ECONOMIC GROWTH WITH GIFTS IN THE FAMILY

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Abstract

There are two basic models of economic growth. In the overlapping-generations model, there is no private transfer decision of households. In the model with an infinite-lived representative consumer, there is a complete harmony of all generations who behave as a unique agent. We propose another approach with gifts in the family and a unique head of the family. It differs from the joy of giving model in two points: parents take into account wealth of the children and gifts are two-sided, from parents to children and from children to parents.

We first present standard results of the neoclassical growth theory and some recent developments. After that we study three different assumptions on the behavior of households: no gift, gifts in the family and one-sided gifts from parents to children.

1 Neoclassical growth with technical progress

In this chapter, we present some standard notions and results of the neoclassical growth theory. Basic topics are in the four first-sections: the production technology, the Solow growth model, the Golden Rule of capital accumulation and the remuneration of the production factors. The following and last four sections introduce endogenous growth with externalities, optimal growth and the model with an infinitely lived representative consumer, a model of production with fixed costs, and some other allocations of the national product.

1.1 Production technology and technical progress

Constant returns to scale is a basic assumption of the neoclassical production technology. We consider an aggregate production function of two production factors, capital and labor, which is homogeneous of degree one. The technical progress is exogenous and given by a constant growth rate of efficient labor per capita. This is known as Harrod-neutral technical progress (Harrod, 1942).

1.1.1 Production, capital and labor

Aggregate production is defined by a function F_A which is differentiable and *homogeneous of degree one* with respect to aggregate capital K and aggregate labor L. There are constant returns to scale: for any positive real number $x, F_A(xK, xL)$ is equal to $xF_A(K, L)$.

There is an infinite sequence of periods $t, t = 0, 1, 2, 3, \cdots$ Population N_t grows at a constant rate $\nu - 1$, $\nu > 0$ is the growth factor of population: $N_{t+1} = \nu N_t$. Efficient labor per capita h_y grows at a constant rate $\eta - 1$, $\eta > 0 : h_{t+1} = \eta h_t$. *Total efficient labor* in period t is $L_t = N_t h_t$. It satisfies: $L_{t+1} = \nu\eta L_t$. The growth factor $n = \nu\eta$ of efficient labor is the *natural growth factor* (Harrod (1939)).

There is a constant depreciation rate of capital $1 - \delta$ which belongs to the interval $[0, 1]$. Capital after depreciation, δK_t, is added to the aggregate production $F_A(K_t, L_t)$ to define the *total disposable product* Y_t:

$$Y_t \;=\; F_A(K_t, L_t) + \delta K_t \;\equiv\; Y(K_t, L_t) \quad \text{and} \quad Y(K_t, L_t) \;=\; L_t y(k_t) \tag{1.1}$$

$k_t = K_t/L_t$ is the capital-labor ratio, $y(k_t) = Y(k_t, 1)$ is the product-labor ratio. We call $y(\cdot)$ the *product function*.

1.1.2 A standard assumption on the product function

The function $y(\cdot)$ is assumed to be twice continuously differentiable, increasing and strictly concave. We consider the following *standard assumption on the product function* $y(\cdot)$:

$$\forall k > 0,\ y(k) > 0,\ y'(k) > 0,\ \text{and } y''(k) < 0 \tag{1.2}$$

$y'(\cdot)$ and $y''(\cdot)$ are the first order and the second order derivatives of $y(\cdot)$. Any function which is defined and *monotonic* on the interval $\mathbb{R}_+^*$ of positive real numbers admits limits, finite or infinite, when the variable tends to 0 and when it tends to ∞.

We denote these limits: $y(0)$ and $y(\infty)$ for the increasing function $y(\cdot)$, $y'(0)$ and $y'(\infty)$ for the decreasing function $y'(\cdot)$; $y(0)$ and $y'(\infty)$ are finite and non-negative; $y(\infty)$ and $y'(0)$ are positive, finite or infinite.

In the appendix 1.A.1 we show that the standard assumption (1.2) implies that the product-capital ratio $\hat{y}(k) = y(k)/k = Y(K, L)/K$ is a *decreasing function* of k and satisfies for all $k > 0$: $\hat{y}(k) > y'(k)$, i.e. the average productivity of capital is larger than the marginal productivity of capital; moreover, the two limits $\hat{y}(\infty)$ and $y'(\infty)$ which are finite are equal.

1.1.3 An example

In the product function $y(k) = A_1 k^\alpha + A_2 k + A_3$, with $0 < \alpha < 1, A_1 > 0, A_2 > 0, A_3 > 0$, the first term corresponds to a standard Cobb-Douglas function; the second term includes the after-depreciation capital; the third term allows for production with zero capital[1]

For this product function, we have:

$$y'(k) \ = \ \alpha A_1 k^{\alpha-1} + A_2 \quad \text{and} \quad \hat{y}(k) \ = \ A_1 k^{\alpha-1} + A_2 + A_3 k^{-1}$$

[1]Such a product function, with $\alpha = 1/2$, is obtained with a production function with a constant elasticity of substitution $\sigma_F = 2$:

$$y(k) \ = \ A(bk^{1/2} + 1 - b)^2 + \delta k = (Ab^2 + \delta)k + 2Ab(1-b)k^{1/2} + A(1-b)^2$$

The study of the CES production functions is made in Appendix 1.A.2.

The limits are:

$$\begin{aligned} y(0) &= A_3,\ y'(0) = \infty,\ \hat{y}(0) = \infty \\ y(\infty) &= \infty,\ y'(\infty) = A_2,\ \hat{y}(\infty) = A_2 \end{aligned}$$

The two decreasing functions $y'(k)$ and $\hat{y}(k)$ are represented on figure 1.1

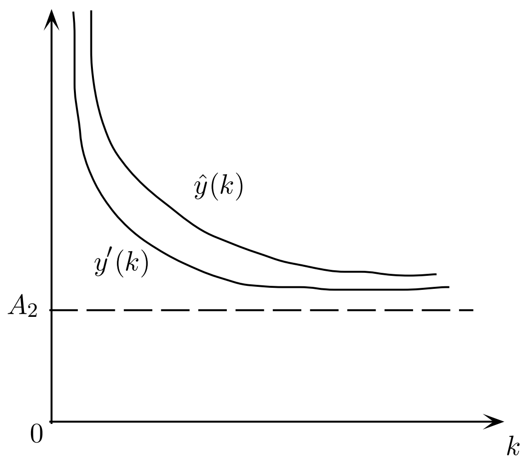

Figure 1.1

1.2 Dynamics with a constant savings rate: the Solow model

In his celebrated paper of 1956, R.M. Solow studies the dynamics of the capital-labor ratio when savings are proportional to production and he discusses many possibilities. We present his main conclusions under the assumption that the product-capital ratio $\hat{y}(k) = y(k)/k$ is a decreasing function of the capital-labor ratio k. This assumption simply means that production is increasing with labor[2] and it implies that there always exists a *unique limit* of the dynamics of the capital-labor ratio which is finite of infinite. in the traditional textbook case, the limit is finite and positive, and the long run growth rate is equal to the natural rate.

[2]By definition, $\frac{y(k)}{k} = \frac{Y(k,1)}{k} = Y(1, \frac{1}{k})$ is decreasing if and only if $Y(1, L)$ is an increasing function of L. This property results from the standard assumption (1.2) on the product function $y(\cdot)$.

We assume[3]: $K_{t+1} = sY_t$, with a constant saving rate $s, 0 < s < 1$. This implies:

$$k_{t+1} = \frac{s}{n} y(k_t) \quad \text{and} \quad \frac{k_{t+1}}{k_t} = \frac{s}{n} \hat{y}(k_t). \tag{1.3}$$

Since $y(\cdot)$ is increasing, for any given $k_0 = K_0/L_0 > 0$, the dynamics of k_t are *monotonic* (see in Appendix 1.A.3 a detailed study of monotonic dynamics).

1.2.1 The usual textbook case

The usual textbook case is obtained if $\hat{y}(k)$ takes the value n/s, i.e. if its limits satisfy $\hat{y}(\infty) < n/s < \hat{y}(0)$. Then there exists a unique positive value of the capital-labor ratio k_s^* such that $\hat{y}(k_s^*) = n/s$. It is the unique *steady-state* of the dynamics $k_{t+1} = \frac{s}{n} y(k_t)$, the solution of $k_s^* = \frac{s}{n} y(k_s^*)$, and this steady-state is *globally stable*: for any $k_0 > 0$, the sequence (k_t) solution of (1.3) converges to k_s^*. Figure 1.2 represents the determination of k_s^* and figure 1.3 represents a dynamics of k_t.

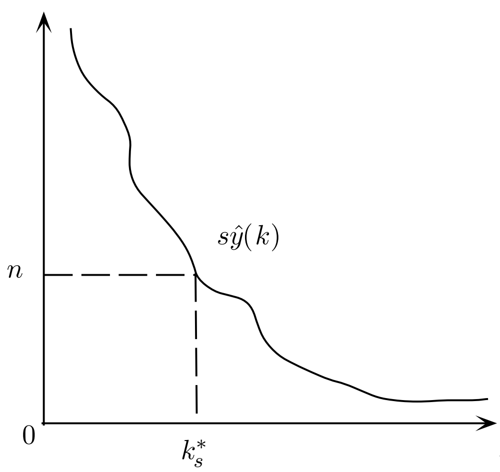

Figure 1.2

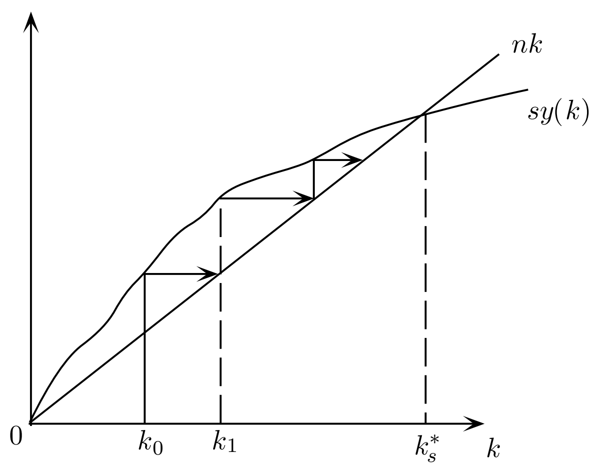

Figure 1.3

[3]With a constant saving-rate s applied to production, we have $K_{t+1} = sF_A(K_t, L_t) + \delta K_t$, and $k_{t+1}/k_t = \frac{1}{n}(s\hat{y}(k_t) + 1 - s)$. The study of this dynamics is quite similar.

