Feasible solutions to a non-linear state-dependent delay control problem

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Abstract

In this paper, a feasibility solution set to a continuous time non-linear control problem is developed. The unique differential state equation contains a state-dependent time delay. The context in which such system appears is real-time electric power consumption management of a large group of electric appliances. In particular, energy consumption of the group is shifted in time in order to deliver flexibility services (e.g., ancillary services, primary frequency control). The objective of the paper is to study the proposed equation system. Developments prove feasibility and stability of some control actions. An acceptable range of reference signals is defined, that the output of the system would be able to follow in a signal tracking framework.

1 Context

In electric power systems, Demand Response (DR) *refers to a wide range of actions which can be taken at the [...]* end-user side *in response to particular conditions within the electricity system* [4]. DR is exploited to respect a strong technical constraint proper to power systems: power generation and power demand must stay equal at all time. Indeed, this constraint imposes power system operators to constantly adjust the output of flexible power assets (e.g., operating reserves) to compensate for unforeseen variations in power consumption and generation.

Most of these adjustments are performed by controlling large and flexible generation units such as gas and hydro power plants. However, power demand may in some cases be sufficiently flexible and controllable to provide similar or higher service quality.

However, as analyzed in [1], large shares of the available flexible demand are scattered around in the system. Residential power demand, much harder to access than high-voltage connected industrial power demand, could indeed cover up to 90% of the power increase and around 30% of the power decrease potential. In fact, all thermal (e.g. refrigerators,

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heat-pumps, electric boilers) and white appliances (e.g. washing-machine) as well as electric vehicles are technically able to shift their consumption in time.

Exploiting this flexibility will have to address the problem of controlling large groups of small loads. We propose here one answer to this problem.

1.1 Initial problem: large homogeneous group of flexible task-driven loads

The control aims at managing the power consumption of a large and homogeneous group of task-driven loads. Following developments make use of five working assumptions.

Firstly, loads are task-driven. That is, a load i starts at time t_s^i, consumes energy at power P_n for a duration T_n and then stops (as shown in Fig.1, *initial*).
Secondly, while a single load may experience several starts during a day, our approach imposes these successive starts to occur independently from each other.
Thirdly, we study homogeneous groups. Such groups contain loads with identical characteristics (P_n,T_n). In such groups, the different load starts (collection of t_s^i) may be considered as a stochastic process S_t, that represents a *density of starts per unit time*. In other words, it is a continuous time form of counting the number of loads starting at each time. Loads that are consuming energy at time t are the *active* loads A_t.
Fourthly, we consider this density of start to be a constant $\bar{s}$: $S_t = \bar{s}\ [s^{-1}]$.
Finally, the control consists in sending a unique signal α_t to all active loads such that their power rate becomes variable $P_t = \alpha_t P_n$. In addition, energy is conserved. Loads will run until they have consumed the initially required amount of energy $E = P_n T_n$.

