Global quadratic optimization on the sets with simplex structure

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Abstract

In the first part of this paper we prove that the global quadratic optimization problem over a simplex can be solved with a constant relative accuracy. In the second part we consider some natural extensions of the result.

1 Introduction

Starting from the seminal paper by Goemans and Williamson [1], we can see an increasing interest to the global quadratic optimization problems over some simple feasible sets. As it was shown in [2], the random hyperplane technique, proposed in [1] for Max-Cut problem, can be applied also to get a constant relative accuracy bound for the problem of maximizing a general quadratic form over a Boolean box. Later on, in the papers [4, 5, 3] it was shown that the feasible set of the problems can be quite complicated (in [3] it can be an arbitrary set, which is convex in squared variables), but we still can find an approximation to the exact solution with a constant relative accuracy. The main technique used in all these papers was *semidefinite relaxation.* In [3] it was also mentioned that the quadratic problems with another feasible set (simplex, for example), might be much more difficult. In this paper we partially confirm and partially reject this hypothesis. We still do not know how to construct a semidefinite relaxation for simplex, which give the constant relative accuracy. However, we show that for this problem we can get such an accuracy using a kind of inner polyhedral approximation.

In this paper we deal with the following definition of relative accuracy. Consider a pair optimization problems

$$\begin{aligned} \text{find } f^* &= \max_x \{ f(x) : \ x \in \mathcal{F} \}, \\ \text{find } f_* &= \min_x \{ f(x) : \ x \in \mathcal{F} \}, \end{aligned}$$

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where $f(x)$ is an arbitrary function and $\mathcal{F}$ is an arbitrary set in R^n. We say that the value ψ approximates f^* with some relative accuracy $\mu \geq 0$ if

$$|f^* - \psi| \leq \mu(f^* - f_*). \tag{1}$$

If there exists $x \in \mathcal{F}$ such that $\psi = f(x)$, we cal ψ the *implementable* approximation. Clearly, in the latter case we always have $\mu \leq 1$, and the relation (1) can be rewritten in the following form:

$$(1-\mu)f^* + \mu f_* \leq \psi.$$

Note that for the problems with complicated feasible set, it can be very difficult even to find an approximation with any finite μ. So, the above concept is useful mainly for the problems with simple $\mathcal{F}$.

All our results on the relative accuracy can be derived from some bounds established for f^* and f_*. For the reader convenience we present here a simple statement on this relation.

Lemma 1 *Let for some estimates ψ^*, ψ_* and some positive factor β we have the following relations:*

$$\psi_* - \beta(\psi^* - \psi_*) \leq f_* \leq \psi_* \leq \psi^* \leq f^* \leq \psi^* + \beta(\psi^* - \psi_*).$$

Then for $\mu = \frac{\beta}{1+\beta}$ we have

$$\begin{aligned} 0 \ \leq \ & f^* - \psi^* \ \leq \ \mu(f^* - f_*), \\ 0 \ \leq \ & \psi_* - f_* \ \leq \ \mu(f^* - f_*). \end{aligned}$$

(Indeed, $(1-\mu)f^* + \mu f_* \leq (1-\mu)(\psi^* + \tau(\psi^* - \psi_*)) + \mu\psi_*) = \psi^*$, etc.)

The paper is organized as follows. In Section 2, using a straightforward reasoning, we show how to get constant relative accuracy bounds for the problem of maximizing an arbitrary quadratic form on the standard simplex in R^n. In the next Section 3 we give a matrix interpretation of this result. In Section 4, for general convex sets we introduce a concept of inner polyhedral approximation. This object provides us with an approximation of the support function for the set of a constant relative accuracy. We show that for the inner approximations there exists a simple calculus. As an example of applications, we consider a problem of maximizing a quadratic form over a boundary of l_1-box. In the last Section 5 we show that the above results can be extended onto the problems with general quadratic objective function and the feasible set formed as a convex hull of a finite number of points.

2 Bounds for the simplex

Consider the problem:

$$\begin{aligned} \text{find} \quad f^* \ &= \max_x \{ f(x) = \langle Qx, x\rangle : \ \langle e, x\rangle = 1, \ x \geq 0 \in R^n \}, \\ \text{find} \quad f_* \ &= \min_x \{ f(x) = \langle Qx, x\rangle : \ \langle e, x\rangle = 1, \ x \geq 0 \in R^n \}, \end{aligned} \tag{2}$$

where Q is an indefinite matrix with the elements $Q^{(i,j)}$. In what follows we denote $e \in R^n$ the vector of all ones and e_i the ith coordinate vector in R^n.

Our goal is to estimate the optimal values f_* and f^* of the problem (2) with a constant relative accuracy. Define

$$q^* = \max_{i,j} \; Q^{(i,j)}, \quad q_* = \min_{i,j} \; Q^{(i,j)},$$

$$d^* = \max_{i} \; Q^{(i,i)}, \quad d_* = \min_{i} \; Q^{(i,i)}.$$

Since $f(e_i) = Q^{(i,i)}$ we have

$$f_* \le d_* \le d^* \le f^*. \tag{3}$$

Denote

$$\tau^* = \max\left\{d^*, \, \tfrac{d_*+q^*}{2}\right\}, \quad \tau_* = \min\left\{d_*, \, \tfrac{d^*+q_*}{2}\right\}.$$

The main result of this section is as follows.