We have only assumed that $y(\cdot)$ is increasing and that $\hat{y}(\cdot)$ is decreasing.[4] An important feature of the textbook case, is that the *long run growth rate is equal to the natural rate* and does not depend on the saving rate:

$$\lim_{t\to\infty} \frac{Y_{t+1}}{Y_t} = \frac{L_{t+1}y(k_s^*)}{L_t y(k_s^*)} = n \quad \text{and} \quad \lim_{t\to\infty} \frac{K_{t+1}}{K_t} = n$$

But long run growth properties are different if k_t tends to ∞ (as in example 3 in Solow (1956)).

1.2.2 Unlimited increase of k_t and endogenous growth

Assume now $\hat{y}(\infty) \geq n/s$. This equivalently means that the average productivity of capital is always larger than $n/s : \forall k > 0, \hat{y}(k) > n/s$. Then, for any $k_0 > 0$, the sequence (k_t), solution of (1.3), increases indefinitely and tends to ∞. Indeed, for all $t \geq 0$, we have: $k_{t+1}/k_t = \frac{s}{n}\hat{y}(k_t) > 1$.

When the limit $s\hat{y}(\infty)$ of the growth factor of k_t is larger than n, the long run growth rate $s\hat{y}(\infty) - 1$ is larger that the natural rate $n - 1$ and *it does depend on the saving rate*. We shall say that economic growth is *endogenous* when the long run growth rate depends on the savings behavior. In the recent theory of endogenous growth, it is often assumed that there are externalities in the aggregate production function (see section 1.5 below). But there are also studies of endogenous growth without externalities (Rebelo, 1991).

1.2.3 The decrease of k_t with limit 0

The last possibility is $\hat{y}(0) \leq n/s$, which is equivalent to: $\forall k > 0, \hat{y}(k) < n/s$. Then, for any $k_0 > 0$, the sequence (k_t) solution of (1.3) decreases and tends to 0. Indeed, for all $t \geq 0$, we have: $k_{t+1}/k_t = \frac{s}{n}\hat{y}(k_t) < 1$.

Here again, the long run growth factor $s\hat{y}(0)$ depends on the saving rate, and it is smaller than the natural growth factor when $s\hat{y}(0) < n$. Sometimes, one says that 0 is a poverty trap.

Remark: To obtain the textbook usual case for all saving rates, $0 < s < 1$, it is often assumed that the marginal productivity of capital $y'(k)$ satisfies the Inada conditions: $y'(0) = \infty$ and $y'(\infty) = 0$. In fact, it is sufficient that these conditions are satisfied by the average productivity of capital $\hat{y}(k)$. In particular, to exclude a zero limit of k_t, it is sufficient to assume $\hat{y}(0) = \infty$.

[4] The study of economic growth by Swan (1956) is based on the properties of the product-capital ratio with a Cobb-Douglas production function.

The possibility of production without capital: $y(0) > 0$ implies this condition and excludes a zero limit of k_t.

1.3 The Golden Rule of capital accumulation

Many authors have studied the consumption and the capital accumulation along the growth paths at the natural rate.[5] Total consumption in period t is equal to the difference between the total product Y_t and the next period capital stock K_{t+1}.

Thus the consumption per unit of efficient labor $\hat{c}_t$ is equal to $y(k_t) - nk_{t+1}$. When $k_t = k$ is constant, $\hat{c} = y(k) - nk$ is the *consumption per unit of efficient labor* along the natural rate growth path corresponding to the constant value k of the capital-labor ratio.

Under the standard assumption (1.2), $\hat{c}(k) = y(k) - nk$ is a strictly concave function, and its derivative $\hat{c}'(k) = y'(k) - n$ decreases from $y'(0) - n$ to $y'(\infty) - n$ when k increases from 0 to ∞.

1.3.1 The Golden Rule capital-labor ratio

Under assumption (1.2), there is an interior maximum of $\hat{c}(k)$ if and only if the marginal productivity of capital $y'(k)$ takes the value n, i.e. its limits satisfy: $y'(\infty) < n < y'(0)$.

The solution k_{GR} of $y'(k) = n$ leading to the maximum of consumption per unit of efficient labor is called the *Golden Rule capital-labor ratio*. This ratio k_{GR} is the steady-state of the Solow dynamics (1.3) when the saving rate $s = s_{GR}$ is equal to the ratio of the marginal productivity on the average productivity of capital at $k = k_{GR}$. Indeed, the steady-state condition for $k_s^* = k_{GR}$ is

$$s \; = \; \frac{nk_{GR}}{y(k_{GR})} \; = \; \frac{y'(k_{GR})}{\hat{y}(k_{GR})} \; \equiv \; s_{GR}$$

s_{GR} is smaller than 1, since we have: $y'(k) < \hat{y}(k), \forall k > 0$ (section 1.1.2)

Remark: The condition of existence of the Golden Rule, $y'(\infty) < n < y'(0)$, excludes an unlimited increase of k_t in the Solow dynamics. Indeed, we have: $\hat{y}(\infty) = y'(\infty)$ (section 1.1.2), and $y'(\infty) < n$ implies: $sy'(\infty) < n$ for all savings rate $s, 0 < s < 1$. This condition also implies that the function $\hat{c}(k)$ becomes negative after some level $\tilde{k}$ of the capital-labor ratio (this is

[5] See in particular, Phelps (1961).

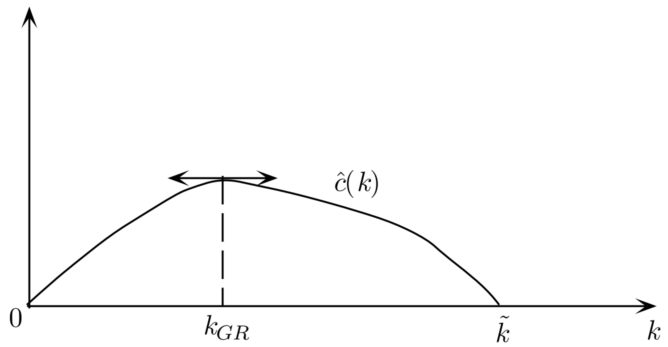

Figure 1.4

because the limit $\hat{y}(\infty)$ of $y(k)/k$ when k tends to ∞ is smaller than n). We represent the curve $\hat{c}(k)$ on figure 1.4.

1.3.2 Over-accumulation of capital

We say that a natural rate growth path, $k_t = k$, is in *over-accumulation of capital* when a decrease in k increases $\hat{c}(k)$, i.e. $\hat{c}'(k) < 0$, or $y'(k) < n$. Given such a growth path, it is possible to increase consumption (per unit of efficient labor) at all dates, since it it sufficient to reduce investment and k at one date and to maintain it constant after this change.

Under the assumption for existence of $k_{GR}, y'(\infty) < n < y'(0)$, k is in over-accumulation if and only if k is larger than k_{GR}. For the steady-state k_s^* is the Solow dynamics (1.3), we have: $k_s^* > k_{GR}$ if and only if $s > s_{GR}$.

As shown by figure 1.5, lowering k from k_s^* increases the consumption $\hat{c}(k)$. The golden rule capital-labor ratio k_{GR} can be reached in one step.

1.3.3 Under-accumulation of capital

A natural rate growth path: $k_t = k$ is in *under-accumulation of capital* when a decrease in k diminishes $\hat{c}(k)$, i.e. $\hat{c}'(k) > 0$ or $y'(k) > n$. It is then not possible to increase consumption at

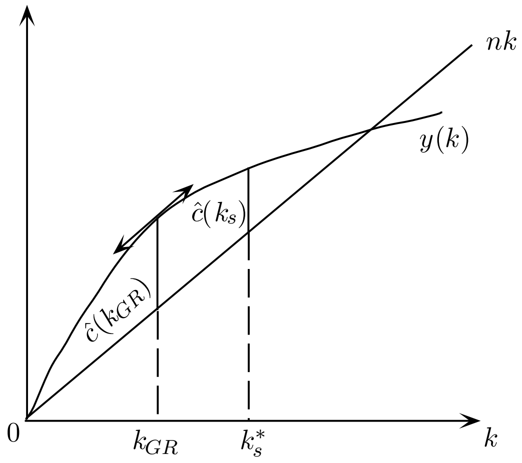

Figure 1.5

all dates: any increase in $\hat{c}(k)$ decreases the following capital stock and the level of stationary consumption per unit of efficient labor.

More precisely, the stationary consumption $\hat{c}(k)$ is *Pareto-optimal* when k is in under-accumulation: it is impossible to increase it at one period without decreasing it at some later period[6] (a detailed proof is in Appendix 1.A.4). Capital accumulation along a growth path $k_t = k$ which is in over-accumulation, it is said *inefficient*. When it is in under-accumulation, it is said *efficient*.

Remark: When the condition of existence of the Golden Rule is not satisfied, then either $y'(\infty) \geq n, \hat{c}(k)$ is increasing for all $k > 0$, and all natural rate growth paths are efficient (in under-accumulation), or $y'(0) \leq n$, $\hat{c}(k)$ is decreasing for all $k > 0$ and all natural rate growth pathes are inefficient (in over-accumulation).

1.4 Remuneration of the production factors

1.4.1 Remuneration of the production factors at their marginal productivities

A differentiable function which is homogeneous of degree 1 satisfies the Euler identity and its partial derivatives are homogeneous of degree 0 (see Appendix 1.A.5).

[6]The sequences of individual consumptions $\hat{c}(k)h_t$ and of total consumptions $\hat{c}(k)L_t$ are also Pareto-optimal, because h_t and L_t are exogenous.

