ANALYSIS OF A NON-LINEAR MODEL OF MONETARY DYNAMICS

by

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0. Introduction

It is a well established result that descriptive dynamic macroeconomic models can exhibit dynamic instability. In this paper we shall be particularly concerned with the dynamic instability as it arises in models of monetary dynamics. Such instability was first observed in the model developed by Cagan (1956). In this model the money market cleared instantaneously and inflationary expectations adjusted adaptively to the actual rate of inflation. For relatively high speeds of adjustment the model exhibits instability. More general models allowing in addition for monetary market disequilibrium such as those developed by Goldman (1972) and Hadji michalakis (1971) and allowing for capital accumulation as in Sidsauski (1967), Shell et al (1969), Nagatani (1970) and Hadjimichalakis (1971) continue to exhibit the same basic dynamic instability property. Yarrow (1977) showed that the stability properties of such models are also sensitive to the specification of the money demand function.

Burmeister (1980) emphasizes that saddle point instability can arise in dynamic macroeconomic models when capital gains and losses are properly accounted for. Thus asymptotic convergence to a steady state equilibrium can be obtained in such models only from certain initial conditions. The worrisome feature of this type of saddle point instability is that it is very difficult to find any convincing economic mechanism which can select the initial values so that the economy is on the stable manifold of the saddle point. Much recent research is directed towards justifying the choice of the stable arm of the saddle as the path followed by the economy, Burmeister (1980).

The main thrust of the present paper is in a different direction. Here we concentrate in the interaction of the non-linearities and time lags inherent in many dynamic macroeconomic models. Non linearities arise through the functional forms of the various components of such models, such as money demand functions and investment functions. Time lags arise through the assumption of sluggish adjustment in various markets. Our main point is that while an equilibrium point may exhibit local instability the dynamic paths diverging from the equilibrium point may be converging to a stable limit cycle on which the macroeconomic quantities are fluctuating in a regular fashion.

In particular we consider a version of the orginal Cagan model in which inflationary expectations are formed with a time lag and the money market adjusts sluggishly. We show, when account is taken of the non-linear nature of the money demand function that the equilibrium point is globally asymptotically stable for relatively long lags in the formation of expectations but that for relatively short lags the equilibrium point is locally unstable with all dynamic paths tending to a stable limit cycle. In this latter case all dynamic paths are oscillatory and bounded inespective of the initial conditions. In this framework the expectations lag plays the role of a bifurcation parameter in that the qualitative nature of the dynamics of the model change as the expectations lag passes through a critical value.

In section 1 we introduce a non-linear money demand function which is a function of the expected rate inflation and give some economic justification for its form. In section 2 we assume inflationary expectations are formed adaptively and that the money market clears instantaneously.

For slowly adapting expectations we have the usual result that expectations converge to the equilibrium zero value. However for rapidly adapting expectations we find that the non-linear effects of the money demand function change the traditional picture which shows inflationary expectations diverging to plus or minus infinity. Now we find that expectations diverge from zero but tend to an upper or lower bound in finite time. In section 3 we allow for sluggish price adjustment due to excess demand in the goods market. Here we find that the non-linear money demand function together with the lagged adjustment of expectations and prices combine in such a way as to yield asymptotic stability for slow adjustment of expectations and a limit cycle for rapid adjustment of expectations.

Macroeconomic models of the type we construct here which take account of the interaction of non-linearities and time lags provide a mechanism for the generation of business cycles. Such models, unlike linear models are capable of generating persistent oscillations for a wide range of parameter values. There is already a rich literature on non-linear models of the business cycle which concentrate on various aspects of the goods market such as a non-linear accelerator, we refer in particular to the works of Goodwin (1951), Kaldor (1940) and the survey article by Ichimura (1955). To our knowledge there have been no analyses showing that models of monetary dynamics can exhibit similar behaviour.

1. The Money Demand Function

We consider an economy producing one non-durable good and in which there is only one asset, money. The level of homogeneous output is assumed fixed.

Since the ouput is non-durable, saving can only be done through the holding of money. The desired money holding at time t will depend on the expected rate of change of prices.

Let M^d, P and π denote respectively the nominal money demand, price of homogeneous output and expected proportionate rate of change of prices all at time t. Thus

(1) $\pi = E(\dot{P}/P)$

where E denotes expectation at time t and . denotes differentiation with respect to time.

In this simple economy we assume that there is a fixed transactions demand for money which we denote by T. It is the asset demand for money which is of interest and importance in our model.

Letting A denote the asset demand for real money balances we postulate that a certain fraction of (physical) wealth will be held as real money balances and that fraction will depend on π, the expected proportionate rate of changes of prices, i.e.

(2) $A = \alpha(\pi)W$

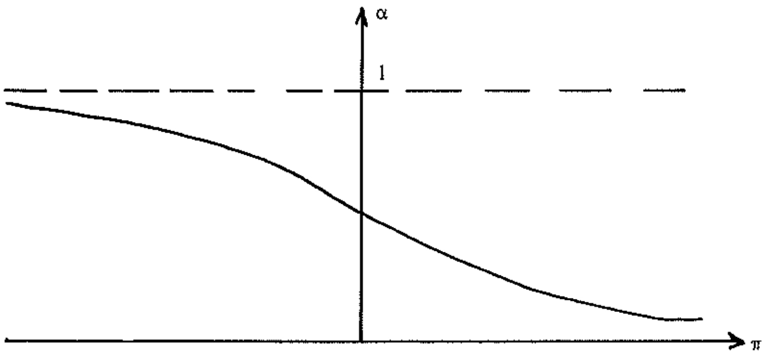
when W represents (physical) wealth remaining after transactions demand has been substracted from fixed output. Since we are assuming fixed physical output in this model we may henceforth set $W = 1$. We further postulate that

(3) $\alpha'(\pi) < 0.$

We would justify the sign of the derivative in (3) by arguing that as π increases, nominal money holdings lose purchasing power in terms of real physical goods, thus decreasing the asset demand for real money. Alternatively we could point to certain empirical work justifying the sign in (3) e.g. Sargent (1977) and Sargent and Wallace (1973).

