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Impartiality, priority, and solidarity in the theory of justice*

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Abstract

The ethic of 'priority' is a compromise between the extremely compensatory ethic of 'welfare equality' and the needs-blind ethic of 'income equality'. We propose an axiom of priority, and characterize resource-allocation rules that are impartial, prioritarian, and solidaristic. They comprise a class of rules which equalize across individuals some index of resources and welfare. Consequently, we provide an ethical rationalization for the many applications in which such indices have been used (e.g., the 'human development index,' 'index of primary goods,' etc.)

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1 Introduction

In this paper, we study the ethics of resource allocation in a basic and common problem. There is a resource, available in given quantity, to be allocated among individuals, each of whom possesses a capability to transform the resource into some given valued outcome, and the achievements of individuals, with regard to that outcome, are interpersonally comparable.

In many resource allocation problems of this sort, there are two focal points of distribution: to distribute the available resource equally among all who need it, and to distribute the resource among the population so as to equalize the outcomes among them. Often, however, the 'equal-resource' allocation seems too harsh: it does not take into account the differential ability of individuals to convert the resource into the desired outcome. On the other hand, often the equal-outcome allocation seems too extreme: it may require giving the lion's share of the resource to very 'handicapped' individuals, ones with poor outcome functions, and this may appear to be unfair to those who are more fortunate. This is a familiar criticism of the Rawlsian maximin allocation.

The philosopher Derek Parfit, partly as a reaction to the extremism of Rawlsian maximin, coined the term 'priority' for the view that lies 'in between' the equal-resource and the equal-outcome view. He proposed that the right view is to give *priority* to those who are less capable of transforming resource into outcome. (Parfit did not work on a formal domain of problems, so we are paraphrasing here.) We formalize Parfit's view as an axiom of allocation rules indicating that no individual can dominate another in both resources and outcomes. In particular, individuals with a worse capability of transforming resources into outcomes are allocated more resources, a desirable feature according to Sen (1973). It is intuitively clear that priority admits a large class of possible resource allocations –just think of the possible compromises between the equal-resource and the equal-outcome allocation. Prioritarianism includes –it would seem as polar cases– those two allocations or allocation rules.

There is a second principle that we believe characterizes fairness in many problems, which we call *solidarity*. The idea is that, if an allocation rule is fair, then when new individuals join a society (e.g., through birth or immigration) then the resources allocated to all the *original* members should change in the same direction. Intuitively, if the new members bring with them a lot of resources, then everyone in the original population should gain, and if they bring with them few or no resources, then everyone in the original population should chip in some resource to help them. The solidarity axiom has been used in different forms by Thomson (1983), Roemer (1986), Moulin (1987), Sprumont (1996) and Fleurbaey and Maniquet (1999), among others.

Finally, we believe that fairness requires *impartiality*. This means that in deciding how to allocate the resource, we ignore all attributes of persons that are irrelevant, according to our moral standard, to the problem at hand. For instance, if the problem is one of allocating scarce rescuer time to saving earthquake victims, we ignore the victim's religion and race (though perhaps not his age). In reality, impartiality is a very strong requirement. Suppose the issue is to distribute educational resources to children, who have different capacities to transform them into future earning power. One might say that impartiality requires that the allocation rule ignore the wealth of parents. However, the allocation rule that allocates such resources in the United States surely does not ignore parental wealth. We state the impartiality axiom as a domain axiom of the allocation rules.

Here are some examples where, we believe, the axioms of solidarity, priority and impartiality either apply in common practice, or, arguably, morality suggests that they should apply:

1. Allocation of parental time to children. In this example, we assume that the (predicted) success of each child is a function of the parental time she receives. Solidarity says that when a new child arrives in the family, parental time to all the other children changes in the same way (decreases, here); priority says that, generally, a parent should devote more time to children who are less able, but not to the extent of rendering those children more successful than more able children.

2. Distribution of a parent's estate among children. The same ideas apply as in #1. Here, equal division of the estate is commonly done, which is consistent with solidarity and priority.

3. Allocation of the budget of educational finance. Here, we assume that the predicted future wage of a child of a given type is a function of the amount she receives in educational finance. Priority says that we should devote at least as much educational resource to children who have inferior abilities to transform the resource into future earning power.

4. The Americans with Disabilities Act requires employers to spend extra resources to enable disabled workers to perform adequately on the job.

The reader can doubtless supply many more examples.

An example that does not satisfy priority is triage on the battlefield. The resource is physician time and the individual capability to transform the resource is the probability of survival of a wounded soldier. The army's goal is to maximize the number of soldiers who survive, and can return to battle; it will devote no resources to badly wounded soldiers for whom the probability of survival is very poor. Note that fairness is not the issue here, but maximizing the effectiveness of the army. Note also that, unlike previous examples, military decisions usually involve externalities which may well explain why our analysis does not apply in this case.

Consider, however, the victims of an earthquake, where the resource is scarce rescuer time and the individual capability to transform the resource is the probability-of-survival. Should the allocation of rescuer time satisfy priority, or is this a case like triage? It depends whether our objective is fairness towards individuals or to maximize the number of people saved. If it is fairness, then priority applies. It is intuitively clear that the most 'conservative' prioritarian practice would be to give all victims equal time; the most 'radical' would be to allocate the time among victims so that all have an equal probability of survival. Priority says that in no case do we allocate so much rescuer time to a more badly trapped victim that we increase his survival probability above the survival probability of a less badly trapped victim.