The total product function $Y(K, L)$ is differentiable and homogeneous of degree 1. Thus we have: $Y(K, L) = KY_1'(K, L) + LY_2'(K, L)$, and the partial derivatives satisfy: $Y_i'(K, L) = Y_i'(k, 1)$, where $k = K/L$ is the capital-labor ratio. *The total product is equal to the sum of the two factors' costs* when K and L are remunerated at their marginal productivities. In terms per unit of efficient labor, we have

$$y(k) \;=\; kR(k) + w(k), \quad \text{where } k = K/L \tag{1.4}$$

and

$$R(k) \;=\; Y_1'(k, 1) = y'(k) \quad \text{and} \quad w(k) = Y_2'(k, 1) = y(k) - ky'(k) = k(\hat{y}(k) - y'(k)). \tag{1.5}$$

The standard assumption (1.2) on the product function $y(\cdot)$ implies that $R(k) = y'(k)$ is *decreasing* and $w(k)$ is *increasing* (its derivative is: $w'(k) = -ky''(k) > 0$). The limit $w(0)$ is equal to $y(0)$ because the limit of the difference $y(k) - w(k) = ky'(k)$ is equal to 0 when k tends to 0 (Appendix 1.A.1.). And the limit when k tends to ∞ of $w(k)/k$ is $\hat{y}(\infty) - y'(\infty) = 0$ (see section 1.1.2).

1.4.2 Competitive behavior of firms

Consider a set of firms indexed by $j \in J$ and endowed with the same production function; the latter is differentiable and homogeneous of degree 1: $F_j(K_j, L_j) = F(K_j, L_j)$; we also assume that the rate of capital depreciation $1 - \delta$ is the same.

Firms are price takers: w and r denote respectively the unit wage and the interest rate. Given the cost of labor w and the cost of capital $r + 1 - \delta$ of capital, they maximize profit:

$$F(K_j, L_j) - wL_j - (r + 1 - \delta)K_j \;=\; Y(K_j, L_j) - wL_j - RK_j$$

$Y(K, L) = F(K, L) + \delta K$ is the total product function in the aggregate economy (with the aggregate production function F), and $R = 1 + r$ is the gross rate of returns.

The maximum profit problem of firm j admits finite positive solutions $K_j > 0, L_j > 0$ if and only if $k_j = K_j/L_j$ satisfies the two first order conditions: $R = Y_1'(k_j, 1)$ and $w = Y_2'(k_j, 1)$, and the corresponding profit is equal to 0 (by the Euler identity). This implies the equality of the capital-labor ratios: $k_j = k$, for all firm j.

Thus, if the total efficient labor L and the total capital stock K are given, the equilibrium with competitive firms satisfies: $\forall j, k_j = k = K/L, \quad R = Y_1'(k,1) = R(k)$ and $w = Y_2'(k,1) = w(k)$. In the aggregate economy we have the relations (1.4) and (1.5), no matter the size and the number of the different firms j.

This static equilibrium analysis applies at each period t. The assumption of a given total capital stock $K_t = \sum_j K_{jt}$ can be interpreted as resulting from investment decisions made in the preceding period. When K_{jt} is already installed in firm j (one period time to build capital), then firm j simply maximizes its revenue net of wage cost which implies: $Y_2'(K_{jt}, L_{jt}) = w_t$. This in turn implies the equalities: $k_{jt} = K_{jt}/L_{jt} = K_t/L_t = k_t$ and one assumes that the corresponding net revenue is distributed to the capital owners. From the Euler identity, it results that the gross return on capital is $R_t = Y_1'(k_t, 1) = R(k_t)$, and the relations (1.4) and (1.5) hold in the aggregate economy at period t.

1.4.3 Dynamics with different savings rates on capital and on labor incomes

Following Kaldor (1955-6), many authors have assumed that the savings rate on capital income s_R is larger than the savings rate on wage income s_w. With different savings rates, the dynamics of the capital stock satisfies: $K_{t+1} = s_R R_t K_t + s_w w_t L_t$ and the dynamics of the capital-labor ratio are:

$$nk_{t+1} \;=\; s_R k_t R(k_t) + s_w w(k_t) \;=\; s_w y(k_t) + (s_R - s_w) k_t y'(k_t)$$

We assume $s_R > s_w$. Since $k_{t+1}/k_t = \frac{1}{n}(s_w \hat{y}(k_t) + (s_R - s_w) y'(k_t))$ is a decreasing function of k_t, there exists a unique positive steady state when the limits satisfy: $s_w \hat{y}(0) + (s_R - s_w) y'(0) > n > s_R y'(\infty)$. But dynamics are not necessarily monotonic (they are if $ky'(k)$ is non-decreasing) and the steady state may be unstable. In Appendix (1.A.6), we study the standard example[7] with $s_R = 1$ and $s_w = 0$ in the case of a CES production function (local stability imposes a not to low elasticity of substitution).

When savings are based on wage income ($s_R = 0 < s_w$), the ratio $k_{t+1}/k_t = (\frac{s_w}{n}) w(k_t)/k_t$ is not in general a monotonic function of k_t and there may exist several steady states. This type of dynamics occurs in the OLG model studied in section 2.

[7]With $s_R = 1$ an $s_w = 0$, the steady-state, solution of $y'(k) = n$, is the Golden Rule capital-labor ratio $k = k_{GR}$.

1.5 Endogenous growth with externalities

In order to study the implications of learning by doing, Arrow (1962) introduces externalities: the total sum of past investments has a positive effect on the productivity of labor. In a macroeconomic model, Frankel (1962) has studied a simplified form of learning by doing. This approach has been developped by Romer (1986, 1989) initiating the recent theory of endogenous growth.[8]

1.5.1 Individual firms' behavior and the aggregate economy

The external effect is given by a function $E(K_t, L_t)$ of the aggregate levels of capital K_t and efficient labor L_t. For individual firms j (assumed to be small), this effect operates as an *externality*: $E_t = E(K_t, L_t)$ is considered by each firm j as independent from its decisions. The production function of firm j (including capital after depreciation) is given by:

$$Y_{jt}(K_j, L_j) \;=\; Y(K_j, L_j, E_t)$$

Y is assumed to be differentiable and homogeneous of degree 1 with respect to the two first arguments: K_j and L_j and is the same for all firms. The competitive behavior of firms is defined exactly as in section 1.4.2 with a *fixed* level of E_t. Profit maximization by firm j leads to: $Y_1'(k_{jt}, 1, E_t) = R_t$, $Y_2'(k_{jt}, 1, E_t) = w_t$, and $k_{jt} = K_{jt}/L_{jt} = k_t$ for all j. Profits are all equal to 0.

In the aggregate economy, for given total capital K_t and total labor L_t, we have with $k_t = K_t/L_t$:

$$R_t \;=\; Y_1'(k_t, 1, E_t), \quad w_t = Y_2'(k_t, 1, E_t) \quad \text{and} \quad E_t = E(K_t, L_t).$$

The remunerations of the production factors satisfy: $R_t k_t + w_t \;=\; Y(k_t, 1, E_t)$. In general, $E_t = E(k_t L_t, L_t)$ depends on both k_t and L_t, and if L_t is not constant, the product function $y_t(\cdot)$ is time-dependent:[9] $y_t(k) = Y(k, 1, E(kL_t, L_t))$.

[8] d'Autumne and Michel (1993) study endogenous growth in the Arrow-learning by doing model.

[9] The assumption of a constant $L_t = L$ (i.e. $n = 1$) is often made in studies with endogenous growth. It is also possible to obtain a long run growth path if $y_t(k)$ admits some limit $y_\infty(k)$ when t tends to ∞ (Romer, 1986).

1.5.2 Study with stationary externalities

Externalities are stationary if $E_t = E(k_t)$ is a function of the capital-labor ratio which is the same at each period. In this case, we have in the aggregate economy, with $k_t = K_t/L_t$:

$$y(k_t) \;=\; Y(k_t, 1, E(k_t)), \quad R_t = Y_1'(k_t, 1, E(k_t)) \equiv R(k_t), \quad w_t = Y_2'(k_t, 1, E(k_t)) \equiv w(k_t). \tag{1.6}$$

The factors of production are remunerated at their *private* marginal productivities, i.e. with fixed externality. The *social* marginal productivities take into account the externality, and we have, assuming that $E(k)$ is differentiable:

$$y'(k_t) \;=\; Y_1'(k_t, 1, E(k_t)) + Y_3'(k_t, 1, E(k_t))E'(k_t).$$

Generally, $R(k)$ is different from $y'(k)$. In the economy with stationary externalities, the allocation of capital income and labor income satisfies the accounting relation (1.4): $y(k) = kR(k) + w(k)$, but not the marginal conditions (1.5). As we shall see, it is possible to include in our study the case of stationary externalities: we will only assume that $R(\cdot)$ is decreasing and $w(\cdot)$ is increasing.

1.5.3 A Cobb-Douglas example

Consider the Cobb-Douglas example with externality $E(k)$:

$$Y(K, L, E(k)) \;=\; E(k)K^{\alpha}L^{1-\alpha} + \delta K$$

which leads to the aggregate product function $y(k) = E(k)k^{\alpha} + \delta k$.

The externality $E(k)$ allows a long run growth rate which is finite and larger than the natural rate if the limit of the product-capital ratio $\hat{y}(\infty)$ finite and larger than n. This is obtained with a Cobb-Douglas externality: $E(k) = A_1 + A_2 k^{\lambda}$ only if[10] $\lambda = 1 - \alpha$. In that case, $\hat{y}(k) = A_1 k^{\alpha-1} + A_2 + \delta$, its limit $\hat{y}(\infty)$ is equal to $A_2 + \delta$, and it is larger than n if $A_2 > n - \delta$.