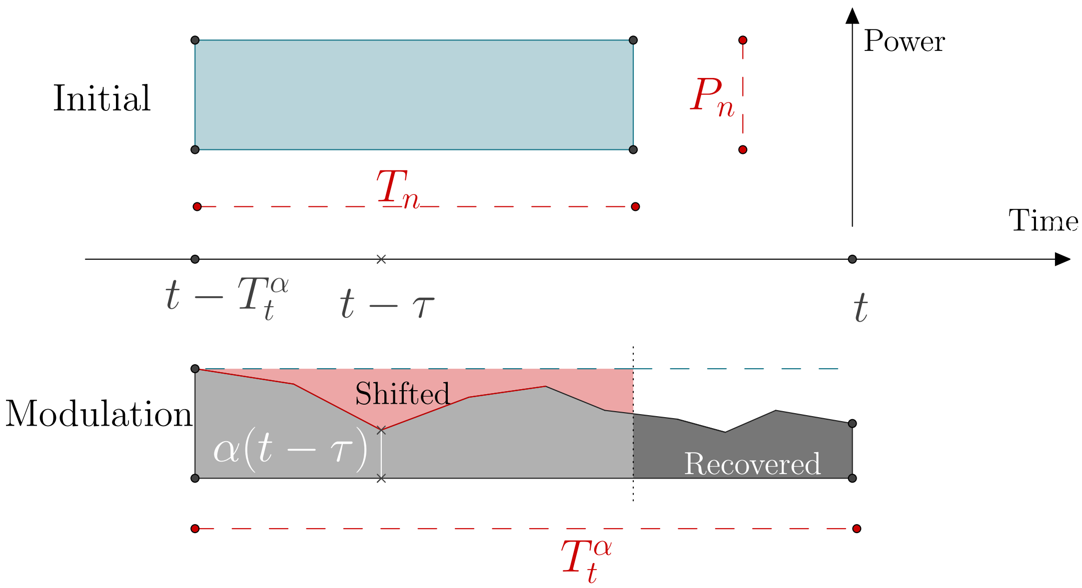

Figure 1: A single task-driven load (t_s^i, P_n, T_n) and its power profile before (intial) and after modulation. The control input $\alpha(t)$ is presented at time $(t-\tau)$ with $P_n = 1$.

Fig. 1 shows an example of control input α_t and the impact on the power shape of a single load *stopping* at time t. The load has consumed its energy E at another rate than initially planned, and therefore was active for a duration $T_t^\alpha \neq T_n$, the *dynamic duration*. As the control signal is unique and the group homogeneous, any load starting at time $t - T_t^\alpha$ (or stopped at t) experiences the same power profile.

The energy conservation constraint (1) defines implicitly the dynamic duration T_t^α.

$$\int_{t-T_t^\alpha}^{t} \alpha_\tau P_n d\tau = E \tag{1}$$

The active loads at time t are those loads that have started later than the load that just stopped.

$$A_t = \int_{t-T_t^\alpha}^{t} S_\tau d\tau = \bar{s} T_t^\alpha \quad [-]$$

The output of the system is the total power of the group P_t^{tot}, in other words the power consumption of all active loads.

$$P_t^{tot} = A_t \, \alpha_t P_n = \alpha_t P_n \bar{s} T_t^\alpha \quad [kW] \tag{2}$$

The challenge in controlling such large groups is to link individual control decisions to a global group's reaction. Due to the uniqueness of the control signal and the energy conservation constraint, group's dynamics boil down to a unique state equation sufficient to define the output dynamics.

The time derivative of equation 1 form with equation 2 a non-linear system $\mathcal{S}$.

$$\mathcal{S} = \begin{cases} \dfrac{dT_t^\alpha}{dt} = 1 - \dfrac{\alpha_t}{\alpha(t - T_t^\alpha)} \\ P_t^{tot} = \alpha_t P_n \bar{s} T_t^\alpha \end{cases} \tag{3}$$

This system holds two non-linear equations, and the state evolution equation is a delay differential equation in which the delay T_t^α is itself state of the system.

There are ways to linearize the above control problem (state-bin models, [2] for TCL control). However, as linearization is only valid locally (around an equilibrium state), constant feedback will be required to track non-linear states' evolution. Hence, loads should be capable of sending data very regularly to a control center. The research goal being to minimize the communication requirements, linearization is not a valid option. There exist techniques to decrease communication needs in a linearized framework, as shown in [3], which degrade significantly service quality.

The flexibility of the group is defined as

1. In the short run, group's ability to provide a power deviation $\Delta P_t^{tot} = P_t^{tot} - P^{eq}$ from initial situation. That is, without control intervention $\alpha_t = 1 \; \forall t$, $T_t^\alpha = T_n$ and $P_t^{tot} = \bar{s} T_n P_n = \bar{s} E = P^{eq}$.