It is clear that α varies between the limits 1 (all physical wealth held in money) and 0 (no physical wealth held in money). Thus the function $\alpha(\pi)$ must have the general shape shown in figure 1

Figure 1



We have preferred to draw α approaching its upper and lower limits asymptotically since this function is an aggregation of the decisions of many individual agents.

It is also possible to justify the general shape of the function $\alpha(\pi)$ by appealing to a utility maximizing framework. We may postulate the typical economic agent at any point in time as facing the decision problem of how much of physical wealth to consume in the current period and how much to hold as money which may be used to purchase consumption in the next period. By taking this decision so as to maximize a two period utility function an asset demand function of the type shown in figure 1 emerges (see appendix 1)

Summing the transactions demand and asset demand we may write the demand for real money balances as

$$\frac{M^d}{P} = T + \alpha(\pi) \quad . \tag{4}$$

Taking logarithms of equation (4) we may write the money demand function as

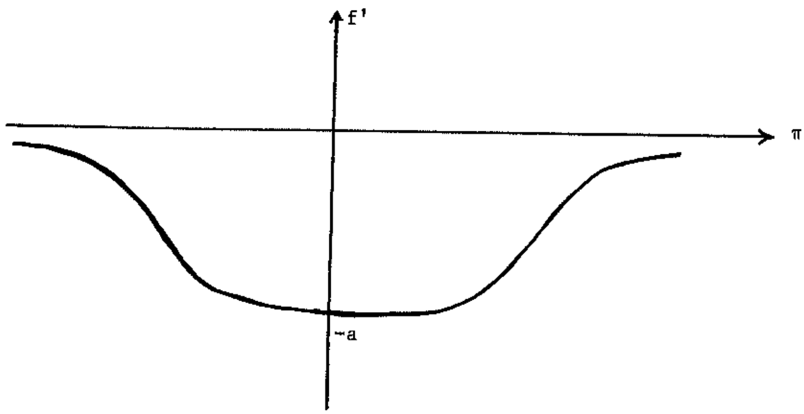
(5) $m^d = p + f(\pi)$,

where $m^d = \log M^d$, $p = \log P$, $\pi = E(\dot{p})$, and

(6) $f(\pi) = \log (T + \alpha(\pi))$.

The function $f(\pi)$ will also have the general shape shown in figure 1, but varying between the limits $\log (T + 1)$ and $\log T$. We have deliberately drawn $\alpha(\pi)$ (and hence $f(\pi)$) as having its middle section almost linear as this would seem to be in keeping with the earlier cited empirical work which indicates that these functions are close to linear over a good part of their domains. It follows that the general shape of $f'(\pi)$ must be as shown in figure 2. We put $f'(0) = -a (a>0)$

Figure 2



and we have drawn $f'(\pi)$ very flat around this value in keeping with our comment that $f(\pi)$ is almost linear over its middle section. For technical reasons we assume that the function $f'(\pi)$ assumes its minimum at $\pi = 0$.

2. The Formation of Expectations

We shall assume that expectations of price changes are formed from the knowledge of all past price changes. Restricting ourselves to a linear combination of past price changes we could write quite generally

$$(7) \qquad \pi(t) = \int_{-\infty}^{t} w(t-s)\, \dot{p}\,(s)ds,$$

showing $\pi(t)$ as a weighted average of price changes at past points in time. Different mechanisms for the formation of expectations are obtained according to the form of the continuously distributed weighting function $w(t)$. For example an exponentially declining weighting function of the form

$$(8) \qquad w(t) = \frac{e^{-t/\tau}}{\tau}$$

would give exponentially declining weight to price changes at past values of time. Noting that

$$(9) \qquad \int_0^\infty w(t)dt = 1 \quad \text{and} \quad \int_0^\infty t\, w(t)dt = \tau$$

we see that τ may be interpreted as the mean time lag of the weighting function $w(t)$ defined by (8). Thus if $w(t)$ were replaced by an equivalent discrete weighting function concentrated at the mean of $w(t)$ then (7) would become

$$(10) \qquad \pi(t) = \dot{p}(t-\tau)$$

which may be regarded as an averaged discrete version of (7). Equation (10) states that the expected price change at time t is equal to the actual price change τ time units previously.

This interpretation captures in a very simple form the effect of lags in the formation of expectations. The simplest conceptual model involving lags in the formation of expectations and the non-linearity of the money demand function would involve using (10). However the resulting model would require the analysis of a non-linear differential-difference equation for which there are few established stability results.

Since (10) and (7) (with weighting function 8) are equivalent in an averaged sense we do our analysis using continusouly distributed lags which result in non-linear differential equations which are mathematically more tractable.

By time differentiation we see that (1) and (8) are equivalent to the differential equation

(11) $\dot{\pi} = \frac{1}{\tau}\,(\dot{p} - \pi)$,

which is the familiar adaptive expectations mechanism with speed of adjustment $1/\tau$.

It is not difficult to show that in the limit as $\tau \to 0$, so that there is no lag in the formation of expectations, equations (1), (8) reduce to the perfect foresight case

(12) $\pi = \dot{p}$.

Of course many other forms of weighting function could be chosen and justified on economic grounds. For example

(13) $w(t) = \dfrac{t\, e^{-t/\tau}}{\tau^2}$

which is zero at $t = 0$, rises to a maximum at $t = \tau$ and then declines asymptotically to zero. This weighting function gives maximum weight to information τ time units in the past. One argument justifying this form of weighting function would be that economic agents give less weight to more recent information because they are waiting to see whether a new trend will confirm itself.