The corresponding remunerations of the production factors (at their private marginal productivities) are

[10] If $\lambda < 1 - \alpha, \hat{y}(\infty) = \delta$ and if $\lambda > 1 - \alpha, \hat{y}(\infty) = \infty$.

$$R(k) = \alpha E(k)k^{\alpha-1} + \delta = \alpha A_1 k^{\alpha-1} + \alpha A_2 + \delta \tag{1.7}$$

$$w(k) = (1-\alpha)E(k)k^{\alpha} = (1-\alpha)A_1 k^{\alpha} + (1-\alpha)A_2 k \tag{1.8}$$

The function $R(\cdot)$ is decreasing and the function $w(\cdot)$ is increasing. Interestingly, when k tends to ∞, the limit of $w(k)/k$ equal to $(1-\alpha)A_2$; it is positive (this limit is equal to 0 in an economy without externality, see §1.4.1).

Another type of externality applies to the human capital h_t when h_t is endogenous (depending on the educational choices of agents), this will be studied in chapter X.

1.6 Optimal growth and the model with an infinite-lived representative consumer

We shall often consider the following utility function of consumption with $\sigma > 0$:

$$u(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma} \text{ if } \sigma > 0, \sigma \neq 1, \text{ and if } \sigma = 1 : u(c) = \ln c \tag{1.9}$$

The marginal utility: $u'(c) = c^{-1/\sigma}$ is defined for all $\sigma > 0$. We shall call it the *CE utility* function.[11]

1.6.1 The Ramsey optimal growth model

In his celebrated article of 1928, Ramsey studies optimal growth in continuous time with an utility function of aggregate consumption C_t and a disutility of aggregate labor L_t. We present a simplified version with $L_t = L$ constant.

The aggregate production[12] $F(K_t, L)$ is equal to the sum of C_t and dK_t/dt, the increase in the capital stock K_t. Ramsey excludes social discounting which he call "a practice ethically indefensible" and he defines the maximum of stationary welfare, the "bliss" $\hat{U}$ equal to the upper-bound with respect to K of the utility $U(F(K, L))$. This upper-bound is assumed to be finite.

[11]We shall see that, with this CE utility function, the elasticity of the optimal consumption ratio c_{t+1}/c_t with respect to the marginal productivity of capital $y'(k_{t+1})$ is constant and equal to σ.

[12]This aggregate productio function is net of depreciation.

For example, if $F(K, L)$ tends to ∞ when K tends to ∞, we have $\tilde{U} = U(\infty)$ and with a CE utility function (1.7), $\tilde{u} = u(\infty)$ is finite if and only if $0 < \sigma < 1$, and then $\tilde{u}(\infty) = 0$.

To solve the optimal growth problem: maximize $\int_0^\infty (U(C_t) - \tilde{U})dt$, subject to $dK_t/dt = F(K_t, L) - C_t$, Ramsey considers a change of variable from time to capital. Then the problem becomes simply to maximize at each date $(U(C_t) - \tilde{U})/(F(K_t, L) - C_t)$ with respect to C_t. This leads to the celebrated Keynes-Ramsey rule:

$$U'(C_t)(F(K_t, L) - C_t) \; = \; \tilde{U} - U(C_t).$$

At any time, the marginal utility of savings should be equal to the increase in utility which is needed to reach the bliss point.

In the CE example with $0 < \sigma < 1$, the Keynes-Ramsey rule is: $C_t = (1 - \sigma)F(K_t, L)$ and corresponds to a constant savings rate. This savings rate crucially depends on the concavity of the *social* utility of *aggregate* consumption.

1.6.2 Optimal growth with social discounting

It is now a current practice to consider a social planner who maximizes the discounted infinite sum of the utilities of individual consumptions c_t. With a discount factor γ, this sum is:

$$\sum_{t=0}^{\infty} \gamma^t u(c_t) \; = \; \sum_{t=0}^{\infty} \gamma^t u(h_t \hat{c}_t)$$

where $\hat{c}_t$ is the consumption per unit of efficient labor and satisfies: $\hat{c}_t = y(k_t) - nk_{t+1}$; $k_t = K_t/L_t$ is the capital-labor ratio and $y(\cdot)$ is the product function which is assumed to satisfy (1.2).

After substitution of $\hat{c}_t$, k_{t+1} appears in only two terms of the infinite sum:

$$\gamma^t u(h_t(y(k_t) - n(k_{t+1})) + \gamma^{t+1} u(h_{t+1}(y(k_{t+1}) - nk_{t+2}))$$

At optimal growth (with $k_{t+1} > 0$), the maximum with respect to k_{t+1} leads to

$$-h_t n u'(c_t) + \gamma h_{t+1} y'(k_{t+1}) u'(c_{t+1}) = 0.$$

The marginal rate of substitution of individual consumptions $u'(c_t)/(\gamma u'(c_{t+1}))$ is equal to the rate of economic transformation[13] viz. $y'(k_{t+1})/\nu$. The assumption of a CE utility function

[13] With $Y_t = N_t c_t + K_{t+1}$ and $Y(K_{t+1}, L_{t+1}) = N_{t+1}c_{t+1} + K_{t+2}$, we have: $-\frac{\partial c_t}{\partial K_{t+1}} = \frac{1}{N_t}$, $\frac{\partial c_{t+1}}{\partial K_{t+1}} = \frac{1}{N_{t+1}} Y_1'(k_{t+1}, 1)$, and their ratio is equal to $\frac{1}{\nu} y'(k_{t+1})$.

implies existence of an optimal growth path with a constant growth rate.

1.6.3 Optimal growth with a CE utility function

With the CE utility function (1.7), we have $u'(c) = c^{-1/\sigma}$ and the first order optimality conditions are: $c_{t+1}/c_t = (\gamma y'(k_{t+1})/\nu)^\sigma$. Incorporating the growth factor ν of h_t in the discounting, for $\sigma \neq 1$:

$$\gamma^t u(h_t \hat{c}_t) \;=\; \gamma^t h_t^{1-1/\sigma} u(\hat{c}_t) \;=\; \hat{\gamma}^t u(\hat{c}_t) h_0^{1-1/\sigma}, \text{ with } \hat{\gamma} = \gamma \nu^{1-1/\sigma}$$

we obtain the standard optimization problem (in units per efficient labor):

$$\text{maximize} \sum_{t=0}^{\infty} \hat{\gamma}^t u(\hat{c}_t) \quad \text{subject to} \quad k_{t+1} \equiv \frac{1}{n}(y(k_t) - \hat{c}_t). \tag{1.10}$$

The same problem is obtained if $\sigma = 1$, with $\hat{\gamma} = \gamma$: there is then an additive term in the objective which is independent from $\hat{c}_t$ and which converges if $\gamma < 1$.

There are three possibilities (for details see Appendix 1.A.7).

- *The convergence to the modified Golden Rule* $k^*(\hat{\gamma})$ solution of $y'(k^*(\hat{\gamma}) = \frac{n}{\hat{\gamma}}$. If and only if $y'(0) < n/\hat{\gamma} < y'(\infty)$ this solution exists. If in addition[14] $\hat{\gamma} < 1$, then for any initial initial capital-labor ratio $k_0 > 0$, the optimal growth path exists, is unique and the sequence k_t converges monotonically to $k^*(\hat{\gamma})$.

- *The unlimited increase of* k_t *and of* $\hat{c}_t$. If $y'(\infty) \geq n/\hat{\gamma}$ and $\hat{\gamma}$ satisfies[15]: $\hat{\gamma} < (y'(\infty)/n)^{1/\sigma - 1}$, the optimal growth path exists, is unique, and k_t and $\hat{c}_t$ increase and tend to ∞. The long run growth rate is larger than the natural rate when $y'(\infty) > n/\hat{\gamma}$.

- *The last possibility* occurs if $y'(0) \leq n/\hat{\gamma}$ (finite $y'(0)$ and low $y'(0)$ or low $\hat{\gamma}$). A similar analysis applies and the optimal capital-labor ratio k_t tends to 0 (possibly in finite time).

It is worth nothing the similarities of those three possibilities with those of the Solow growth model.

[14] The optimality conditions include in addition to the first order marginal conditions, a limit condition when t tends to ∞, the transversality condition (Michel 1990). This condition imposes a restriction on the discount factor $\hat{\gamma}$.

[15] See footnote 14

1.6.4 The model with an infinite-lived representative consumer

Such a consumer represents a *dynasty of altruistic agents* who transmits wealth to their descendants. They maximize the same objective function as the social planner. But agents are price-takers. In period t, the income per capita is $R_t x_t + w_t h_t$, where x_t is the (inherited) current wealth. The dynamics of x_t are

$$\nu x_{t+1} \; = \; R_t x_t + w_t h_t - c_t.$$

Any agent of the dynasty should forecast accurately all future prices. In the representative consumer's problem, the marginal optimality conditions are: $u'(c_t)/(\gamma u'(c_{t+1})) \; = \; R_{t+1}/\nu$.

The dynamic equation of wealth is equivalent to the dynamics of aggregate capital if all prices are perfectly expected. Indeed, with $x_t = k_t h_t$, the dynamics of wealth imply

$$\nu k_{t+1} h_{t+1} \; = \; h_t(R_t k_t + w_t) - c_t \; = \; h_t y(k_t) - c_t.$$

In the economy *without externality*, the gross return on savings R_{t+1} is equal $y'(k_{t+1})$, and the marginal optimality conditions are also equivalent. Then the optimal solution of representative consumer's problem coincides with the social planer's solution.

In the economy *with externalities*, this is not the case since the *private* marginal productivity of capital R_{t+1} is then different from the *social* marginal productivity $y'(k_{t+1})$. The social planner's problem takes into account all external aggregate effects. In order to decentralize the social optimum, it is necessary to subsidize the return on savings for equalizing it to the social return.

1.7 Production with fixed costs

If return to scale are not constant, then the size of firms matters.

Consider first the example of a convex-concave production function G_j of one factor L (figure 1.6). Assume that there exists a level of activity L^* at which the marginal productivity $G_j'(L^*)$ is equal to the average productivity $G_j(L^*)/L^*$. Then at this level, the profit of the firm is zero when the production factor is remunerated at its marginal productivity. We shall study this problem with two production factors and the assumption of a fixed cost. In the one factor production model, production with a fixed cost corresponds to a simplified form $\tilde{G}_j$ of a convex-concave production technology (figure 1.7).

