

Consistent Collusion-Proofness and Correlation in Exchange Economies

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Abstract

We present a feasible strategic market mechanism with finitely many agents whose Nash, semi-strong Nash and coalition-proof Nash equilibria fully implement the Walrasian equilibria. We define a strategic equilibrium concept, called correlated semi-strong equilibrium, and show that the Walrasian equilibria can be implemented by these equilibria, and also by the coalition-proof correlated equilibria of our mechanism. We show that these two concepts, suitably modified with transfers, fully implement the Pareto optimal allocations. We extend to semi-strong correlated equilibria Aumann's Bayesian interpretation of correlated equilibria.

Keywords: Coalition-proofness, Correlation, Semi-strong equilibrium, Implementation.

JEL Classification: C72 D51 + implementation

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1 Introduction

The objective of implementation theory, as stated in Jackson [13], is to characterize the set of social choice correspondences that are obtainable as equilibrium outcomes when individuals interact through some game-form making strategic use of their knowledge. Specifically, given a social choice correspondence F which maps profiles of characteristics to allocations, does there exist a game-form G such that for any profile R of characteristics, the players' outcome from the game is precisely $F(R)$?

For a pure exchange economy with at least three players but a finite number of players, we present a strategic market mechanism which is individually feasible (a player's final allocation belongs to his consumption set), and collectively feasible when no player goes bankrupt (markets clear). A strategy of a player consists in quoting a price and announcing a vector of net demands to every other player in the game.

Using this strategic market mechanism, we show that the Walrasian equilibria can be fully implemented via the pure Nash and semi-strong Nash equilibria (Kaplan [14]). We also obtain full implementation of Walrasian equilibria with coalition-proof Nash equilibria (Bernheim *et al* [3]). Consequently, our mechanism is such that outcomewise, the Nash, semi-strong Nash and coalition-proof Nash equilibria are all equivalent.

In addition, we introduce the concepts of correlated semi-strong and correlated strong equilibria and compare them to other notions through examples, namely the coalition-proof correlated equilibrium concept of Ray [19]. The correlated semi-strong and coalition-proof correlated equilibria of the strategic market mechanism that we present both implement the Walrasian equilibria. With transfers, these solution concepts fully implement the Pareto optimal allocations. Moreover, we show that Aumann's Bayesian interpretation of correlated equilibria [2] can be extended to semi-strong correlated equilibria.

The argument that drives the results stated above is essentially the following: given a Walrasian equilibrium or, more generally, a Pareto optimal allocation, if a coalition profitably deviates, it must do so by modifying its price quoting strategies. However, and this is the important observation, such a deviation is not internally consistent as a member of the coalition would then benefit from an arbitrage opportunity. The hypotheses on preferences required to get these results are minimal.

There are several ways of defining the notion of coalition-proofness in the literature, especially when players are allowed to correlate their actions. In this paper, we adopt the concept of coalition-proof correlated equilibrium of Ray [19], which rests on the coalition-proof Nash equilibrium notion of Bernheim *et al* [3]. As defined in Bernheim *et al* [3], "an agreement is a coalition-proof Nash equilibrium (CPNE) if and only if it is Pareto efficient within the class of self-enforcing agreements; an agreement is self enforcing if and only if no proper coalition of players, taking the actions of its complement as fixed, can agree to deviate in a way that makes all of its members better off." The notion of Ray

[19] uses extensively that of a CPNE: “a coalition-proof correlated equilibrium of a game is a pair consisting of a correlation device and a CPNE of the game extended by the device.”

In Ray [19], deviations by coalitions take place *ex ante*, before the players learn their actions recommended by the correlation device, and a coalition that deviates cannot construct a new correlation device. Correlated semi-strong and correlated strong equilibria, as we define them in this paper, share these characteristics as well. In the spirit of Ray, they are the natural extensions of the semi-strong Nash (Kaplan [14]) and strong Nash equilibria (Aumann [1]). Recall that the strong Nash equilibrium concept is defined in terms of deviations by every possible coalition, and not only unilateral deviations as for Nash equilibria. A semi-strong equilibrium is defined in terms of deviations by coalitions that are Nash equilibria in the reduced game. Using these notions, we truly study implementation problems in a *complete* information setting.

Two papers are related to this one. In [11], Hurwicz presents a strategic market mechanism which is collectively but not individually feasible, anonymous (the strategy sets of the players do not depend on their characteristics), smooth, and whose Nash equilibria implement the Walrasian equilibria. The author does not present any implementation results using coalition-proofness.

The mechanism presented by Schmeidler in [20] is collectively but not individually feasible, discontinuous and anonymous. It allows the author to obtain implementation results in strong Nash equilibria.

In both of these games, players are not allowed to go bankrupt, and both require that there be at least three players in the game.

Dubey [4] presents a strategic market mechanism that allows the players to exchange out of equilibrium, in contrast with Schmeidler [20] where the no-trade is the generic outcome out of equilibrium. However, the no-trade equilibrium is always a Nash equilibrium of Dubey’s game and hence, his strategic market game is not competitive (a game being competitive when its Nash equilibria fully implement the Walrasian equilibria). Indeed, the no-trade is a Walrasian equilibrium only when the initial endowments are Pareto optimal, which is generically not the case.

In this paper, we assume that the mechanism designer ignores the preferences of the agents but knows their initial endowments. Indeed, Hurwicz, Maskin and Postlewaite [12] have shown that there exists no Pareto efficient strategic market mechanism with a finite number of players that does not depend on the initial endowments and which is individually and collectively feasible (any agent’s final allocation belongs to his consumption set and markets clear).

It is also well-known that if an economy admits a Walrasian equilibrium on the boundary of its feasible set, then the Walrasian correspondence is not Maskin monotonic (see Jackson [13] for details on the Maskin monotonicity) and thus not Nash implementable. We avoid this problem by requiring that all the players strictly prefer their initial endowments to any allocation on the boundary of their consumption set (Hong [10]).

In order not to require the mechanism designer to know the feasible set

(that is, the aggregate endowments), it is standard to do as in Postlewaite and Wettstein [17], and require the agents to reveal their endowments as part of their strategy, and furthermore add appropriate incentive constraints on the mechanism to get truthful revelation in equilibrium. However, one then classically forbids the players to overstate their true endowments (implicitly, since they have to show physically their endowments). As a consequence, the mechanism depends on the true endowments. From our point of view, nothing substantial would have been gained by doing so here and the notations would have been considerably complicated. We therefore implicitly assume that the mechanism designer does know the relevant characteristics of the economy when he constructs the game.

Finally, we observe that the Nash equilibria of our mechanism fully implement the constrained Walrasian equilibria under weaker conditions on the initial endowments than those required to implement the Walrasian equilibria. Observe that the constrained Walrasian equilibria share the same efficiency properties as the Walrasian equilibria, and exist even in cases where Walrasian equilibria do not exist. In [17], Postlewaite and Wettstein present a strategic market mechanism that is feasible, continuous and implements in Nash equilibria the constrained Walrasian equilibria. However, they do not address the question of correlation and coalition-proofness. In the appendix, we show by means of an example that the constrained Walrasian equilibria cannot be implemented in semi-strong Nash in Postlewaite and Wettstein's game.

Before presenting the details of the model, it may be useful to add a few words about Shapley-Shubik games. In a completely different context, Shapley and Shubik [21] present a strategic market game which is individually and collectively feasible, and such that all allocations induced by the Nash equilibria of the game are individually rational, in the sense that for any player, the utility derived from his final allocation is larger than that obtained from his initial endowment. This is so as a player can choose to play the no-trade. However, the no-trade equilibrium is always a Nash equilibrium of the game and hence, their strategic market game is not competitive. More generally, generically, no Nash equilibrium is Pareto efficient in this kind of game (Dubey and Rogawski [5]). However, when the number of players tends to infinity, the allocations induced by the non-trivial Nash equilibria of this game converge to the Walrasian equilibria (Dubey and Shubik [6]). The main departure between that type of literature and our approach is that we want to get efficiency even for finite economies.

The paper is organized as follows. In section 2, we present the economy, the game-form and the solution concepts. In section 3, we present a full implementation result using Nash and semi-strong Nash equilibria. In section 4, we present implementation results involving correlation and some examples to relate to one another the different solution concepts introduced. Section 5 extends to semi-strong correlated equilibria the Bayesian interpretation known for correlated equilibria. In section 6, we get full implementation of Pareto optimal allocations via correlated semi-strong equilibria and coalition-proof correlated

equilibria with transfers.