Finally, we point out that the form (7) is only one possible use of past information. For instance price expectations might be taken as a function of past prediction errors. A mechanism reflecting this type of expectations formation would be

(14) $\dot{\pi}(t) = \int_{\infty}^{t} w(t-s)\ (\dot{p}(s) - \pi(s))ds$,

showing the rate of change of expectations as a continuously distributed weighted average of past prediction errors.

In our subsequent analysis we shall use (7) and (8) to describe the formation of expectations. However we should point out that the alternative mechanisms which we have described above will lead to results qualitatively similar to those found in sections (3) and (4).

3. Instantaneous Adjustment of the Money Market

In this section we shall analyse the interaction of lags in the formation of price expectations and the non-linear money demand function under the assumption that the money market clears instantaneously.

Thus assuming a constant money supply, m, we have by (5) that

(15) $$p = m - f(\pi).$$

We assume that expectations of future price changes are an exponentially declining weighted average of past price changes as given by (7) and (8). However it is mathematically most convenient to express this mechanism in the form of equation (11). Combining (11) and (15) we find that expectations satisfy the non-linear differential equation

(16) $$\dot{\pi} = \frac{-\pi}{(\tau + f'(\pi))} .$$

The traditional analysis of the inflationary process described by the differential equation (16), such as the analysis of Cagan (1956) assumes $f(\pi)$ is a negatively sloped linear function so that $f'(\pi)$ may be replaced by $f'(0) = -a(a>0)$. In this case (16) reduces to

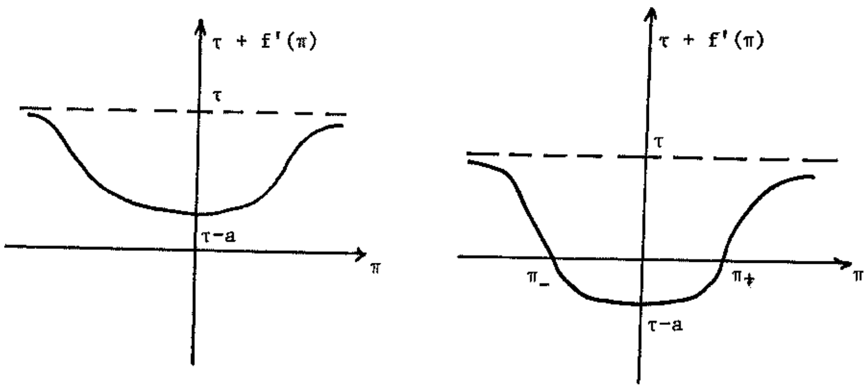
(17) $$\dot{\pi} = \frac{-\pi}{(\tau-a)} .$$

Equation (17) displays the classical result that for $\tau>a$ inflationary expectations are stable and for $\tau<a$ inflationary expectations are unstable. We recall that $\tau>a(<a)$ may be interpreted as (i) giving more (less) weight to past price changes in equation (19), (ii) as larger (shorter) lags in the formation of expectations in equation(10) or (iii) as slower (faster) speed of adjustment in equation (11).

However, it does not seem to be stressed enough in the traditional analysis that equation (11) describes the <u>local</u> behaviour of (16) around the equilibrium point $\pi = 0$. Closer analysis of (16) reveals a more complex global picture of the motion of π.

The postulated form of $f'(\pi)$ shown in figure 2 leads to the graphs of $\tau + f'(\pi)$ shown in figure 3. We distinguish

Figure 3

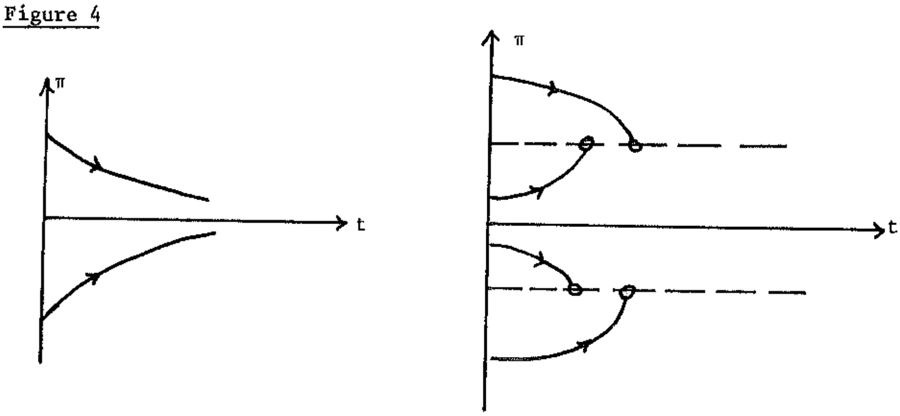


(a) $\tau-a>0$ (b) $\tau-a<0$

between two cases (a) $\tau-a > 0$ so that $\tau + f'(\pi) > 0$ for all π, and (b) $\tau - a< 0$ so that $\tau + f'(\pi) \leq 0$ for $\pi_- \leq \pi \leq \pi_+$ and $\tau + f'(\pi) > 0$ for all other π.

Consider first of all the case (a) in which $\tau-a > 0$. We see from (16) that in this case $\dot{\pi} < 0$ for all π and so the global dynamic behaviour of π is as predicted by the local analysis of (17). The dynamic fath of $\pi(t)$ from arbitrary initial values is shown in figure 4(a). Consider for example an initial value of π above the equilibrium level. The expected rate of inflation adjusts in a decreasing manner, with the demand for money decreasing to its equilibrium level.