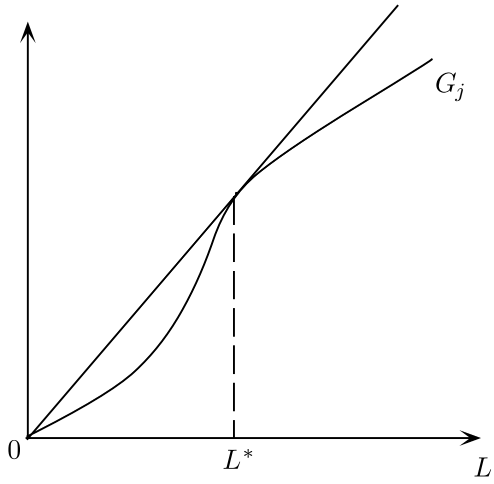

Figure 1.6

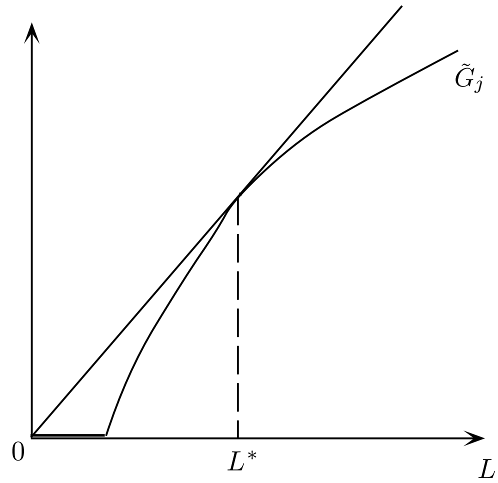

Figure 1.7

1.7.1 Perfect competition with free entry

We consider a function $F(K, L)$ which is concave, differentiable and homogeneous of degree $\lambda, 0 < \lambda < 1$, and a fixed cost $\xi > 0$ (or sunk cost, see e.g. Mas-Colell and al, 1995).

For any firm j, the production function, with capital K_j and labor L_j is:

$$F_j(K_j, L_j) \;=\; \begin{cases} F(K_j, L_j) - \xi \;\text{ if positive} \\ 0 \text{ if } F(K_j, L_j) \leq \xi. \end{cases}$$

Given the costs of labor w and of capital $r + 1 - \delta$ (sum of interest rate r and depreciation rate $1 - \delta$), the maximization of profit $F_j(K_j, L_j) - wL_j - (r + 1 - \delta)K_j$ implies when production is positive: $F_2'(K_j, L_j) = w$ and $F_1'(K_j, L_j) = r + 1 - \delta$. Using the Euler identity (Appendix 1.A.5), we obtain that the maximum of profit is equal to:

$$F(K_j, L_j) - \xi - L_j F_2' - K_j F_1' \;\equiv\; (1 - \lambda) F(K_j, L_j) - \xi.$$

Perfect competition with *free entry* is defined by the condition of zero maximum profit of all firms. There is no limit to the number of firms, and additional firms enter the market as long as a positive profit is possible.

For a given total capital stock K and a total labor supply L, free entry determines the size and the number χ of producing firms by the zero maximum profit condition:

$$K_j \;=\; K/\chi, \quad L_j = L/\chi, \quad \text{and} \quad \xi = (1-\lambda)F(K/\chi, L/\chi) \equiv (1-\lambda)\chi^{\lambda}F(K,L).$$

The competitive prices are: $w = F_2'(K/\chi, L/\chi)$ and $R = 1 + r = F_1'(K/\chi, L/\chi) + \delta$.

In this definition, $\chi = bF(K,L)^{1/\lambda}$, with $b = (1-\lambda)^{1/\lambda}\chi^{-1/\lambda}$, is a function of K and L which is homogeneous of degree 1. In general, $\chi(K,L)$ is not an integer, but it is used as a good approximation when the number of firms is large.

1.7.2 The aggregate economy

The total product with capital K and efficient labor L, including capital after depreciation δK is: $Y = \chi(F(K/\chi, L/\chi) - \chi) + \delta K$. This is a function of K and L which is homogeneous of degree 1:

$$Y(K,L) \;=\; \lambda b^{1-\lambda}F(K,L)^{1/\lambda} + \delta K \;\equiv\; Ly(k)$$

$k = K/L$ is the capital-labor ratio.

This total product has the following *efficiency property*: the maximum with respect to χ of possible total products $\chi^{1-\lambda}(F(K,L) - \xi) + \delta K$ is obtained with the free entry condition $(1-\lambda)\chi^{-\lambda}F(K,L) = \xi$. Moreover, the gross return on savings R is equal to the marginal productivity of capital in the aggregate economy $Y_1'(K,L) = y'(k)$. Indeed, since $F_1'(K,L)$ is homogeneous of degree $\lambda - 1$ (Appendix 1.A.5), we have:

$$R \;=\; \chi^{1-\lambda}F_1'(K,L) + \delta = b^{1-\lambda}F(K,L)^{1/\lambda - 1}F_1'(K,L) + \delta = Y_1'(K,L).$$

As a consequence, all the preceding studies made for the standard approach with an aggregate production function homogeneous of degree 1 apply to this economy with fixed cost. The only difference is that the size and the number of producing firms is determined (instead of section 1.4.2).

1.7.3 A Cobb-Douglas example

Consider the function $F(K,L) = AK^{\alpha_1}L^{\alpha_2}$, with $\alpha_1 > 0$, $\alpha_2 > 0$ and $\lambda = \alpha_1 + \alpha_2 < 1$. The free entry condition with χ identical firms leads to:

$$\xi \;=\; bF(K,L)^{1/\lambda} \;=\; bA^{1/\lambda}K^{\alpha}L^{1-\alpha},$$

where $b = (1-\lambda)^{1/\lambda}\xi^{-1/\lambda}$, $\alpha = \alpha_1/\lambda$ and $1-\alpha = \alpha_2/\lambda$. In the aggregate economy, total product is:

$$Y(K,L) \;=\; \lambda b^{1-\lambda}A^{1/\lambda}K^{\alpha}L^{1-\alpha} + \delta K \;\equiv\; Ly(k).$$

The gross return on savings is:

$$R(k) \;=\; y'(k) \;=\; \alpha\lambda b^{1-\lambda}A^{1/\lambda}k^{\alpha-1} + \delta.$$

1.8 Some other allocations of the national product

1.8.1 A collective negociation allocation

McDonald and Solow (1981) consider a collective bargain on both employment and wage. The total capital stock K and the total efficient labor L are given. We assume a reservation wage $\underline{w}$ (in efficient units).

The objective of workers is to maximize $W = (w-\underline{w})L^e$, where L^e is the employed efficient labor. The objective of firms is to maximize $\Pi = Y(K,L^e) - wL^e$. With a power index θ of workers ($0<\theta<1$), the Nash bargaining solution is defined by maximizing $W^{\theta}\Pi^{1-\theta}$ with respect to w and L^e, subject to $L^e \le L$.

The maximum with respect to w yields: $w = \underline{w} - \theta(y^e - \underline{w})$ where $y^e = y(K/L^e)$. With this value of w, the derivative of $W^{\theta}\Pi^{1-\theta}$ with respect to L^e has the same sign as $Y_2'(K,L^e) - \underline{w}$.

Thus, if the reservation wage $\underline{w}$ is smaller or equal to the full employment marginal productivity of labor $Y_2'(K,L)$, then solution of the bargaining problem leads to full employment: $L^e = L$. The corresponding allocation of the national gross product is:

$$w = \underline{w} + \theta(y(k)-\underline{w}) \equiv w(k) \quad \text{and} \quad R(k) = \frac{1}{k}(y(k)-w) = (1-\theta)(\frac{y(k)-\underline{w}}{k}) \tag{1.11}$$

where $k = K/L$ is the full-employment capital-labor ratio. This allocation satisfies (1.4), $w(\cdot)$ is increasing, and $R(\cdot)$ is decreasing on the set of non-negative k such that $Y_2'(k,1) = y(k) - ky'(k) \ge \underline{w}$.

In the case of unemployment, it is possible to obtain the allocation accounting relation (1.4) by assuming that unemployment benefits are equal to the current wage.[16]

1.8.2 Non-competitive behavior on the capital market

Considering the economy with fixed cost (section 1.7), we assume that firms are competitive on the labor market, but that capital owners are not and choose the size of the firms. The revenue of the firm j net of wage w, with $w = F_2'(K_j, L_j)$, is $F(K_j, L_j) - \xi - L_j F_2'(K_j, L_j)$.

Given the total capital stock K and the total efficient labor L, the total net income of χ identical firms is

$$\Pi \;=\; \chi(F(K/\chi, L/\chi) - \xi) - LF_2'(K/\chi, L/\chi).$$

With the homogeneity properties (F of degree λ and F_2' of degree $\lambda - 1$), we obtain:

$$\Pi \;=\; \chi^{1-\lambda}(F(K, L) - LF_2'(K, L)) - \xi.$$

The number $\tilde{\chi}$ (and the size $K_j = K/\tilde{\chi}, L_j = L/\tilde{\chi}$) which maximizes this net revenue satisfies: $(1-\lambda)\tilde{\chi}^{-\lambda}(F(K, L) - LF_2'(K, L)) = \xi$. This number $\tilde{\chi}$ is smaller than the number χ of firms in the competitive economy with free entry, since we have:

$$\frac{\tilde{\chi}}{\chi} \;=\; \left(1 - \frac{LF_2'(K, L)}{F(K, L)}\right)^{1/\lambda} < 1.$$

Also the corresponding wage is smaller, since it satisfies:

$$\tilde{w} \;=\; F_2'(K\tilde{\chi}, L/\tilde{\chi}) \;=\; \tilde{\chi}^{1-\lambda} F_2'(K, L).$$

In this economy, the total product $\tilde{Y}(K, L) = \tilde{\chi}(F(K; L) - \chi) + \delta K$ is homogeneous of degree one, but it is lower than the competitive product $Y(K, L)$ (which is the maximum feasible product). The corresponding allocation of the national product satisfies (1.4), $\tilde{w}(k) < w(k)$ and $\tilde{R}(k) > R(k)$ (by definition of $\tilde{\chi}$).