In this paper, we characterize the set of allocation rules that jointly satisfy impartiality, priority, and solidarity. The intuitions that we have hinted at are verified: there is a large class of such rules, and the equal-resource and equal-outcome rules are polar cases in that class, on the 'conservative' and 'radical' ends. The admissible class turns out to involve *indices* of resources and outcomes. To be precise, we will show that the three axioms require us to equalize *some index* of resources and outcomes, at the highest possible level. In particular, these rules pay equal attention to resources *and* outcomes in an explicit way; they are not 'welfarist' rules that consider only the pattern of outcomes that resource allocations generate. As such, this work is a contribution to non-welfarist social-choice theory. Indeed, the non-welfarist aspect of our approach is evident in the priority axiom: for that axiom implements a special moral concern for the *amount of resource* that a person receives. (That claim is vague, but we hope the reader agrees with us.)

The rest of the paper is organized as follows. In Section 2 we present the preliminaries of the model. In Section 3 we present the axiomatic theory of resource allocation involving the concepts of impartiality, priority, and solidarity. In Section 4 we characterize the family of rules satisfying these three notions. In Section 5, we relate our result with previous work in the literature on distributive justice. Finally, Section 6 concludes. All proofs have been relegated to an Appendix.

2 The model

Let $\mathbb{I}$ represent a population of all potential individuals (a continuum set) and let $\mathcal{I}$ be the family of all finite subsets of $\mathbb{I}$. Let $N = \{1, 2, ..., n\} \in \mathcal{I}$ be a set of individuals with generic elements i and j. Individuals derive welfare from a resource, called *wealth*. We assume that $\mathbb{I} \times \mathbb{R}_+$ is endowed with a complete order. The expression $(i, W) \succsim (j, W')$ is read: "individual i equipped with wealth W enjoys a welfare level at least as high as individual j equipped with wealth W'". We assume that this order is continuous in W, and satisfies that, for any $i, j \in N$ and $W \in \mathbb{R}_+$ there is a wealth level W' such that $(i, W) \sim (j, W')$. We further assume that for any pair $i, j \in N$, $(i, 0) \sim (j, 0)$. A wealth level of zero can be thought of as inducing death, which is an equally bad outcome for all individuals. Finally, we assume that welfare is strictly increasing in wealth for every individual.

It is convenient to represent this interpersonally level comparable welfare ordering as follows. Fix a particular individual and call her individual 0. For any other individual i define a function $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ where for each $W \in \mathbb{R}_+$, $u_i(W)$ is such that $(0, u_i(W)) \sim (i, W)$. In other words, $u_i(W)$ is the wealth that 0 must receive in order that she enjoy the same level of welfare as individual i enjoys with wealth W. We say that an individual i is *disabled* with respect to another individual j if the former needs more wealth than the latter one to reach the same level of welfare, i.e., if $u_i \geq u_j$.

The assumptions on $\succsim$ tell us that for all i, u_i is a continuous, strictly-increasing, and unbounded function satisfying that $u_i(0) = 0$.[1] We shall also assume that $\{u_i : i \in \mathbb{I}\}$ constitutes a *dense domain*, i.e., the graphs of these functions cover the positive quadrant.[2]

We define an **economy** e as a triple (N, u, W), where $N \in \mathcal{N}$ is the set of individuals, $u = (u_i)_{i \in N}$ is the profile of utility functions (defined as above) for individuals in N, and $W \in \mathbb{R}_+$ represents the available wealth. The family of all economies is $\mathcal{E}$.

An **allocation rule** is a function F that associates to each economy

[1] Note that the functions u_i comprise a profile of utility functions for individuals which measure utility in a level comparable way.

[2] This is not an essential assumption. The model and the ensuing results would also hold by dropping this assumption at the only cost of having a more sophisticated notation.

$e = (N, u, W) \in \mathcal{E}$ a unique point $F(e) = (F_i(e))_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} F_i(e) = W$. That is, an allocation rule indicates how to distribute the wealth available in an economy among its members. Examples of rules are the following. First, the rule that awards each agent the same amount:

Equal-Resource *rule (ER):* $ER_i(N, u, W) = \frac{W}{n}$.

An alternative to the equal-resource rule is obtained by focusing on the welfare levels individuals achieve, as opposed to what resources they receive, and choosing the vector at which these welfare levels are equal.

Equal-Welfare *rule (EW):* $EW_i(N, u, W) = u_i^{-1}(\lambda)$, *where* $\lambda > 0$ *is chosen so that* $\sum_{i \in N} u_i^{-1}(\lambda) = W$.

Note that, for all $i \in I$, u_i^{-1} is a continuous, strictly-increasing, and unbounded function satisfying that $u_i^{-1}(0) = 0$. From here, it follows that EW is well-defined.

3 The axioms

We now present the axioms we want rules to satisfy. These axioms will reflect the three notions discussed above: *impartiality*, *priority* and *solidarity*.

Impartiality is modeled by the domain axiom, as defining rules on the class of economies $\mathcal{E}$ we are excluding much information about persons that we consider ethically irrelevant.

We now turn to *priority*. Our axiom of *priority* says that no agent can dominate another agent both in resources and welfare.

Priority *(PR) Let* $e = (N, u, W) \in \mathcal{E}$ *and* $i, j \in N$ *such that* $F_i(e) < F_j(e)$. *Then* $u_i(F_i(e)) \geq u_j(F_j(e))$.

Note that this axiom guarantees that disabled agents receive at least as much wealth as abler ones: we discriminate positively towards the disabled. In other words, priority implies the *weak equity* axiom introduced by Sen (1973). On the other hand, the axiom also says that the obligation towards the unfortunate is limited, as a disabled person is never resourced to the extent that her welfare exceeds that of an able agent. It is also straightforward to show that priority implies a weak version of anonymity which says that individuals that are equally able are rewarded equally. This is usually referred as *symmetry*.

We conclude with *solidarity*. Here we rely upon a literature which has formulated various solidarity axioms in the past twenty years. Alternative versions of solidarity have been considered in different contexts like fair division (e.g., Thomson, 1983; Roemer, 1986), compensation problems (e.g., Moulin, 1987; Fleurbaey and Maniquet, 1999), cost and surplus-sharing (e.g., Keiding and Moulin, 1991; Chun, 1999), or collective choice (e.g., Sprumont, 1996; Ehlers and Klaus, 2001). Our notion of solidarity says that the arrival of immigrants, whether or not accompanied by changes in the available wealth, should affect all original agents in the same direction: all gain or all lose, or all receive the same as before.