Figure 4



(a) $\tau-a>0$ (b) $\tau-a<0$

The case $\tau < a$ is more interesting. For initial values of π satisfying $\pi_- < \pi_0 < \pi_+$, $\tau + f'(\pi) < 0$, so that $\dot{\pi} > 0$ and π moves away from the origin. However along any such path the limit of $\dot{\pi}$ as $\pi \to \pi_+$ (π_-) is $+\infty$ ($-\infty$), and so the lines $\pi = \pi_+$, $\pi = \pi_-$ are reached in finite time. For initial values satisfying $\pi_0 > \pi_+$, $\pi_0 < \pi_-$ the dynamic motion is reversed since now $\tau + f'(\pi) > 0$ and $\dot{\pi} < 0$. Again the lines $\pi = \pi_+$ and $\pi = \pi_-$ are reached in finite time. The dynamic motion of $\pi(t)$ under the assumption $\tau - a < 0$ is shown in figure 4(b).

We have drawn figure 4 by appealing to intuitive notions. A detailed mathematical derivation is given in Appendix 2.

The case $\tau < a$, which under the traditional local analysis seems to imply an even increasing rate of inflation, leads when analyzed as a *non linear* differential equation to inflationary expectations being stuck at some upper or lower levels. These levels depend essentially on the inflection points in the money demand function. Consider an initial value of π above the zero level, but below π_+. Because of the short lag in expectations (i.e. rapid adjustment), there tends to be overadjustment in expectations and so expectations start to rise. However at the same time demand for money initially falls rapidly but ultimately levels off as the lower limit of the money demand function is approached. This rapid slackening off of the money demand causes the level of expectations to become "stuck" at π_+. Similar reasoning can be applied to the various other possible initial values i.e. $\pi_0 > \pi_+$ $0 > \pi_0 > \pi_-$, $\pi_0 < \pi_-$.

The above story of expectations being "stuck" forever at upper or lower bounds seems unrealistic. The essential cause of such a simple and unrealistic result is that the differential equation (16) is of first order and so precludes an important and very rich source of dynamic behaviour, namely limit cycles. We feel that any simple model hoping to capture the important dynamic elemets of the real economy must contain at least two differential equations to allow for this important class of dynamic motions. By allowing for disequilibrium in the money market we obtain just such a model to which we now direct our attention. We now find that in the case of too rapid adjustment instead of being "stuck" at upper or lower limits, expectations tend rather to oscillate between these extremes.

4. Lagged Adjustment of the Money Market

The process of lagged adjustment in the money market is captured by the differential equation

$$(18) \qquad \dot{p} = \beta\,(m - p - f(\pi)), \qquad (\beta > 0)$$

which shows the rate of price change as a positive linear function of the excess supply of money. Equation (18) can also be interpreted as reflecting inflationary pressure arising from excess demand in the goods market, which in the present model by Walras' law translates into excess supply of money.

We continue to assume that expectations are formed according to equation (11) which becomes

$$(19) \qquad \dot{\pi} = -\frac{\beta}{\tau}\,p - \frac{(\beta\, f(\pi)+\pi)}{\tau} + \frac{\beta m}{\tau}\,.$$

The dynamic motion of π and p is determined by the non-linear differential system (18), (19). The equilibrium $(\bar{p}, \bar{\pi})$ of this differential system is given by

$$(20) \qquad \begin{aligned} \bar{\pi} &= 0\,, \\ \bar{p} &= m - f(0). \end{aligned}$$

Let us first perform a local stability analysis of the differential system (18), (19) around its equilibrium $(\bar{\pi}, \bar{p})$. Using Δp, $\Delta\pi$ to denote deviations from these equilibrium values the local linearisation of (18), (19) assumes the form

$$(21.a) \qquad \Delta\dot{p} = -\beta\,\Delta p + a\beta\,\Delta\pi,$$

$$(21.b) \qquad \Delta\dot{\pi} = -\frac{\beta}{\tau}\,\Delta p + \frac{(a\beta-1)}{\tau}\,\Delta\pi\,.$$

Denoting by A the matrix of the differential system (21) we readily find that $\det(A) = \beta/\tau > 0$ implying that the two eigen values of A are either both positive, both negative or complex. Since trace $(A) = \frac{a\beta-(1+\beta\tau)}{\tau}$ we have

$$\begin{array}{ll} \text{local stability for} & \tau > a - 1/\beta \\ \text{local instability for} & \tau < a - 1/\beta \end{array} \tag{22}$$

So if we were to rely solely on the local linear analysis we would conclude that expectations are stable if they are formed with a sufficiently long lag (i.e. adjust slowly) and unstable if they are formed with too short a lag (i.e. adjust too rapidly). We are however able to extract a further important piece of information from the local linear analysis. We set $\varepsilon = a\beta - (1+\beta\tau)$ and notice that $\varepsilon=0$ is the switchover point between local stability and local instability. The eigenvalue of the matrix are a function of ε and so we denote them by $\lambda(\varepsilon)$. It is a straightforward matter to verify the following : -

$$\begin{array}{l} \text{(i) } \lambda(0) \text{ is pure complex ,} \\ \text{(ii) } R\left(\left.\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon=0}\right) \neq 0. \end{array} \tag{23}$$

where $R(z)$ denotes the real part of the complex number z. The conditions (23) allow us to invoke the Hopf bifurcation theorem (see appendix 3) to assert that the non-linear system, (18), (19) has a limit cycle either for $\varepsilon < 0$, or $\varepsilon > 0$ (but not both) and ε small.

The possible dynamic motions of the non linear-system (18), (19) for ε close to zero that emerge from the above local linear analysis are displayed in the (p,π) phase planes shown in figures 5 and 6.