In the Cobb-Douglas example (section 1.7.3), we have:

[16]This may be done in an economy with complementary production factors: $F_A(K, L) = \min\{A_1 K, A_2 L\}$, and inelastic labor supply L.

$$
\begin{aligned}
\tilde{\chi} &= b(1-\alpha_2)^{1/\lambda}A^{1/\lambda}K^{\alpha}L^{1-\alpha} = (1-\alpha_2)^{1/\lambda}\chi \\
\tilde{w}(k) &= \tilde{\chi}^{1-\lambda}F_2'(K,L) \;=\; (1-\alpha_2)^{1/\lambda-1}w(k) \\
\tilde{R}(k) &= \frac{1}{k}(\tilde{y}(k)-\tilde{w}(k)) \;=\; (1-\alpha_2)^{1/\lambda}\lambda b^{1-\lambda}A^{1/\lambda}k^{\alpha-1}+\delta.
\end{aligned}
$$

APPENDIX of Section 1

1.A.1 Study of the product-capital ratio $\hat{y}(k) = y(k)/k$

The first order derivative of $\hat{y}(k)$ is: $\hat{y}'(k) = \frac{1}{k^2}(y'(k)k - y(k))$. We apply the mean value theorem for derivatives to $y(\cdot)$ on the interval $[0, k]$: there exists some fraction $\theta, 0 < \theta < 1$, which depends on k and such that $y(k) - y(0) = ky'(\theta k)$. On figure 1.A.1, the tangent to the curve $y(\cdot)$ at θk is parallel to AB.

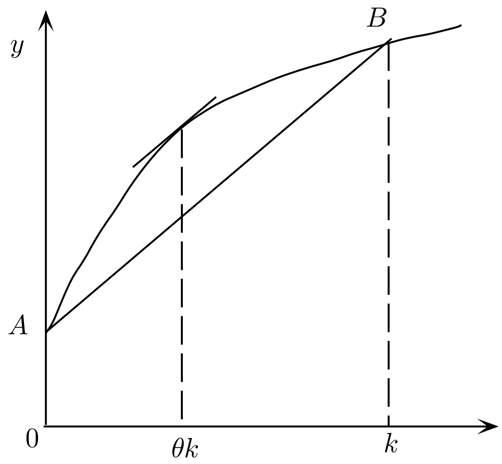

Figure 1.A.1

We also have: $y'(\theta k) > y'(k)$ since $y'(\cdot)$ is decreasing. This implies: $y(k) - ky'(k) > y(0) \geq 0$, the derivative $\hat{y}'(k)$ is negative, $\hat{y}(\cdot)$ is decreasing and satisfies: $\hat{y}(k) - y'(k) > y(0)/k$, $\hat{y}(k) > y'(k)$ for all $k > 0$. This strict inequality implies: $\hat{y}(0) \geq y'(0)$, these limits being finite of infinite. The inequality $y(k) - y(0) > ky'(k) > 0$ imply: $\lim_{k \to 0} ky'(k) = 0$. If $y(0) > 0$, the limit $\hat{y}(0)$ of $y(k)/k$ when k tends to 0 is infinite. If $y(0) = 0$, we have: $y(k)/k = y'(\theta k)$ and the two limit $\hat{y}(0)$ and $y'(0)$ are equal, both being finite or infinite.

To study the limits when k tends to ∞, we apply the mean value theorem for derivatives to $y(\cdot)$ on the interval $[k, 2k]$. We have: $y(2k) - y(k) = ky'(k + \theta_1 k)$ with some fraction $\theta_1, 0 < \theta_1 < 1$. It implies

$$2\hat{y}(2k) - \hat{y}(k) \;=\; \frac{1}{k}(y(2k) - y(k)) \;=\; y'(k + \theta_1 k).$$

Taking the limits when k tends to ∞, we obtain: $2\hat{y}(\infty) - \hat{y}(\infty) = y'(\infty)$ and, since these limits are finite: $\hat{y}(\infty) = y'(\infty)$.

1.A.2 Study of the production function, with constant elasticity of substitution

We consider the CES production function

$$A(\alpha K^\lambda + (1-\alpha)L^\lambda)^{1/\lambda} \;=\; AL(\alpha k^\lambda + 1 - \alpha)^{1/\lambda}$$

with $A > 0$, $0 < \alpha < 1$, $\lambda < 1$ and $\lambda \neq 0$. The elasticity of substitution is $\sigma_F = 1/(1-\lambda)$. The corresponding product function (with a depreciation rate $1-\delta$ of the capital stock) is: $y(k) = A(\alpha k^\lambda + 1 - \alpha)^{1/\lambda} + \delta k$ and we have:

$$\begin{aligned}
y'(k) &= \alpha k^{\lambda-1} A(\alpha k^\lambda + 1 - \alpha)^{1/\lambda - 1} + \delta = \alpha A(\alpha + (1-\alpha)k^{-\lambda})^{1/\lambda-1} + \delta > 0 \\
y''(k) &= -(1-\alpha)(1-\lambda)k^{-\lambda-1}\alpha A(\alpha + (1-\alpha)k^{-\lambda})^{1/\lambda-2} < 0 \\
\hat{y}(k) &= y(k)/k = A(\alpha + (1-\alpha)k^{-\lambda})^{1/\lambda} + \delta
\end{aligned}$$

To study the limits when k tends to 0 and k tends to ∞, we distinguish the two cases: $\lambda > 0$ (high substitution: $\sigma_F > 1$) and $\lambda < 0$ (low subsitution: $\sigma_F < 1$).

If $0 < \lambda < 1$, the limit of k^λ when k tends to 0 (respectively to ∞) is equal to 0 (respectively to ∞) and we have:

$$y(0) \;=\; A(1-\alpha)^{1/\lambda}, \quad y(\infty) = \infty, \quad y'(0) = \infty = \hat{y}(0), \quad y'(\infty) = A\alpha^{1/\lambda} + \delta = \hat{y}(\infty).$$

If $\lambda < 0$, we have:

$$y(0) = 0, \quad y(\infty) = A(1-\alpha)^{1/\lambda}, \quad y'(0) = A\alpha^{1/\lambda} + \delta = \hat{y}(0), \quad y'(\infty) = \delta = \tilde{h}aty(\infty).$$

In the limit case $\lambda \to 0$ ($\sigma_F = 1$), we have the Cobb-Douglas production function (example 1.1.3).

1.A.3 Monotonic dynamics

We consider a function $\ell(\cdot)$ defined continuous and non-decreasing on some interval I of the set $I\!R$ of real numbers with values in $I : \forall x \in I, \ell(x) \in I$. Given $x_0 \in I$, the time path $(x_t)_{t \geq 0}$ which is solution of: $\forall t \geq 0, x_{t+1} = \ell(x_t)$ is uniquely defined and belongs to I. A *steady state* $x \in I$ is a solution of $\ell(x) = x$.

Property. *Given any $x_0 \in I$, the sequence $(x_t)_{t\geq 0}$ is monotonic. This sequence either converges to a steady stade $\tilde{x} \in I$, or it goes to a boundary of I. It never goes from one side of a steady state to another.*

The proof is simple. If for example $x_t \geq x_{t-1}$, then we have $\ell(x_t) \geq \ell(x_{t-1})$ and $x_{t+1} \geq x_t$. By induction, the inequality between the two first terms x_0 and $x_1 = \ell(x_0)$ hold for all $t \geq 0$. Any monotonic sequence admits a limit which is finite or infinite. Either the limit belongs to I and it is a steady state (by continuity of ℓ) or it is a boundary point of I. If $\tilde{x}$ is a steady state, the inequality: $x_0 \leq \tilde{x}$ implies $x_1 = \ell(x_0) \leq \tilde{x} = \ell(\tilde{x})$ and by induction: $\forall t \geq 0, x_t \leq \tilde{x}$.

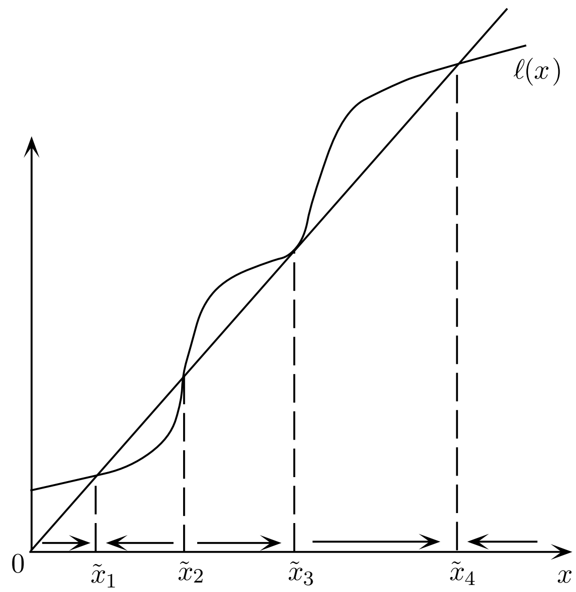

Figure 1.A.2

This property allows to study graphically the dynamics. The intersections of the curve $\ell(\cdot)$ with the 45°line are the steady states of the dynamics: $x_{t+1} = \ell(x_t)$. On the intervals where $\ell(x) > x$, the sequence is increasing (on figure 1.A.2, the intervals $(0, \tilde{x}_1), (\tilde{x}_2, \tilde{x}_3), (\tilde{x}_3, \tilde{x}_4)$).

On the intervals where $\ell(x) < x$, the sequence is decreasing (on figure 1.A.2, the intervals $(\tilde{x}_1, \tilde{x}_2)$ and $(\tilde{x}_4, \infty)$). In general, the steady states are alternatively stable (with derivative $\ell'(\tilde{x}) < 1$) and unstable (with derivative $\ell'(\tilde{x}) > 1$). But it may exist a tangent point: on figure 1.A.2, $\tilde{x}_3$ is stable from the left and unstable from the right. The local stability sufficient condition: $|\ell'(\tilde{x}_i)| < 1$ is equivalent to $\ell'(\tilde{x}_i) < 1$ for a non-decreasing function ℓ (such that $\ell'(x) \geq 0, \forall x$).