Solidarity *(SL). Let* $e = (N, u, W) \in \mathcal{E}$ *and* $e' = (N', u', W') \in \mathcal{E}$, *such that* $N' \subseteq N$. *Let* $F_{N'}(e)$ *denote the projection of* $F(e)$ *onto the set of coordinates corresponding to* N'. *Then either* $F(e') = F_{N'}(e)$, $F(e') > F_{N'}(e)$ *or* $F(e') < F_{N'}(e)$.[3]

Solidarity can be decomposed in two axioms that appear frequently in the literature: *resource monotonicity* and *consistency*. Resource monotonicity (e.g., Roemer, 1986) says that when a bad or good shock comes to an economy, all its members should share in the calamity or windfall. Formally,

Resource Monotonicity *(RM). Let* $e = (N, u, W) \in \mathcal{E}$ *and* $e' = (N', u', W') \in \mathcal{E}$, *such that* $N' = N$, $u' = u$ *and* $W' > W$. *Then* $F(e') > F(e)$.

Consistency says that if a sub-group of individuals secedes with the resource allocated to it under a rule, then in the smaller economy the rule allocates the resource in the same way.[4] Formally,

Consistency *(CY). Let* $e = (N, u, W) \in \mathcal{E}$. *Let* $N' \subset N$ *and* $e' = (N', u', W')$, *where* $u' = (u_i)_{i \in N'}$ *and* $W' = \sum_{i \in N'} F_i(e)$. *Then* $F_i(e) = F_i(e')$, *for all* $i \in N'$.

It is straightforward to show that the axiom of solidarity is equivalent to the combination of resource monotonicity and consistency. It can also be shown that resource monotonicity implies the following technical axiom that will be useful in the ensuing discussion:

Resource Continuity *(RC). If* $W^n \to W$ *then* $F(N, u, W^n) \to F(N, u, W)$.

[3] Note that for $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, ..., y_n) \in \mathbb{R}^n$, we write $x > y$ if $x_i > y_i$ for all $i = 1, ..., n$.

[4] The reader is referred to Young (1994), Roemer (1996) or Thomson (2004) for the many applications that exist in the literature on distributive justice concerning this notion.

4 The characterization result

Both the equal-resource and the equal-welfare rules presented above satisfy impartiality, priority, and solidarity. In this section, we identify all the remaining existing rules satisfying these properties. To do so, it is worth noting first that the equal-resource and the equal-welfare rules are somehow extreme rules among those satisfying impartiality and priority. More precisely,

Proposition 1 *Let $e = (N, u, W) \in \mathcal{E}$. Let i (j) be the ablest (disablest) individual in N. Then, for all rules F satisfying impartiality and priority, we have:*
(i) $ER_i(e) \geq F_i(e) \geq EW_i(e)$
(ii) $ER_j(e) \leq F_j(e) \leq EW_j(e)$

Proposition 1 says that the equal-resource rule is the best (worst) impartial and prioritarian rule for the ablest (disablest) agent, whereas the equal-welfare rule is the best (worst) impartial and prioritarian rule for the disablest (ablest) agent. This suggests that the set of all rules satisfying impartiality, priority, and solidarity should consist of all rules resulting from a compromise between the equal-resource and the equal-welfare rule. Our proposal for this compromise is the following.

Let Φ be the class of all functions $\varphi : \mathbb{R}^2_{++} \cup \{(0,0)\} \to \mathbb{R}_+$, continuous on its domain and non-decreasing, such that $\inf\{\varphi(x,y)\} = \varphi(0,0) = 0$ and for all $(x,y) > (z,t)$, $\varphi(x,y) > \varphi(z,t)$. Let φ be a function in the class Φ. For all $i \in \mathbb{I}$ define the function $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ that determines the $\varphi-$value agent i achieves, depending on the wealth she receives, i.e., $\psi_i(w) = \varphi(w, u_i(w))$ for all $w \in \mathbb{R}_+$. Then, we define the corresponding *index-egalitarian* rule as the rule that equalizes the $\varphi-$value across individuals in an economy.

Index-egalitarian *rule (E^φ): $E^\varphi_i(e) = \psi_i^{-1}(\lambda)$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \psi_i^{-1}(\lambda) = W$.*

Note that, for all $i \in I$, ψ_i^{-1} is a continuous, strictly-increasing, and unbounded function satisfying that $\psi_i^{-1}(0) = 0$. From here, it follows that E^φ is well defined. Note also that applied in this manner to an agent's wealth and welfare, φ can be considered to be a generalized index of wealth and welfare. So the rules just defined equalize a generalized index of wealth and welfare.[5] The equal-resource rule is the E^{φ_1} rule, where $\varphi_1(x,y) = x$. The equal-welfare rule is the E^{φ_2} rule, where $\varphi_2(x,y) = y$.

[5] We are indebted in a major way to Klaus Nehring, who suggested the E^φ rules.

All the rules within the family $\{E^{\varphi}\}_{\varphi\in\Phi}$ satisfy impartiality, priority, and solidarity. More remarkably, there is no other rule satisfying these properties simultaneously, as the next result shows.

Theorem 2 *A rule F satisfies impartiality, priority, and solidarity if and only if $F \in \{E^{\varphi}\}_{\varphi\in\Phi}$.*

The reader might note that although we chose a specific profile of utility functions to represent the interpersonal ordering $\succsim$, the class of rules characterized in Theorem 2 is independent of this choice.

