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Addendum

On deformations of C*-algebras by actions of Kählerian Lie groups

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We show that two approaches to equivariant deformation of C*-algebras by actions of negatively curved Kählerian Lie groups, one based on oscillatory integrals and the other on quantizations maps defined by dual 2-cocycles, are still equivalent despite the nonunitarity of our 2-cocycles.

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0. Introduction

The paper [2] relies in a crucial way on unitarity of the cocycle Ω_θ, but it turns out that Ω_θ is only a coisometry. This can be seen already from [2, Theorem A.1]: the map Φ is injective but not surjective, contrary to the claim there that it is a diffeomorphism (see Lemma 2.3). In fact, it seems doubtful that already the simplest Kählerian Lie group G, the connected component of the $ax+b$ group over the reals, supports any nontrivial dual unitary 2-cocycle. At the very least, such a cocycle must be such that the corresponding deformation of the function algebra $L^\infty(G)$ is not a type I factor, see [3, Remark 2.12]. The entire nonconnected $ax+b$ group does support a nontrivial dual cocycle though [3].

The unitarity of Ω_θ was stated for the first time in [4] as a consequence of the identity from [1, Proposition 8.47], which therefore is also wrong. The origin of the mistake is similar: a change of variables closely related to Φ is not bijective, see erratum to [1].

Although the dual cocycle Ω_θ is not unitary, it still has some nice properties that allow one to develop a deformation theory similar to [4]. Within the framework of this extended theory the main results of [2] remain true as stated, with essentially identical proofs. This is what will be explained in this erratum.

1. Deformation by Nonunitary Cocycles

We follow the conventions of [4]. Let G be a locally compact quantum group and $\hat{W}$ be the multiplicative unitary of the dual quantum group. Assume we are given an element $\Omega \in L^\infty(\hat{G}) \bar{\otimes} L^\infty(\hat{G})$ satisfying the cocycle identity

$$(\Omega \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \hat{\Delta})(\Omega).$$

We assume in addition that:

(1) the element Ω is coisometric: $\Omega\Omega^* = 1$;
(2) there exists a unitary $X \in L^\infty(\hat{G})$ such that $(XJ)^2 = 1$ and $\Omega = (X \otimes X)(\hat{R} \otimes \hat{R})(\Omega_{21}^*)\,\hat{\Delta}(X)^*$;
(3) we have $(\hat{\Delta} \otimes \iota)(\Omega)(\iota \otimes \hat{\Delta})(\Omega^*) = (\Omega^* \otimes 1)(1 \otimes \Omega)$.

Recall that $\hat{R}$ denotes the unitary antipode on $L^\infty(\hat{G})$, so that $\hat{R}(x) = Jx^*J$.

We then let

$$\tilde{J} := XJ, \quad L := (\tilde{J} \otimes \hat{J})\Omega\hat{W}^*(J \otimes \hat{J}),$$

and define, for $\nu \in \mathcal{K}^* = \mathcal{B}(L^2(G))_*$, a *quantization map*

$$T_\nu \colon L^\infty(G) \to \mathcal{B}(L^2(G)) \quad \text{by } T_\nu(x) := (\iota \otimes \nu)(L(x \otimes 1)L^*).$$

Given a left action $\alpha\colon A \to M(C_0(G) \otimes A)$ on a C*-algebra A, we define the Ω-*deformation* of A as the C*-subalgebra $A_\Omega \subset M(\mathcal{K} \otimes A)$ generated by the elements $(T_\nu \otimes \iota)\alpha(a)$ for $a \in A$ and $\nu \in \mathcal{K}^*$. Note that although this definition depends on the unitary X, a different choice would give an isomorphic C*-algebra.

The definition of A_Ω as such does not require conditions (1)–(3): we could take any unitary in $L^\infty(\hat{G})$ for X in order to define L. However, without additional assumptions it is not clear how reasonable this definition is, since already for the function algebra there is another more transparent definition. We can define a product $\star_\Omega$ on the Fourier algebra $A(G) := L^\infty(\hat{G})_*$ of G by

$$\omega \star_\Omega \nu = (\omega \otimes \nu)(\hat{\Delta}(\cdot)\Omega^*).$$

It is straightforward to check that the cocycle identity for Ω can be written as

$$(\hat{\Delta} \otimes \iota)(\hat{W}\Omega^*)\Omega_{12}^* = (\hat{W}\Omega^*)_{13}(\hat{W}\Omega^*)_{23},$$

see [4, Sec. 2.1]. This implies that we have a representation

$$\pi_\Omega \colon (A(G), \star_\Omega) \to \mathcal{B}(L^2(G)) \text{ defined by } \pi_\Omega(\omega) := (\omega \otimes \iota)(\hat{W}\Omega^*).$$