2. In the long run, the ability of the group to maintain a certain power deviation and effectively shift energy in time $E^{sh} = -\int \Delta P_t^{tot} \, dt \quad [kJ]$.

2 Determining the feasible solutions set

The objective of this section is to prove that the system may reach stable states and describe the feasible trajectories that the output may follow. We also determine bounds on the amount of energy that may be shifted by the group E^{sh}.

For convenience, some general notations are considered and the system is normalized ($P_n = 1$,$\bar{s} = 1$,$T_n = C$). The above system is transformed to a dimensionless single input ($u_t = \alpha_t$) single output ($y_t = \frac{P_t^{tot}}{\bar{s}P_nT_n}$) and single state ($x_t = \frac{T_t^{\alpha}}{T_n}$) and takes the following form.

$$\mathcal{S} = \begin{cases} \dot{x}_t = 1 - \dfrac{u_t}{u_{(t-x_t)}} & (4) \\ y_t = x_t u_t & (5) \end{cases}$$

with : $x(0) = x_0;\ u_t \in [u_m, u_M]\,\forall t;\ 0 < u_m \leq 1;\ 1 \leq u_M < \infty;\ u_t = v_t(\forall t \leq 0)$.

Equation 4 is the time derivative of the energy conservation constraint and defines the state/delay x_t. Indeed, eq.(6) is the equivalent to eq. (1).

$$\int_{t-x_t}^{t} u_\tau d\tau = C\ \forall t \tag{6}$$

Note that, without control intervention, the natural input is constant $u_t = 1$. Equation 6 shows that the natural state value is $x_t = C$ and therefore, natural output is $y_t = C$.

Let's now highlight, through three different propositions, the feasibility and limitations of this control problem.

- Proposition 1 shows that, as long as past input belong to a bounded interval, imposing a constant input leads the system to stabilize to an equilibrium in finite time.
- Proposition 2 defines the potential energy of the group, and shows that it represents an equivalent state of the system, which may only evolve when the output $y(t)$ differs from its natural value C.
- Proposition 3 states that it is always possible to impose the output to take its natural value C and that bounds on past inputs imposes future inputs to stay bounded and converge to a constant value. We compute the asymptotic value which is inversely proportional to the potential energy of the system just before the output was switched to C. Finally, input bounds imply the potential energy to be bounded, what limits the energy shifting potential of the group $E^{sh} = -\int (y_t - C)\, dt$.

Proposition 1. *Let (u_e, x_e, y_e) be an equilibrium point of the system (u_t, x_t, y_t). For any constant $C \in \mathcal{R}_+ \setminus \{0\}$, any previous inputs $u_t = v_t$ $(\forall t \leq 0)$ $: \mathcal{R} \rightarrow [0 < u_m \leq 1, 1 \leq$*

$u_M < \infty$] and the associated initial state $x(0)$ s.t.

$$\{x(0)| \int_{-x(0)}^{0} \upsilon_\tau d\tau = C\}$$

the system always reaches its equilibrium $(u_e, x_e = \frac{C}{u_e}, y_e = C)$ if $u_t = u_e (\forall t \geq 0)$ in finite time $t_e \leq \frac{C}{u_e}$.

Proof. Firstly, the bounds on the input u_t imply through equation (4) a bounded time derivative of the state x_t.

$$(1 - \frac{u_M}{u_m}) \leq \dot{x}_t \leq (1 - \frac{u_m}{u_M}) < 1 \quad \forall t \tag{7}$$

Secondly the constant C and the bounded input function u_t imply bounds on any value of x_t. Indeed, equation (6) has to be verified for all time t and all values of u_t. Extreme values of the integral of u_t define extreme values of x_t.

$$\int_{t-x_m}^{t} u_M d\tau = C \quad \& \quad \int_{t-x_M}^{t} u_m d\tau = C \quad \forall t \tag{8}$$

This gives therefore limits on x_t.

$$0 < \frac{C}{u_M} = x_m \leq x_t \leq x_M = \frac{C}{u_m} < \infty \quad \forall t \tag{9}$$

Furthermore, as $y_t = u_e x_t$, the output signal is also bounded for any time t.

$$0 < \frac{C u_m}{u_M} \leq y_t \leq \frac{C u_M}{u_m} < \infty \quad \forall t \tag{10}$$

The system is therefore stable in the *Bounded-input Bounded-output* sense for any input $u_t \in [u_m, u_M]$.