2 The model

2.1 The economy

Notations $|S|$ denotes the cardinality of the set S ; $\text{int}S$ and ∂S respectively denote the interior and the boundary of S ; $B_L(x, \epsilon) := \{y \in \mathbb{R}^L : \|y - x\| \leq \epsilon\}$.

Consider a pure exchange economy $\mathcal{E} = (X_i, u_i, \omega_i)_{i \in I}$ with N agents $i \in I = \{1, \dots, N\}$ and L commodities $h \in \{1, \dots, L\}$. For each agent i , let $X_i \subset \mathbb{R}^L$ be his consumption set. The utility function of agent i is $u_i : X_i \rightarrow \mathbb{R}$ and $\omega_i \in X_i$ denotes his initial endowment. The aggregate endowment is $\bar{\omega} := \sum_i \omega_i$. Let $\hat{X}_i := \{x \in \mathbb{R}_+^L : x \leq \bar{\omega}\}$.

Assumption 1

- $N \geq 3$, and for all $i \in I$, $X_i = \mathbb{R}_+^L$.
- For all $i \in I$, u_i is quasi concave on $\hat{X}_i$.
- (local non satiation) For all $x \in \hat{X}_i$, for all $\epsilon > 0$, there exists $y \in B_L(x, \epsilon) \cap \hat{X}_i$ such that $u_i(y) > u_i(x)$.
- (boundary condition) For all $x \in \partial X_i \cap \hat{X}_i$, $u_i(\omega_i) > u_i(x)$ ¹.

Definition 1

- $(p^*, (x_i^*)_i)$ is a *Walrasian equilibrium* if and only if $p^* \neq 0$ and
 - for all $i \in I$, $x_i^* \in \text{argmax}_x u_i(x)$ under the constraint $p^*x \leq p^*\omega_i$,
 - $\sum_i x_i^* = \sum_i \omega_i$.
- $(p^*, (x_i^*)_i)$ is a *constrained Walrasian equilibrium* if and only if $p^* \neq 0$ and
 - for all $i \in I$, $x_i^* \in \text{argmax}_x u_i(x)$ under the constraints $p^*x \leq p^*\omega_i$ and $x \leq \sum_i \omega_i$,
 - $\sum_i x_i^* = \sum_i \omega_i$.

We denote by $WE(\mathcal{E})$ the set of Walrasian equilibria of the economy $\mathcal{E}$ and $CWE(\mathcal{E})$ the set of constrained Walrasian equilibria. Note that $CWE(\mathcal{E})$ induces Pareto optimal allocations and $WE(\mathcal{E}) \subset CWE(\mathcal{E})$.

¹ This condition implies that for all $i \in I$, $\omega_i >> 0$.

2.2 The game-form

A *game* Γ is defined by $\Gamma = (A_i, \phi_i)_{i=1, \dots, N}$ where for all $i \in I$, A_i denotes the set of pure actions of player i , while $\phi_i : A = \prod_{h \in I} A_h \rightarrow \mathbb{R}_+^L$ denotes the strategic outcome function of player i .

A_i will be defined as the set of prices proposed by player i , $p_i \in \mathbb{R}_+^L$, and the vector of net demands addressed by player i to player $j \neq i$, $z_i = (z_j^i)_{j \neq i} \in (\mathbb{R}^L)^{N-1}$:

$$A_i = \left\{ (p_i, z_i) \in \mathbb{R}_+^L \times (\mathbb{R}^L)^{N-1} \right\}.$$

The *strategic budget constraint* of player i is

$$\sum_{j \neq i} p_j (z_j^i - z_i^j) \leq 0. \quad (1)$$

Note that this constraint does not depend on the price quoted by player i . Let $\psi_i : A \rightarrow \mathbb{R}^L$ be the *auxiliary function* defined by

$$\psi_i(a) = \begin{cases} \sum_{j \neq i} (z_j^i - z_i^j) & \text{if (1) is satisfied} \\ -\omega_i & \text{otherwise} \end{cases}$$

We can now define the *strategic outcome function* of player i , $\phi_i : A \rightarrow \mathbb{R}_+^L$:

$$\phi_i(a) = \text{proj}_i \circ \text{proj}_F((\omega_k + \psi_k(a))_{k=1, \dots, N})$$

where

$$F = \left\{ (x_i)_i \in \prod_{i \in I} X_i : \sum_i x_i \leq \sum_i \omega_i \right\}$$

is the *feasible set*, which is convex, compact and nonempty;

$$\text{proj}_F : (\mathbb{R}^L)^N \rightarrow F$$

is the orthogonal projection over F and

$$\text{proj}_i : (\mathbb{R}^L)^N \rightarrow \mathbb{R}^L$$

is the canonical projection on the i th coordinate.

The *payoff function* of player i with respect to the actions, $g_i : A \rightarrow \mathbb{R}$, is defined by $g_i(a) = u_i \circ \phi_i(a)$.

Remark The mechanism is feasible, in the sense that any strategy profile induces an outcome such that demand does not exceed supply (given by the initial endowments) and the agents receive a bundle which belongs to their consumption set. Moreover, the mechanism is not balanced out of equilibrium if some player goes bankrupt. In this way, part of the initial resources are wasted. Finally, it is not individually rational out of equilibrium since if someone is kept in the red, he necessarily gets zero.

2.3 Solution concepts

We now recall a first definition of a coalition-proof Nash equilibrium (Bernheim, Peleg and Whinston [3]).²

Definition 2

- If $N = 1$, $a^* \in A$ is a *coalition-proof Nash equilibrium* (CPNE) if and only if

$$a^* \in \operatorname{argmax}_{a \in A} g_i(a).$$

- For $N > 1$, suppose that the notion of CPNE has been defined for less than N players. Then
 - $\bar{a} \in A$ is *self-enforcing* if for all coalition $S \subset I$, $\bar{a}_S = (\bar{a}_i)_{i \in S}$ is a CPNE in the game $\Gamma^{\bar{a}, S}$ with outcome function for player $i \in S$ defined by $\phi_i^{\bar{a}}(a_S) := \phi_i(a_S, \bar{a}_{-S})$ ³ for all $a_S \in A_S = \prod_{i \in S} A_i$.
- $a^* \in A$ is a CPNE if it is self-enforcing and Pareto-undominated among the self-enforcing strategy vectors, namely,

$$\text{for all } \bar{a} \in A \text{ self-enforcing, } \exists i_o \in I \text{ such that } g_{i_o}(\bar{a}) \leq g_{i_o}(a^*). \quad (2)$$

We now present a definition of a CPNE equivalent to that just defined.

Definition 3 Let $\bar{a} \in A$ and $S \subseteq I$, $S \neq \emptyset$.

- An *internally consistent improvement* (ICI) of S upon $\bar{a}$ is defined by induction on the cardinality of S :
 - If $S = \{i\}$, $a_i \in A_i$ is an ICI of $\{i\}$ upon $\bar{a}$ if and only if $g_i(a_i, \bar{a}_{-i}) > g_i(\bar{a})$.
 - If $|S| > 1$, then $a_S \in A_S = \prod_{i \in S} A_i$ is an ICI of S upon $\bar{a}$ if and only if

$$\text{for all } i \in S, g_i(a_S, \bar{a}_{-S}) > g_i(\bar{a}), \quad (3)$$

and there exists no coalition $S' \subseteq S$, $S' \neq \emptyset$ which has an ICI upon $(a_S, \bar{a}_{-S})$.

- $a^* \in A$ is a CPNE if and only if there exists no $S \subseteq I$, $S \neq \emptyset$ which has an ICI upon a^* .

² $X \subseteq Y$ (resp. $X \subset Y$) means that X is included in and possibly equal to (resp. strictly included in) Y .

³ $_{-S} = I \setminus S$, $a_{-S} = (a_i)_{i \in -S}$

Remark A *strict* CPNE is defined as a CPNE with requirements (2) and (3) replaced respectively by

- There exists no $\bar{a} \in A$ self-enforcing such that $g_i(\bar{a}) \geq g_i(a^*)$, for all $i \in I$, with strict inequality for some $i_o \in I$.
- For all $i \in S$, $g_i(a_S, \bar{a}_{-S}) \geq g_i(\bar{a})$ and for some $i_o \in S$, the strict inequality holds.