Theorem 1 *We have*

$$\tau_* - 2(\tau^* - \tau_*) \le q_* \le f_* \le \tau_* \le \tau^* \le f^* \le q^* \le \tau^* + 2(\tau^* - \tau_*). \tag{4}$$

The relative accuracy of the approximations τ^* *and* τ_* *is at least* $\frac{2}{3}$.

In order to prove this result we need several auxiliary statements.

Lemma 2

$$q_* \le f_* \le f^* \le q^*.$$

Proof:
Indeed, for any $x \ge 0$, $\langle e, x\rangle = 1$ we have

$$f(x) = \sum_{i,j} x^{(i)} x^{(j)} Q^{(i,j)} \le \sum_{i,j} x^{(i)} x^{(j)} q^* = q^*.$$

Similarly, we get $f(x) \ge q_*$. $\square$

Lemma 3 *Assume that the matrix* Q *has non-negative diagonal elements. Then*

$$f^* \ge \tfrac{1}{2} q^* \ge 0. \tag{5}$$

Proof:
If we have $d^* = q^*$ then $f^* = q^*$ in view of Lemma 2 and inequality (3). Hence in this case inequality (5) holds.

Assume that $d^* < q^*$. Then for any i we have

$$0 \le Q^{(i,i)} < q^*.$$

Let us fix a pair of indices (i, j) such that

$$q^* = Q^{(i,j)}.$$

Clearly we have $i \neq j$. Denote

$$\alpha = q^* - Q^{(i,i)} > 0, \quad \beta = q^* - Q^{(j,j)} > 0.$$

Consider the point

$$\bar{x} = \gamma e_i + (1-\gamma) e_j$$

with $\gamma = \frac{\beta}{\alpha+\beta} \in (0,1)$. Then

$$\begin{aligned} f^* \geq f(\bar{x}) &= \gamma^2 Q^{(i,i)} + (1-\gamma)^2 Q^{(j,j)} + 2\gamma(1-\gamma) q^* \\ &= \tfrac{\beta^2 Q^{(i,i)} + \alpha^2 Q^{(j,j)} + 2\alpha\beta q^*}{(\alpha+\beta)^2} \\ &= q^* + \tfrac{\beta^2 (Q^{(i,i)} - q^*) + \alpha^2 (Q^{(j,j)} - q^*)}{(\alpha+\beta)^2} \\ &= q^* - \tfrac{\alpha\beta^2 + \alpha^2\beta}{(\alpha+\beta)^2} = q^* - \tfrac{\alpha\beta}{\alpha+\beta}. \end{aligned}$$

Recall that the diagonal entries of Q are non-negative. Therefore

$$\tfrac{1}{\alpha} + \tfrac{1}{\beta} = \tfrac{1}{q^* - Q_{ii}} + \tfrac{1}{q^* - Q_{jj}} \geq \tfrac{2}{q^*}.$$

Hence, $\frac{\alpha\beta}{\alpha+\beta} \leq \frac{1}{2} q^*$. □

Lemma 4 *For an arbitrary matrix Q we have*

$$f^* \geq \tfrac{1}{2}(d_* + q^*).$$

Proof:
Let τ be an arbitrary real value. Consider the matrix

$$Q(\tau) = Q + \tau e e^T.$$

For such a matrix all characteristics of the problem (2),

$$f_*,\ f^*,\ q_*,\ q^*,\ d_*,\ d^*,$$

become functions of τ. Moreover, it is clear that

$$f^*(\tau) = f^* + \tau, \quad q^*(\tau) = q^* + \tau, \quad d^*(\tau) = d^* + \tau,$$

$$f_*(\tau) = f_* + \tau, \quad q_*(\tau) = q_* + \tau, \quad d_*(\tau) = d_* + \tau.$$

Let us choose $\tau = -d_*$. Then all diagonal entries of the matrix $Q(\tau)$ become non-negative. Therefore, using Lemma 3 we have:

$$f^*(\tau) \geq \tfrac{1}{2} q^*(\tau) = \tfrac{1}{2} q^* - \tfrac{1}{2} d_*.$$

On the other hand, $f^*(\tau) = f^* + \tau = f^* - d_*$. □

We are ready now to prove Theorem 1.

Proof of Theorem 1. Indeed, in view of Lemma 2, Lemma 4 and inequality (3) we have

$$d^* \leq f^* \leq q^*, \quad f^* \geq \tfrac{1}{2}(d_* + q^*).$$

Therefore

$$q^* \geq f^* \geq \max\{d^*, \tfrac{1}{2}(d_* + q^*)\} = \tau^*.$$

Applying the same reasoning to the matrix $-Q$ we get

$$-d_* \leq -f_* \leq -q_*, \quad -f_* \geq \tfrac{1}{2}(-d^* - q_*).$$

Thus,

$$q_* \leq f_* \leq \min\{d_*, \tfrac{1}{2}(d^* + q_*)\} = \tau_*.$$

Hence, the middle inequalities in (4) are proved. Let us estimate the gap $\tau^* - \tau_*$. Indeed,