Moreover, the system will reach an equilibrium if the input is maintained to a constant value $u_t = u_e$ $(\forall t \geq 0)$. Starting from t=0, and before a threshold time t_e, we have that $x_t < t$ $(\forall t < t_e)$. As $\dot{x}_t < 1$ we can derive that $\exists t_e$ such that $x(t_e) = t_e$.

Equation (6) expressed at threshold time t_e gives the following.

$$\int_{t_e - x(t_e)}^{t_e} u_\tau d\tau = \int_{0}^{t_e} u_e d\tau = u_e t_e = C \tag{11}$$

Therefore, the threshold time $t_e = \frac{C}{u_e} < \infty$ and defines the equilibrium delay/state $x(t_e) = t_e = x_e$.

Furthermore, as expressed by (4), $\dot{x}(t \geq t_e) = 0$. Indeed, $u_t = u(t - t_e) = u_e$ $(\forall t \geq t_e)$. The equilibrium output may also be computed by $y_e = x_e u_e = C$. This result is crucial and shows that steady state may only be reached when the output is constant and equal to its natural value C. □

Proposition 2. *Considering system $\mathcal{S}$, the potential energy function E_t is a positive definite function.*

$$
\begin{aligned}
E_t &= \int_{t-x_t}^{t} (C - \int_{\theta}^{t} u_\tau d\tau) d\theta \quad > 0 && (12)\\
&= Cx_t - \int_{t-x_t}^{t} (\int_{\theta}^{t} u_\tau d\tau) d\theta && (13)\\
&= \int_{t-x_t}^{t} (\int_{t-x_t}^{\theta} u_\tau d\tau) d\theta && (14)
\end{aligned}
$$

The potential energy E_t is conserved if and only if $y_t = C$.

Proof. Firstly, E_t is trivially positive definite. Indeed, we have,

$$C = \int_{t-x_t}^{t} u_\tau \, d\tau$$

and, as $u_t > 0$, $\forall t$,

$$C - \int_{\theta}^{t} u_\tau d\tau > 0, \ \forall \theta > (t - x_t).$$

Secondly, let's define

$$f(t, \theta) = \int_{\theta}^{t} u_\tau \, d\tau$$

and its primitive with respect to θ, denoted as F_t.

Similarly U_t is the primitive of u_t at time t. Therefore, we have

$$f(t, \theta) = U_t - U_\theta.$$

We compute below the time derivative $\dot{E}_t$ of the potential energy.

$$
\begin{aligned}
\dot{E}_t &= C\dot{x}_t - \frac{d}{dt}(\int_{t-x_t}^{t} (\int_{\theta}^{t} u_\tau d\tau) d\theta) && (15)\\
&= C\dot{x}_t - \frac{d}{dt} g_t && (16)
\end{aligned}
$$

where we need to compute the time derivative of function g_t.