5 Related literature

In this section, we relate our result with previous work on the literature on distributive justice. The simplest model in this literature is the widely-studied *rationing model*. This model (first formalized by O'Neill, 1982, and recently surveyed by Moulin, 2002 and Thomson, 2003) describes a situation in which a given amount of wealth has to be allocated among a group of agents, when the available amount is not enough to satisfy all their *claims* (the only individual characteristic in the model). Formally, a rationing problem is described by a triple (N, c, W), where N is the set of agents, $c \in \mathbb{R}^n_+$ is a vector of claims, and $W \in \mathbb{R}_+$ represents the amount of wealth to be divided. The very notion of rationing problem requires $\sum_{i\in N} c_i \geq W > 0$. An allocation rule is a function F that associates with every (N, c, W) a unique point $F(N, c, W) \in \mathbb{R}^n$ such that $0 \leq F(N, c, W) \leq c$ and $\sum_{i\in N} F_i(N, c, W) = W$. One of the most influential results in this literature is due to Young (1987), who characterizes the set of all rules satisfying symmetry, resource continuity and consistency. This set comprises the so-called *parametric rules*. Formally, a rule F is parametric (in the rationing model) if there exists a function $f : [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$, where $[a, b] \subseteq \mathbb{R}$, continuous and weakly monotonic in its first argument, such that:

(i) For all $x \in \mathbb{R}_+$, $f(a, x) = 0$;

(ii) For all $x \in \mathbb{R}_+$, $f(b, x) = x$; and

(iii) Given $e = (N, c, W)$, there exists $\lambda_e \in [a, b]$ such that $F_i(e) = f(\lambda_e, c_i)$ for all $i \in N$.

The economic environments we use in this paper are not rationing problems, but they are quite related and therefore a similar approach to Young (1987) can be followed to construct parametric rules in our model. Let $\mathcal{V}$

denote the set of utility functions, i.e., $\mathcal{V} = \{v : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, strictly increasing and s.t. $v(0) = 0$, $\lim_{x\to\infty} v(x) = \infty\}$.

We say that a rule F is *parametric* (in the model of our paper) if there exists a function $f : \mathbb{R}_+ \times \mathcal{V} \to \mathbb{R}_+$, continuous and strictly monotonic in its first argument, such that:

(i) For all $v \in \mathcal{V}$, $f(0, v) = 0$.

(ii) For all $v \in \mathcal{V}$, $\lim_{\lambda\to\infty} f(\lambda, v) = \infty$; and

(iii) Given $e = (N, u, W) \in \mathcal{E}$, there exists $\lambda_e \in \mathbb{R}_+$ such that $F_i(e) = f(\lambda_e, u_i)$ for all $i \in N$.

Adapting to our context an argument introduced by Young (1987), we can show the following result:

Proposition 3 *If a rule satisfies impartiality, priority, and solidarity then it is parametric.*

We show now, by means of an example, that the converse of Proposition 3 is not true. For each agent $i \in \mathbb{I}$ we define her *unity claim* as $u_i^{-1}(1)$, i.e., the wealth that i must receive to enjoy the same level of welfare as individual 0 enjoys with wealth 1. We consider now the rule that allocates the wealth of an economy proportionally to agents' unity claims in the economy. Formally, for all $e = (N, u, W) \in \mathcal{E}$,

$$P(e) = \lambda \cdot (u_i^{-1}(1))_{i\in N},$$

where $\lambda = \frac{W}{\sum_{i\in N} u_i^{-1}(1)}$. Then, P is a parametric rule that fails to satisfy priority.[6] To show this, for all $x \in \mathbb{R}_+$, let $u_1(x) = x/2$ and $u_2(x) = x^2$. Consider $e = (\{1, 2\}, \{u_1, u_2\}, 2)$. Then, $P(e) = (4/3, 2/3)$ and $u_1(4/3) = 2/3 > 4/9 = u_2(2/3)$.

The above and Theorem 2 show that the index-egalitarian rules are parametric, but the converse is not true. This makes our characterization result genuinely different from previous results in the literature.

6 Recapitulation

We have presented a characterization, in simple environments, of allocation rules that satisfy impartiality, priority, and solidarity. Our characterization result shows that the combination of these three notions is equivalent to a

[6] Its parametric representation would be $f(\lambda, u_i) = \lambda \cdot u_i^{-1}(1)$.

kind of egalitarianism, where the equality in question is equality of a conception of well-being that is some general *index* of welfare and resources, where the index is not determined without further assumptions. Two classical distribution rules are polar (and even 'dual') in the class of index-egalitarian rules –the equal-resource and equal-welfare allocation rules.

Our result shows, in particular, that prioritarianism, at least in conjunction with solidarity, does not preclude equality, but it modifies the equalisandum from 'welfare' to an index of welfare and resources. Using indices of resources and outcomes to measure the success of an allocation procedure is a fairly common practice. The UNDP's human development indicator is an index of a country's GDP, literacy rate, and infant mortality rate. John Rawls (1971) worked, famously, with an index of primary goods: some of those 'goods' were resources, and some 'outcomes.' Amartya Sen (1980, 1992) has written of using an index of functionings as a possible measure of a person's welfare. In these examples, the social welfare supremum is thought to be the allocation of resources that equalizes the index in question, at the highest possible level. This is our characterization theorem.

7 Appendix 1: the proof of the theorem

It is not difficult to show that all the E^{φ} rules are impartial and satisfy PR and SL. Conversely, let F be an impartial rule that satisfies PR and SL. Then, as mentioned above, F satisfies RM, CY and RC.