$$\begin{aligned}
\tau^* - \tau_* &= \max\{d^*, \tfrac{1}{2}(d_* + q^*)\} - \min\{d_*, \tfrac{1}{2}(d^* + q_*)\}, \\
&= \tfrac{1}{2}\left[d^* + \tfrac{1}{2}(d_* + q^*) + |d^* - \tfrac{1}{2}(d_* + q^*)|\right] \\
&\quad -\tfrac{1}{2}\left[d_* + \tfrac{1}{2}(d^* + q_*) - |d_* - \tfrac{1}{2}(d^* + q_*)|\right] \\
&= \tfrac{1}{4}(q^* - q_*) + \tfrac{1}{4}(d^* - d_*) + \tfrac{1}{2}|d^* - \tfrac{1}{2}(d_* + q^*)| + \tfrac{1}{2}|d_* - \tfrac{1}{2}(d^* + q_*)| \\
&\geq \tfrac{1}{4}(q^* - q_*) + \tfrac{1}{4}(d^* - d_*) + \tfrac{1}{2}|\tfrac{3}{2}(d^* - d_*) - \tfrac{1}{2}(q^* - q_*)| \\
&\geq (\tfrac{1}{4} + \tfrac{1}{4} \cdot \tfrac{1}{3})(q^* - q_*) = \tfrac{1}{3}(q^* - q_*).
\end{aligned}$$

Thus,

$$q^* \leq q_* + 3(\tau^* - \tau_*) \leq \tau_* + 3(\tau^* - \tau_*) = \tau^* + 2(\tau^* - \tau_*).$$

It remains to use Lemma 1. $\square$

3 Matrix interpretation

Let us present a geometric interpretation for the above results. Denote S^n the space of symmetric $n \times n$-matrices. For X and Y from S^n we can introduce the standard inner product

$$\langle X, Y \rangle_M = \sum_{i,j} X^{(i,j)} Y^{(i,j)}.$$

Then we can represent our objective function in the following form:

$$f(x) = \langle Qx, x \rangle = \langle Q, xx^T \rangle_M.$$

Thus, the problem (2) is equivalent to the following convex problem in the space of matrices:

$$\begin{aligned} f^* = & \max_{X \in S^n} \{ \langle Q, X \rangle_M : \ X \in \mathcal{P}_S \}, \\ f_* = & \min_{X \in S^n} \{ \langle Q, X \rangle_M : \ X \in \mathcal{P}_S \}, \end{aligned} \tag{6}$$

where $\mathcal{P}_S = \operatorname{Conv}\{ xx^T : \ \langle e, x \rangle = 1, \ x \geq 0 \}$. Hence, in order to approximate the values f_* and f^* we need to approximate the convex set $\mathcal{P}_S$. Let us show how we can do that.

Denote

$$e_{i,j} = \tfrac{1}{2}(e_i + e_j), \quad i, j = 1, \ldots, n.$$

Clearly, $e_{i,j} = e_{j,i}$, but we keep two variants of the notation for convenience reason. Denote

$$\begin{aligned} \psi^* \ &= \max_{i,j} \ f(e_{i,j}), \qquad \psi_* \ = \min_{i,j} \ f(e_{i,j}), \\ d^* \ &= \max_{i} \ f(e_{i,i}), \qquad d_* \ = \min_{i} \ f(e_{i,i}). \end{aligned}$$

Denote $E_{i,j} = e_{i,j} e_{i,j}^T \in S^n$. We need the following auxiliary result.

Lemma 5 *For any $x \in R^n$ we have*

$$\langle e, x \rangle \cdot D(x) + xx^T = 2 \sum_{i,j} x^{(i)} x^{(j)} E_{i,j},$$

where $D(x) \in S^n$ is a diagonal matrix with the diagonal elements $x^{(i)}$.

Proof:
Denote the matrix in the left-hand side of this equality by A and the matrix in the right-hand side by B. For any $i \neq j$ we have

$$\begin{aligned} A^{(i,j)} = & \ \ x^{(i)} x^{(j)}, \\ B^{(i,j)} = & \ \ 2(x^{(i)} x^{(j)} E_{i,j}^{(i,j)} + x^{(j)} x^{(i)} E_{j,i}^{(i,j)}) = x^{(i)} x^{(j)}. \end{aligned}$$

Since the off-diagonal elements of the matrices A and B coincide, it remains to show that $Ae = Be$. Indeed,

$$\begin{aligned} Ae \ &= \langle e, x \rangle x + x \langle x, e \rangle = 2 \langle e, x \rangle x, \\ Be \ &= 2 \sum_{i,j} x^{(i)} x^{(j)} E_{i,j} e = 2 \sum_{i,j} x^{(i)} x^{(j)} e_{i,j} = \sum_{i,j} x^{(i)} x^{(j)} (e_i + e_j) = 2 \langle e, x \rangle x. \quad \Box \end{aligned}$$

Using this lemma, we can show that the polyhedron

$$\hat{\mathcal{P}}_S = \operatorname{Conv}\{ E_{i,j}, \ i = 1, \ldots n, \ j = i, \ldots, n \},$$

(which is a simplex in S^n), provides us with a good inner approximation of the set $\mathcal{P}_S$.