Firstly, we may develop g_t.

$$
\begin{aligned}
g_t &= \int_{t-x_t}^{t} f(t, \theta) d\theta && (17)\\
&= U_t \int_{t-x_t}^{t} d\theta - \int_{t-x_t}^{t} U_\theta \, d\theta && (18)\\
&= x_t U_t - \int_{t-x_t}^{t} U_\theta \, d\theta && (19)
\end{aligned}
$$

Therefore,

$$\frac{d}{dt}g_t = \dot{x}_t U_t + x_t \dot{U}_t - U_t - U_{(t-x_t)}(1-\dot{x}_t) \tag{20}$$

Recalling that $U_t - U_{(t-x_t)} = \int_{t-x_t}^{t} u_\tau d\tau = C$, the time derivative of g_t may be simplified to what follows.

$$\begin{aligned} \frac{d}{dt}g_t &= (U_t - U_{(t-x_t)})(\dot{x}_t - 1) + x_t u_t && (21)\\ &= C(\dot{x}_t - 1) + x_t u_t && (22)\\ &= C(\dot{x}_t - 1) + y_t && (23) \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{E}_t &= C\dot{x}_t - (C(\dot{x}_t - 1) + y_t) && (24)\\ \dot{E}_t &= C - y_t && (25) \end{aligned}$$

We have that $\dot{E}_t = 0$ whenever $y_t = C$.

Furthermore, the potential energy of the group is maintained at its natural value $E^{eq} = \dfrac{C^2}{2}$, when the group is in its initial, uncontrolled, state ($u_t = 1$, $x_t = C, \forall t$).

The potential energy is directly linked to the shifted energy $E_t^{sh} = -\int (y_t - C)\, dt$,

$$E_t = \frac{C^2}{2} + E_t^{sh}$$

□

Proposition 3. *Considering the system $\mathcal{S}$: for any bounded past input $u_{t<0} = \nu_t \in [u_m, u_M]$, it is always possible to impose future output to stick to the natural output $y_t = C$ $\forall t \geq 0$. Furthermore, the system $\mathcal{S}$ converges asymptotically toward an equilibrium $(u_e, x_e = \frac{C}{u_e}, y_e = C)$. In addition, Future input values are insured to stay within the bounds of past values ν_t.*

The amount of energy that the group is able to shift is limited.

$$-\frac{C^2}{2} < \frac{C^2}{2}\frac{1-u_M}{u_M} \leq E^{sh} \leq \frac{C^2}{2}\frac{1-u_m}{u_m}$$

Proof. Firstly, equation (28) shows that u_t will remain $\forall t > 0$ within it bounds.

$$\begin{aligned} C &= u_t\, x_t && (26)\\ &= \int_{t-x_t}^{t} u_\tau\, d\tau && (27) \end{aligned}$$

Implies $\forall t \geq 0$

$$u_t = \frac{1}{x_t}\int_{t-x_t}^{t} u_\tau d\tau \tag{28}$$

Which leads to

$$u_m \leq u_t \leq u_M \quad \forall t \tag{29}$$

We note that u_t corresponds at any time $t \geq 0$ to the average value of past inputs u_t computed on the *active time window* $[t - x_t, t[$. Therefore, we may impose $y_t = C$ for any initial condition as long as $v_t \in [u_m, u_M]$. Furthermore, we can show that u_t and x_t are asymptotically converging to constant values.

Indeed, the system is asymptotically stable in the sense of Lyapunov. Let us first define the positive semi-definite function $V(x, u, t) \geq 0$.

$$\begin{aligned} V(x,u,t) &= \int_{t-x_t}^{t} (u_\tau - u_t)^2 d\tau && (30) \\ &= \int_{t-x_t}^{t} u_\tau^2 d\tau + u_t^2 x_t - 2u_t \int_{t-x_t}^{t} u_\tau d\tau && (31) \\ &= \int_{t-x_t}^{t} u_\tau^2 d\tau - C u_t && (32) \end{aligned}$$

The function takes zero value only if the input stays constant in the active time window, such that all values $\{u_\tau | \tau \in [t - x_t, t]\}$ are equal. Furthermore, the time derivative $\dot{V}(x, u, t)$ of $V(x, u, t)$ is negative semi-definite, as shown below.

