Springer Proceedings in Mathematics & Statistics

Helena Albuquerque
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Non-Associative Algebras and Related Topics

NAART II, Coimbra, Portugal, July 18–22, 2022

Springer

Springer Proceedings in Mathematics & Statistics

Volume 427

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ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-031-32706-3 ISBN 978-3-031-32707-0 (eBook)
https://doi.org/10.1007/978-3-031-32707-0

Mathematics Subject Classification: 17-XX, 17AXX, 46-XX, 46HXX, 68P25, 68P30

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Alberto Elduque at the University of Zaragoza

Organization

Non-Associative Algebras and Related Topics: NAART II, Coimbra, Portugal, July 18–22, 2022, was a joint organization of CMUC-Centre for Mathematics of the University of Coimbra, CMUP-Centre for Mathematics of the University of Porto and CMA-UBI—Centre of Mathematics and Applications of the University of Beira Interior.

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NAART II group photo, Department of Mathematics, University of Coimbra

Preface

On October 6, 2020, Alberto Elduque turned 60. Several months before, a big group of friends (and colleagues) had started the preparation of a conference, to be held in July 2020, to celebrate this special birthday, including the publication of some proceedings that would gather those results presented during the conference whose authors agreed to contribute.

Everything was planned and agreed with Springer, when the COVID-19 pandemic completely changed the scenario… and our lives. The complicated sanitary situation and the difficulties to travel obliged us to postpone the conference (and consequently its proceedings) for two years.

One of the (minor) consequences of COVID was that instead of a nice conference, listening to math talks and having the chance to hug Alberto after singing to him "Happy birthday to you, happy birthday dear Alberto…", we had to organize, exactly on October 6, 2020, and with the "needed cooperation and complicity" of Alberto's close relatives, a short online meeting with a small number of people. But at least we had the chance to offer an unexpected present to Alberto and congratulate him on the screen. And we even had a birthday cake, real for a few people and virtual for the rest!

Fortunately, two years later (in July 2022), the NAART II Conference in honor of Alberto Elduque took place in Coimbra as it had been planned. And this time, we all enjoyed the birthday cake! It is true that we had some very dear people absent, but they were so present in our hearts that we feel that they have also congratulated Alberto in the way he deserves.

From very early on in his academic career, Alberto always sought to maintain collaborations with mathematicians from different parts of the world, an attitude that reveals a willingness to learn and an openness to share knowledge, the engine of scientific advancement. The fruits of these collaborations were innumerable, and the influence that his works have on the development of certain themes within non-associative algebra is remarkable. The NAART II Conference allowed many of us to witness both the importance of such results and their influence on the work of other algebraists (particularly, but not exclusively, on his Ph.D. students).

Alberto is a brilliant mathematician, a wonderful professor and a great human being. It is not easy to combine these three qualities in the same person. During the conference, it was clear how much Alberto is loved, respected and valued. He has advised many Ph.D. students, creating a recognized Algebra group, and his contributions to the field of Non-associative Algebra are numerous and outstanding. He is always ready to talk about mathematics, to explain anything to the newer generations, to help anyone who needs it, to collaborate in any activity, even if it demands a lot of time and effort, and always with a friendly smile on his face!

During the online meeting in 2020, Alberto's wife, Pili, thanked him for "making their life together so easy-going". We think this beautiful phrase expresses perfectly what Alberto is like. It is easy to learn with him, it is easy to teach or to work with him, it is easy to research or to collaborate with him, and it is easy to be with him.

This book includes several papers that were presented in some of the conference talks and shows the affection of Alberto's teachers, fellows, students and collaborators.

We would like to express our sincere gratitude to all those who collaborated to make this book a reality: authors, referees and the editorial staff… The work of all of them was essential. Special thanks go to all the speakers present at the conference, in particular those who made an effort to participate online, since they could not be in Coimbra. And we extend our thanks to all the participants in the NAART II Conference who traveled to Coimbra, some of them from very distant places, to be with Alberto, to share some emotive moments with him and to remember many previous experiences, enjoying listening to math talks and discussing mathematics.