Theorem 2 *For an arbitrary matrix Q we have*

$$\begin{aligned} 2\psi_* - \psi^* \quad & \leq 2\psi_* - d^* \leq f_* \leq \psi_* \leq d_* \\ & \leq d^* \leq \psi^* \leq f^* \leq 2\psi^* - d_* \leq 2\psi^* - \psi_*. \end{aligned} \tag{7}$$

The relative accuracy of the approximations ψ^* and ψ_* is at least $\frac{1}{2}$.

Proof:
Clearly, $\psi^* \leq f^*$. On the other hand, let us fix an arbitrary $x \geq 0$ with $\langle e, x\rangle = 1$. Denote $\alpha_{i,j} = x^{(i)}x^{(j)} \geq 0$. It is clear that

$$\sum_{i,j} \alpha_{i,j} = 1.$$

Therefore, in view of Lemma 5, we have

$$\begin{aligned} f(x) \quad & = \langle Q, xx^T\rangle_M = \langle Q, 2\sum_{i,j} x^{(i)}x^{(j)}E_{i,j} - D(x)\rangle_M \\ & = \sum_{i,j} \alpha_{i,j} f(e_{i,j}) - \sum_i Q^{(i,i)}x^{(i)} \leq 2\psi^* - d_*. \end{aligned}$$

The rest inequalities are straightforward. It remains to apply Lemma 1. □

Note that the bound, which we have now for f^*,

$$\psi^* \leq f^* \leq 2\psi^* - d_*, \tag{8}$$

is better than that of Theorem 1. However, in particular problems the point $\bar{x}$ used in the proof of Lemma 3, can provide us with a sharper estimate.

Inequality (8) has a simple geometric interpretation. Let us define the linear operations for the sets in a usual way:

$$z \in \alpha A + \beta B \quad \Leftrightarrow \quad z = \alpha x + \beta y,\ x \in A,\ y \in B.$$

Denote

$$\mathcal{D} = \mathrm{Conv}\,\{e_i e_i^T,\ i = 1, \ldots, n\} \subset \hat{\mathcal{P}}_S.$$

Corollary 1

$$\hat{\mathcal{P}}_S \subset \mathcal{P}_S \subset \hat{\mathcal{P}}_S^+ \equiv 2\hat{\mathcal{P}}_S - \mathcal{D} \subset 2\hat{\mathcal{P}}_S - \hat{\mathcal{P}}_S. \tag{9}$$

Note that the set $\hat{\mathcal{P}}_S^+$ is a polyhedron in S^n with $n^2(n+1)/2$ vertices.

4 Inner polyhedral approximations

The form of the inclusion (9) suggests the following definition.

Definition 1 *Let $\mathcal{P}$ be a closed convex bounded set in R^n. The pair of sets $\mathcal{A} = \{A^+, A^-\}$ is called an* inner approximation *of $\mathcal{P}$ if*

$$A^+ \subseteq \mathcal{P}, \quad A^- \subseteq \mathcal{P}$$

and there exists a constant $\mu = \mu(\mathcal{A}) \geq 0$ such that

$$\mathcal{P} \subseteq \mathrm{Conv}\,((1+\mu)A^+ - \mu A^-).$$

The constant μ is called the relative error *of the approximation $\mathcal{A}$.*

From (9) we can see that A^+ and A^- are not necessarily different.

An inner approximation $\mathcal{A}$ generates the following *estimate functions*:

$$\begin{aligned} \psi^+_{\mathcal{A}}(c) &= \max_{x \in A^+ \bigcup A^-} \langle c, x \rangle, \\ \psi^-_{\mathcal{A}}(c) &= \min_{x \in A^+ \bigcup A^-} \langle c, x \rangle. \end{aligned}$$

Clearly, $\psi^+_{\mathcal{A}}(c) \geq \psi^-_{\mathcal{A}}(c)$ for any $c \in R^n$.

The reason for Definition 1 can be seen from the following result.

Lemma 6 *Let $\mathcal{A}$ be an inner approximation of $\mathcal{P}$ with the relative error μ. Then for any objective vector $c \in R^n$ we have the following relations:*

$$\begin{aligned} \psi^-_{\mathcal{A}}(c) - \mu(\psi^+_{\mathcal{A}}(c) - \psi^-_{\mathcal{A}}(c)) \quad &\leq \min_{x \in \mathcal{P}} \langle c, x \rangle \ \leq \psi^-_{\mathcal{A}}(c) \leq \psi^+_{\mathcal{A}}(c) \\ &\leq \max_{x \in \mathcal{P}} \langle c, x \rangle \ \leq \psi^+_{\mathcal{A}}(c) + \mu(\psi^+_{\mathcal{A}}(c) - \psi^-_{\mathcal{A}}(c)). \end{aligned}$$

Proof:
Indeed, since $\mathcal{P} \subseteq (1+\mu)A^+ - \mu A^-$, we have

$$\begin{aligned} \max_x \{\langle c, x \rangle : \ x \in \mathcal{P}\} \quad &\leq \max_x \{\langle c, x \rangle : \ x \in (1+\mu)A^+ - \mu A^-\} \\ &= (1+\mu) \max_x \{\langle c, x \rangle : \ x \in A^+\} + \mu \max_x \{-\langle c, x \rangle : \ x \in A^-\} \\ &\leq (1+\mu)\psi^+_{\mathcal{A}}(c) - \mu \min_x \{\langle c, x \rangle : \ x \in A^-\} \\ &\leq (1+\mu)\psi^+_{\mathcal{A}}(c) - \mu\psi^-_{\mathcal{A}}(c). \end{aligned}$$

The rest inequalities are straightforward or can be proved in a similar way. □

Let us show how the quality of the inner approximation changes under some standard operations with convex sets. However, in order to do that, we need to introduce one more concept.