$$\begin{aligned} \dot{V}(x,u,t) &= \frac{d}{dt}\left(\int_{t-x_t}^{t} u_\tau^2 - C u_t \right) && (33) \\ &= \left(\frac{d}{dt} \int_{t-x_t}^{t} u_\tau^2 \right) - C \dot{u}_t && (34) \end{aligned}$$

Because $y_t = x_t\, u_t$ is set to C, we have $x_t = \dfrac{C}{u_t}$ and $\dot{x}_t = -\dfrac{C \dot{u}_t}{u_t^2}$ such that $C \dot{u}_t = -u_t^2 \dot{x}_t$. Therefore,

$$\begin{aligned} \dot{V}(x,u,t) &= \left(\frac{d}{dt} \int_{t-x_t}^{t} u_\tau^2 d\tau \right) + u_t^2 \dot{x}_t && (35) \\ &= \left(u_t^2 - u_{(t-x_t)}^2 (1 - \dot{x}_t) \right) + u_t^2 \dot{x}_t && (36) \\ &= u_t^2 \left[(1 - \frac{u_{(t-x_t)}^2}{u_t^2}) + \dot{x}_t \ (1 + \frac{u_{(t-x_t)}^2}{u_t^2}) \right] && (37) \end{aligned}$$

Recalling that $\dot{x}_t = 1 - \frac{u_t}{u(t-x_t)}$, we have that $\left(\frac{u_t}{u(t-x_t)}\right)^2 = \left(1-\dot{x}_t\right)^2$.

$$\begin{aligned}
\dot{V}(x,u,t) &= u_t^2\left[(1-\frac{1}{(1-\dot{x}_t)^2}) + \dot{x}_t\,(1+\frac{1}{(1-\dot{x}_t)^2})\right] && (38)\\
&= \frac{u_t^2}{(1-\dot{x}_t)^2}\left[(1-\dot{x}_t)^2 - 1 + \dot{x}_t(1-\dot{x}_t)^2 + \dot{x}_t\right] && (39)\\
&= \frac{u_t^2}{(1-\dot{x}_t)^2}(\dot{x}_t-1)\left[(\dot{x}_t-1) + \dot{x}_t(\dot{x}_t-1) + 1\right] && (40)\\
&= \frac{u_t^2}{(1-\dot{x}_t)^2}(\dot{x}_t-1)\dot{x}_t^2 && (41)\\
&= \left(\frac{u_t\dot{x}_t}{(1-\dot{x}_t)}\right)^2(\dot{x}_t-1) \quad \leq \quad 0 && (42)
\end{aligned}$$

Indeed, $(\dot{x}_t - 1)$ is always negative, as equation (4) shows, and the condition $u_t > 0$ insures $\dot{x} < 1$.

Consequently, the function $V(x,u,t)$ must decrease and tend asymptotically to zero as soon as $y_t = C$.

The time derivative $\dot{V}(x,u,t)$ will converge asymptotically to zero. This implies that $\dot{x}_t$ will also tend to zero. As $\dot{x}_t = 0$ implies $u_t = u_{(t-x_t)}$, steady state situation is reached if and only if $u_t = u_{(t-x_t)}$ for all successive time t, meaning that the input will be constant.

The above development shows that u_t will converge to a constant value u_e. Therefore, the system converges to $(u_t = u_e, x_t = \frac{C}{u_e}, y_t = C)$. As we have seen, there exist a state E_t, the *potential energy* of the system $\mathcal{S}$, that remains constant when $y_t = C$ (proposition 2). The energy E_0 at time $t = 0$ defines the asymptotic equilibrium.

$$E_0 \quad = \quad Cx_0 - \int_{-x_0}^{0}(\int_{\theta}^{0} \upsilon_\tau d\tau)d\theta \tag{43}$$

$$E_t|_{u_e,x_e}E_0 \quad \Leftrightarrow \quad Cx_e - u_e\frac{x_e^2}{2} = Cx_0 - \int_{-x_0}^{0}(\int_{\theta}^{0} \upsilon_\tau d\tau)d\theta \tag{44}$$

$$\frac{Cx_e}{2} = E_0 \quad \Leftrightarrow \quad x_e = \frac{2E_0}{C} \quad \Leftrightarrow \quad u_e = \frac{C^2}{2E_0} \tag{45}$$