This book is divided into four parts. The first one is dedicated to Lie algebras, superalgebras and groups. The second one is devoted to Leibniz algebras. In the third one, results about associative and Jordan algebras and other related structures are included. Finally, papers considering other non-associative structures appear in Part four. Alberto has contributed in the past to the four subjects that define the parts of this book, mainly the first and fourth. In this case, his contribution appears in Part 4. So, his contribution is the finishing touch to this book. The rest of the papers (Alberto's being an exception) appear in the corresponding part following alphabetical order of the first author's surname.

Needless to say, this book is dedicated to Alberto; it is a book for Alberto, a very small gift compared to what Alberto deserves. But we want it to be also a gift for Pili and Eva, their daughter. We hope that, as mathematicians, they will both appreciate it. Their presence in Coimbra with Alberto was, apart from the best tribute to Alberto, a pleasure for all of us participants in the conference, and it allowed us to share some time with them, get to know them in some situations and verify that, very often, beside a great man there is a great family supporting his activity. Congratulations Alberto on your birthday, on your family and on your human and mathematical achievements and success!

Coimbra, Portugal — Helena Albuquerque
Coimbra, Portugal — Jose Brox
Oviedo, Spain — Consuelo Martínez
Coimbra, Portugal — Paulo Saraiva
December 2022

Contents

Biderivations of Low-Dimensional Leibniz Algebras

Manuel Mancini

Abstract In this paper we give a complete classification of the Leibniz algebras of biderivations of right Leibniz algebras of dimension up to three over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$. We describe the main properties of such class of Leibniz algebras and we also compute the biderivations of the four-dimensional *Dieudonné Leibniz algebra* $\mathfrak{d}_1$. Eventually we give an algorithm for finding derivations and anti-derivations of a Leibniz algebra as pair of matrices with respect to a fixed basis.

Keywords Leibniz algebra · Lie algebra · Derivation · Biderivation

1 Introduction

Leibniz algebras were first introduced by J.-L. Loday in [1] as a non-antisymmetric version of Lie algebras, and many results of Lie algebras were also established in the frame of Leibniz algebras. Earlier, such algebraic structures had been considered by A. Blokh, who called them D-algebras [2], for their strict connection with the derivations. Leibniz algebras play a significant role in different areas of mathematics and physics.

In [3] derivations of low-dimensional Leibniz algebras have been classified and studied. After two short preliminary sections, in this paper we aim to study and classify the Leibniz algebras of biderivations of low-dimensional Leibniz algebras. Independently of its intrinsic interest, biderivations find concrete applications in

Supported by University of Palermo and by the “National Group for Algebraic and Geometric Structures, and their Applications” (GNSAGA—INdAM).

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H. Albuquerque et al. (eds.), *Non-Associative Algebras and Related Topics*,
Springer Proceedings in Mathematics & Statistics 427,
https://doi.org/10.1007/978-3-031-32707-0_8

representation theory (cf. [4, 5]), (sub-) Riemannian geometry and control theory (see [6], and the bibliography therein), just to give two instances.

The first section is devoted to some background material on Leibniz algebras which will be useful for the rest of the paper. We address the reader to [7, 8] for more details.

In Sect. 3 we give the definitions of *derivation*, *anti-derivation* and *biderivations* for a right Leibniz algebra, which have been first introduced by J.-L. Loday in [1]. We also recall the main properties of the right and left adjoint maps and, as an example, we show how to compute the Leibniz algebra of biderivations of the four-dimensional *Dieudonné Leibniz algebra* $\mathfrak{d}_1$ (cf. [9]).

In the last section we classify the Leibniz algebra of biderivations of two and three-dimensional right Leibniz algebras over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$. We use the classification of low-dimensional complex Leibniz algebras and their derivations (cf. [3, 7]) and we give an algorithm for finding derivations and anti-derivations of a Leibniz algebras as pair of matrices with respect to a fixed basis.