Let $i \in \mathbb{I}$ be given and $\alpha \in \mathbb{R}_+$. Let $E(F, i, \alpha)$ be the domain of economies for which individual i obtains an amount of wealth α, under rule F. Formally:

$$E(F, i, \alpha) = \{e = (N, u, W) \in \mathcal{E} : F_i(e) = \alpha\}.$$

Let $C(F, i, \alpha)$ be the set of points in the plane which are achieved as wealth-welfare ordered pairs under the action of F on individuals who are members of economies in $E(F, i, \alpha)$. Formally:

$$C(F, i, \alpha) = \{(F_j(e), u_j(F_j(e))) \text{ for } e = (N, u, W) \in E(F, i, \alpha),\ j \in N\}.$$

Since F satisfies PR and RC, it follows that $C(F, i, \alpha)$ is not empty. Our aim is to show that the family of curves $\{C(F, i, \alpha) : \alpha \in \mathbb{R}_+\}$ is the isoquant map of an appropriate function $\varphi \in \Phi$ and therefore to show that $F = E^{\varphi}$.

We show first that any $C(F, i, \alpha)$ is downward sloping, i.e., if (a, b), $(a', b') \in C(F, i, \alpha)$ and $a' > a$ then $b' \leq b$. Suppose, to the contrary, that $b' > b$. By definition, there exist $e = (N, u, W)$ and $e' = (N', u', W') \in$

$E(F,i,\alpha)$ and $j \in N$, $k \in N'$ such that $(a,b) = (F_j(e), u_j(F_j(e)))$ and $(a',b') = (F_k(e'), u_k(F_k(e')))$. Let $e^j = (\{i,j\},(u_i,u_j),\alpha + a)$ and $e^k = (\{i,k\},(u_i,u_k),\alpha + a')$. Then, by CY, $F(e^j) = (\alpha, a)$ and $F(e^k) = (\alpha', a')$. Now, by PR and RC, there is a wealth W^* such that $e^* = (\{i,j,k\},(u_i,u_j,u_k),W^*) \in E(F,i,\alpha)$. Then, $F_i(e^*) = F_i(e^j) = F_i(e^k) = \alpha$. Thus, by SL applied to e^* and e^j, $F_j(e^*) = F_j(e^j) = a$ and, by SL applied to e^* and e^k, $F_k(e^*) = F_k(e^k) = a'$. Therefore, $F_j(e^*) = a$, $F_k(e^*) = a'$ and so $F_j(e^*) < F_k(e^*)$. Thus, by PR, $u_j(F_j(e^*)) \geq u_k(F_k(e^*))$. However, we also know, by hypothesis, that $b = u_j(F_j(e^*)) < u_k(F_k(e^*)) = b'$, a contradiction.

We show now that $\{C(F,i,\alpha) : \alpha \in \mathbb{R}_+\}$ is a collection of disjoint sets. Let $\alpha_1 > \alpha_2$. Suppose $(a,b) \in C(F,i,\alpha_1) \cap C(F,i,\alpha_2)$. Let $e_1 = (N_1,u_1,\alpha_1) \in E(F,i,\alpha_1)$, $e_2 = (N_2,u_2,\alpha_2) \in E(F,i,\alpha_2)$ and $j \in N_1$, $k \in N_2$ such that $(a,b) = (F_j(e_1), u_j(F_j(e_1))) = (F_k(e_2), u_k(F_k(e_2)))$. Consider the economies $\widehat{e}_1 = (\{i,j\},(u_i,u_j),a+\alpha_1)$ and $\widehat{e}_2 = (\{i,k\},(u_i,u_k),a+\alpha_2)$. CY implies that $F_i(\widehat{e}_1) = \alpha_1$ and $F_i(\widehat{e}_2) = \alpha_2$. By RC and PR, there is $W > a + \alpha_2$ such that $\widetilde{e}_2 = (\{i,k\},(u_i,u_k),W) \in E(F,i,\alpha_1)$. By RM, applied to $\widehat{e}_2$ and $\widetilde{e}_2$ we know that $F_k(\widetilde{e}_2) > F_k(\widehat{e}_2) = a$. Therefore, $(a,b) < (F_k(\widetilde{e}_2), u_k(F_k(\widetilde{e}_2))) \in C(F,i,\alpha_1)$. This contradicts the fact that $C(F,i,\alpha_1)$ is downward sloping.

Next, we show that if $\alpha_1 > \alpha_2$ then $C(F,i,\alpha_1)$ lies above $C(F,i,\alpha_2)$, i.e.,

(i) For all $(a,b) \in C(F,i,\alpha_2)$ there exists $(a',b') \in C(F,i,\alpha_1)$ such that $(a,b) < (a',b')$.

(ii) There is no $(a'',b'') \in C(F,i,\alpha_2)$ and $(a,b) \in C(F,i,\alpha_1)$ such that $(a'',b'') < (a,b)$.

(i) Let $(a,b) \in C(F,i,\alpha_2)$. Then, there exists $e = (\{i,j\},(u_i,u_j),W) \in E(F,i,\alpha_2)$ such that $F_j(e) = a$ and $u_j(F_j(e)) = b$. Since $F_i(e) = \alpha_2$, by a similar argument to the one in the above paragraph, we find a wealth value $W^* > W$ such that $e^* = (\{i,j\},(u_i,u_j),W^*) \in E(F,i,\alpha_1)$. Let $(a',b') = (F_j(e^*), u_j(F_j(e^*)))$. Then, $(a',b') \in C(F,i,\alpha_1)$. Furthermore, since F satisfies RM and u_j is strictly increasing, $(a',b') > (a,b)$.

(ii) Conversely, let $(a,b) \in C(F,i,\alpha_1)$. Suppose there were a point $(a'',b'') \in C(F,i,\alpha_2)$ such that $(a'',b'') > (a,b)$. We know, by (i), that there is a point $(a''',b''') \in C(F,i,\alpha_1)$ such that $(a''',b''') > (a'',b'')$. Thus, $(a''',b''') > (a,b)$, which contradicts that $C(F,i,\alpha_1)$ is downward sloping.