Definition 2 *An inner approximation $\mathcal{A}$ is called* regular *if* $\mathrm{Conv}\,(A^+) \bigcap \mathrm{Conv}\,(A^-) \neq \emptyset$.

The necessity of this concept can be seen from the following trivial fact.

Lemma 7 *Let A and B be closed and bounded in R^n. Assume that*

$$\mathrm{Conv}\,(A)\bigcap \mathrm{Conv}\,(B) \neq \emptyset.$$

Then the sets

$$S(\mu) = \mathrm{Conv}\ ((1+\mu)A - \mu B)\,, \quad \mu \geq 0,$$

are monotone *in μ in the following sense: $S(\mu_1) \subseteq S(\mu_2)$ for any $\mu_2 \geq \mu_1 \geq 0$.*

Proof:
Let us fix an arbitrary $c \in R^n$. Since $\mathrm{Conv}\,(A)\bigcap \mathrm{Conv}\,(B) \neq \emptyset$, we have

$$\max_x\{\langle c, x\rangle :\ x \in A\} \geq \min_x\{\langle c, x\rangle :\ x \in B\}.$$

Therefore

$$\max_x\{\langle c, x\rangle :\ x \in S(\mu_1)\} = \max_x\{\langle c, x\rangle :\ x \in (1+\mu_1)A - \mu_1 B\}$$

$$= (1+\mu_1)\max_x\{\langle c, x\rangle :\ x \in A\} + \mu_1 \max_x\{\langle c, x\rangle :\ x \in -B\}$$

$$= \max_x\{\langle c, x\rangle :\ x \in A\} + \mu_1\left(\max_x\{\langle c, x\rangle :\ x \in A\} - \min_x\{\langle c, x\rangle :\ x \in B\}\right)$$

$$\leq \max_x\{\langle c, x\rangle :\ x \in A\} + \mu_2\left(\max_x\{\langle c, x\rangle :\ x \in A\} - \min_x\{\langle c, x\rangle :\ x \in B\}\right)$$

$$= \max_x\{\langle c, x\rangle :\ x \in S(\mu_2)\}.$$

□

For the reader convenience let us present in an explicit form some useful facts on the convex hulls. The proof of that statement can be found in Appendix.

Lemma 8 *1. For arbitrary closed bounded sets A and B from R^n we have*

$$\mathrm{Conv}\,(A+B) = \mathrm{Conv}\,(A) + \mathrm{Conv}\,(B).$$

2. For arbitrary closed bounded sets A_1, B_1 and A_2, B_2 from R^n we have

$$\mathrm{Conv}\,(A_1+B_1, A_2+B_2) \subseteq \mathrm{Conv}\,(A_1\bigcup A_2) + \mathrm{Conv}\,(B_1\bigcup B_2).$$

3. For arbitrary closed bounded sets A_1, B_1 from R^{n_1} and A_2, B_2 from R^{n_2} we have

$$\mathrm{Conv}\,((A_1+B_1)\times(A_2+B_2)) = \mathrm{Conv}\,(A_1\times A_2) + \mathrm{Conv}\,(B_1\times B_2).$$

Now we can speak about the operations with the convex sets.

Theorem 3

1. Projection. *Let $\mathcal{A}$ be an inner approximation of the set $\mathcal{P}$ and $B(x)$ be a linear operator from R^n to R^m. Then $\mathcal{A}_+ = \{B(A^+), B(A^-)\}$ is an inner approximation for the set $\mathcal{P}_+ = \{y = B(x) \in R^m : \; x \in \mathcal{P}\}$ with the relative error $\mu_+ = \mu(\mathcal{A})$. If $\mathcal{A}$ is regular, then $\mathcal{A}_+$ is also regular.*
2. Convex hull. *Let $\mathcal{A}_i$ be regular inner approximations of the sets $\mathcal{P}_i$, $i = 1, 2$. Then*
$$\mathcal{A} = \Big\{A^+ = A_1^+ \bigcup A_2^+, A^- = A_1^- \bigcup A_2^-\Big\}$$
is a regular inner approximation for the set $\mathcal{P} = \mathrm{Conv}\,(\mathcal{P}_1, \mathcal{P}_2)$.
3. Summation. *Let $\mathcal{A}_i$ be regular inner approximations of the sets $\mathcal{P}_i$, $i = 1, 2$. Then*
$$\mathcal{A} = \Big\{A^+ = A_1^+ + A_2^+, A^- = A_1^- + A_2^-\Big\}$$
is a regular inner approximation for the set $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$.
4. Direct product. *Let $\mathcal{A}_i$, $i = 1, 2$, be regular inner approximations of the sets $\mathcal{P}_1 \subset R^{n_1}$ and $\mathcal{P}_2 \subset R^{n_2}$. Then*
$$\mathcal{A} = \Big\{A^+ = A_1^+ \times A_2^+, A^- = A_1^- \times A_2^-\Big\}$$
is a regular inner approximation for the set $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$.