2 Leibniz Algebras

We assume that $\mathbb{F}$ is a field with $\mathrm{char}(\mathbb{F}) \neq 2$. For the general theory we refer to [7].

Definition 1 A *right Leibniz algebra* over $\mathbb{F}$ is a vector space L over $\mathbb{F}$ endowed of a bilinear map (called *commutator* or *bracket*) $[-,-] : L \times L \to L$ which satisfies the *right Leibniz identity*

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] \;\; \forall x, y, z \in L.$$

In the same way we can define a left Leibniz algebra, using the left Leibniz identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \;\; \forall x, y, z \in L.$$

A Leibniz algebra that is both left and right is called *symmetric Leibniz algebra*. From now on we assume that $\dim_{\mathbb{F}} L < \infty$.

Clearly every Lie algebra is a Leibniz algebra and every Leibniz algebra with skew-symmetric commutator is a Lie algebra. Thus it is defined an adjunction (see [10]) between the category **LieAlg** of the Lie algebras and the category **LeibAlg** of the Leibniz algebras. The left adjoint of the inclusion i : **LieAlg** $\to$ **LeibAlg** is the functor π : **LeibAlg** $\to$ **LieAlg** that associates with every Leibniz algebra L the quotient $L/\,\mathrm{Leib}(L)$, where $\mathrm{Leib}(L) = \langle [x, x] \mid x \in L \rangle$ is called the *Leibniz kernel* of L. We observe that $\mathrm{Leib}(L)$ is the smallest bilateral ideal of L such that $L/\,\mathrm{Leib}(L)$ is a Lie algebra. Moreover $\mathrm{Leib}(L)$ is an abelian algebra.

We define the left and the right center of a Leibniz algebra

$$\mathrm{Z}_l(L) = \{x \in L \mid [x, L] = 0\}\,, \;\; \mathrm{Z}_r(L) = \{x \in L \mid [L, x] = 0\}\,,$$

and we observe that they coincide when L is a Lie algebra. The *center* of L is $\mathrm{Z}(L) = \mathrm{Z}_l(L) \cap \mathrm{Z}_r(L)$. In the case of symmetric Leibniz algebras, the left center and the right center are bilateral ideals, but in general $\mathrm{Z}_r(L)$ is an ideal of the right Leibniz algebra L, meanwhile the left center is not even a subalgebra.

Finally we recall the definitions of nilpotent and solvable Leibniz algebras.

Definition 2 Let L be a right Leibniz algebra over $\mathbb{F}$ and let

$$L^{(0)} = L, \;\; L^{(k+1)} = [L^{(k)}, L], \quad \forall k \geq 0,$$

be the *lower central series of* L. L is n*−step nilpotent* if $L^{(n-1)} \neq 0$ and $L^{(n)} = 0$.

Definition 3 Let L be a right Leibniz algebra over $\mathbb{F}$ and let

$$L^0 = L, \;\; L^{k+1} = [L^k, L^k], \quad \forall k \geq 0,$$

be the *derived series of* L. L is n*−step solvable* if $L^{n-1} \neq 0$ and $L^n = 0$.

3 Derivations, Anti-Derivations and Biderivations

In this section we recall the definitions of *derivation*, *anti-derivation* and *biderivation* for right Leibniz algebras and we show an example of computation of the biderivations algebra.

The definition of *derivation* is the same as in the case of Lie algebras.