Let $(a,b) \in \mathbb{R}^2_+$ and $i \in \mathbb{I}$ be given. By the assumption of dense domain, and the above, there exists a unique $\alpha \in \mathbb{R}_+$ such that $(a,b) \in C(F,i,\alpha)$. Define then the function $\varphi : \mathbb{R}^2_{++} \cup \{(0,0)\} \to \mathbb{R}_+$ by $\varphi(a,b) = \alpha$, where

$\alpha \in \mathbb{R}_+$ is the unique number for which $(a, b) \in C(F, i, \alpha)$. We show that $\varphi \in \Phi$.

Since F satisfies PR, it is straightforward to show that $\varphi(0, 0) = 0$ and $\varphi(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2_+$.

Let $x, x', y \in \mathbb{R}_{++}$ such that $x < x'$. If $\varphi(x, y) > \varphi(x', y)$ then $C(F, i, \varphi(x, y))$ lies above $C(F, i, \varphi(x', y))$. In such a case, since $(x', y) \in C(F, i, \varphi(x', y))$, there exists $(z, t) \in C(F, i, \varphi(x, y))$ such that $(x', y) < (z, t)$. Then, $(z, t) > (x, y)$. This contradicts that $C(F, i, \varphi(x, y))$ is downward slopping. Similarly, we show that $\varphi(x, y) \leq \varphi(x', y)$ for all $x, x', y \in \mathbb{R}_{++}$ such that $y < y'$.

Let $(x, y), (z, t) \in \mathbb{R}^2_+$ such that $(x, y) > (z, t)$. By downward sloppiness, $\varphi(x, y) \neq \varphi(z, t)$. If $\varphi(x, y) < \varphi(z, t)$ then $C(F, i, \varphi(z, t))$ lies above $C(F, i, \varphi(x, y))$. We have, however, that $(x, y) \in C(F, i, \varphi(x, y))$, $(z, t) \in C(F, i, \varphi(z, t))$ and $(x, y) > (z, t)$, which represents a contradiction.

Finally, we show that φ is continuous on $\mathbb{R}^2_+$. Let $\{(a_n, b_n)\} \to (a, b)$. We must show that $\{\alpha_n\} = \{\varphi(a_n, b_n)\} \to \alpha = \varphi(a, b)$. If such is not the case, then there exists a subsequence $\{\alpha_{k_n}\}$ that converges to $\overline{\alpha} \neq \alpha$. We assume that $\overline{\alpha} < \alpha$.[7] Then, for k_n sufficiently large, $\alpha_{k_n} < \frac{\alpha + \overline{\alpha}}{2} < \alpha$, and therefore, $C(F, i, \alpha_{k_n})$ lies below $C(F, i, \frac{\alpha + \overline{\alpha}}{2})$ and this one lies below $C(F, i, \alpha)$. In particular, there exists a ball B, about $(a, b) \in C(F, i, \alpha)$ which lies above $C(F, i, \frac{\alpha + \overline{\alpha}}{2})$. Since $(a_n, b_n) \to (a, b)$, it follows that for large k_n, $(a_{k_n}, b_{k_n}) \in B$. On the other hand, $(a_{k_n}, b_{k_n}) \in C(F, i, \alpha_{k_n})$ which, for large k_n, lies below $C(F, i, \frac{\alpha + \overline{\alpha}}{2})$. This represents a contradiction.

The proof of the theorem concludes by showing that $F = E^{\varphi}$, i.e., $F(N, u, W) = E^{\varphi}(N, u, W)$ for all $(N, u, W) \in \mathcal{E}$. Fix $e = (N, u, W) \in \mathcal{E}$. Two cases are distinguished.

Case 1: $i \in N$.

Let $\lambda = F_i(e)$. Then, $(F_j(e), u_j(F_j(e))) \in C(F, i, \lambda)$ for all $j \in N$. By definition of φ, $\varphi(F_j(e), u_j(F_j(e))) = \lambda$, for all $j \in N$. Thus, $\psi_j(F_j(e)) = \lambda$ for all $j \in N$. Since $\sum_{j \in N} F_j(e) = W$, it follows that $F(e) = E^{\varphi}(e)$.

Case 2: $i \notin N$.

Pick two agents $j, k \in N$. Let $w_j = F_j(e)$ and $w_k = F_k(e)$. By PR and RC, there are two economies $\widehat{e} = (\{i, j\}, (u_i, u_j), \widehat{W})$ and $\widetilde{e} = (\{i, k\}, (u_i, u_k), \widetilde{W})$ such that $w_j = F_j(\widehat{e})$ and $w_k = F_k(\widetilde{e})$. Let $\widehat{w}_i = F_i(\widehat{e})$ and $\widetilde{w}_i = F_i(\widetilde{e})$.

[7] The proofs for the cases $\alpha < \overline{\alpha} < \infty$ and $\overline{\alpha} = \infty$ are similar.

We show that $C(F,j,w_j) = C(F,i,\widehat{w}_i)$.

Let $(a,b) \in C(F,j,w_j)$. Then, there exists $l \in \mathbb{I}$ such that $b = u_l(a)$ and $(w_j, a) = (F_j(e^2), F_l(e^2))$, where $e^2 = (\{j,l\},(u_i,u_l), w_j + a)$. By PR and RC, there exists W^3 such that $F_i(e^3) = \widehat{w}_i$, where $e^3 = (\{i,j,l\},(u_i,u_j,u_l), W^3)$. Let $\widehat{e}^* = (\{i,j\},(u_i,u_j), \widehat{w}_i + F_j(e^3))$. Then, by CY, $F(\widehat{e}^*) = (\widehat{w}_i, F_j(e^3))$. Thus, by RM, $F_j(e^3) = w_j$. Let $e^{2*} = (\{j,l\},(u_j,u_l), w_j + F_l(e^3))$. Then, by CY, $F(e^{2*}) = (w_j, F_l(e^3))$. Thus, by RM, $F_l(e^3) = a$. Thus, $F(e^3) = (\widehat{w}_i, w_j, a)$. Consequently, $e^3 \in E(F,i,\widehat{w}_i)$ and $(a,b) \in C(F,i,\widehat{w}_i)$, showing that $C(F,j,w_j) \subseteq C(F,i,\widehat{w}_i)$. The proof of the converse inclusion goes along the same lines.

