambda^2(\Omega)$ defines the heat flux. Fourier's law is used as closure equation

$$j_q = \star_{\Gamma_{qq} T^2} d\left(\frac{1}{T}\right), \tag{34}$$

where $\Gamma_{qq} \in \Lambda^0(\Omega)$ is the heat diffusion coefficient. The plasma temperature $T \in \Lambda^0(\Omega)$ is the averaged plasma temperature. We assume the local equilibrium assumption hold. Therefore the Gibbs' fundamental relation maps the internal energy $u \in \Lambda^3(\Omega)$ to the temperature $T \in \Lambda^0(\Omega)$ and the entropy density $s \in \Lambda^3(\Omega)$ such that $\delta u = T\delta s$. The entropy balance equations reads

$$\frac{\partial s}{\partial t} = -d\left(\frac{j_q}{T}\right) + j_q \wedge d\left(\frac{1}{T}\right) + \frac{1}{T} E \wedge j + \frac{\sigma_u}{T}. \tag{35}$$

The second and third terms in the right–hand side of equation (35) refer to the irreversible entropy source terms $\sigma_s \in \Lambda^3(\Omega)$, associated to the heat diffusion losses, and the Joule effect, respectively. The internal energy balance equation (33) is subject to the following boundary condition $tr(j_q)$ that represents the heat flux flowing through the plasma boundary $\partial\Omega$.

3.3 Coupled formulation

The coupled plasma model (Vu et al., 2016) is defined by the Maxwell's equations (28) and (29), the internal energy balance equation (33), or alternatively the entropy balance equation (35). The main coupling present in this model is the Joule effect associated to the plasma resistivity η. The total energy density $e \in \Lambda^3(\Omega)$ is the sum of the electro–magnetic energy density $e_{\mathrm{em}} \in \Lambda^3(\Omega)$, and the internal energy $u \in \Lambda^3(\Omega)$, which time derivative is given by the extended Gibb's relation $\delta e = T\delta s + H \wedge \delta B + E \wedge \delta D$. The time derivative along the system trajectory is given by

$$\frac{\partial e}{\partial t} = -d\left(j_q + E \wedge H\right) + E \wedge j_{\mathrm{ext}} + \overline{\sigma}_u, \tag{36}$$

where we have used equations (32) and (33). The total energy function $\mathbb{E} : \Lambda^3(\Omega) \to \mathbb{R}$ is chosen as the integral of the total energy density e over the volume Ω, such that

$$\mathbb{E} = \frac{1}{2} \int_{\Omega} (D \wedge E + B \wedge H) + \int_{\Omega} u(s), \tag{37}$$

and the total entropy function $\mathbb{S} : \Lambda^3(\Omega) \to \mathbb{R}$ is defined as $\mathbb{S} = \int_\Omega s(u)$. The electro–magnetic part of the dynamics is governed by the magnetic and electric fields. They are chosen as state variables. For the thermal part of the system, we are free to chose either the internal energy density of the entropy density. Each choice leads to an appropriate definition of the system. The GENERIC formulation is not unique and both an energy and an entropy formulations are candidates for modelling the dynamics, as illustrated in the next Sections.

4. ENTROPY FORMULATION

The entropy density is chosen as one of the state variables, with the electric and the magnetic fields such that the state vector x is defined as $x^{(s)} = (s,\, D,\, B)^\top \in \Lambda^3(\Omega) \times \Lambda^2(\Omega) \times \Lambda^2(\Omega) = \mathcal{X}^{(s)}$, where $\mathcal{X}^{(s)}$ denotes the state space of the entropy formulation. The total energy and entropy functions are now defined by

$$\mathbb{E} = \int_\Omega e(u, D, B) \qquad (\in \mathbb{R}), \tag{38}$$

and

$$\mathbb{S} = \int_\Omega s \qquad (\in \mathbb{R}). \tag{39}$$

As the entropy density is an independent state variable, the entropy density in the total entropy function does not depend on other state variables. Functional derivatives of the potential functions (38) and (39) are now obtained as

$$\frac{\delta \mathbb{E}}{\delta x^{(s)}} = \begin{pmatrix} T \\ E \\ H \end{pmatrix} \quad \left(\in \Lambda^0(\Omega) \times \Lambda^1(\Omega) \times \Lambda^1(\Omega)\right), \tag{40}$$

$$\frac{\delta \mathbb{S}}{\delta x^{(s)}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \left(\in \Lambda^3(\Omega) \times \Lambda^0(\Omega) \times \Lambda^0(\Omega)\right). \tag{41}$$

Let us now detail the conservative and dissipative parts of the dynamics as functions of total energy and total entropy gradients.