Definition 4 Let L be a Leibniz algebra over $\mathbb{F}$. A *derivation* of L is a linear map $d : L \to L$ such that

$$d([x, y]) = [d(x), y] + [x, d(y)] \;\; \forall x, y \in L.$$

Remark 1 Fixed a basis $\{e_1, \dots, e_n\}$ of L, where $n = \dim_{\mathbb{F}} L$, a linear map $d : L \to L$ is a derivation if and only if

$$d(\left[e_i, e_j\right]) = \left[d(e_i), e_j\right] + \left[e_i, d(e_j)\right] \;\; \forall i, j = 1, \dots, n.$$

The right multiplications are particular derivations called *inner derivations* and an equivalent way to define a right Leibniz algebra is to say that the (right) adjoint map $\mathrm{ad}_x = [-, x]$ is a derivation, for every $x \in L$. Meanwhile, for a right Leibniz algebra, the left adjoint maps are not derivations in general.

With the usual bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$, the set $\mathrm{Der}(L)$ is a Lie algebra and the set $\mathrm{Inn}(L)$ of all inner derivations of L is an ideal of $\mathrm{Der}(L)$. Furthermore, $\mathrm{Aut}(L)$ is a Lie group and the associated Lie algebra is $\mathrm{Der}(L)$.

The definitions of *anti-derivation* and *biderivation* for a Leibniz algebra have been first given by J.-L. Loday in [1].

Definition 5 An *anti-derivation* of a right Leibniz algebra L is a linear map $D : L \to L$ such that

$$D([x, y]) = [D(x), y] - [D(y), x], \quad \forall x, y \in L.$$

For a left Leibniz algebra we have to ask that

$$D([x, y]) = [x, D(y)] - [y, D(x)], \quad \forall x, y \in L.$$

We observe that in the case of Lie algebras, there is no difference between a derivation and an anti-derivation. Moreover one can check that, for every $x \in L$, the left adjoint map $\mathrm{Ad}_x = [x, -]$ defines an anti-derivation.

Remark 2 The set of anti-derivations of a Leibniz algebra L has a $\mathrm{Der}(L)-$module structure with the action

$$d \cdot D := [D, d] = D \circ d - d \circ D,$$

for every $d \in \mathrm{Der}(L)$ and for every anti-derivation D.

Remark 3 Let $D : L \to L$ be an anti-derivation. Then, for every $x \in L$, we have

$$D([x, x]) = [D(x), x] - [D(x), x] = 0,$$

thus $D(\mathrm{Leib}(L)) = 0$.

Definition 6 Let L be a right Leibniz algebra. A *biderivation* of L is a pair

$$(d, D)$$

where d is a derivation and D is an anti-derivation, such that

$$[x, d(y)] = [x, D(y)], \quad \forall x, y \in L. \tag{1}$$

The set of all biderivations of L, denoted by $\mathrm{Bider}(L)$, has a Leibniz algebra structure with the bracket

$$[(d, D), (d', D')] = (d \circ d' - d' \circ d, D \circ d' - d' \circ D), \quad \forall (d, D), (d, D') \in \mathrm{Bider}(L),$$

and it is possible to define a Leibniz algebra morphism

$$L \to \mathrm{Bider}(L)$$

by

$$x \to (-\mathrm{ad}_x, \mathrm{Ad}_x), \quad \forall x \in L$$

The pair $(-\operatorname{ad}_x, \operatorname{Ad}_x)$ is called *inner biderivation* of L and the set of all inner biderivations forms a Leibniz subalgebra of $\operatorname{Bider}(L)$.

Now we give an example of computation of the biderivations of a Leibniz algebra.

Example 1 Let $\mathfrak{d}_1$ be the four-dimensional *Dieudonné Leibniz algebra* (see [9] for more details), i.e. $\mathfrak{d}_1$ has basis $\{e_1, e_2, e_3, z\}$ and non-zero commutators

$$[e_1, e_3] = [e_2, e_3] = -[e_3, e_1] = [e_3, e_2] = z.$$

We want to find the Leibniz algebra $\operatorname{Bider}(\mathfrak{d}_1)$ of biderivations of $\mathfrak{d}_1$. The derivations of the Dieudonné Leibniz algebra $\mathfrak{d}_n$ over a field $\mathbb{F}$, with $\operatorname{char}(\mathbb{F}) \neq 2$, have been found in [11] and it turns out that

$$\operatorname{Der}(\mathfrak{d}_1) = \left\{ \begin{pmatrix} x & 0 & \alpha & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ a_1 & a_2 & a_3 & x+y \end{pmatrix} \middle| x, y, \alpha, a_1, a_2, a_3 \in \mathbb{F} \right\}$$