How is the algebra $\pi_\Omega(A(G))$ related to the deformation $C_0(G)_\Omega$ of $C_0(G)$ with respect to the left action of G on itself by translations? This can be answered under conditions (1)–(3), thanks to the following result, cf. [4, Identity (3.1)].

Proposition 1.1. *If an element* $\Omega \in L^\infty(\hat{G}) \mathbin{\bar{\otimes}} L^\infty(\hat{G})$ *satisfies conditions* (2) *and* (3), *then*

$$L_{23}\hat{W}_{12}L_{23}^* = (\hat{W}\Omega^*)_{12}L_{13}. \tag{1.1}$$

Proof. By the definition of L we have to prove that

$$\begin{aligned}(J \otimes \tilde{J} \otimes \hat{J})\Omega_{23}\hat{W}_{23}^*(J \otimes J \otimes \hat{J})\hat{W}_{12}(J \otimes J \otimes \hat{J})\hat{W}_{23}\Omega_{23}^*(J \otimes \tilde{J} \otimes \hat{J})\\ = \hat{W}_{12}\Omega_{12}^*(\tilde{J} \otimes J \otimes \hat{J})\Omega_{13}\hat{W}_{13}^*(J \otimes J \otimes \hat{J}).\end{aligned}$$

As $(J \otimes \hat{J})\hat{W} = \hat{W}^*(J \otimes \hat{J})$ and $\hat{W}_{23}\hat{W}_{12}\hat{W}_{23}^* = \hat{W}_{12}\hat{W}_{13}$, this is equivalent to

$$\begin{aligned}(J \otimes \tilde{J} \otimes \hat{J})\Omega_{23}(J \otimes J \otimes \hat{J})\hat{W}_{12}\hat{W}_{13}(J \otimes J \otimes \hat{J})\Omega_{23}^*(1 \otimes X \otimes 1)\\ = \hat{W}_{12}\Omega_{12}^*(\tilde{J} \otimes J \otimes \hat{J})\Omega_{13}\hat{W}_{13}^*,\end{aligned}$$

hence,

$$\begin{aligned}(J \otimes \tilde{J} \otimes \hat{J})\Omega_{23}(J \otimes J \otimes \hat{J})\hat{W}_{12}(J \otimes J \otimes \hat{J})\hat{W}_{13}^*\Omega_{23}^*(1 \otimes X \otimes 1)\\ = \hat{W}_{12}\Omega_{12}^*(\tilde{J} \otimes J \otimes \hat{J})\Omega_{13}\hat{W}_{13}^*.\end{aligned}$$

Multiplying by $\hat{W}_{12}^*(1 \otimes 1 \otimes J\hat{J})$ on the left and by $\hat{W}_{13}(J \otimes X^*J \otimes J)$ on the right and using that $XJ = JX^*$, we get

$$\begin{aligned}\hat{W}_{12}^*(J \otimes \tilde{J} \otimes J)\Omega_{23}(J \otimes J \otimes \hat{J})\hat{W}_{12}(J \otimes J \otimes \hat{J})\hat{W}_{13}^*\Omega_{23}^*\hat{W}_{13}(J \otimes J \otimes J)\\ = \Omega_{12}^*(\tilde{J} \otimes XJ \otimes J)\Omega_{13}(J \otimes J \otimes J).\end{aligned}$$

Since, we can replace $\hat{J}$ by J in this expression, we get

$$\begin{aligned}\hat{W}_{12}^*(1 \otimes X \otimes 1)(\hat{R} \otimes \hat{R})(\Omega^*)_{23}\hat{W}_{12}(\hat{R} \otimes \hat{R} \otimes \hat{R})(\hat{W}_{13}^*\Omega_{23}\hat{W}_{13})\\ = \Omega_{12}^*(X \otimes X \otimes 1)(\hat{R} \otimes \hat{R})(\Omega^*)_{13}.\end{aligned}$$