Definition 4 Let $\bar{a} \in A$ and $S \subseteq I$, $S \neq \emptyset$.

- Let $\Gamma^{\bar{a}, S}$ be the game induced by S with outcome function for all $a \in A_S$ and for all $i \in S$ defined by $\phi_i^{\bar{a}}(a) := \phi_i(a, \bar{a}_{-S})$. Then $NE(\Gamma^{\bar{a}, S}) = \{a^* \in A_S : a^* \text{ is a pure Nash equilibrium of the game } \Gamma^{\bar{a}, S}\}$.
- A strategy $a^* \in A$ is a *semi-strong Nash equilibrium* (ssNE) if for every coalition $\emptyset \neq S \subseteq I$ and every $a_S \in NE(\Gamma^{a^*, S})$, there exists $i \in S$ such that $g_i(a^*) \geq g_i(a_S, a_{-S}^*)$ (Kaplan [14]).
- A strategy $a^* \in A$ is a *strong Nash equilibrium* (sNE) if and only if for all coalition $S \subseteq I$ and for all strategy $a_S \in A_S$, there exists an agent $i \in S$ such that $g_i(a^*) \geq g_i(a_S, a_{-S}^*)$ (Aumann [1]).

3 Full Nash and semi-strong Nash implementation

Proposition 1 Under Assumption 1, the game Γ with complete information is such that

- for all $a^* \in NE(\Gamma)$, $p_i^* = \bar{p}$ for all i and $(\bar{p}, (\phi_i(a^*))_{i \in I}) \in WE(\mathcal{E})$;
- for all $(p^*, (x_i^*)_{i \in I}) \in WE(\mathcal{E})$, there exists $a^* \in NE(\Gamma)$ such that for all i , $p_i^* = p^*$ and for all $i \in I$, $\phi_i(a^*) = x_i^*$.

Proof We first show that every Nash equilibrium induces a Walrasian equilibrium (“ $NE(\Gamma) \subseteq WE(\mathcal{E})$ ”).

Let $a^* = (a_i^*)_i \in NE(\Gamma)$, where $a_i^* = (p_i^*, (z_j^{*i})_{j \neq i})$; then a^* is individually rational:

$$\text{for all } i \in I, g_i(a^*) \geq u_i(\omega_i).$$

Indeed, suppose there exists $i_o \in I$ such that $g_{i_o}(a^*) < u_{i_o}(\omega_{i_o})$. Then by playing the strategy $\bar{a}_{i_o} = (\bar{p}_{i_o}, (\bar{z}_j^{i_o})_{j \neq i_o})$ defined by $\bar{p}_{i_o} = p_{i_o}^*$ and for all $j \neq i_o$

$$\bar{z}_j^{i_o} = \sum_{h \notin \{i_o, j\}} (z_h^{*j} - z_j^{*h}) + z_{i_o}^{*j}$$

player i_o can get $\phi_{i_o}(a_{-i_o}^*, \bar{a}_{i_o}) = \omega_{i_o}$. By definition and assumption,

$$g_{i_o}(a_{-i_o}^*, \bar{a}_{i_o}) = u_{i_o}(\phi_{i_o}(a_{-i_o}^*, \bar{a}_{i_o})) = u_{i_o}(\omega_{i_o}) > g_{i_o}(a^*),$$

which contradicts the definition of a Nash equilibrium. Consequently, each player satisfies his strategic budget constraint. For if i_o violates his strategic budget constraint, then $\phi_{i_o}(a^*) = 0$ as $\psi_{i_o}(a^*) = -\omega_{i_o}$ and $X_i = \mathbb{R}_+^L$ for all i , in which case the boundary condition is not met.

Now, for all i , $\phi_i(a^*) \in \text{int}(\hat{X}_i)$. Indeed, suppose $\phi_{i_o}(a^*) \in \partial(\hat{X}_{i_o})$ for some $i_o \in I$. Then $\phi_i(a^*) \in \partial(X_i)$ for some $i \in I$ possibly different from i_o . But then the boundary condition is violated.

Now, we show that for all $i \in I$, $p_i^* = \bar{p} \in \mathbb{R}_{++}^L$. Suppose that there exists j, j' such that $p_j^{*h} > p_{j'}^{*h} \geq 0$ for some commodity $h \in \{1, \dots, L\}$. Some player $i \notin \{j, j'\}$ is such that $\phi_i(a^*) \in \text{int}(\hat{X}_i)$, as we have seen. By the local non satiation hypothesis, for every $\epsilon > 0$ there exists $y_i \in \hat{X}_i$ such that $\|y_i - \phi_i(a^*)\| \leq \epsilon$ and $u_i(y_i) > u_i(\phi_i(a^*))$.

Consider the following strategy for player i : $a_i = (\tilde{p}_i, (\tilde{z}_j^i)_{j \neq i})$ where

- $\tilde{p}_i = p_i^*$,
- $\tilde{z}_k^i = z_k^{*i}$, for all $k \notin \{i, j, j'\}$
- $\tilde{z}_{j'}^i = z_{j'}^{*i} + \frac{p_{j'}^*(y_i - \phi_i(a^*))}{(p_j^{*h} - p_{j'}^{*h})} I_{\{h\}} + y_i - \phi_i(a^*)$
- $\tilde{z}_j^i = z_j^{*i} - \frac{p_j^*(y_i - \phi_i(a^*))}{(p_j^{*h} - p_{j'}^{*h})} I_{\{h\}}$

where $I_{\{h\}} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^L$ with nonzero h -th coordinate. This strategy allows player i to obtain y_i , a contradiction.

Now, since every player $i \in I$ satisfies his strategic budget constraint, then

$$\sum_i \phi_i(a^*) = \sum_i (\psi_i(a^*) + \omega_i).$$

Let us verify that for all $i \in I$,

$$\phi_i(a^*) \in \text{argmax } u_i(x),$$

under the constraint $\bar{p}x \leq \bar{p}\omega_i$. Suppose that there exists $i_o \in I$ for which there exists $y_{i_o} \in X_{i_o} = \mathbb{R}_+^L$ with $\bar{p}y_{i_o} \leq \bar{p}\omega_{i_o}$ and $u_{i_o}(y_{i_o}) > u_{i_o}(\phi_{i_o}(a^*))$. By quasi concavity and local nonsatiation of u_{i_o} , every element

$$z \in [\phi_{i_o}(a^*), y_{i_o}] = \{\lambda \phi_{i_o}(a^*) + (1 - \lambda)y_{i_o} : \lambda \in [0, 1]\}$$

is such that $u_{i_o}(z) > u_{i_o}(\phi_{i_o}(a^*))$. For z close enough to $\phi_{i_o}(a^*)$, then $z \in \hat{X}_{i_o}$ since $\phi_{i_o}(a^*) \in \text{Int}(\hat{X}_{i_o})$. Moreover, $\bar{p}z \leq \bar{p}\omega_{i_o}$. Hence using the strategy $\bar{a}_{i_o} = (\tilde{p}_{i_o}, (\tilde{z}_j^{i_o})_{j \neq i_o})$ which consists in playing the following price and quantity setting actions when he observes $\bar{p}$:

- $\tilde{p}_{i_o} = p_{i_o}^* = \bar{p}$,
- $\tilde{z}_j^{i_o} = z_j^{*i_o} + z - \phi_{i_o}(a^*)$ for some j ,
- $\tilde{z}_k^{i_o} = z_k^{*i_o}$ for all $k \notin \{i_o, j\}$

allows player i_o to get z as a final outcome. Indeed, $(\phi_k(a^*))_{k \in I}$ belongs to the interior of the feasible set F ; for z close enough to $\phi_{i_o}(a^*)$, we get that $(\psi_k(a_{-i_o}^*, \bar{a}_{i_o}) + \omega_k)_{k \in I}$ still belongs to F . If no player goes bankrupt in spite of the deviation of i_o , then $\phi_{i_o}(a_{-i_o}^*, \bar{a}_{i_o}) = z$. If player $j \neq i_o$ goes bankrupt, then he gets zero and player i_o still gets z .

We now show that $WE(\mathcal{E}) \subseteq NE(\Gamma)$.