Now let $D \in \operatorname{gl}(\mathfrak{d}_1)$ be an anti-derivation, then D is represented, with respect to the basis $\{e_1, e_2, e_3, z\}$, by a matrix of the type

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ A_1 & A_2 & A_3 & 0 \end{pmatrix}$$

where $D(z) = 0$ because $\operatorname{Leib}(\mathfrak{d}_1) = [\mathfrak{d}_1, \mathfrak{d}_1] = \mathbb{F}z$, and the entries $a_{ij}, i, j = 1, 2, 3$, must satisfy the following equations

$$a_{31} + a_{32} = 0,$$

$$a_{11} + a_{21} + a_{33} = 0,$$

$$a_{12} + a_{22} - a_{33} = 0,$$

$$-a_{12} - a_{22} - a_{33} = 0.$$

Thus a general anti-derivation of $\mathfrak{d}_1$ is represented by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{11} + a_{21} - a_{12} & a_{23} & 0 \\ a_{31} & -a_{31} & a_{11} + a_{21} & 0 \\ A_1 & A_2 & A_3 & 0 \end{pmatrix}$$

and, by applying the condition (1), we obtain

$$a_{31} = 0, \ \alpha = a_{13} - a_{23}, \ y = a_{11} + a_{21}$$

and

$$x = a_{11} - a_{21} = a_{11} + a_{21} - 2a_{12}.$$

We conclude that

$$\mathrm{Bider}(\mathfrak{d}_1) = \left\{ \left(\begin{pmatrix} x & 0 & \alpha & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ a_1 & a_2 & a_3 & x+y \end{pmatrix}, \begin{pmatrix} \frac{y+x}{2} & \frac{y-x}{2} & \alpha+\beta & 0 \\ \frac{y-x}{2} & \frac{y+x}{2} & \beta & 0 \\ 0 & 0 & y & 0 \\ A_1 & A_2 & A_3 & 0 \end{pmatrix} \right) \middle| x, y, \alpha, \beta, a_i, A_j \in \mathbb{F} \right\}$$

and the inner biderivations are represented by the pairs of matrices of type

$$\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_1 & a_3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & -a_1 & A_3 & 0 \end{pmatrix} \right)$$

4 Biderivations of Low-Dimensional Leibniz Algebras

Now we want to study in detail the Leibniz algebras of biderivations of low-dimensional non-Lie Leibniz algebras over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$. There is no non-trivial Leibniz algebra in dimension 1, thus we start with two-dimensional Leibniz algebras.

4.1 Biderivations of Two-Dimensional Leibniz Algebras

Let $\dim_{\mathbb{F}} L = 2$, i.e. $L = \mathbb{F}e_1 + \mathbb{F}e_2$, then, as shown in [12] by C. Cuvier, up to isomorphism we have only two non-Lie Leibniz algebra structures on L.

1. L_1 : nilpotent Leibniz algebra with non-trivial bracket $[e_2, e_2] = e_1$;
2. L_2 : solvable Leibniz algebra with the table of multiplication $[e_1, e_2] = [e_2, e_2] = e_1$.

Notice that L_1 is a symmetric Leibniz algebra, meanwhile L_2 is only a right Leibniz algebra.