Analogously, it can be shown that $C(F,k,w_k) = C(F,i,\widetilde{w}_i)$. Thus, it follows that $(w_j, u_j(w_j)) \in C(F,i,\widehat{w}_i) \cap C(F,i,\widetilde{w}_i)$, which implies that $\widehat{w}_i = \widetilde{w}_i$. Let $\lambda = \widehat{w}_i = \widetilde{w}_i$. Then, $C(F,j,w_j) = C(F,i,\lambda) = C(F,k,w_k)$. This shows, in particular, that $(F_l(e), u_l(F_l(e))) \in C(F,i,\lambda)$ for all $l \in N$. From here, the proof of Case 1 concludes. ■

8 Appendix 2: Auxiliary statements

To save space, we have included in this second appendix, formal proofs of some statements made within the body of the text.

Statement 1 A rule satisfies solidarity if and only if it satisfies consistency and resource monotonicity.

Proof. Let F be an allocation rule satisfying SL. It is straightforward to show that F satisfies RM. Let us see that it also satisfies CY. Let $e = (N,u,W) \in \mathcal{E}$ be given and fix $N' \subset N$. Denote $w = (F_i(e))_{i\in N'}$ and consider the new economy $e' = (N', u', W')$, where $u' = (u_i)_{i\in N'}$ and $W' = \sum_{i\in N'} F_i(e)$. By SL we either have $F(e') = w$, $F(e') < w$ or $F(e') > w$. Now, since F is an allocation rule then it allocates the wealth of each economy completely, i.e., $\sum_{i\in N'} F_i(e') = W' = \sum_{i\in N'} F_i(e)$. Thus, $F(e') = w$, which shows that F satisfies CY.

Conversely, let F be an allocation rule satisfying RM and CY. Let $e = (N,u,W)$ and $e' = (N',u',W') \in \mathcal{E}$, such that $N' \subseteq N$. Consider the auxiliary economy $\widehat{e} = (N',u',\widehat{W}) \in \mathcal{E}$, where $\widehat{W} = \sum_{i\in N'} F_i(e)$. Then, by CY, $F(\widehat{e}) = F_{N'}(e)$. By RM, one of the following three possibilities happens: $F(\widehat{e}) = F(e')$, $F(\widehat{e}) > F(e')$ or $F(\widehat{e}) < F(e')$, which shows that F satisfies SL. $\blacksquare$

Statement 2 If a rule satisfies resource monotonicity then it also satisfies resource continuity.

Proof. Let F be an allocation rule satisfying RM. Let $e = (N, u, W) \in \mathcal{E}$ be given and consider a sequence of positive real numbers $\{W^n\} \to W$. For each n, let $e^n = (N, u, W^n) \in \mathcal{E}$. We show that $\lim_{n\to\infty} F(e^n) = F(e)$, i.e., $\lim_{n\to\infty} F_i(e^n) = F_i(e)$ for all $i \in N$.

Assume, by contradiction, that $\lim_{n\to\infty} F_i(e^n) \neq F_i(e)$ for some $i \in N$. Then, there exists $\varepsilon > 0$ such that for all $n_0 \in \mathbb{Z}_+$ there exists $n > n_0$ such that $|F_i(e^n) - F_i(e)| > \varepsilon$. Now, since $\{W^m\} \to W$, it follows that there exists $m_0 \in \mathbb{Z}_+$ such that for all $m > m_0$, $|W^m - W| \leq \varepsilon$. In particular, for $n_0 = m_0$, it follows that there exists $n > n_0$ for which

$$|F_i(e^n) - F_i(e)| > |W^n - W| \,. \tag{1}$$

We now distinguish three cases:
Case 1: $W^n = W$.
In this case, $e^n = e$ and therefore $|F_i(e^n) - F_i(e)| = 0 = |W^n - W|$, which contradicts (1).
Case 2: $W^n > W$.
In this case, by RM, $F_k(e^n) > F_k(e)$ for all $k \in N$. Then, (1) becomes

$$W - F_i(e) > W^n - F_i(e^n),$$

or, equivalently,

$$\sum_{k\neq i} F_k(e) > \sum_{k\neq i} F_k(e^n).$$

Now, since $F_k(e^n) > F_k(e)$ for all $k \in N$, it follows that $\sum_{k\neq i} F_k(e) < \sum_{k\neq i} F_i(e^n)$, which represents a contradiction.
Case 3: $W^n < W$.
In this case, by RM, $F_k(e^n) < F_k(e)$ for all $k \in N$. Then, (1) becomes

$$W - F_i(e) < W^n - F_i(e^n),$$

or, equivalently,

$$\sum_{k\neq i} F_k(e) < \sum_{k\neq i} F_k(e^n).$$

Now, since $F_k(e^n) < F_k(e)$ for all $k \in N$, it follows that $\sum_{k \neq i} F_k(e) > \sum_{k \neq i} F_i(e^n)$, which represents a contradiction.

Statement 3 If a rule satisfies priority and solidarity then it is parametric.

Proof. Let F be an impartial rule satisfying PR and SL. Then, F also satisfies CY, RM and RC. We show that F is parametric.