Recalling that the coproduct on $L^\infty(\hat{G})$ is given by $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$, we obtain

$$\begin{aligned}\hat{\Delta}(X)_{12}(\hat{\Delta} \otimes \iota)(\hat{R} \otimes \hat{R})(\Omega^*)(\hat{R} \otimes \hat{R} \otimes \hat{R})(\iota \otimes \hat{\Delta})(\Omega)_{213} \\ = \Omega^*_{12}(X \otimes X \otimes 1)(\hat{R} \otimes \hat{R})(\Omega^*)_{13}.\end{aligned}$$

Applying $\hat{R} \otimes \hat{R} \otimes \hat{R}$ and flipping the first two factors, we then obtain

$$(\iota \otimes \hat{\Delta})(\Omega)(\hat{\Delta} \otimes \iota)(\Omega^*)\,\hat{\Delta}(X)_{12} = \Omega^*_{23}(X \otimes X \otimes 1)(\hat{R} \otimes \hat{R})(\Omega^*)_{21},$$

where we used that $(\hat{R} \otimes \hat{R})\,\hat{\Delta} = \hat{\Delta}^{\mathrm{op}}\,\hat{R}$ and $\hat{R}(X) = X$. By virtue of assumption (2) this is equivalent to

$$(\iota \otimes \hat{\Delta})(\Omega)(\hat{\Delta} \otimes \iota)(\Omega^*) = \Omega^*_{23}\Omega_{12},$$

which is condition (3). □

Corollary 1.2 (cf. [4, Proposition 3.1]). *If Ω is a dual cocycle satisfying conditions* (1)–(3), *then*

$$\overline{\pi_\Omega(A(G))} = [T_\nu(x) \colon x \in C_0(G),\ \nu \in \mathcal{K}^*]. \tag{1.2}$$

It follows that $\overline{\pi_\Omega(A(G))}$ is a nondegenerate C^*-algebra of operators on $L^2(G)$ and

$$C_0(G)_\Omega = V(\overline{\pi_\Omega(A(G))} \otimes 1)V^*,$$

where $V = (\hat{J} \otimes \hat{J})\hat{W}(\hat{J} \otimes \hat{J})$.

We remind that the square brackets denote the norm closure of the linear span.

Proof. Slicing the first and the third legs of (1.1) and remembering that L_{13} is coisometric we immediately get the first statement. Next, the right-hand side of (1.2) is a self-adjoint space of operators, since the maps T_ν are completely positive for positive ν. On the other hand, the left-hand side is an algebra. Hence, both sides give a C*-algebra. This C*-algebra acts nondegenerately on $L^2(G)$, because $C_0(G)$ acts nondegenerately and $T_\nu(1) = \nu(1)1$.

Since, the first leg of L is in $L^\infty(\hat{G})$ while the first leg of V is in $L^\infty(\hat{G})'$, we have

$$V(T_\nu(x) \otimes 1)V^* = (T_\nu \otimes \iota)(V(x \otimes 1)V^*) = (T_\nu \otimes \iota)\Delta(x)$$

for $x \in C_0(G)$. This proves the last statement of the corollary. □

2. Dual Cocycles on Kählerian Lie Groups

As in [2, Sec. 3], let G be a Kählerian Lie group, $\theta \in \mathbb{R}^*$ and Ω_θ be the dual cocycle defined by the kernel K_θ. We want to check that Ω_θ satisfies conditions

(1)–(3). The operator Ω_θ is coisometric by [2, Theorem A.1] and injectivity of the map Φ there (see Lemma 2.3). Condition (2) is also satisfied for $X = 1$, since $\overline{K_\theta(g,h)} = K_\theta(h,g)$. As for (3), the cocycle identity implies that

$$(\Omega_\theta^*\Omega_\theta \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega_\theta)(\iota \otimes \hat{\Delta})(\Omega_\theta^*) = (\Omega_\theta^* \otimes 1)(1 \otimes \Omega_\theta).$$

It follows that if we let $F_\theta := \Omega_\theta^*\Omega_\theta$ then condition (3) is equivalent to

$$(F_\theta \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega_\theta)(\iota \otimes \hat{\Delta})(F_\theta) = (\hat{\Delta} \otimes \iota)(\Omega_\theta)(\iota \otimes \hat{\Delta})(F_\theta).$$

We will show that the following stronger property holds.