1.A.4 Efficiency of a natural rate growth path in under-accumulation of capital

We consider k in under-accumulation and we show that if we increase consumption at some date, say $\hat{c}_0 > \hat{c}(k)$, then it is impossible to maintain all the following consumptions $\hat{c}_t \geq \hat{c}(k)$. By assumption, k_1 is smaller than k:

$$nk_1 = y(k) - \hat{c}_0 < y(k) - \hat{c}(k) = nk, \text{ and } k_1 < k.$$

At point t, if $\hat{c}_t \geq \hat{c}(k)$, we have: $nk_{t+1} = y(k_t) - \hat{c}_t \leq y(k_t) - \hat{c}(k)$; and if $k_t < k$, then $\hat{c}(k_t) < \hat{c}(k)$ because k is in under-accumulation. By induction: $nk_{t+1} \leq y(k_t) - \hat{c}(k) < y(k_t) - \hat{c}(k_t) = nk_t$. If it exists, the decreasing sequence (k_t) admits a limit $\underline{k}, 0 \leq \underline{k} < k$, which verifies

$$n\underline{k} \leq y(\underline{k}) - \hat{c}(k) < y(\underline{k}) - \hat{c}(\underline{k}) = n\underline{k}.$$

Since $\underline{k} < \underline{k}$ is impossible, such a sequence does not exist.

1.A.5 Homogeneous functions

Consider a real number λ and a function $F(K, L)$ defined and differentiable on the set of positives values of K and L. The function F is said homogeneous of degree λ if it satisfies:

$$F(xK, xL) = x^{\lambda} F(K, L) \quad \text{for all } K > 0, L > 0 \text{ and } x > 0.$$

Differentiating with respect to x leads to:

$$KF_1'(xK, xL) + LF_2'(xK, xL) = \lambda x^{\lambda - 1} F(K, L).$$

With $x = 1$, this yields the Euler identity: $KF_1'(K,L) + LF_2'(K,L) = \lambda F(K,L)$. Differentiating the definition identity with respect to K leads to:

$$xF_1'(xK, xL) \;=\; x^\lambda F_1'(K,L) \quad \text{and} \quad F_1'(xK,xL) = x^{\lambda-1}F_1'(K,L).$$

The partial derivative F_1' is homogeneous of degree $\lambda - 1$. Similarly F_2' is homogeneous of degree $\lambda - 1$.

1.A.6 Dynamics with savings equal to capital return

The dynamics of the capital-labor ratio are: $nk_{t+1} = k_t y'(k_t)$, and with a CES production function (using the notations of 1.A.2):

$$k_{t+1} \;=\; \frac{\alpha}{n} A k_t^\lambda (\alpha k_t^\lambda + 1 - \alpha)^{1/\lambda - 1} + \delta k_t.$$

In the high substition case ($0 < \lambda < 1$), dynamics are monotonic and, if the Golden Rule exists ($y'(0) > n > y'(\infty)$), k_t converges to k_{GR} (by the property of monotonic dynamics, 1.A.3).

In the low substitution case ($\lambda < 0$), dynamics of k_t are more complicated. In the special case where $\delta = 0$ and $A = \alpha/n$, the Golden Rule capital-labor ratio is $k_{GR} = 1$, and the local stability condition of $k_{GR} : dk_{t+1}/dk_t = \lambda(1-\alpha)+\alpha > -1$ is equivalent to $\lambda > -(1+\alpha)/(1-\alpha)$. Convergence to the steady state k_{GR} is then excluded when substitution is very low: the local stability condition implies $\sigma_F > \frac{1-\alpha}{2}$.

1.A.7 Optimal growth with discounting and a CE utility function

A more detailed study is in de la Croix and Michel (2002) chapter 2. For the standard optimization problem (1.8), we define the Lagrangian of period t which equal to the sum of current utility $u(\hat{c}_t)$ and the increase in Shadow value (with shadow price p_t) of the capital-labor ratio:

$$L_t \;=\; u(\hat{c}_t) + \hat{\gamma} p_{t+1} k_{t+1} - p_t k_t \;=\; u(\hat{c}_t) + \hat{\gamma} p_{t+1} \frac{1}{n}(y(k_t) - \hat{c}_t) - p_t k_t.$$

Maximizing L_t with respect to $\hat{c}_t$ and k_t leads to $u'(\hat{c}_t) = \hat{\gamma} p_{t+1} \frac{1}{n}$ and $p_t = \hat{\gamma} p_{t+1} \frac{1}{n} y'(k_t)$. And the transversality condition (Michel, 1990) is $\lim_{t\to\infty} \hat{\gamma}^t p_t k_t = 0$. With a CE utility function, we have: $\hat{c}_t = (n/\hat{\gamma} p_{t+1})^\sigma$. We distinguish three possibilities:

a) A steady-state $\hat{c}_t = \hat{c}, k_t = k, p_t = p$ satisfy these conditions if and only if

1. $y'(k) = n/\hat{\gamma}$ admits a solution $k = k^*(\hat{\gamma})$ (i.e. $y'(0) > \frac{n}{\gamma} > y'(\infty)$) and

2. $\hat{\gamma} < 1$ (for the transversality condition $\lim \hat{\gamma}^t pk = 0$ to be satisfied).

In that case, for any $k_0 > 0$, the optimal growth path exists, the sequence k_t is monotonic and converges to $k^*(\hat{\gamma})$.

In the simple example with $\sigma = 1$ and $y(k) = Ak^\alpha$, the conditions: $\hat{c}_t = \frac{n}{\hat{\gamma} p_{t+1}}$, $nk_{t+1} = Ak_t^\alpha - \hat{c}_t$ and $p_t = \hat{\gamma} p_{t+1} \frac{1}{n} \alpha A k_t^{\alpha-1}$ are satisfied by $\hat{c}_t = (1 - \alpha\hat{\gamma})Ak_t^\alpha$, $nk_{t+1} = \alpha\hat{\gamma}Ak_t^\alpha$ and $p_t = \alpha/(1 - \alpha\hat{\gamma})k_t$. The optimal dynamics of k_t are the Solow dynamics (1.3) with $s = \alpha\hat{\gamma}$.

b) If $y'(\infty) \geq n/\hat{\gamma}$, then for any $k_t > 0$, $\hat{\gamma} y(k_t) > n$, and along an optimal path: $p_t > p_{t+1}$. The decreasing sequence p_t necessarily tends to 0 (there exists no positive steady state), $\hat{c}_t$ increases and tends to ∞. Then k_t also increases and tends to ∞. The long run growth factor of p_t is $n/(\hat{\gamma} y'(\infty))$, which implies that the long run growth factor of $\hat{c}_t$ and of k_t is $(\frac{\hat{\gamma} y'(\infty)}{n})^\sigma \equiv G_k$. The transversality condition is satisfied if and only if in the long run the growth factor of $\hat{\gamma}^t p_t k_t$ is smaller than 1, i.e. $\hat{\gamma}\frac{n}{\hat{\gamma} y'(\infty)} G_k < 1$.

Thus the transversality condition is equivalent to $G_k < \frac{y'(\infty)}{n}$ and to $\hat{\gamma} < (y'(\infty)/n)^{1/\sigma - 1}$. Note that this condition is also equivalent to $\hat{\gamma} G_k^{1-1/\sigma} < 1$ which means that the objective function $\sum_{t=0}^{\infty} \hat{\gamma} u(\hat{c}_t)$ converges with the CE utility function (1.7) when the growth factor of $\hat{c}_t$ is equal to G_k.

c) If $y'(0) \leq n/\hat{\gamma}$, a similar study applies to an interior optimal solution (with $k_t > 0, \forall t) : p_t$ is increasing and tends to ∞, $\hat{c}_t$ and k_t decrease and tend to 0 when the transversality condition is satisfied: $\hat{\gamma} < (y'(0)/n)^{1/\sigma - 1}$. if $y(0) > 0$, then k_t becomes equal to 0 in finite time and then optimal consumption is equal to $y(0)$.

2 Households' behavior and intertemporal equilibrium

We study economic growth in an economy with finite-lived agents. There is an infinite number of periods $t = 0, 1, 2, \cdots$

Households live two periods, or more precisely they make decisions in two periods.[17] We call them *young agents* in the first period and *old agents* in the second period of decisions of their life.

The number of agents young in period t, N_t, grows at a constant rate $\nu - 1$: $N_t = \nu N_{t-1}$, $\nu > 0$. Each of them is endowed by h_t units of efficient labor, and h_t grows at a constant rate $\eta - 1 : h_t = \eta h_{t-1}, \eta > 0$. Their labor supply is inelastic, and the total efficient labor in period t, $N_t h_t$, grows at the natural rate $n - 1, n = \nu\eta$.

Old agents do not work and are retired. This overlapping generations approach in which there are at each period two types of decision-makers, young and old agents, leads to the specific analysis of the first period $t = 0$ (see below section 2.1.2).

2.1 The budget constraints

2.1.1 The two period-lived agents

Every agent young in period $t, t \geq 0$, supplies h_t units of efficient labor, and his wage income is $w_t h_t$, where w_t is the wage per unit of efficient labor. His total first period income is: $\omega_t = w_t h_t + x_t$, $\omega_t > 0$, and either $x_t > 0$, or $x_t < 0$, or $x_t = 0$. We shall consider the *three possibilities*: either he receives a gift $x_t > 0$ from his parent, or he makes a gift $-x_t > 0$ to his parent, or there is no gift: $x_t = 0$. The agent young in t consumes c_t and saves s_t. His budget constraint in the first period of life is:

$$\omega_t \; = \; w_t h_t + x_t \; = \; c_t + s_t. \tag{2.1}$$

In period $t + 1$, this agent is old and receives the proceeds of his savings, $R_{t+1}s_t$, where R_{t+1} is the factor of interest. He consumes d_{t+1} and his budget constraint in the second period of life is:

[17] We may assume that agents live three periods, but they make no decision in the childhood-period. Consumption of this period is assumed to be included in parents' consumption.

$$R_{t+1}s_t = d_{t+1} + \nu x_{t+1}. \tag{2.2}$$

x_{t+1} is a gift and ν is the number of his children.