Figure 5

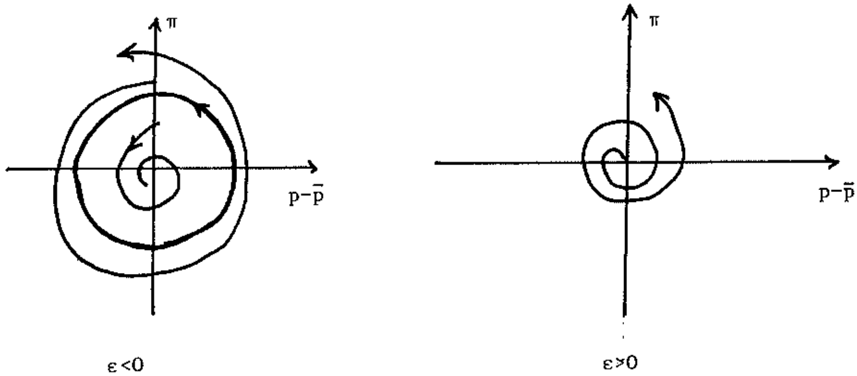
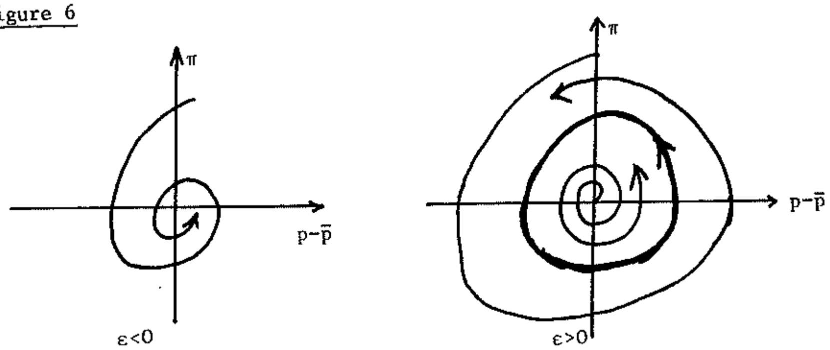


Figure 6



Consider first figure 5. For $\varepsilon < 0$ (i.e. $\tau > a - 1/\beta$ or a relatively long lag in the formation of expectations) prices and expectations are unstable around the limit cycle ; converging to the equilibrium $(\bar{p}, \bar{\pi})$ from initial values within the limit cycle and diverging away from the limit cycle from initial values outside the limit cycle. For $\varepsilon > 0$ (i.e. $\tau < a - 1/\beta$ or a relatively short lag in the formation of expectations) prices and expectations diverge from $(\bar{p}, \bar{\pi})$ from all initial values.

In figure 6 the picture is reversed. For $\varepsilon < 0$ prices and expectations converge to the equilibrium point $(\bar{p}, \bar{\pi})$ from all initial values. For $\varepsilon > 0$, there is convergence to the limit cycle from initial values both within and outside the limit cycle.

On the limit cycle prices and expectations fluctuate incessantly around the equilibrium point $(\bar{p},\bar{\pi})$. These fluctuations are due to the interaction of the non linear money demand function, the lag in the formation of expectations and the sluggish adjustment of the money market. To determine whether the limit cycle is stable and on which side of $\varepsilon = 0$ it occurs involves an asymptotic analysis of the non-linear system (18), (19) using techniques outlined in Andronov et al. (1966). Details of this analysis are given in Appendix 4 where we find that figure 6 describes the dynamic behaviour of p and π and we derive a first order approximation to the period and amplitude of the limit cycle. In particular we find that the period of the limit cycle depends on a, β and τ whilst the amplitude of the limit cycle is inversely related to the third derivative of the money demand function.

We should emphasize that in such asymptotic analysis it must be assumed that ε is close to zero. However for the non-linear differential system (18), (19) we are able to say much more since this is a two dimensional system. For two-dimensional non-linear differential systems a large number of stability results are available which enable us to analyse (18), (19) completely without making any assumptions on the magnitude of ε.

We state our main results concerning the stability of the non-linear differential equations (18), (19) in the following two propositions :

Proposition 1 : In the case $\tau > a - 1/\beta$, the equilibrium point $(\bar{p}, \bar{\pi})$ of the non-linear differential system (18), (19) is globally asymptotically stable.

Proof : Denote the right hand sides of (18) and (19) by $\phi_1(p, \pi)$ and $\phi_2(p, \pi)$ respectively. Note that

$$Q_1 = \frac{\partial\phi_1}{\partial p} + \frac{\partial\phi_2}{\partial\pi} = -\frac{\beta}{\tau}\left(\tau + \frac{1}{\beta} + f'(\pi)\right)$$

and since $-a$ is the lowest value of $f'(\pi)$ we have by the conditions of the proposition that $Q_1 < 0$.

Next we note that

$$Q_2 \equiv \frac{\partial\phi_1}{\partial p} \cdot \frac{\partial\phi_2}{\partial\pi} - \frac{\partial\phi_1}{\partial\pi}\ \frac{\partial\phi_2}{\partial p} = \frac{\beta}{\tau} > 0$$

and also that

$$Q_3 \equiv \frac{\partial\phi_1}{\partial\pi}\frac{\partial\phi_2}{\partial p} = \frac{\beta^2}{\tau}\ f'(\pi) \neq 0 \text{ everywhere.}$$

Thus all the sufficient conditions given by Olech (1963) (namely, $Q_1 < 0$, $Q_2 > 0$ and $Q_3 \neq 0$ everywhere) for the global asymptotic stability of the equilibrium point $(\bar{p}, \bar{\pi})$ are satisfied. This terminates the proof of proposition 1.

<u>Proposition 2</u> : In the case $\tau < a - 1/\beta$, the non-linear differential system (18), (19) has a unique stable limit cycle.