Proposition 2.1. *We have* $(F_\theta \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega_\theta) = (\hat{\Delta} \otimes \iota)(\Omega_\theta)(\iota \otimes \hat{\Delta})(F_\theta)$.

We don't have any conceptual explanation of this property. It is possible that this is true for any dual cocycle defined using a unitary quantization map as in [3, Sec. 2.2]. In this case the identity is not very difficult to verify by a direct computation as follows.

First of all, it is clearly enough to consider elementary Kählerian Lie groups.

Lemma 2.2. *For any* $x \in \mathbb{R}$ *and* $\theta \in \mathbb{R}^*$, *we have, with* $g := (x, 0, 0) \in G$:

$$(\lambda_g \otimes \lambda_g)\Omega_\theta = \Omega_{e^{-2x}\theta}(\lambda_g \otimes \lambda_g).$$

The proof is straightforward.

It follows that it suffices to prove the proposition for $\theta = \pm 2$. In fact, by passing from G to $G \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$, where α is the involutive automorphism defined at the Lie algebra level by $H \mapsto H$, $E \mapsto -E$, $X_i \mapsto X_{i+d}$ and $X_{i+d} \mapsto X_i$ for $i = 1, \dots, d$, we can similarly relate Ω_2 and Ω_{-2}. Thus, it is enough to consider $\theta = -2$.

From now on we write Ω and F instead of Ω_{-2} and F_{-2}. The operator $\mathcal{F}_{P,-2}\colon L^2(G) \to L^2(G)$ from [2] is up to a phase factor the partial Fourier transform $\mathcal{F}_P$ in P coordinate:

$$(\mathcal{F}_P f)(q,p) = (2\pi)^{-(d+1)/2} \int_P e^{-i\,p.p'} f(q,p')\,dp'.$$

Therefore by [2, Theorem A.1] we have

$$\Omega = (\mathcal{F}_P^* \otimes \mathcal{F}_P^*)U_\Phi(\mathcal{F}_P \otimes \mathcal{F}_P),$$

where

$$(U_\Phi\varphi)(g_1,g_2) = \mathrm{Jac}_\Phi(g_1,g_2)^{1/2}\varphi(\Phi(g_1,g_2)),$$

and the map $\Phi\colon G\times G\to G\times G$ is defined by

$$\Phi(g_1,g_2):=\Big(a_1-\frac{1}{2}\operatorname{arcsinh}(e^{-2a_2}t_2),\frac{\alpha^2}{\alpha'}n_1+\frac{t_2}{2\alpha'}m_1+\frac{\alpha t_2}{2\alpha'}n_2-\frac{\alpha}{\alpha'}m_2,$$
$$\frac{t_1^2t_2}{8\alpha'}n_1+\frac{\alpha^2}{\alpha'}m_1-\frac{\alpha t_1t_2}{4\alpha'}n_2+\frac{\alpha t_1}{2\alpha'}m_2,t_1;$$
$$a_2+\frac{1}{2}\operatorname{arcsinh}(e^{-2a_1}t_1),-\frac{\alpha t_1}{2\alpha'}n_1+\frac{\alpha}{\alpha'}m_1+\frac{\alpha^2}{\alpha'}n_2+\frac{t_1}{2\alpha'}m_2,$$
$$\frac{\alpha t_1t_2}{4\alpha'}n_1-\frac{\alpha t_2}{2\alpha'}m_1+\frac{t_1t_2^2}{8\alpha'}n_2+\frac{\alpha^2}{\alpha'}m_2,t_2\Big),$$

with

$$\alpha=\frac{1}{2}(e^{2a_1}+(e^{4a_1}+t_1^2)^{1/2})^{1/2}(e^{2a_2}+(e^{4a_2}+t_2^2)^{1/2})^{1/2},\quad \alpha'=\alpha^2+\frac{1}{4}t_1t_2.$$

(Note that these formulas slightly differ from those in [2], correcting typos and the erroneous claim that m_1 and m_2 remain unchanged).