Let $(p^*, (x_i^*)_i) \in WE(\mathcal{E})$. Consider the strategy profile $a^* = (a_i^*)_i$ defined for all $i \in I$ by $a_i^* = (p_i^*, (z_j^{*i})_{j \neq i})$ where $p_i^* = p^*$ and $z_j^{*i} = \frac{1}{N}(x_i^* - \omega_i)$. We then see that $a^* \in NE(\Gamma)$ and for all $i \in I$, $\phi_i(a^*) = x_i^*$. Indeed, player i has no incentive to deviate in prices as his own price does not affect his final outcome (recall that the budget constraint of agent i does not depend on his own price). In addition, player i has no incentive not to satisfy his budget constraint since he then receives zero. The Walrasian equilibrium being individually rational, he is better off satisfying his constraint. $\square$

Remark The reader who dislikes the boundary condition for its strong implications on the endowments should note that the above result is true for constrained Walrasian equilibria, once the boundary condition in Assumption 1 is replaced with the assumption that for all $i \in I$, $\omega_i > 0$ and preferences are strictly monotone. Indeed, any $CWE(\mathcal{E})$ can be obtained as a $NE(\Gamma)$, by the same argument as above since it makes no use of the boundary condition. Now, to show that any Nash equilibrium induces a constrained Walrasian equilibrium, the following modifications of the proof are in order:

- No player goes bankrupt, since the no-trade is preferred to 0.
- Every player fixes the same price. Otherwise, there is an arbitrage opportunity for some player i which is not saturated by some allocation which is feasible (if he were saturated, then he would possess all of the endowments, hence some other player would not hold anything; but we are at a Nash equilibrium, and consequently every player is individually rational).

The rest of the proof follows *verbatim*. We shall use again constrained Walrasian equilibria in the proof of Proposition 5.

Proposition 2 *Under Assumption 1, the semi-strong equilibria of Γ fully implement every Walrasian equilibrium of $\mathcal{E}$.*

Sketch of the proof Take $(p^*, (x_i^*)_i) \in WE(\mathcal{E})$ and consider $b^* = (b_i^*)_i$ defined for all $i \in I$ by $b_i^* = (p_i^*, (z_j^{*i})_{j \neq i})$, where $p_i^* = p^*$ and $z_j^{*i} = \frac{1}{N}(x_i^* - \omega_i)$.

By the Pareto optimality of $(x_i^*)_i$ and feasibility, the grand coalition has no improvement b upon b^* . Now, assume $S \subset I$ has an improvement b upon b^* . Then there must exist $i \in S$ such that when playing b_i instead of b_i^* , i quotes a different price vector $p_i^* \neq p^*$. Hence some player $i_o \in S \setminus \{i\}$ faces an arbitrage opportunity in $\Gamma^{b^*, S}$ if b is played. Indeed, $|S| \geq 2$ since b^* is a Nash equilibrium. Now by using some strategy $\tilde{b}_{i_o}$, i_o could improve his payoff. Hence $b \notin NE(\Gamma^{b^*, S})$ and therefore is not an improvement.

The converse argument follows the proof of Proposition 1. $\square$

Remark In the proof of Proposition 3', we show in appendix that $WE(\mathcal{E}) \subseteq CPNE(\Gamma)$. As $CPNE(\Gamma) \subseteq NE(\Gamma) = WE(\mathcal{E})$, then $WE(\mathcal{E}) = CPNE(\Gamma)$. Hence our strategic market game is such that outcomewise, the Nash, semi-strong Nash and coalition-proof Nash equilibria are all equivalent. In general, the following inclusions hold, $ssNE(\Gamma) \subseteq CPNE(\Gamma) \subseteq NE(\Gamma)$, but may be strict.

4 Correlation

We now recall the notion of a coalition-proof correlated equilibrium introduced by Ray [19] and give a definition of correlated semi-strong and correlated strong equilibria. We present some examples to relate to one another the solution concepts presented so far and give an implementation result using the strategic market mechanism presented above.

Definition 5

- A *correlation device* d consists of finite sets of messages, $\{M_1, \dots, M_N\}$, and a probability distribution P over $M = \prod_{i \in I} M_i$. The device selects private messages for each player according to P .
- The extended game Γ_d is the game where the correlation device d sends private messages to the players and then the game Γ is played.
- A pure strategy σ_i for player i in the game Γ_d is a mapping $\sigma_i : M_i \rightarrow A_i$.
- The payoff to player i when the pure strategy vector $\sigma = (\sigma_i)_i$ is played is

$$U_i(\sigma) = \sum_{m \in M} P(m) u_i \circ \phi_i((\sigma_i(m_i))_i).$$

Definition 6

- A *coalition-proof correlated equilibrium* (CPCE) of the game Γ is a pair (d, σ) such that σ is a CPNE of the extended game Γ_d .
- A *correlated semi-strong equilibrium* (CssE) of the game Γ is a pair (d, σ) with d a correlation device and σ a ssNE of the game Γ_d .

- A *correlated strong equilibrium* (CsE) of the game Γ is a pair (d, σ) with d a correlation device and σ a sNE of the game Γ_d .

Remarks

- In Aumann's [2] Bayesian interpretation of correlated equilibria, uncertainty bears on strategies. Here, typically, the space of strategies of a player is not finite. Hence to impose $|M_i| < \infty$ for all i is a real restriction, that could however be suppressed without modifying any results of this paper, except for section 5. One can in fact define in all generality (Forges [8]) a correlation device d as a probability space $(C, \mathcal{C}, P)$ together with sub σ -fields $(\mathcal{C}_i)_{i \in I}$ of $\mathcal{C}$. The extension Γ_d of the game Γ by the device d is the game where nature first selects $c \in C$ according to P , each player $i \in I$ is then informed of the events in $\mathcal{C}_i$ which contain c , and finally the game Γ is played. In the game Γ_d , a pure strategy σ_i for player i is a $\mathcal{C}_i$ -measurable mapping $\sigma_i : C \rightarrow (A_i, \mathcal{B}(A_i))$. The payoff to player i when $\sigma = (\sigma_i)_{i \in I}$ is played is $E^P(u_i(\phi_i(\sigma)))$. The probability measure π_i induced on A_i by σ_i and P is defined as follows:

$$\text{for all } B \in \mathcal{B}(A_i), \pi_i(B) = P((\sigma_i)^{-1}(B)).$$

- As pointed out in Ray [19], coalition-proofness depends on the structure of the correlation device. The revelation principle, which normally allows one to restrict attention to canonical representations, namely the equilibria consisting of a canonical device and the obedient strategy, does not hold for CPCE.
- Example 2 in Ray [19] can be used to show that the revelation principle does not hold for CSE. Recall simply that in a 2-player game, CPCE and CSE are equivalent notions.
- Deviations by coalitions take place *ex ante*, before the players learn the actions recommended by the correlation device, and a coalition which deviates cannot construct a new correlation device. Without this restriction, the game is no longer with complete information, but with incomplete information (see [9]).
- Ray has shown in [19] that every pure CPNE is a direct CPCE. It is also true that any pure ssNE is a direct CSE. This property will be used in a crucial way in the next proposition.
- For $N = 2$, ssNE and CPNE are equivalent notions, as are CSE and CPCE. Ray [19] proves the existence of such equilibria. However, existence is not guaranteed for $N \geq 3$, as he shows in a counterexample.

For the sake of completeness, the next example reminds that CPNE, ssNE and sNE are distinct notions, as are their correlated counterparts.

Example 1

Consider the 3-player game shown below, where player A chooses rows, player B chooses columns and player C chooses matrices:

	B_1	B_2		B_1	B_2
A_1	1, 1, -5	-5, -5, 0	A_1	-1, -1, 5	-5, -5, 0
A_2	2, -5, 0	0, 0, 10	A_2	-5, -5, 0	-2, -2, 0
	C_1			C_2	

The two Nash equilibria of this game are $N_1 = (A_2, B_2, C_1)$ and $N_2 = (A_1, B_1, C_2)$. Note that N_2 is not a sNE (as it is Pareto dominated by N_1), nor a ssNE. Holding the action played by C fixed to C_1 , (A_2, B_2) is the unique Nash equilibrium (note that to play A_2 is a dominant strategy for player A). (A_1, B_1) is a profitable deviation from (A_2, B_2) which is not a Nash equilibrium; hence (A_2, B_2, C_1) is a ssNE but not a sNE.