<u>Proof</u> : By differentiating (18) and (19) and eliminating p and all of its derivatives we obtain for π the second order non-linear differential equation

$$(24)\quad \ddot{\pi} + \frac{\beta}{\tau}\left(\frac{1+\tau\beta}{\beta} + f'(\pi)\right)\dot{\pi} + \frac{\beta}{\tau}\pi = 0.$$

The differential equation (24) is of the general form

$$(25)\quad \ddot{\pi} + u(\pi)\dot{\pi} + v(\pi) = 0 \ ,$$

where $u(\pi) = \beta((1+\tau\beta)/\beta + f'(\pi))/\tau$, $v(\pi) = \beta\pi/\tau$, which is of the Liénard type (see Gandolfo (1980)).

Levinson and Smith (1942) have shown that (25) has a unique limit cycle if the following conditions are satisfied :-

(i) $u(\pi)$ and $v(\pi)$ are differentiable ;

(ii) there exist two positive numbers π_1, π_2 such that $u(\pi) < 0$, for $-\pi_1 < \pi < \pi_2$ and $u(\pi) \geq 0$ otherwise,

(iii) $\pi\, v(\pi) > 0$ for $\pi \neq 0$,

(iv) $\lim_{\pi\to\infty} U(\pi) = \lim_{\pi\to\infty} V(\pi) = \infty$, where

$$U(\pi) = \int_0^\pi u(\pi)d\,\pi \text{ and } V(\pi) = \int_0^\pi v(\pi)d\,\pi$$

It is a relatively simple matter to verify that all of these conditions are satisfied by the differential equation (24). The crucial condition is (ii) which given the assumed form of $f'(\pi)$ is certainly satisfied for $\tau < a - 1/\beta$.

The stability of the limit cycle follows by considering the local linearisation of (24) around $\pi = 0$, which as we have already seen displays phase curves moving away from the origin when $\tau < a - 1/\beta$. Then we consider (24) for large π, when $f'(\pi) \simeq 0$ which has phase curves moving towards the origin. We thus establish that the phase curves of (24) move away from the origin for small π and towards the origin for large π. Hence the unique limit cycle is stable. This completes the proof of proposition 2.

It may be of interest to concentrate on the time path of say $\pi(t)$ (the time path of $P(t)$ is analogous) under the conditions of proposition 2, that is $\tau < a - 1/\beta$. If $\bar{A}$ is the maximum amplitude of the limit cycle oscillation then for initial values within the limit cycle, the path of $\pi(t)$ is a shown in figure 7, with the oscillations growing towards the periodic limit cycle as $t\to\infty$. On the other hand for initial values outside the limit cycle , the path of $\pi(t)$ is as shown in figure 8 with the oscillations shinking towards to the periodic limit cycle as $t\to\infty$. Thus sustained oscillations are observed whatever the set of initial values.

Figure 7

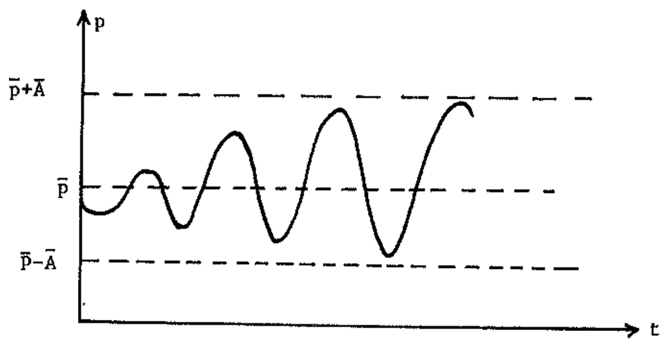
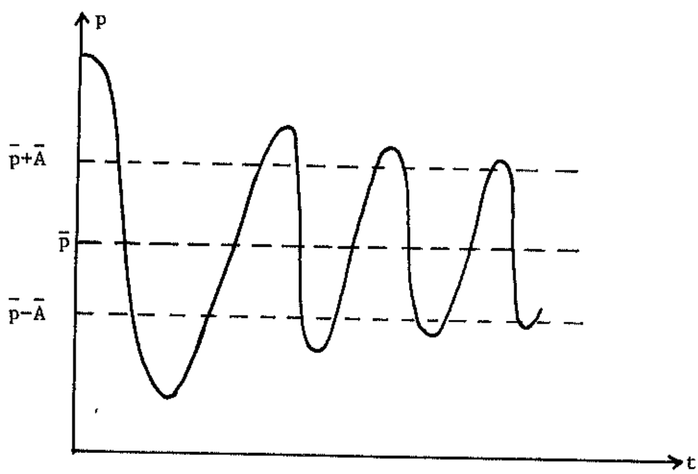


Figure 8



5. Conclusion

We have shown that a model having a money demand function depending non-linearly on the expected rate of inflation with inflationary expectations being formed adaptively is globally asymptotically stable for relatively long lags in the formation of expectations and exhibits sustained oscillations for relatively short lags in the formation of expectations.

Our analysis shows the importance of taking careful account of the interaction of non-linearities and time lags in dynamic macro-economic models. Analyses concluding at a demonstration of local instability of an equilibrium point may need to be pursued a little further, perhaps using the Hopf bifurcation theorem to test for the existence of limit cycles.

The occurrence of limit cycles in non-linear dynamic macro-economic models provides one mechanism for the generation of business cycles. Further work along the lines initiated here would require a richer model allowing capital accumulation and a more sophisticated financial sector. The differential system governing the dynamic motion of such extended models will generally be of dimension larger than two and it will probably not be possible to obtain a global picture of the phase space as was the case in the present model. In larger models we will have to be content with an asymptotic analysis close to the bifurcation parameters (the ε of section 4 above).

Our analysis could also have some bearing on the general saddlepoint instability problem referred to in the introduction. The nature of this problem for a model of dimension n is illustrated in figure 9. An equilibrium point E lies at the intersection of a stable manifold of dimension $s(R_s^-)$ and an unstable manifold of dimension $n-s(R_{n-s}^+)$.