In Items 2 – 4 the relative error of the inner approximation $\mathcal{A}$ is
$$\mu = \max\{\mu(\mathcal{A}_1), \mu(\mathcal{A}_2)\}.$$

Proof:
The proof of Item 1 is straightforward. Let us prove Items 2 – 4. It is clear that in all cases we have $A^+ \subseteq \mathcal{P}$ and $A^- \subseteq \mathcal{P}$. Let $\bar{x}_i \in \mathrm{Conv}\,(A_i^+) \bigcap \mathrm{Conv}\,(A_i^-)$, $i = 1, 2$. Then, in Item 2 we have
$$\begin{aligned} \bar{x}_1 \; &\in \mathrm{Conv}\,\Big(\mathrm{Conv}\,(A_1^+), \mathrm{Conv}\,(A_2^+)\Big) = \mathrm{Conv}\,(A_1^+ \bigcup A_2^+), \\ \bar{x}_1 \; &\in \mathrm{Conv}\,\Big(\mathrm{Conv}\,(A_1^-), \mathrm{Conv}\,(A_2^-)\Big) = \mathrm{Conv}\,(A_1^- \bigcup A_2^-). \end{aligned}$$
In Item 3
$$\begin{aligned} \bar{x}_1 + \bar{x}_2 \; &\in \mathrm{Conv}\,(A_1^+) + \mathrm{Conv}\,(A_2^+) = \mathrm{Conv}\,(A_1^+ + A_2^+), \\ \bar{x}_1 + \bar{x}_2 \; &\in \mathrm{Conv}\,(A_1^-) + \mathrm{Conv}\,(A_2^-) = \mathrm{Conv}\,(A_1^- + A_2^-). \end{aligned}$$
Finally, in Item 4 we have
$$\begin{aligned} (\bar{x}_1, \bar{x}_2) \; &\in \mathrm{Conv}\,(A_1^+) \times \mathrm{Conv}\,(A_2^+) = \mathrm{Conv}\,(A_1^+ \times A_2^+), \\ (\bar{x}_1, \bar{x}_2) \; &\in \mathrm{Conv}\,(A_1^-) \times \mathrm{Conv}\,(A_2^-) = \mathrm{Conv}\,(A_1^- \times A_2^-). \end{aligned}$$
Thus, in all cases $\mathrm{Conv}\,(A^+) \bigcap \mathrm{Conv}\,(A^-) \neq \emptyset$. Moreover, since in all cases $\mathcal{A}_i$ are regular, we can consider them as inner approximations for $\mathcal{P}_i$ with the relative accuracy
$$\mu = \max\{\mu(\mathcal{A}_1), \mu(\mathcal{A}_2)\}.$$

Further, in Item 2 in view of Lemma 8 we get the following:

$$\begin{aligned}
\mathcal{P} &= \mathrm{Conv}\,(\mathcal{P}_1, \mathcal{P}_2) \subseteq \mathrm{Conv}\,\Big((1+\mu)A_1^+ + \mu(-A_1^-), (1+\mu)A_2^+ + \mu(-A_2^-)\Big) \\
&\subseteq \mathrm{Conv}\,\Big((1+\mu)(A_1^+ \bigcup A_2^+)\Big) + \mathrm{Conv}\,\Big((-\mu A_1^-)\bigcup(-\mu A_2^-)\Big) \\
&= \mathrm{Conv}\,\Big((1+\mu)(A_1^+ \bigcup A_2^+) - \mu(A_1^- \bigcup A_2^-)\Big).
\end{aligned}$$

In Item 3 we have

$$\begin{aligned}
\mathcal{P} &= \mathcal{P}_1 + \mathcal{P}_2 \subseteq \mathrm{Conv}\,\Big((1+\mu)A_1^+ + \mu(-A_1^-)\Big) + \mathrm{Conv}\,\Big((1+\mu)A_2^+ + \mu(-A_2^-)\Big) \\
&= \mathrm{Conv}\,\Big((1+\mu)A_1^+ + \mu(-A_1^-) + (1+\mu)A_2^+ + \mu(-A_2^-)\Big) \\
&= \mathrm{Conv}\,\Big((1+\mu)(A_1^+ + A_2^+) - \mu(A_1^- + A_2^-)\Big).
\end{aligned}$$

In Item 4, using Lemma 8, we have

$$\begin{aligned}
\mathcal{P} &= \mathcal{P}_1 \times \mathcal{P}_2 \subseteq \mathrm{Conv}\,\Big((1+\mu)A_1^+ + \mu(-A_1^-)\Big) \times \mathrm{Conv}\,\Big((1+\mu)A_2^+ + \mu(-A_2^-)\Big) \\
&= \mathrm{Conv}\,\Big(((1+\mu)A_1^+) \times ((1+\mu)A_2^+)\Big) + \mathrm{Conv}\,\Big((-\mu A_1^-) \times (-\mu A_2^-)\Big) \\
&= \mathrm{Conv}\,\Big((1+\mu)(A_1^+ \times A_2^+) - \mu(A_1^- \times A_2^-)\Big).
\end{aligned}$$