The game presented in introduction in Bernheim, Peleg and Whinston [3], which inspired the above game, gives an example of a CPNE which is not a ssNE and hence, of a CPCE which is not a Csse. It is identical to the game above, but for the entry (A_2, B_1, C_1) which reads $(-5, -5, 0)$.

Example 2

We now observe that for the no-trade Walrasian equilibrium associated to the following economy, the equivalence ssNe=CPCE=Csse holds. We therefore observe that in some cases, correlation does not modify the strategic equilibrium outcomes. Moreover, in this example, the unique ssNE is not a sNE.

Let $I = \{1, 2, 3, 4\}$, $L = 2$ and $X_i = \mathbb{R}_+^2$. For all $i \in I$, let $u_i(x, y) = x^\alpha y^{1-\alpha}$, with $\alpha = 1/2$.

For each player, the marginal rate of substitution between x and y is

$$\frac{\alpha}{1-\alpha} \frac{y}{x} = \frac{p^x}{p^y}.$$

Let $\omega_i = (1/2, 1/2)$ for all i . Then there exists a unique Walrasian equilibrium, the no-trade equilibrium, with $p^* = (p^x, p^y) = (1/2, 1/2)$ and $x_i = \omega_i$ for all $i \in I$.

Note that the CPCE corresponding to the no-trade Walrasian equilibrium is efficient. Indeed, if it were not Pareto optimal, then the grand coalition could play the no-trade, whatever the message $m \in M$ delivered by the device. This would represent a profitable deviation, which is self-enforcing.

By Proposition 2, the no-trade equilibrium is a semi-strong equilibrium, hence a correlated semi-strong equilibrium (trivially).

We now exhibit a coalition with a profitable deviation that is not a Nash equilibrium to show that the no-trade equilibrium is a ssNE but not a sNE. Assume $p_1^h \neq p^{*h} = p_3^h$ is observed by player 2 and $p_2^h \neq p^{*h} = p_4^h$ is observed

by player 1, for $h \in \{x, y\}$. By playing the following strategy a_i , player $i \in S = \{1, 2\}$ would get $\bar{\omega}/2 > \omega_i$: $a_i = (\tilde{p}_i, (\tilde{z}_j^i)_{j \neq i})$ where

- $\tilde{p}_i = p_i^*$,
- $\tilde{z}_k^i = 0$ for all $k \notin \{i, j, j'\}$
- $\tilde{z}_{j'}^i = \frac{p_{j'} \hat{y}_i}{(p_j^h - p_{j'}^h)} I_{\{h\}} + \bar{\omega}/2 - \omega_i$
- $\tilde{z}_j^i = -\frac{p_{j'} \hat{y}_i}{(p_j^h - p_{j'}^h)} I_{\{h\}}$

where $I_{\{h\}} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^L$ with nonzero h -th coordinate, $\hat{y}_i = \bar{\omega}/2$ and $j' = j + 2$.

Proposition 3 *Assume Assumption 1 holds. Every Walrasian equilibrium can be implemented as a correlated semi-strong equilibrium (CsSE).*

Proof Let $(p^*, (x_i^*)_{i \in I})$ be a Walrasian equilibrium. Consider the trivial correlation device d^* with $|M_i| = 1$ for all $i \in I$, and let $\bar{\sigma} = (\bar{\sigma}_i)_i$ be such that for all $i \in I$, $\bar{\sigma}_i(m_i) = (p_i^*, (z_j^{*i})_{j \neq i})$ where $p_i^* = p^*$ and $z_j^{*i} = \frac{1}{N}(x_i^* - \omega_i)$ for all j . We show that $\bar{\sigma}$ is a ssNE, and hence a CsSE: there is no coalition $S \subset I$ which has a profitable deviation $\bar{\sigma}_S$ upon $\bar{\sigma}$ which is a Nash equilibrium. Indeed, in order to improve his payoff, player $j \in S$ must change his price quoting strategy. Suppose there exists $j \in S$ such that $\bar{\sigma}_{S,j}(m_j) = (p_j, (z_k^{*j})_{k \neq j})$ with $p_j \neq p^*$. Assume wlog $p_j^h < p^{*h}$ for some $h \in \{1, \dots, L\}$. Then there exists $j' \in I \setminus S$ and $i \in S \setminus \{j\}$ such that $p_{j'} = p^*$ and i faces an arbitrage opportunity. Note that S is a strict subset of I as $(x_i^*)_i$ is Pareto optimal and $\{j\}$ is a strict subset of S as a Walrasian equilibrium is a Nash equilibrium.

As $(p^*, (x_i^*)_{i \in I})$ is a Walrasian equilibrium, the boundary condition implies that $x_i^* \in \text{int} \hat{X}_i$ for all i .

By the local non satiation hypothesis, for every $\epsilon > 0$ there exists $y_i \in \hat{X}_i$ such that $\|y_i - x_i^*\| \leq \epsilon$ and $u_i(y_i) > u_i(x_i^*)$.

Consider the following strategy for player i : $\sigma_i(m_i) = (\tilde{p}_i, (\tilde{z}_j^i)_{j \neq i})$ where

- $\tilde{p}_i$ is chosen according to $\bar{\sigma}_S$,
- $\tilde{z}_k^i = z_k^{*i}$, for all $k \notin \{i, j, j'\}$
- $\tilde{z}_j^i = z_j^{*i} + \frac{p_j(y_i - x_i)}{(p_{j'}^h - p_j^h)} I_{\{h\}} + y_i - x_i$
- $\tilde{z}_{j'}^i = z_{j'}^{*i} - \frac{p_j(y_i - x_i)}{(p_{j'}^h - p_j^h)} I_{\{h\}}$

where $I_{\{h\}} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^L$ with nonzero h -th coordinate, $(z_l^{*i})_{l \neq i}$ is the quantity setting allocation of player i according to $\bar{\sigma}$ (resp. $\bar{\sigma}_S$) if $i \in I \setminus S$ (resp. S), and x_i is the final allocation received by i when $\bar{\sigma}$ is played by $I \setminus S$

and $\bar{\sigma}_S$ is played by S . Player i gets $y_i \in \hat{X}_i$, and thus the improvement by coalition S is not a Nash equilibrium. Hence $\bar{\sigma}$ is a ssNE. $\square$

Remark We show in appendix (Proposition 3') that every Walrasian equilibrium can be implemented as a CPCE.

5 Semi-strong Bayesian rationality

In this section, we show that the result of Aumann [2] can be extended to correlated semi-strong equilibria.

Let G be an N -person game in strategic form. Let A_i be the set of pure actions of player $i \in I = \{1, \dots, N\}$ and $g_i(a)$ be the payoff to i of an N -tuple $a = (a_i)_{i \in I}$. Let $A = \prod_{i \in I} A_i$ and $g = (g_i)_{i \in I}$. Assume exogenously given

- a finite set Ω with generic element ω ;
- for each player i , a probability measure q_i on Ω ;
- for each player i , a partition $\mathcal{P}_i$ of Ω .

The set Ω represents the set of all possible states of the world, and each ω denotes a specific state. The probability measure q_i is i 's prior on Ω . The partition $\mathcal{P}_i$ is i 's information partition. If the true state of the world is $\omega \in P \in \mathcal{P}_i$, then i knows that some element of P is the true state of the world but does not know which one it is.

Common prior assumption All priors q_i are the same: there is a probability measure q on Ω such that $q = q_1 = \dots = q_N$.

Let $a_i^*(\omega)$ be the action chosen by i in the game G at the state ω of the world. Each a_i^* is an exogenously specified function from Ω to A_i . We assume that each player i knows which action he chooses, namely, a_i^* is measurable with respect to $\mathcal{P}_i$.

Definition 7 Let $a^*(\omega) = (a_1^*(\omega), \dots, a_N^*(\omega))$ be the N -tuple of actions chosen at state ω .

- A player i is *Bayes rational at ω* if the action chosen maximizes his expected payoff with respect to q given his information:

$$E(g_i(a^*)|\mathcal{P}_i)(\omega) \geq E(g_i(a_i, a_{-i}^*)|\mathcal{P}_i)(\omega), \quad \text{for all } a_i \in A_i.$$

- The set of players I is *semi-strongly Bayes rational at ω* if there exists no coalition S and strategy τ_S such that

- for all $i \in S$, i is Bayes rational at ω by playing $\tau_{S,i}$ when for all $j \in S \setminus \{i\}$, j plays $\tau_{S,j}$ and for all $j \notin S$, j plays a_j^* ;

– for all $i \in S$,

$$E(g_i(\tau_S, a_{-S}^*) | \mathcal{P}_i)(\omega) \geq E(g_i(a^*) | \mathcal{P}_i)(\omega)$$

with strict inequality for some $i \in S$.