Let $u_i \in \mathcal{V}$. Let u_0 be the identity function, i.e., $u_0(x) = x$ for all $x \in \mathbb{R}_+$. For every $\lambda \in \mathbb{R}_+$, define

$$t = f(\lambda, u_i) \iff (t, \lambda) = F(\{i, 0\}, (u_i, u_0), t + \lambda).$$

We have the following:

1. f **is well defined**. Let $u_i \in \mathcal{V}$ and $\lambda \in \mathbb{R}_+$ be given. Then, by PR and RC, there exists $W \geq \lambda$ such that $F_0(\{i, 0\}, (u_i, u_0), W) = \lambda$. Note that, by RM, W is unique. Let $t = W - \lambda \geq 0$. Then, $(t, \lambda) = F(\{i, 0\}, (u_i, u_0), t + \lambda)$ and therefore, $t = f(\lambda, u_i)$.

2. f **is continuous in its first component**. Let $u_i \in \mathcal{V}$ and $\{\lambda^n\} \to \lambda$. Let $t^n = f(\lambda^n, u_i)$ and $t = f(\lambda, u_i)$. We have to show that $\{t^n\} \to t$. Assume, by contradiction, that $\{t^n\} \nrightarrow t$. Then, there exists a subsequence $\{t^{n_k}\}$, such that, either $\{t^{n_k}\} \to \widehat{t} < t$, or $\{t^{n_k}\} \to +\infty$.[8] Consider the corresponding subsequence $\{\lambda^{n_k}\}$. Then, in the former case, $(t^{n_k}, \lambda^{n_k}) \to (\widehat{t}, \lambda)$. Since F satisfies RC, it follows that $(t^{n_k}, \lambda^{n_k}) = F(\{i, 0\}, (u_i, u_0), t^{n_k} + \lambda^{n_k}) \to F(\{i, 0\}, (u_i, u_0), \widehat{t} + \lambda)$. Thus, $(\widehat{t}, \lambda) = F(\{i, 0\}, (u_i, u_0), \widehat{t} + \lambda))$ and therefore, $\widehat{t} = f(\lambda, u_i) = t$, which is a contradiction. In the latter case, i.e., if $\{t^{n_k}\} \to \infty$, $(t^{n_k}, \lambda^{n_k}) = F(\{i, 0\}, (u_i, u_0), t^{n_k} + \lambda^{n_k}) \to (\infty, \lambda)$. Thus, for all $M > 0$, there exists k_0 such that for all $k > k_0$, $t^{n_k} > M$ and $\lambda^{n_k} < \lambda + 1$. Let $M = \max\{u_i^{-1}(\lambda + 1), \lambda + 1\}$ and $k > k_0$. Then, $t^{n_k} > \lambda^{n_k}$ and therefore, by PR, $u_i(t^{n_k}) \leq u_0(\lambda^{n_k}) = \lambda^{n_k} < \lambda + 1 \leq u_i(M)$, which contradicts the fact that u_i is strictly increasing.

3. f **is strictly monotonic in its first component**. Let $u_i \in \mathcal{V}$ and $\lambda < \lambda'$. Let $t = f(\lambda, u_i)$ and $t' = f(\lambda', u_i)$. Then, $(t, \lambda) = F(\{i, 0\}, (u_i, u_0), t + \lambda)$ and $(t', \lambda') = F((\{i, 0\}, (u_i, u_0), t' + \lambda')$. Since F satisfies RM, it follows that $t < t'$.

[8] The case in which $\widehat{t} > t$ is analogous.

4. $f(0, u_i) = 0$ **for all** $u_i \in \mathcal{V}$. If, on the contrary, $f(0, u_i) = t > 0$ for some $u_i \in \mathcal{V}$, then $(t, 0) = F(\{i, 0\}, (u_i, u_0), t + 0)$. By PR, $u_i(t) \leq u_0(0) = 0$, a contradiction.

5. $\lim_{\lambda \to \infty} f(\lambda, u_i) = \infty$ **for all** $u_i \in \mathcal{V}$**.** Assume, by contradiction, that there exists $u_i \in \mathcal{V}$ and $M > 0$ such that, for all λ, $f(\lambda, u_i) < M$. Let $\lambda_0 = \max\{u_i(M), M\}$ and $t_0 = f(\lambda_0, u_i)$. Then, $t_0 < M \leq \lambda_0$. Now, by definition of f, $(t_0, \lambda_0) = F(\{i, 0\}, (u_i, u_0), t_0 + \lambda_0)$. Thus, by PR, $u_i(t_0) \geq u_0(\lambda_0) = \lambda_0 \geq u_i(M) > u_i(t_0)$, where the last inequality follows from the fact that u_i is strictly increasing. This is obviously a contradiction.

6. **Given** $e = (N, u, W) \in \mathcal{E}$**, there exists** $\lambda_e \in \mathbb{R}_+$ **such that** $F_i(e) = f(\lambda_e, u_i)$ **for all** $i \in N$**.** Let $e = (N, u, W) \in \mathcal{E}$ be given and denote $x = F(e)$. Let $j \notin N$ such that $u_j = u_0$, and for each $W' > 0$ construct $e'(W') = (N \cup \{j\}, (u, u_j), W')$. By PR and RC, there exist W^{12}, W^{13} such that $F_1(e'(W^{12})) + F_2(e'(W^{12})) = x_1 + x_2$ and $F_1(e'(W^{13})) + F_3(e'(W^{13})) = x_1 + x_3$. Let $\overline{e}_k = (\{1, k\}, (u_1, u_k), x_1 + x_k)$, for $k = 2, 3$. By CY, $F(\overline{e}_k) = (x_1, x_k)$ for $k = 2, 3$. Thus, $F_1(e'(W^{12})) = x_1 = F_1(e'(W^{13}))$ and therefore, by RM, $W^{12} = W^{13}$. Proceeding further with this argument, we can show that there exist values W^* and λ^* such that $F(e'(W^*)) = (x, \lambda^*)$. By CY, $(x_i, \lambda^*) = F(\{i, j\}, (u_i, u_0), x_i + \lambda^*)$ for all $i \in N$. Thus, $x_i = f(\lambda^*, u_i)$ for all $i \in N$. ■

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