To ensure stability initial conditions must be chosen on R_s^- and it is difficult to find an economic mechanism which would always guarantee that initial conditions lie on the stable manifold. However it is possible that on R_{n-s}^+ the equilibrium point E is locally unstable but with trajectories moving out from E towards a stable limit cycle around E as in figure 10. In this case stability (in the orbital sense) is ensured from any initial point in the space spanned by R_s^- and R_{n-s}^+.

Figure 9

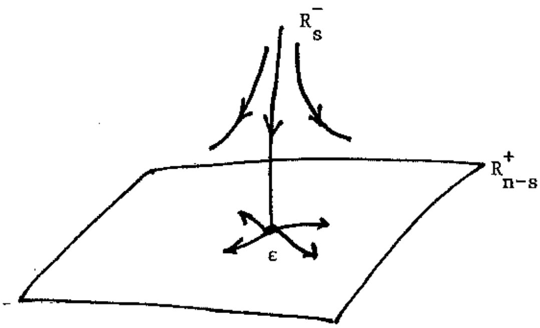
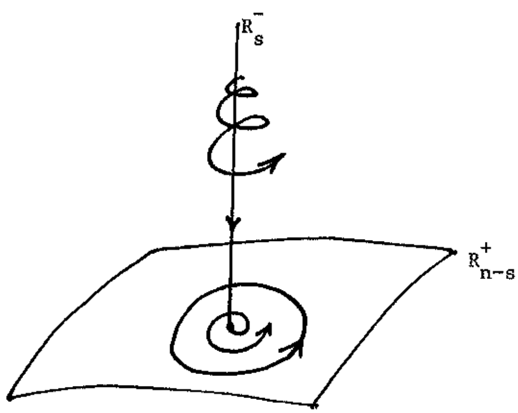


Figure 10



Appendix 1

Consider a two period framework in which the typical economic agent receives as income in period 1 a fixed quantity Y of a non-durable good. Let p_1, p_2 be the prices of the good in periods 1 and 2 respectively with $p_2=(1+\mu)p_1$, where μ is the expected rate of inflation. We envisage the agent as receiving his endowment of good in period 1 only and he seeks his allocation of consumption over the two periods so as to maximise a discounted utility function. In this framework money is needed to carry consumption from period 1 to period 2. Thus if M is the nominal quantity of money held over from period 1, this quantity of nominal money will purchase an amount M/p_2 of physical good in period 2. Since an amount of nominal money M in period 1 may be obtained for the quantity M/p_1 of physical good, the typical agents' two period maximisation problem may be written : -

Seek M, the amount of nominal money demanded in period 1 so as to maximise the 2 period utility function

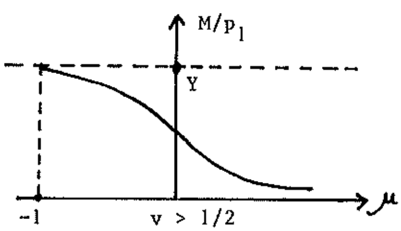
$$V = u(Y - M/p_1) + \rho\, u(M/p_2),$$

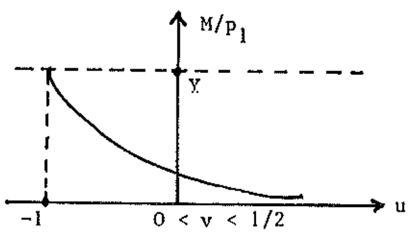
when ρ is a discount factor satisfying $0 < \rho < 1$.

Assuming a utility function of the form $u(x) = x^v/v$, $0 < v < 1$, it is a relatively simple matter to show that the real money demand in period 1 is given by

$$\frac{M}{p_1} = Y/(1 + \rho^{\frac{1}{v-1}}\ (1+u)^{\theta}) \quad , \qquad \theta = \frac{v}{1-v} \ .$$

Plotting M/p_1 as a function of μ we distinguish two cases (i) $v > \frac{1}{2}$ and, (ii) $v < 1/2$. (Ignoring the border line case $v = 1/2$)





Concentrating in the case, $v > 1/2$ would obtain the general shape of figure 1 for the money demand function if we bear in mind that the discrete time inflation rate μ and the continuous time inflation rate π are related via $\pi = \log(1 + \mu)$.

Appendix 2

We wish to consider the behaviour of the differential equation (16)

$$(16) \qquad \dot{\pi} = \frac{-\pi}{\tau + f'(\pi)}$$

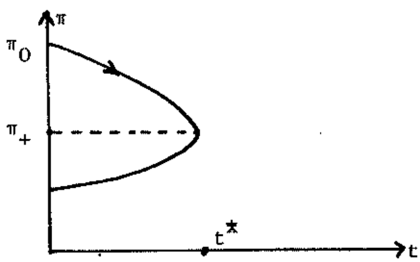
close to the points π_+ , π_- at which $\tau + f'(\pi) = 0$. Consider in particular the solution close to $\pi = \pi_+$. Expanding $\tau + f'(\pi)$ in a Taylor series around $\pi = \pi_+$ we find that (16) is approximated by the differential equation

$$\dot{\pi} = \frac{-\pi}{f''(\pi_+)\ (\pi - \pi_+)}$$

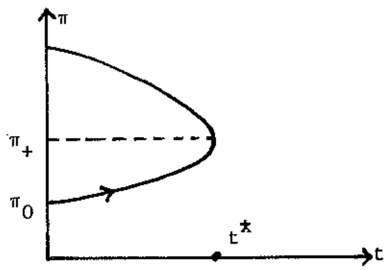
where under the assumptions we have made, $f''(\pi_+)>0$ (we ignore the special case $f''(\pi_+) = 0$). Solving this differential equation for t as a function of π we find that

$$t = f''(\pi_+)\ (\pi_+ \log \frac{\pi}{\pi_0} - (\pi - \pi_0).$$

Plotting t as a function of π and then inverting we obtain for π as a function of t the graphs shown below



$(\pi_0 > \pi_+)$



$(\pi_0 < \pi_+)$

Since time t can only move forward it is clear that π reaches π_+ at time t^* i.e. in finite time.