It turns out that

$$\mathrm{Der}(L_1) = \left\{ \begin{pmatrix} 2a & b \\ 0 & a \end{pmatrix} \middle| a, b \in \mathbb{F} \right\}$$

and

$$\mathrm{Der}(L_2) = \left\{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \middle| a \in \mathbb{F} \right\}$$

Moreover it is easy to check that the set of anti-derivations of L_1 and L_2 are both represented by the matrices of the form

$$\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$$

The condition (1) implies that $y = a$ for L_1 and $y = 0$ for L_2, thus we have

$$\mathrm{Bider}(L_1) = \left\{ \left(\begin{pmatrix} 2a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & a \end{pmatrix} \right) \middle| a, b, x \in \mathbb{F} \right\}$$

and

$$\mathrm{Bider}(L_2) = \left\{ \left(\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right) \middle| a, x \in \mathbb{F} \right\}.$$

Finally the inner biderivations of L_1 are represented by the pairs of matrices

$$\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix} \right)$$

meanwhile the biderivations of L_2 are all inner.

4.2 *Biderivations of Three-Dimensional Leibniz Algebras*

Three-dimensional complex Leibniz algebras and their derivations have been classified in [3, 13], meanwhile the more general classification of three-dimensional right Leibniz algebras over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$, can be found in [14, 15].

Let $\dim_{\mathbb{F}} L = 3$ and let $\{e_1, e_2, e_3\}$ be a basis of L over $\mathbb{F}$. The list of non-isomorphic three-dimensional right Leibniz algebras over $\mathbb{F}$ is the following.

Leibniz algebra	**Non-zero brackets**
L_1	$[e_1, e_3] = -2e_1,\ [e_2, e_2] = e_1,\ [e_3, e_2] = -[e_2, e_3] = e_2$
$L_2(\alpha),\ \alpha \neq 0$	$[e_1, e_3] = \alpha e_1,\ [e_3, e_2] = -[e_2, e_3] = e_2$
L_3	$[e_3, e_2] = -[e_2, e_3] = e_2,\ [e_3, e_3] = -e_1$
L_4	$[e_2, e_2] = e_1,\ [e_3, e_3] = e_1$
L_5	$[e_2, e_2] = e_1,\ [e_3, e_3] = -e_1$
$L_7(\alpha),\ \alpha \neq 0$	$[e_2, e_2] = [e_2, e_3] = e_1,\ [e_3, e_3] = \alpha e_1$
L_8	$[e_2, e_3] = e_1$
L_9	$[e_1, e_3] = e_2,\ [e_2, e_3] = e_1$
L_{10}	$[e_1, e_3] = e_2,\ [e_2, e_3] = -e_1$
$L_{12}(\alpha),\ \alpha \neq 0$	$[e_1, e_3] = e_2,\ [e_2, e_3] = \alpha e_1 + e_2$
L_{13}	$[e_1, e_3] = e_1,\ [e_2, e_3] = e_2$
L_{14}	$[e_1, e_3] = e_2,\ [e_3, e_3] = e_1$
L_{15}	$[e_1, e_3] = e_1 + e_2,\ [e_3, e_3] = e_1$

Here we use the same notation of [15], but we do not report the algebras $L_6(\alpha)$ and $L_{11}(\alpha)$, where $\alpha \neq 0$, which are isomorphic to L_4 and L_9 respectively. We want to extend the results of [3] by completing the classification of the Lie algebras of derivations of three-dimensional Leibniz algebras over a general field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$, and by finding the biderivations of this class of Leibniz algebras.

Remark 4 In this section we use the following algorithm for finding derivations and anti-derivations. Let L be a Leibniz algebra and let $(d, D) \in \mathrm{Bider}(L)$. Then, for every $x, y \in L$, we have

$$d([x, y]) = [d(x), y] + [x, d(y)], \quad D([x, y]) = [D(x), y] - [D(y), x]$$

if and only if

$$(d \circ \mathrm{ad}_y)(x) = (\mathrm{ad}_y \circ d)(x) + \mathrm{ad}_{d(y)}(x), \quad (D \circ \mathrm{ad}_y)(x) = (\mathrm{ad}_y \circ D)(x) - \mathrm{Ad}_{D(y)}(x),$$

thus

$$[d, \mathrm{ad}_y] = \mathrm{ad}_{d(y)}, \quad [D, \mathrm{ad}_y] = -\mathrm{Ad}_{D(y)}.$$