4.1 Reversible dynamics

The reversible dynamics associated to the balance equations (28), (29) and (35) are associated to the gradient of the total energy gradient (40) and the operator L. The approach carried out to build the conservative operator L is the following. The operator has to embed the conservative parts of the dynamics. The Maxwell's equations (28) and (29) are defined with a skew–symmetric structure that can be easily identified, and we retrieve the Poisson bracket operator (Marsden et al., 2001, chap. 8). In the entropy balance equation (35) the first term $d\left(j_q/T\right)$ is usually seen as a conservative element but with the chosen Fourier's phenomenological law (34), this term is strictly dissipative. We expect the first row and the first column of the operator L to be full of zeros since the entropy contribution in the conservative part is null. One gets the following bulk L Poisson operator:

$$L^{(s)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -d(\cdot) \\ 0 & d(\cdot) & 0 \end{pmatrix}, \tag{42}$$

where all exterior derivative operators are acting on the element on their right sides. Applying the adjoint operator of $L^{(s)}$ or checking its skew–symmetry (11), one obtains after integration by part the following boundary Poisson operator:

$$L^{\partial(s)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & tr(\cdot) \\ 0 & 0 & 0 \end{pmatrix}, \tag{43}$$

where $tr(\cdot)$ denote the trace operator on the boundary $\partial\Omega$.

4.2 Irreversible dynamics

The dissipative bracket gather all resistive phenomena involved in the dynamics. One should expect to find in the operator M the plasma resistivity, heat dissipativity, gathered in the irreversible entropy production. After identification of all resistive elements in the balance equations, one gets:

$$M^{(s)} = \begin{pmatrix} M_{11}^{(s)} & -\star_{1/\eta} E & 0 \\ -\star_{1/\eta} E & T\star_{1/\eta}(\cdot) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

where $M_{11}^{(s)}$ is the following operator

$$M_{11}^{(s)} = -\frac{1}{T} d\left(j_q\right) + E \wedge j_{\mathrm{ohm}}. \tag{45}$$

The symmetry analysis of the dissipative operator $M^{(s)}$ as highlighted in (21) leads to the following boundary dissipative bracket

$$M^{\partial(s)} = \begin{pmatrix} tr(\star_{\Gamma_{qq}T^2} d(\cdot)) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{46}$$

4.3 Balance equations

The bulk state balance equation is given by

$$\frac{\partial x^{(s)}}{\partial t} = L^{(s)} \cdot \frac{\delta \mathbb{E}}{\delta x^{(s)}} + M^{(s)} \cdot \frac{\delta \mathbb{S}}{\delta x^{(s)}} + B^{(s)} u_d, \tag{47}$$

with the distributed input operator

$$B^{(s)} u_d(t) = \begin{pmatrix} 1 & \frac{1}{T} E \wedge (\cdot) \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sigma_u}{T} \\ j_{\mathrm{ext}} \end{pmatrix}. \tag{48}$$

The total energy balance equations is then

$$\frac{d\mathbb{E}}{dt} = -\int_{\partial\Omega} tr(E) \wedge tr(H) - \int_{\partial\Omega} tr\left(J_q\right) + \int_{\Omega} \frac{\delta \mathbb{E}}{\delta x^{(s)}} B^{(s)} u_d,$$

where we recover equation (36) under an integral form. The total entropy balance equation is, according to (27), given by:

$$\begin{aligned} \frac{d\mathbb{S}}{dt} = &\int_{\Omega} \left[\begin{pmatrix} d\left(\frac{1}{T}\right) & E \end{pmatrix} \begin{pmatrix} \star_{\Gamma_{qq}T^2} & 0 \\ 0 & \star_{1/\eta} \end{pmatrix} \begin{pmatrix} d\left(\frac{1}{T}\right) \\ E \end{pmatrix} \right] \\ &- \int_{\partial\Omega} tr\left(\frac{J_q}{T}\right) + \int_{\Omega} \frac{\delta \mathbb{S}^{(s)}}{\delta x^{(s)}} B^{(s)} u_d, \end{aligned} \tag{49}$$

we recover the expected integral formulation of the total entropy balance equations (35). This is indeed what we where looking for at the first stage of the analysis.