The analysis of the solution of the differential equation (16) in the vicinity of $\pi = \pi_-$ is entirely analogous.

Appendix 3

Consider the differential system

$$\dot{x} = f(x, \alpha) \quad , x \in R^2$$

such that $f(0,\alpha) = 0$ for all sufficiently small α. The Hopf bifurcation theorem states : suppose the linearized equation about $x = 0$ has eigenvalues $\gamma(\alpha) \pm i\, w(\alpha)$ where $\gamma(0) = 0$, $w(0) = w_0 \neq 0$, and if in addition the eigenvalues cross the imaginary axis with non zero speed so that $\gamma'(0) \neq 0$. Then for α small there exists a one parameter family of periodic solutions of $\dot{x} = f(x,\alpha)$ bifurcating from the zero solution. In particular a limit cycle exists in exactly one of the cases (i) $\alpha < 0$, (ii) $\alpha = 0$, (iii) $\alpha > 0$. However further analysis is required to determine the specific type of bifurcation. The Hopf bifurcation theorem is discussed in Marsden and McCracken (1976).

Appendix 4

Using ψ to denote $\dot{\pi}$ the non-linear differential system (18), (19) can be written

(ia) $\dot{\psi} = -\frac{\beta}{\tau}\left(\frac{1+\tau\beta}{\beta} + f'(\pi)\right)\psi - \frac{\beta}{\tau}\pi,$

(ib) $\dot{\pi} = \psi$.

We have already defined $\varepsilon = \beta(a - (1+\beta\tau)/\beta)$ and we have established by the Hopf bifurcation theorem that $\varepsilon = 0$ is a bifurcation value. Following established perturbation techniques we define new variables $\Psi(t)$ and $\Pi(t)$ such that

$$\psi(t) = \sqrt{|\varepsilon|}\,\Psi(t),\quad \pi(t) = \sqrt{|\varepsilon|}\,\Pi(t).$$

Substituting into (i) we find that Ψ and Π satisfy the differential system

(iia) $\dot{\Psi} = -\frac{\beta}{\tau}\left(\frac{1+\tau\beta}{\beta} + f'(\sqrt{|\varepsilon|}\,\Pi)\right)\Psi - \frac{\beta}{\tau}\,\Pi,$

(iib) $\dot{\Pi} = \Psi$

Notice that when $\varepsilon = 0$ (the bifurcation point) the system is equivalent to the local linearisation (equation (21) of the main text) of the original system at the bifurcation point. The solution of (ii) at this point is of the form

(iii) $\Psi(t) = A\cos\left(\sqrt{\frac{\beta}{\tau}}\,t + \theta\right),\ \Pi(t) = A\sin\left(\sqrt{\frac{\beta}{\tau}}\,t + \theta\right),$

where A is a constant amplitude and θ is the phase angle. The technique, known as the method of averaging, for approximating the solution of (ii) for ε close to zero consists in seeking a solution in the form

(iv) $\Psi(t) = A(t)\cos\left(\sqrt{\frac{\beta}{\tau}}\,t + \theta(t)\right),\ \Pi(t) = A(t)\sin\left(\sqrt{\frac{\beta}{\tau}}\,t + \theta(t)\right),$

and then finding differential equations satisfied by $A(t)$ and $\theta(t)$. The method of averaging is explained in Andronov et al. (1966). Application of this technique yields that to order ε, $\theta(t)$ = constant and $A(t)$ satisfies the differential equation

(v) $\frac{dA}{dt} = F(A).$

where

(vi) $F(A) = \frac{A^2}{\tau} \int_0^{2\pi} (1 - \frac{\beta}{\varepsilon} (a + f' (\sqrt{|\varepsilon|} A \cos \phi))) \sin^2 \phi \ d\phi.$

Expanding $f'(\sqrt{|\varepsilon|} A \cos \phi)$ up to order ε we find that

(vii) $F(A) = \pi \frac{A^2}{\tau} (1 - \frac{|\varepsilon|}{\varepsilon} \beta \frac{f'''(0)}{8} A^2).$

To order ε the amplitude of the limit cycle is given by the non-zero solution of $F(A) = 0$. Clearly from (vii) this latter equation can only have the non zero solution

(viii) $\bar{A} = \sqrt{\frac{8}{\beta f'''(0)}}$

for $\varepsilon > 0$. It is a simple matter to verify from (v) that $\dot{A} > 0$ for $0 < A < \bar{A}$ and $\dot{A} < 0$ for $\bar{A} < A$, hence $\bar{A}$ is an asymptotically stable equilibrium point of the differential equation (v). Thus we have a stable limit cycle for $\varepsilon > 0$ and the amplitude of the limit cycle is $\bar{A}$. From arbitrary initial points the amplitude of oscillations at time t is $A(t)$ to order ε with $A(t) \to \bar{A}$ as $t \to \infty$. It is possible to obtain higher order approximations to $A(t)$ and $\theta(t)$ but this is not necessary for our purposes.

Finally we point out that by approximating $f'(\sqrt{|\varepsilon|}\ \pi)$ up to order ε in equation (ii) (and using the fact that $f''(0) = 0$) we obtain the Van der Pol equation, see Andronov et al. (1966). This non-linear differential equation is the one of the simplest and earliest investigated which can generate limit cycles.

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