Bounds on inputs and therefore on u_e (i.e., initial conditions are bounded) define bounds on E_t. Finally, this imposes limits on the amount of energy that may be shifted by the group.

$$\frac{C^2}{2u_M} \leq E_t \leq \frac{C^2}{2u_m} \quad \Leftrightarrow \quad \frac{C^2}{2}\frac{1-u_M}{u_M} \leq E^{sh} \leq \frac{C^2}{2}\frac{1-u_m}{u_m} \tag{46}$$

Furthermore, physical limits on the upper bound of $u < \infty$ or similarly the positive definite character of E_t imply $E^{sh} > -C^2/2$.

□

3 Conclusions

This paper explores the analytic solution set to a non-linear delay differential equations system.

As developed in proposition 1, there exist an infinite number of steady state where the input is a constant $u_e \in [u_m, u_M]$, the output is necessarily equal to a natural value $y = x\,u = C$ and the unique state is also constant $x_e = C/u_e$.

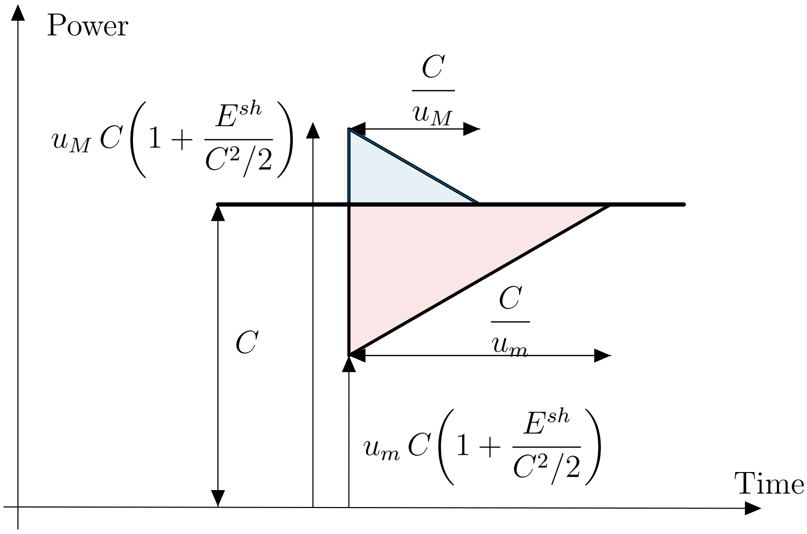

Figure 2: Static view of possible output trajectories (shaded area), starting from an equilibrium point $y_e = C$ and $E_0 = C^2/2 + E^{sh}$.

As illustrated on Fig 2, there exist a space of future possible solutions starting from any of these initial states, where $E^s h = E_t - \frac{C^2}{2}$ is defined in proposition 2 the amount of energy that was shifted by the group to reach its initial steady-state solution.

Due to proposition 3, it is always possible to stabilize to the natural output $y = C$, while maintaining the input within its bounds.

More particular solutions, needed for instance if the output should track an external reference signal r_t, require numerical integration. However, the output y_t will only be able to track a reference signal r_t that respect the following conditions.

$$u_m\,C\left(1+\frac{2R_t}{C^2}\right) \leq \quad r_t \quad \leq u_M\,C\left(1+\frac{2R_t}{C^2}\right) \tag{47}$$

$$\frac{C^2}{2}\,\frac{1-u_M}{u_M} \leq \quad R_t \quad \leq \frac{C^2}{2}\,\frac{1-u_m}{u_m} \tag{48}$$

where $R_t = R_0 + \int_0^t (C - r_\tau)\,d\tau$.

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