5. ENERGY FORMULATION

We now consider the energy density as a state variable $x^{(u)} = (u,\, D,\, B)^\top \in \Lambda^3(\Omega) \times \Lambda^2(\Omega) \times \Lambda^2(\Omega) = \mathcal{X}^{(u)}$ where $\mathcal{X}^{(u)}$ is the state space. The total energy and the total entropy real–valued functions are now

$$\mathbb{E} = \int_{\Omega} e(u,\, D,\, B), \qquad \mathbb{S} = \int_{\Omega} s(u), \tag{50}$$

respectively. The total energy density being an independent state variable, the total energy function is only function of the total energy density and the total entropy function is a function of the internal energy u. The functional derivatives of the total energy and entropy (50) are given by

$$\frac{\delta \mathbb{E}}{\delta x^{(u)}} = \begin{pmatrix} 1 \\ E \\ H \end{pmatrix} \in \Lambda^3(\Omega) \times \Lambda^1(\Omega) \times \Lambda^1(\Omega), \tag{51}$$

and

$$\frac{\delta \mathbb{S}}{\delta x^{(u)}} = \frac{1}{T} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \Lambda^0(\Omega) \times \Lambda^0(\Omega) \times \Lambda^0(\Omega). \tag{52}$$

5.1 Conservative part

With the energy formulation, the Poisson operator is defined by identifying the conservative part of the total energy balance equation (36) and of the Maxwell's equations (29) and (28). One obtains

$$L^{(u)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -d(\cdot) \\ 0 & d(\cdot) & 0 \end{pmatrix}. \tag{53}$$

We retrieve the same bulk Poisson bracket operator as the functional derivative (51) is similar to the one obtained for the entropy formulation (40). The boundary operator is defined as

$$L^{\partial(u)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & tr(\cdot) \\ 0 & 0 & 0 \end{pmatrix}, \tag{54}$$

and follows from the test of the symmetry property of the bulk Poisson operator. One should note that the internal energy is a degenerate functional of the conservative bracket.

5.2 Dissipative part

The dissipative bracket, for the case where the internal energy is set as a state variable, is given by

$$M^{(u)} = \begin{pmatrix} -d \star_{\Gamma_{qq} T^2} d(\cdot) & -T \star_{1/\eta} E & 0 \\ -T \star_{1/\eta} E & T \star_{1/\eta} (\cdot) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{55}$$

such that its associate boundary operator is deduced form the symmetry property of the full dissipative bracket is

$$M^{\partial(u)} = \begin{pmatrix} tr(\star_{\Gamma_{qq} T^2} d(\cdot)) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{56}$$

Since the dissipative bracket is a second order differential operator, the resulting dissipative boundary bracket is a first order one.

5.3 Balance equations

The bulk state balance equation is given by

$$\frac{\partial x^{(u)}}{\partial t} = L^{(u)} \cdot \frac{\delta \mathbb{E}}{\delta x^{(u)}} + M^{(u)} \cdot \frac{\delta \mathbb{S}}{\delta x^{(u)}} + B^{(u)} u_d, \tag{57}$$

with the distributed input operator

$$B^{(u)} u_d(t) = \begin{pmatrix} 1 & E \wedge (\cdot) \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_u \\ j_{\text{ext}} \end{pmatrix}. \tag{58}$$

Hence, using an energy formulation, we recover the expected total energy (36), and entropy balance equations (35).

Remark 2. Other state variables could be considered instead of the entropy or the internal energy. For example, in the work of (Jelić et al., 2006), the total energy is considered to model dissipative electro–magnetism within the GENERIC formulation. The resulting model representation is similar to ours. The major difference is that the total energy e being function of the electric and magnetic fields, the functional derivative of the total entropy function has no non–zero elements and the resulting Poisson bracket is more complex than the one computed in our energy representation. Nevertheless, the dissipative operator can embedded more complex dissipative phenomena.

6. CONCLUSION

We have presented the open GENERIC formulation (Öttinger, 2006) enabling the formulation of boundary and distributed open multi–physics systems. The representation approach was illustrated through two equivalent formulations of a simplified Tokamak model. Future areas for investigation include stability and passivity analysis of the Tokamak model using the GENERIC formulation following (Vincent et al., 2018) and derivations of structure–preserving discretization schemes with application to the Tokamak GENERIC model following (Cardoso-Ribeiro et al., 2017).

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