Fixed a basis $\{e_1, \ldots, e_n\}$ of L, we have that the biderivation (d, D) is represented by a pair of $n \times n$ matrices $((d_{i,j})_{i,j}, (D_{i,j})_{i,j})$ and for every $i = 1, \ldots, n$

$$[d, \mathrm{ad}_{e_i}] = \mathrm{ad}_{d(e_i)}, \quad [D, \mathrm{ad}_{e_i}] = -\mathrm{Ad}_{D(e_i)},$$

which are equations in the entries of the matrices representing d and D. By solving this set of equations, and after imposing the compatibility condition *(1)*, we find the matrices $(d_{i,j})_{i,j}, (D_{i,j})_{i,j}$.

A straightforward application of the above algorithm produces the following complete classification of biderivations of three-dimensional right Leibniz algebras over

$\mathbb{F}$. In particular we find that the dimension of these biderivation algebras lies between three and six and there are only two parameterized families of Leibniz algebra of biderivations of three-dimensional Leibniz algebras over $\mathbb{F}$.

Theorem 1 *Let $\mathbb{F}$ be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$*. The Leibniz algebras of biderivations of three-dimensional right Leibniz algebras over $\mathbb{F}$ can be described as follows.*

– $\mathrm{Bider}(L_1) = \left\{ \left(\begin{pmatrix} 2x & y & 0 \\ 0 & x & y \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -y & a \\ 0 & x & y \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_2(\alpha)) = \left\{ \left(\begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & y & z \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, z, a, b \in \mathbb{F} \right\}$,

if $\alpha \neq -1$ *and*

$\mathrm{Bider}(L_2(-1)) = \left\{ \left(\begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & b \\ 0 & y & z \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, z, b \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_3) = \left\{ \left(\begin{pmatrix} 0 & 0 & y \\ 0 & x & z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & x & z \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, z, a \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_4) = \left\{ \left(\begin{pmatrix} 2x & y & z \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \right) \middle| x, y, z, a, b \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_5) = \left\{ \left(\begin{pmatrix} 2x & y & t \\ 0 & x & -z \\ 0 & z & x \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & x & -z \\ 0 & z & x \end{pmatrix} \right) \middle| x, y, z, t, a, b \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_7(\alpha)) = \left\{ \left(\begin{pmatrix} \gamma x & y & z \\ 0 & x & \frac{x}{2} \\ 0 & -\frac{x}{2\alpha} & (\gamma - 1)x \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & x & \frac{x}{2} \\ 0 & -\frac{x}{2\alpha} & (\gamma - 1)x \end{pmatrix} \right) \middle| x, y, z, a, b \in \mathbb{F} \right\}$,

where $\gamma = \dfrac{4\alpha - 1}{2\alpha}$;

– $\mathrm{Bider}(L_8) = \left\{ \left(\begin{pmatrix} x+y & z & t \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}, \begin{pmatrix} 0 & z & t \\ 0 & 0 & a \\ 0 & 0 & y \end{pmatrix} \right) \middle| x, y, z, t, a \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_9) = \left\{ \left(\begin{pmatrix} x & y & 0 \\ y & x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_{10}) = \left\{ \left(\begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_{12}(\alpha)) = \left\{ \left(\begin{pmatrix} x & \alpha y & 0 \\ y & x+y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_{13}) = \left\{ \left(\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_{14}) = \left\{ \left(\begin{pmatrix} 2x & 0 & y \\ y & 3x & z \\ 0 & 0 & x \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & x \end{pmatrix} \right) \middle| x, y, z, a, b \in \mathbb{F} \right\}$;

– $\mathrm{Bider}(L_{15}) = \left\{ \left(\begin{pmatrix} x & 0 & x \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\}$.

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