□

Let us demonstrate how the above concepts work on some example. Consider the following problem:

$$\begin{aligned}
\text{find } F^* &= \max_{x\in R^n}\{\langle Qx, x\rangle : \sum_{i=1}^n |x^{(i)}| = 1\}, \\
\text{find } F_* &= \min_{x\in R^n}\{\langle Qx, x\rangle : \sum_{i=1}^n |x^{(i)}| = 1\}.
\end{aligned} \tag{10}$$

In the space of matrices S^n this problem becomes as follows:

$$\begin{aligned}
\text{find } F^* &= \max_{X\in S^n}\{\langle Q, X\rangle_M : X \in \mathcal{P}\}, \\
\text{find } F_* &= \min_{X\in S^n}\{\langle Q, X\rangle_M : X \in \mathcal{P}\},
\end{aligned} \tag{11}$$

where $\mathcal{P} = \mathrm{Conv}\,\left(xx^T : \sum_{i=1}^n |x^{(i)}| = 1\right)$. Denote by σ_k, $k = 1, \dots, 2^n$ all vectors with the components ± 1 in R^n and let $\sigma_1 = e$. Then the feasible set of the problem (11) can be represented as follows:

$$\mathcal{P} = \mathrm{Conv}\,(\mathcal{P}_k,\ k = 1, \dots, 2^n),$$

where $\mathcal{P}_k = \mathrm{Conv}\left(xx^T : \langle \sigma_k, x\rangle = 1,\ D(\sigma_k)x \geq 0\right)$. In Section 3 we have shown that the set $\mathcal{P}_1$ admits an inner approximation $\mathcal{A}_1 = (A_1^+, A_1^-)$ (see (9)) with

$$\begin{aligned} A_1^+ &= \{E_{i,j},\ i = 1, \ldots, n,\ j = i, \ldots, n\}, \\ A_1^- &= \{e_i e_i^T,\ i = 1, \ldots, n\}, \end{aligned}$$

and $\mu(\mathcal{A}_1) = 1$. Since $A_1^- \subset A_1^+$, this approximation is regular. From the symmetry reasons it is clear that all other $\mathcal{P}_k$ also admit regular inner approximations $\mathcal{A}_k$ defined as follows:

$$A_k^+ = \{E_{i,j}^k,\ i = 1, \ldots, n, j = i, \ldots, n\}, \quad A_k^- = A_1^-,$$

where $E_{i,j}^k = e_{i,j}^k (e_{i,j}^k)^T$ and

$$e_{i,j}^k = \tfrac{1}{2}\left(\sigma_k^{(i)} e_i + \sigma_k^{(j)} e_j\right).$$

Note that $\mu(\mathcal{A}_k) = 1$. Therefore, using the statement of Item 2 in Theorem 3, we can get the following inner approximation $\mathcal{A} = (A^+, A^-)$ for the set $\mathcal{P}$:

$$A^+ = \bigcup_{k=1}^{2^n} A_k^+, \quad A^- = A_1^-, \quad \mu(\mathcal{A}) = 1.$$

However, note that the number of different elements in the set A^+ is quite small. This set is comprised by all possible symmetric matrices of the form

$$\tfrac{1}{4}(e_i \pm e_j)(e_i \pm e_j)^T.$$

Thus, we have proved the following theorem.

Theorem 4 *Let $f(x) = \langle Qx, x\rangle$. Denote*

$$\psi^* = \max_{i,j}\ f(\tfrac{1}{2}(e_i \pm e_j)), \quad \psi_* = \min_{i,j}\ f(\tfrac{1}{2}(e_i \pm e_j)).$$

then

$$2\psi_* - \psi^* \leq F_* \leq \psi_* \leq \psi^* \leq F^* \leq 2\psi^* - \psi_*.$$

In the next section we show that this result can be also obtained from some general statement (see Theorem 6).

5 Some extensions

Let us consider now some natural extension of the above results. First of all, let us look at the following problem:

$$\begin{aligned} \text{find } f^* &= \max_x \{f(x) = \langle Qx, x\rangle + 2\langle b, x\rangle :\ \langle e, x\rangle = 1,\ x \geq 0 \in R^n\}, \\ \text{find } f_* &= \min_x \{f(x) = \langle Qx, x\rangle + 2\langle b, x\rangle :\ \langle e, x\rangle = 1,\ x \geq 0 \in R^n\}. \end{aligned} \tag{12}$$

Theorem 5 *Define* $\psi^* = \max_{i,j} f(e_{i,j})$ *and* $\psi_* = \min_{i,j} f(e_{i,j})$. *Then*

$$2\psi_* - \psi^* \leq f_* \leq \psi_* \leq \psi^* \leq f^* \leq 2\psi^* - \psi_*.$$

The relative accuracy of the values ψ^*, ψ_* *is at least* $\frac{1}{2}$.

Proof:
Indeed, define $\hat{Q} = Q + be^T + eb^T$. Then, for any x feasible for (12) we have

$$f(x) = \langle Qx, x\rangle + 2\langle b, x\rangle = \langle Qx, x\rangle + 2\langle b, x\rangle \cdot \langle e, x\rangle = \langle \hat{Q}x, x\rangle.$$

Thus, the statement of the theorem follows from Theorem 2.