Proposition 4 *If I is semi-strongly Bayes rational at each ω , then the distribution of the strategy N -tuple $a^* : \Omega \rightarrow \prod_i A_i$ is a correlated semi-strong equilibrium distribution.*

The proof is given in appendix.

6 Transfers

In Proposition 3, we have shown that $WE(\mathcal{E}) \subseteq C_{ss}E(\Gamma)$, and in Proposition 3' in appendix, we show $WE(\mathcal{E}) \subseteq CPCE(\Gamma)$. In order to get full implementation, we need to allow for transfers and substitute the whole set of Pareto optimal allocations to the set of Walrasian equilibria.

We now define the notions of coalition-proof correlated equilibrium and correlated semi-strong equilibrium with transfers, and obtain full implementation of all Pareto optimal allocations with these solution concepts applied to our strategic market mechanism.

For all $i \in I$, let M_i be the finite space of messages of player i , with generic element m_i .

Definition 8

- A *correlated contract*, (d, y) , is a correlation device $d = (M_1, \dots, M_N, P)$ together with an N -tuple of transfers $y = (y_i)_{i \in I} \in \mathbb{R}^N$ which satisfies $\sum_{i \in I} y_i = 0$.
- Γ_y is the modified game obtained from Γ by replacing $\psi_i(\cdot)$ for all $i \in I$ with $\psi_i^y(\cdot)$ defined by

$$\psi_i^y(a) = \begin{cases} \sum_{j \neq i} (z_j^i - z_i^j) & \text{if } \sum_{j \neq i} p_j(z_j^i - z_i^j) \leq y_i \\ -\omega_i & \text{otherwise} \end{cases}$$

- The corresponding outcome function is defined by

$$\phi_i^y(a) = \text{proj}_i \circ \text{proj}_F((\omega_k + \psi_k^y(a))_{k=1, \dots, N}).$$

- The extended game $\Gamma_{d,y}$ is the game where the correlation device d sends private messages to the players, and then the game Γ_y is played.
- A pure strategy σ_i for player i in the game $\Gamma_{d,y}$ is a mapping $\sigma_i : M_i \rightarrow A_i$.

- The payoff to player i when the pure strategy vector $\sigma = (\sigma_i)_i$ is played is

$$U_i(\sigma) = \sum_{m \in M} P(m) u_i \circ \phi_i^y((\sigma_i(m_i))_i).$$

- Let $R : (\Omega, \mathcal{A}, \lambda) \rightarrow (\mathbb{R}_+^L)^N$ be any random allocation, where $(\Omega, \mathcal{A}, \lambda)$ is an arbitrary probability space. R is said to be Pareto optimal if it is *ex post* feasible, namely, $\sum_{i \in I} R_i(\omega) = \sum_{i \in I} \omega_i$ a.s., and there does not exist any other feasible random allocation $R' : (\Omega', \mathcal{A}', \nu) \rightarrow (\mathbb{R}_+^L)^N$ such that $E^\nu(u_i(R'(\omega))) \geq E^\lambda(u_i(R(\omega)))$ for all i , with strict inequality for some $i \in I$.

Remark We call a Pareto optimal random allocation *ex post* feasible in order to distinguish our feasibility notion from the one introduced by Prescott and Townsend [18] where feasibility is only required in expectation.

Definition 9

- A *collusion-proof correlated equilibrium* (cpCE) of the game Γ is a triple (d, y, σ) such that σ is a CPNE of the extended game $\Gamma_{d,y}$, i.e. (d, σ) is a CPCE of the game Γ_y .
- A *correlated semi-strong equilibrium with transfers* (cssE) of the game Γ is a triple (d, y, σ) such that σ is a ssNE of the extended game $\Gamma_{d,y}$, i.e. (d, σ) is a CssE of the game Γ_y .

Assumption 2

- $N \geq 3$, and for all $i \in I$, $X_i = \mathbb{R}_+^L$.
- For all $i \in I$, u_i is quasi concave on $\hat{X}_i$.
- (constrained local non satiation) For all $i \in I$, if $0 \leq x_i^* < \bar{\omega}$, then for all $i \in I$, there exists $0 \leq x_i \leq \bar{\omega}$ such that $u_i(x_i) > u_i(x_i^*)$.

Proposition 5 *Assume Assumption 2 holds and preferences are strongly monotone and continuous. Every cssE induces a deterministic Pareto optimal outcome. Conversely, every Pareto optimal allocation can be obtained as a cssE.*

Proof Consider a Pareto optimal allocation $(x_i^*)_{i \in I}$. By the constrained second welfare theorem (see Appendix), as preferences are strongly monotone and continuous, there is a vector of prices $p^* \neq 0$ and an assignment of wealth levels $w^* = (w_i^*)_{i \in I}$ with $\sum_i w_i^* = p^* \sum_i \omega_i$ such that for all $i \in I$, $x_i^* \in \argmax_x u_i(x)$ under the constraints $p^* x \leq w_i^*$ and $x \leq \sum_i \omega_i$, and $\sum_{i \in I} x_i^* = \sum_{i \in I} \omega_i$. Consider the contract (d^*, y^*) where $y^* = (y_i^*)_i$ with $y_i^* = w_i^* - p^* \omega_i$, and

d^* is trivial, namely $|M_i| = 1$, and the strategy of player i is defined by $\bar{\sigma}_i(m_i) = (p^*, (z_j^{*i})_{j \neq i})$ where $z_j^{*i} = \frac{1}{N}(x_i^* - \omega_i)$. We can show that $\bar{\sigma}$ is a cssE using the proof of Proposition 3. Indeed, Proposition 3 shows that $\bar{\sigma}$ is a ssNE of the game Γ_{d^*, y^*} . Hence $(d^*, \bar{\sigma})$ is a CSE of Γ_{y^*} .

Observe that by Jensen's inequality, every Pareto optimal random allocation must be in fact deterministic.

Now, let (d, y, σ) be a cssE. If the random allocation $(x_i)_{i \in I}$ induced by (d, y, σ) is not Pareto optimal, there exists a deterministic Pareto optimal allocation which Pareto dominates it. (Indeed, if $(x_i)_{i \in I}$ is not deterministic, it is Pareto dominated by $(E(x_i))_{i \in I}$. If this last deterministic allocation is itself not Pareto optimal, then there exists a Pareto optimal deterministic allocation $(\tilde{x}_i)_{i \in I}$ in $\mathcal{E}$ which dominates $(E(x_i))_{i \in I}$, hence dominates $(x_i)_{i \in I}$.)

Using the constrained second welfare theorem, the grand coalition can choose to play a constrained Walrasian equilibrium with transfers which gives an improvement upon (d, y, σ) . □

Remarks

- All the transfers used in the above argument are chosen by the grand coalition, which therefore must know all the characteristics of the economy. This admittedly very strong assumption could be dropped.
- We show in appendix (Proposition 5') that every cpCE induces a deterministic Pareto optimal outcome and every Pareto optimal allocation can be obtained as a cpCE.

7 Conclusion

In this paper, we studied implementation problems in a complete information pure exchange economy, making use of a feasible strategic market mechanism. The solution concepts used are the following: Nash, semi-strong Nash, coalition-proof Nash, coalition-proof correlated (with and without transfers), and correlated semi-strong equilibria (with and without transfers). In a complementary paper [9], we study mechanism design problems in an incomplete information setting.

Some questions remain open. In particular,

- Moreno and Wooders [16] have defined a coalition-proof correlated equilibrium concept intermediate to that of Ray [19] and Einy and Peleg [7]. In their model, deviations occur before the mechanism recommends actions but coalitions are allowed to construct new mechanisms. We leave for further research the task of extending our result to that concept.
- The strategic outcome function used here is discontinuous, as in Schmeidler [20] and Dubey [4]. It would be possible to construct a continuous

strategic outcome function by defining a continuous penalty rule. Whether it is possible or not to get a feasible and smooth Nash competitive mechanism is still an open problem.