Let us first consider the case $d=0$. Then,

$$\Phi(a_1,t_1;a_2,t_2)=\Big(a_1-\frac{1}{2}\operatorname{arcsinh}(e^{-2a_2}t_2),t_1;a_2+\frac{1}{2}\operatorname{arcsinh}(e^{-2a_1}t_1),t_2\Big).$$

Lemma 2.3. *The map Φ is a diffeomorphism of $G\times G$ onto the proper subset*

$$\mathcal{O}:=\{(a_1,t_1;a_2,t_2)\,|\,(1+(e^{-2a_1}t_1+e^{-2a_2}t_2)^2)^{1/2}>e^{-2a_1}t_1-e^{-2a_2}t_2\}\subset G\times G.$$

Proof. Consider the map $\phi\colon\mathbb{R}^2\to\mathbb{R}^2$, $\phi(a_1,a_2):=(\sinh(a_1)e^{-a_2},\sinh(a_2)e^{-a_1})$. It is easy to check that it is a diffeomorphism of $\mathbb{R}^2$ onto the set

$$\{(a_1,a_2)\mid(1+(a_1-a_2)^2)^{1/2}>a_1+a_2\}\subset\mathbb{R}^2.$$

Now, we have $\Phi(a_1,t_1;a_2,t_2)=(b_1,s_1,b_2,s_2)$ if and only if $t_i=s_i$ and

$$\sinh(2b_1-2a_1)=-e^{-2a_2}s_2,\quad \sinh(2b_2-2a_2)=e^{-2a_1}s_1,$$

which in terms of the map ϕ is equivalent to

$$\phi(2b_1-2a_1,2b_2-2a_2)=(-e^{-2b_2}s_2,e^{-2b_1}s_1).$$

This gives the result. □

It follows that $F=\Omega^*\Omega$ is indeed a proper projection, namely,

$$F=(\mathcal{F}_P^*\otimes\mathcal{F}_P^*)M(\mathbb{1}_{\mathcal{O}})(\mathcal{F}_P\otimes\mathcal{F}_P),\tag{2.1}$$

where $M(\mathbb{1}_{\mathcal{O}})$ is the operator of multiplication by the characteristic function $\mathbb{1}_{\mathcal{O}}$ of $\mathcal{O}$.

Next, recall that we have $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$, where the multiplicative unitary $\hat{W}\colon L^2(G \times G) \to L^2(G \times G)$ is defined by

$$(\hat{W} f)(g, h) := f(hg, h).$$

The following lemma is again a straightforward computation.

Lemma 2.4. *We have $\hat{W} = (\mathcal{F}_P^* \otimes \mathcal{F}_P^*)U_\alpha(\mathcal{F}_P \otimes \mathcal{F}_P)$, where $U_\alpha f = f \circ \alpha$ and $\alpha\colon G \times G \to G \times G$ is the diffeomorphism defined by*

$$\alpha(a_1, t_1; a_2, t_2) := (a_1 + a_2, t_1; a_2, t_2 - e^{-2a_1} t_1).$$

Note that the inverse of α is given by

$$\alpha^{-1}(a_1, t_1; a_2, t_2) = (a_1 - a_2, t_1; a_2, t_2 + e^{2(a_2 - a_1)} t_1).$$

It follows that

$$(\hat{\Delta} \otimes \iota)(\Omega) = (\mathcal{F}_P^* \otimes \mathcal{F}_P^* \otimes \mathcal{F}_P^*)(U_\alpha)_{12}^*(U_\Phi)_{23}(U_\alpha)_{12}(\mathcal{F}_P \otimes \mathcal{F}_P \otimes \mathcal{F}_P), \quad (2.2)$$

$$(\iota \otimes \hat{\Delta})(F) = (\mathcal{F}_P^* \otimes \mathcal{F}_P^* \otimes \mathcal{F}_P^*)M(\mathbb{1}_{\alpha_{23}(\mathcal{O}_{13})})(\mathcal{F}_P \otimes \mathcal{F}_P \otimes \mathcal{F}_P). \quad (2.3)$$

From (2.1)–(2.3) we conclude that Proposition 2.1 is equivalent (for $d = 0$) to the following lemma.