To conclude, consider the following general problem:

$$\begin{aligned} \text{find } f^* &= \max_x \{f(x) = \langle Qx, x\rangle + 2\langle b, x\rangle :\ x \in \mathcal{F}\}, \\ \text{find } f_* &= \min_x \{f(x) = \langle Qx, x\rangle + 2\langle b, x\rangle :\ x \in \mathcal{F}\}, \end{aligned} \tag{13}$$

where $\mathcal{F}$ is a convex hull of a finite number of points:

$$\mathcal{F} = \text{Conv}\,\{x_i,\ i = 1, \ldots, m\}.$$

Theorem 6 *Consider the points* $x_{i,j} = \frac{1}{2}(x_i + x_j)$, $i, j = 1, \ldots, m$. *Denote*

$$\psi^* = \max_{i,j}\ f(x_{i,j}), \quad \psi_* = \min_{i,j}\ f(x_{i,j}).$$

Then

$$2\psi_* - \psi^* \leq f_* \leq \psi_* \leq \psi^* \leq f^* \leq 2\psi^* - \psi_*,$$

and the relative accuracy of the values ψ^*, ψ_* *is at least* $\frac{1}{2}$.

Proof:
Indeed, consider the $(n \times m)$ matrix $X = (x_1, \ldots, x_m)$. Then the feasible set and the objective function of the problem (13) can be represented as follows:

$$\begin{aligned} \mathcal{F} &= \left\{ x = X\lambda :\ \lambda \geq 0 \in R^m,\ \sum_{j=1}^m \lambda^{(j)} = 1 \right\}, \\ f(x) &= \langle Qx, x\rangle + 2\langle b, x\rangle = \langle X^T Q X\lambda, \lambda\rangle + 2\langle X^T b, \lambda\rangle. \end{aligned}$$

Thus, all our statements follow from Theorem 5. $\square$

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Appendix: Proof of Lemma 8

1. We need to prove that

$$\mathrm{Conv}\,(A+B) = \mathrm{Conv}\,(A) + \mathrm{Conv}\,(B)$$

for A and B be closed and bounded. Indeed, if $z \in \mathrm{Conv}\,(A+B)$ then

$$z = \alpha(x_1 + y_1) + (1-\alpha)(x_2 + y_2)$$

for some x_1, $x_2 \in A$, y_1, $y_2 \in B$ and $\alpha \in [0,1]$. Therefore

$$z = [\alpha x_1 + (1-\alpha)x_2] + [\alpha y_1 + (1-\alpha)y_2] \in \mathrm{Conv}\,(A) + \mathrm{Conv}\,(B).$$

In the other direction, if $z \in \mathrm{Conv}\,(A) + \mathrm{Conv}\,(B)$ then

$$z = \sum_i \alpha_i x_i + \sum_j \beta_j y_j$$

for finite number of $x_i \in A$ and $y_j \in B$ and

$$\alpha_i \geq 0 : \sum_i \alpha_i = 1, \quad \beta_j \geq 0 : \sum \beta_j = 1.$$

Therefore

$$z = \sum_{i,j} \alpha_i \beta_j (x_i + y_j) \in \mathrm{Conv}\,(A+B)$$

since $x_i + y_j \in A + B$ and $\sum\limits_{i,j} \alpha_i \beta_j = 1$.

2. We need to prove the inclusion

$$\mathrm{Conv}\,(A_1 + B_1, A_2 + B_2) \subseteq \mathrm{Conv}\,(A_1 \bigcup A_2) + \mathrm{Conv}\,(B_1 \bigcup B_2),$$

where A_i and B_i are closed and bounded. Note that any $z \in \mathrm{Conv}\,(A_1 + B_1, A_2 + B_2)$ can be represented as follows:

$$z = \alpha(x_1 + y_1) + (1-\alpha)(x_2 + y_2)$$

for some $x_1 \in A_1$, $y_1 \in B_1$, $x_2 \in A_2$, $y_2 \in B_2$ and $\alpha \in [0,1]$. Therefore,

$$z = [\alpha x_1 + (1-\alpha)x_2] + [\alpha y_1 + (1-\alpha)y_2] \in \mathrm{Conv}\,(A_1 \bigcup A_2) + \mathrm{Conv}\,(B_1 \bigcup B_2).$$

3. We need to prove the identity

$$\mathrm{Conv}\,((A_1 + B_1) \times (A_2 + B_2)) = \mathrm{Conv}\,(A_1 \times A_2) + \mathrm{Conv}\,(B_1 \times B_2)$$

for closed bounded sets A_1, $B_1 \in R^{n_1}$ and A_2, $B_2 \in R^{n_2}$. In view of the statement of the first item, it is enough to prove

$$(A_1 + B_1) \times (A_2 + B_2) = A_1 \times A_2 + B_1 \times B_2.$$

It is clear that the both sides of this relation are exactly as follows:

$$\{(z_1, z_2) : \; z_1 = x_1 + y_1, \; z_2 = x_2 + y_2, \; x_i \in A_i, \; y_i \in B_i\}.$$

$\square$