- The two-player case is largely unexplored in this literature and is covered by none of our results. Hurwicz presents the unique example we are aware of.

8 Appendix

8.1 Implementation in coalition-proof correlated equilibria

Proposition 3' *Assume Assumption 1 holds. Every Walrasian equilibrium can be implemented as a CPCE.*

Proof Let $(p^*, (x_i^*)_{i \in I})$ be a Walrasian equilibrium. Consider the trivial correlation device d^* with $|M_i| = 1$ for all $i \in I$, and let $\bar{\sigma} = (\bar{\sigma}_i)_i$ be such that for all $i \in I$, $\bar{\sigma}_i(m_i) = (p_i^*, (z_j^{*i})_{j \neq i})$ where $p_i^* = p^*$ and $z_j^{*i} = \frac{1}{N}(x_i^* - \omega_i)$ for all j . We show that $\bar{\sigma}$ is a CPNE, and hence a CPCE: there is no coalition $S \subset I$ which has an ICI $\bar{\sigma}_S$ upon $\bar{\sigma}$. Indeed, in order to improve his payoff, player $j \in S$ must change his price quoting strategy. Suppose there exists $j \in S$ such that $\bar{\sigma}_{S,j}(m_j) = (p_j, (z_k^{*j})_{k \neq j})$ with $p_j \neq p^*$. Assume wlog $p_j^h < p^{*h}$ for some $h \in \{1, \dots, L\}$. Then there exists $j' \in I \setminus S$ and $i \in S \setminus \{j\}$ such that $p_{j'} = p^*$ and i faces an arbitrage opportunity. Note that S is a strict subset of I as $(x_i^*)_i$ is Pareto optimal and $\{j\}$ is a strict subset of S as a Walrasian equilibrium is a Nash equilibrium.

As $(p^*, (x_i^*)_{i \in I})$ is a Walrasian equilibrium, the boundary condition implies that $x_i^* \in \text{int} \hat{X}_i$ for all i .

By the local non satiation hypothesis, for every $\epsilon > 0$ there exists $y_i \in \hat{X}_i$ such that $\|y_i - x_i^*\| \leq \epsilon$ and $u_i(y_i) > u_i(x_i^*)$.

Consider the following strategy for player i : $\sigma_i(m_i) = (\tilde{p}_i, (\tilde{z}_j^i)_{j \neq i})$ where

- $\tilde{p}_i$ is chosen according to $\bar{\sigma}_S$,
- $\tilde{z}_k^i = z_k^{*i}$, for all $k \notin \{i, j, j'\}$
- $\tilde{z}_j^i = z_j^{*i} + \frac{p_j(y_i - x_i)}{(p_{j'}^h - p_j^h)} I_{\{h\}} + y_i - x_i$
- $\tilde{z}_{j'}^i = z_{j'}^{*i} - \frac{p_j(y_i - x_i)}{(p_{j'}^h - p_j^h)} I_{\{h\}}$

where $I_{\{h\}} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^L$ with nonzero h -th coordinate, $(z_l^{*i})_{l \neq i}$ is the quantity setting allocation of player i according to $\bar{\sigma}$ (resp. $\bar{\sigma}_S$) if $i \in I \setminus S$ (resp. S), and x_i is the final allocation received by i when $\bar{\sigma}$ is played by $I \setminus S$ and $\bar{\sigma}_S$ is played by S . Player i gets $y_i \in \hat{X}_i$, and thus the improvement by coalition S is not internally consistent. Hence $\bar{\sigma}_I$ is a CPNE. $\square$

Proposition 5' *Assume Assumption 2 holds and preferences are strongly monotone and continuous. Every cpCE induces a deterministic Pareto optimal outcome. Conversely, every Pareto optimal allocation can be obtained as a cpCE.*

Proof Consider a Pareto optimal allocation $(x_i^*)_{i \in I}$. By the constrained second welfare theorem, as preferences are strongly monotone and continuous, there is a vector of prices $p^* \neq 0$ and an assignment of wealth levels $w^* = (w_i^*)_{i \in I}$ with $\sum_i w_i^* = p^* \sum_i \omega_i$ such that for all $i \in I$, $x_i^* \in \operatorname{argmax}_x u_i(x)$ under the constraints $p^* x \leq w_i^*$ and $x \leq \sum_i \omega_i$, and $\sum_{i \in I} x_i^* = \sum_{i \in I} \omega_i$. Consider the contract (d^*, y^*) where $y^* = (y_i^*)_i$ with $y_i^* = w_i^* - p^* \omega_i$ and d^* is trivial, namely $|M_i| = 1$, and the strategy of player i is defined by $\bar{\sigma}_i(m_i) = (p^*, (z_j^{*i})_{j \neq i})$ where $z_j^{*i} = \frac{1}{N}(x_i^* - \omega_i)$. We can show that $\bar{\sigma}$ is a cpCE using the proof of Proposition 3'. Indeed, it shows that $\bar{\sigma}$ is a CPNE of the game Γ_{d^*, y^*} . Hence $(d^*, \bar{\sigma})$ is a CPCE of Γ_{y^*} .

Observe that by Jensen's inequality, every Pareto optimal random allocation must be in fact deterministic.

Now, let (d, y, σ) be a cpCE. If the random allocation $(x_i)_{i \in I}$ induced by (d, y, σ) is not Pareto optimal, there exists a deterministic Pareto optimal allocation which Pareto dominates it. (Indeed, if $(x_i)_{i \in I}$ is not deterministic, it is Pareto dominated by $(E(x_i))_{i \in I}$. If this last deterministic allocation is itself not Pareto optimal, then there exists a Pareto optimal deterministic allocation $(\tilde{x}_i)_{i \in I}$ in $\mathcal{E}$ which dominates $(E(x_i))_{i \in I}$, hence dominates $(x_i)_{i \in I}$.)

Using the constrained second welfare theorem, the grand coalition can choose to play a constrained Walrasian equilibrium with transfers which gives an ICI upon (d, y, σ) . □

8.2 Proof of Proposition 4

Proposition 4 *If I is semi-strongly Bayes rational at each ω , then the distribution of the strategy N -tuple $a^* : \Omega \rightarrow \prod_i A_i$ is a correlated semi-strong equilibrium distribution.*

Proof Let $\Omega = \prod_{i \in I} M_i$, $q \in \Delta(\prod_{i \in I} M_i)$. As I is semi-strongly Bayes rational at each ω , then for all ω there exists no coalition S and strategy τ_S such that for all $i \in S$,

$$E(g_i(\gamma_i, \tau_{S \setminus i}, a_{-S}^*) | \mathcal{P}_i)(\omega) \leq E(g_i(\tau_S, a_{-S}^*) | \mathcal{P}_i)(\omega), \quad (4)$$

for any other action γ_i and

$$E(g_i(a^*) | \mathcal{P}_i)(\omega) \leq E(g_i(\tau_S, a_{-S}^*) | \mathcal{P}_i)(\omega) \quad (5)$$

with strict inequality for some $i \in S$. Hence there exists no S and $(\tau_S^\omega)_{\omega \in \Omega}$ such that equations (4) and (5) hold for all ω , with one strict inequality for at least one player i and one state ω . First, multiply the above inequalities by $q(\mathcal{P}_i(\omega))$

for all i and all ω , then sum each of them over all ω . We then observe that (d, σ) with d the trivial correlation device and $\sigma_i : M_i \rightarrow A_i$ the strategy defined by $\sigma_i(m_i) = a_i^*$ for all i is a correlated semi-strong equilibrium. $\square$

8.3 Constrained second welfare theorem

We here extend to constrained Walrasian equilibria the well-known Second Welfare Theorem, following the proof presented in [15].

Under strongly monotonic preferences, the following local nonsatiation condition holds:

Constrained local nonsatiation condition For all $i \in I$, if $x_i^* < \bar{\omega}$, then for all $i \in I$, there exists $x_i \leq \bar{\omega}$ such that $u_i(x_i) > u_i(x_i^*)$.

In words, the above condition states that if an agent does not possess all of the endowments, then he is not saturated on the feasible set.

Definition An allocation x^* and a price vector $p \neq 0$ constitute a *constrained price quasi-equilibrium with transfers* if there is an assignment of wealth levels $(w_i, \dots, w_N)$ with $\sum_i w_i = p\bar{\omega}$ such that

- 1) for every i , if $x_i \leq \bar{\omega}$ and $u_i(x_i) > u_i(x_i^*)$, then $p x_i \geq w_i$.
- 2) $\sum_i x_i^* = \bar{\omega}$.