Lemma 2.5. *We have*

$$\mathcal{O} \times G = \{(g_1, g_2, g_3) \mid (\alpha_{12} \circ \Phi_{23} \circ \alpha_{12}^{-1})(g_1, g_2, g_3) \in \alpha_{23}(\mathcal{O}_{13})\}.$$

Proof. From Lemma 2.3, we easily get that the points $(b_1, s_1; b_2, s_2; b_3, s_3)$ of $\alpha_{23}(\mathcal{O}_{13})$ are characterized by the inequality

$$(1 + (e^{-2b_1} s_1 + e^{-2b_2} s_2 + e^{-2b_3} s_3)^2)^{1/2} > e^{-2b_1} s_1 - e^{-2b_2} s_2 - e^{-2b_3} s_3.$$

Since

$$\begin{aligned}(\alpha_{12} \circ \Phi_{23} \circ \alpha_{12}^{-1})&(a_1, t_1; a_2, t_2; a_3, t_3)\\ &= \bigg(a_1 - \frac{1}{2}\operatorname{arcsinh}(e^{-2a_3} t_3), t_1; a_2 - \frac{1}{2}\operatorname{arcsinh}(e^{-2a_3} t_3), t_2;\\ &\qquad a_3 + \frac{1}{2}\operatorname{arcsinh}(e^{-2a_1} t_1 + e^{-2a_2} t_2), t_3\bigg),\end{aligned}$$

letting $x_i = e^{-2a_i} t_i$ we see that we have to show that the inequality

$$(1 + (x_1 + x_2)^2)^{1/2} > x_1 - x_2 \quad (2.4)$$

is equivalent to

$$\begin{aligned}&(1 + (e^{\operatorname{arcsinh} x_3} x_1 + e^{\operatorname{arcsinh} x_3} x_2 + e^{-\operatorname{arcsinh}(x_1 + x_2)} x_3)^2)^{1/2}\\ &\qquad > e^{\operatorname{arcsinh} x_3} x_1 - e^{\operatorname{arcsinh} x_3} x_2 - e^{-\operatorname{arcsinh}(x_1 + x_2)} x_3. \quad (2.5)\end{aligned}$$

Put $a := e^{\operatorname{arcsinh}(x_1+x_2)} > 0$ and $b := x_1 - x_2$. Then $x_1 + x_2 = (a - a^{-1})/2$, so (2.4) becomes

$$\frac{a + a^{-1}}{2} > b. \tag{2.6}$$

Similarly, letting $c := e^{\operatorname{arcsinh} x_3} > 0$, (2.5) becomes

$$\frac{ac + a^{-1}c^{-1}}{2} > bc - \frac{a^{-1}c - a^{-1}c^{-1}}{2},$$

which is obviously equivalent to (2.6) for all $a > 0$, $b \in \mathbb{R}$ and $c > 0$. □

This completes the proof of Proposition 2.1 for $d = 0$. The case $d > 0$ can be easily deduced from this. Indeed, from the definition of Φ we see that $\tilde{\mathcal{O}} := \Phi(G\times G)$ is the preimage of the set $\mathcal{O}$ under the map π defined by $\pi(a_1, v_1, t_1; a_2, v_2, t_2) := (a_1, t_1; a_2, t_2)$. It is also easy to check that the transformation $\tilde{\alpha}\colon G \times G \to G \times G$ defining the unitary $(\mathcal{F}_P \otimes \mathcal{F}_P)\hat{W}(\mathcal{F}_P^* \otimes \mathcal{F}_P^*)$ satisfies $\pi \circ \tilde{\alpha} = \alpha \circ \pi$. From this, we see that the coordinates v do not play any role, so the result follows from the case $d = 0$.

Therefore, Ω_θ satisfies assumptions (1)–(3) from Sec. 1, so we get a well-behaved notion of Ω_θ-deformation. The main result of [2] — Theorem 3.2 (Theorem 3.4 in the arXiv version) — then holds as stated, and essentially the only change in the proof is that the operator $\hat{W}_{\Omega_\theta}\Omega_\theta$ gets replaced by the operator $(J\otimes\hat{J})\Omega_\theta\hat{W}^*(J\otimes\hat{J})$ everywhere.

In more detail, [2, Corollary 2.1] (Corollary 2.4 in the arXiv version) and [2, Proposition 2.3] (Proposition 2.8 in the arXiv version) and the discussions preceding them are no longer meaningful and should be removed. The remainder of the paper remains unchanged, with [2, Identity (2.12)] understood as a definition of the operator $\hat{W}_{\Omega_\theta}\Omega_\theta$. Note that the place where we need special properties of Ω_θ is the first identity in the proof of [2, Lemma 2.5] (Lemma 2.10 in the arXiv version), which now holds by Proposition 1.1.

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