Remark A *constrained price equilibrium with transfers* is defined analogously, by replacing condition 1) above with

- 1') for every i , $x_i^* \in \operatorname{argmax}_x u_i(x)$ under the constraints $p x \leq w_i$ and $x \leq \bar{\omega}$.

Proposition 6 *Assume preferences are quasi-concave, continuous, strongly monotone and $\omega_i > 0$ for all i . Then for every Pareto optimal allocation x^* , there is a price vector $p \neq 0$ such that (x^*, p) is a constrained price equilibrium with transfers.*

Proof Let

$$V_i = \left\{ x_i \in \mathbb{R}_+^L : x_i \leq \bar{\omega} \text{ and } u_i(x_i) > u_i(x_i^*) \right\}$$

and

$$V = \sum_i V_i = \left\{ \sum_i x_i : x_i \in V_i \text{ for all } i \right\}.$$

- Every set V_i is convex.
- The sets V and $\{\bar{\omega}\}$ are convex.
- $V \cap \{\bar{\omega}\} = \emptyset$.
- There is $p \neq 0$ and a number r such that $p z \geq r$ for all $z \in V$ and $p\bar{\omega} \leq r$.

- For every i , if $x_i \leq \bar{w}$ and $u_i(x_i) \geq u_i(x_i^*)$, then $p \sum_i x_i \geq r$.
 - Assume $x_i \leq \bar{w}$ and $u_i(x_i) \geq u_i(x_i^*)$ for all i . For all i such that $x_i < \bar{w}$, there exists $\hat{x}_i \leq \bar{w}$ such that $u_i(\hat{x}_i) > u_i(x_i)$ (by local non satiation). If there exists i_o such that $x_{i_o} = \bar{w}$, then take $\hat{x}_{i_o} = x_{i_o}$. Every $\hat{x}_i \in V_i$ and thus $\sum_i \hat{x}_i \in V$ and $p \sum_i \hat{x}_i \geq r$. Taking the limit as $\hat{x}_i \rightarrow x_i$ gives the result.
- $p \sum_i x_i^* = p\bar{w} = r$.
 - We have $p \sum_i x_i^* \geq r$ by the previous step. As $\sum_i x_i^* = \bar{w}$, then $p \sum_i x_i^* \leq r$, and hence the required inequality.
- For every i , if $x_i \leq \bar{w}$ and $u_i(x_i) > u_i(x_i^*)$, then $p x_i \geq p x_i^*$.
 - Assume $x_i \leq \bar{w}$ and $u_i(x_i) > u_i(x_i^*)$; then

$$p \left(x_i + \sum_{k \neq i} x_k^* \right) \geq r = p \left(x_i^* + \sum_{k \neq i} x_k^* \right) = p\bar{w}$$

and hence the result.

- The wealth levels $w_i = p x_i^*$ for $i \in I$ support (x^*, p) as a constrained price quasi-equilibrium with transfers.
- If there is a consumption vector $x'_i \in \mathbb{R}_+^L$ such that $p x'_i < w_i$ (a cheaper consumption for (p, w_i)), then $x_i \leq \bar{w}$ and $u_i(x_i) > u_i(x_i^*)$ imply $p x_i > w_i$.
 - (by contradiction) Suppose that there is an x_i with $x_i \leq \bar{w}$ and $u_i(x_i) > u_i(x_i^*)$ such that $p x_i = w_i$. By the cheaper consumption assumption, there exists $x'_i \in \mathbb{R}_+^L$ such that $p x'_i < w_i$. Then for all $\alpha \in [0, 1)$, we have $p(\alpha x_i + (1 - \alpha)x'_i) < w_i$. For α close to 1, the continuity of the preferences implies that $u_i(\alpha x_i + (1 - \alpha)x'_i) > u_i(x_i^*)$, a contradiction.

□

8.4 Postlewaite and Wettstein versus coalitions

In this section, we show by means of an example that the semi-strong Nash equilibria of the mechanism presented in Postlewaite and Wettstein [17] do not implement the constrained Walrasian allocations.

A strategy of player i in Postlewaite and Wettstein's mechanism consists in announcing an initial endowment profile, v_i , a net trade, z_i , a price vector, p_i , and two positive numbers (r_i, q_i) determining penalties associated with differences in individual announcements and weights assigned to the agents' demands.

In their mechanism, a weighted average of the prices announced by the agents is computed. It is defined in such a way that if all agents other than

agent i announce the same price, then i cannot affect the price as the weight corresponding to his announcement is zero.

The allocation that i receives, α_i , is determined by projecting his announced net trade onto the price hyperplane determined by the weighted average price. A rationing takes place if the allocation is not feasible and moreover, incentives to announce truthfully the endowments are provided.

Consider an exchange economy with three agents and three goods, x, y, z .

Let

$$\begin{aligned} p^x &= p^y = p^z = 1, \\ u_1(x, y, z) &= x + 10^{-5}y + 10^{-10}z, \\ u_2(x, y, z) &= z + 10^{-5}y + 10^{-10}x, \\ u_3(x, y, z) &= y + 10^{-5}x + 10^{-10}z, \end{aligned}$$

and

$$\omega_1 = (1/8, 1/2, 1/8), \quad \omega_2 = (1/8, 1/2, 1/8), \quad \omega_3 = (1/2, 1/4, 1/2).$$

Solving

$$\max u_1(x, y, z) \quad \text{subject to} \quad x + y + z \leq 3/4$$

and $x \leq 3/4, y \leq 5/4, z \leq 3/4$ yields the solution $(x_1^*, y_1^*, z_1^*) = (3/4, 0, 0)$. Similarly, solving

$$\max u_2(x, y, z) \quad \text{subject to} \quad x + y + z \leq 3/4$$

and $x \leq 3/4, y \leq 5/4, z \leq 3/4$ yields $(x_2^*, y_2^*, z_2^*) = (0, 0, 3/4)$. Finally, solving

$$\max u_3(x, y, z) \quad \text{subject to} \quad x + y + z \leq 5/4$$

and $x \leq 3/4, y \leq 5/4, z \leq 3/4$ yields $(x_3^*, y_3^*, z_3^*) = (0, 5/4, 0)$.

Consider a deviation in prices by the coalition composed of players 1 and 2 as follows:

$$p'_1 = p'_2 = (1/2, 2, 1/2)$$

while player 3 still plays $p_3 = (1, 1, 1)$.

We now construct the average price vector, $\bar{p}$, as in [17], using p'_1, p'_2 and p_3 , as follows:

Let

$$a_i = \sum_{j, k \neq i} \|p_j - p_k\|^2, \quad \text{and} \quad a = \sum_i a_i.$$

Define

$$b_i = \begin{cases} a_i/a & \text{if } a > 0 \\ 1/3 & \text{if } a = 0 \end{cases}$$

With the above, the average price vector $\bar{p}$ is defined by $\bar{p} := \sum_i b_i p_i$.

In our example, we find $a_1 = a_2 = 3/2$ and $a_3 = 0$, hence $a = 3$, and $b_1 = b_2 = 1/2$ and $b_3 = 0$, yielding $\bar{p} = (1/2, 2, 1/2)$.

At the price $\bar{p}$, solving

$$\max u_1(x, y, z) \quad \text{subject to} \quad 1/2x + 2y + 1/2z \leq 9/8$$

and $x \leq 3/4$, $y \leq 5/4$, $z \leq 3/4$ yields the solution $(x_1^*, y_1^*, z_1^*) = (3/4, 3/8, 0)$. Similarly, solving

$$\max u_2(x, y, z) \quad \text{subject to} \quad 1/2x + 2y + 1/2z \leq 9/8$$

and $x \leq 3/4$, $y \leq 5/4$, $z \leq 3/4$ yields $(x_2^*, y_2^*, z_2^*) = (0, 3/8, 3/4)$.

The deviation in prices by players 1 and 2 combined with a strategy which consists in declaring a large r_i , with $r_1 = r_2$ (to ensure the desired final allocation) and truthfully revealing endowments is a Nash equilibrium and hence the constrained Walrasian equilibrium cannot be implemented in semi-strong Nash.

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