Either x_{t+1} is the gift which he makes to each of his ν children (if $x_{t+1} > 0$) or $-x_{t+1}$ is the gift which he receives from each of them (if $x_{t+1} < 0$), or there is no gift ($x_{t+1} = 0$).

2.1.2 The first old agents

In period $t = 0$, there are N_{-1} agents who are old in 0. They live only one period, do not work, but receive each an income $R_0 s_{-1}$, which can be interpreted as the proceeds of some past savings s_{-1}. More precisely, a given initial capital stock $K_0 = N_{-1}s_{-1}$ is assumed to be owned uniformly in equal parts by the N_{-1} agents old in period 0, and $R_0 K_0$ is the part of the total national product in period 0 which is received by the capital owners.

The budget constraint of the first old agents is similar to (2.2): $R_0 s_{-1} = d_0 + \nu x_0$, with either $x_0 > 0$, or $x_0 < 0$, or $x_0 = 0$. It is condition (2.2) with $t = -1$.

To study the intertemporal equilibrium, it is necessary to study the life-cycle behavior of the households and to study the equilibrium in each period t with the agents who are alive at this period t (young and old). We also have to define the equilibrium conditions of firms. In the next section, we make only a partial equilibrium analysis and consider later different assumptions on the behavior of the households.

2.2 Assumptions and equilibrium conditions in the aggregate economy

2.2.1 Assumption on the production technology

The production technology is the same at each period, using two production factors, capital K and efficient labor L.

In the basic model of section 1.1.1., we have considered an aggregate production function $F_A(K, L)$ which is homogeneous of degree one. Adding capital after depreciation, δK, we have defined the total disposable product $Y(K, L) = F_A(K, L) + \delta K$. The *product function* $y(\cdot)$ defines the product-labor ratio $y(k) = Y(k, 1)$ as a function of the capital-labor ratio $k = K/L$; and the *average product function* $\hat{y}(\cdot)$ defines the product-capital ratio $\hat{y}(k) = y(k)/k$ as a function of k. These functions are defined on the set $\mathbb{R}_+^*$ of positive real numbers with values in $\mathbb{R}_+^*$. We shall assume:

$$y(\cdot) \text{ and } \hat{y}(\cdot) \text{ are continuous, } y(\cdot) \text{ is increasing and } \hat{y}(\cdot) \text{ is decreasing.} \tag{2.3}$$

In the basic model, assumption (2.3) is a consequence of (1.2). But assumtion (2.3) may hold for other production technologies, like production with stationary externalities (section 1.5.2) or production with fixed cost and decreasing return to scale (section 1.7).

Note that only the assumption (2.3) has been made for studying the dynamics in the Solow model (section 1.2).

2.2.2 Assumption on factor income

Similarly, the studies in chapter 1 of different production technologies and of different aggregate behaviors of workers and capital owners lead to factors' remunerations: $R(k)$ of capital and $w(k)$ of efficient labor, functions of $k = K/L$. We shall assume that these functions satisfy:

$$R(\cdot) \text{ is decreasing and } w(\cdot) \text{ is increasing, and for all, } k > 0 : y(k) = R(k)k + w(k). \tag{2.4}$$

Only in a competitive economy without externality (sections 1.4 and 1.7), R is equal to the marginal productivity of capital: $R(k) = y'(k)$, for all $k > 0$.

The monotony assumptions in (2.3) and (2.4) imply the existence of limits, finite or infinite, when k tends to 0 and when k tends to ∞. The limits $y(0), w(0), \hat{y}(\infty)$ and $R(\infty)$ are finite and non-negative. Each of the limits $y(\infty), w(\infty), \hat{y}(0)$ and $R(0)$ is either positive and finite or equal to ∞.

2.2.3 Equilibrium conditions

We assume that in each period t, the *total capital stock* is equal to the total savings made in the preceding period by the N_{t-1} agents who are old in period t: $K_t = N_{t-1}s_{t-1}$ for all $t \geq 1$. In period $t = 0$, the same relation holds by definition of $s_{-1} = K_0/N_{-1}$, the initial capital stock K_0 being given. Thus the equilibrium condition on the *capital market* in period t is: $K_t = N_{t-1}s_{t-1}$ for all $t \geq 0$.

Efficient labor in period t is supplied inelastically by the N_t agents young in t, each of them supplying h_t units. Thus the *total efficient labor* in period t is $L_t = N_t h_t$. This is the equilibrium condition on the *labor market* in period $t, t \geq 0$.

From the assumption made on factor income, the *equilibrium prices* w_t and R_t satisfy: $w_t = w(k_t)$ and $R_t = R(k_t)$, with $k_t = K_t/L_t$, for all $t \geq 0$.

The total disposable product is $Y_t = L_t y(k_t)$. At equilibrium (on the good market in period t), Y_t is equal to the sum of the consumptions of the N_t agents young in period t, the consumptions of the N_{t-1} agents old in period t and the total investment plus depreciated capital equal to the capital stock of period $t+1, K_{t+1} : Y_t = N_t c_t + N_{t-1} d_t + K_{t+1}$.

Let us summarize these equilibrium conditions for all $t \geq 0$

$$\begin{aligned} K_t &= N_{t-1} s_{t-1} \text{ and } L_t = N_t h_t \\ w_t &= w(k_t) \text{ and } R_t = R(k_t), \text{ with } k_t = K_t/L_t \\ Y_t &= L_t y(k_t) \text{ and } Y_t = N_t c_t + N_{t-1} d_t + K_{t+1} \end{aligned} \tag{2.5}$$

The last condition (the equilibrium condition on the good market) is the consequence of the others, when taking into account the households budget constraints (2.1)-(2.2) and the accounting equality in (2.4): $y(k_t) = R(k_t)k_t + w(k_t)$. Indeed, using the other equilibrium conditions, (2.1), and (2.2) at period t, we have

$$\begin{aligned} N_t c_t + K_{t+1} &= N_t(c_t + s_t) = N_t(w_t h_t + x_t) = L_t w_t + N_t x_t \\ N_{t-1} d_t &= N_{t-1}(R_t s_{t-1} - \nu x_t) = R_t K_t - N_t x_t = R_t L_t k_t - N_t x_t \\ N_t c_t + K_{t+1} + N_{t-1} d_t &= L_t(R(k_t)k_t + w(k_t)) = L_t y(k_t) = Y_t \end{aligned}$$

In order to define the general intertemporal equilibrium in the economy, there remains to analyze the behavior of households. Three different assumptions on this behavior will be considered.

In the economy with *no gift*, we impose: $x_t = 0$ for all $t \geq 0$. This is the standard overlapping generations model with no private transfer decisions of households (in brief, the NG-economy).

In the economy *with gifts in the family* (the FG-economy), we assume that each agent old in t chooses x_t for allocating the family income (equal to the sum of his income $R_t s_{t-1}$ and the wage incomes $\nu w_t h_t$ of his ν children) between his own consumption d_t and the first-period incomes $\omega_t = w_t h_t + x_t$ of his ν children.

In the economy *with parents' gift* (the PG-economy), we have the same choice of the old agents subject to the constraint: $x_t \geq 0$, which excludes gifts from the children to their parent.

2.3 The standard OLG model (the NG-economy)

In the standard OLG model (Allais (1947), Samuelson (1958), Diamond (1965)), there is no gift. Then we have $x_t = 0$ for all $t \geq 0$ in equations (2.1), (2.2). The first old agents make no transfer and simply consume their income: $d_0 \ = \ R_0 s_{-1}, \ s_{-1} = K_0/N_{-1}$ being given.

2.3.1 The savings function

The other households live two periods. In period t, each young agent chooses his consumption c_t and savings s_t subject to (2.1) with $x_t = 0 : c_t + s_t = w_t h_t$. He should expect the gross return on savings R^e_{t+1}. His preferences are represented by an utility function $U(c_t, d_{t+1})$ depending on his consumptions c_t when young and d_{t+1} when old. Expecting $d^e_{t+1} = R^e_{t+1} s_t$, his optimal choice of savings is:

$$s_t \ = \ \mathrm{argmax}_s \ U(w_t h_t - s, R^e_{t+1} s) \ \equiv \ s(w_t h_t, R^e_{t+1}). \tag{2.6}$$

If U is continuous and strictly quasi-concave, the savings function $s(\cdot, \cdot)$ is uniquely defined and continuous.[18] The optimal choice of s_t given by equation (2.6) determines $c_t = w_t h_t - s_t$. The equilibrium value of d_{t+1} is $d_{t+1} = R_{t+1} s_t$, where R_{t+1} is the equilibrium value of the gross return.

2.3.2 Expectations

A usual assumption for expectations is the rational expectations hypothesis, and in a deterministic world, the perfect foresight assumption: $R^e_{t+1} = R_{t+1}$. But with such an assumption, one needs to analyze how the agents shape their expectations, i.e. which is their information set ?

One usually assumes that the agents know the model. Given the current wage income $w_t h_t$, the agents' choice of savings s_t determines the next period capital stock $K_{t+1} = N_t s_t$, the capital-labor ratio $k_{t+1} = K_{t+1}/(N_{t+1} h_{t+1}) = s_t/(n h_t)$ and the equilibrium gross return $R_{t+1} = R(k_{t+1})$.

[18] For a detailed study of the savings function, see de la Croix and Michel (2002) (sections 1.3.3 and 1.8.3). The Inada conditions on U (infinite marginal utility of zero consumptions) imply that both consumptions are positive and $0 < s_t < w_t h_t$. We also assume that the second period consumption good d is a non-Giffen good (Appendix 2.A.1).

The two questions which agents are concerned with are: 1) is there a solution k_{t+1} of $k_{t+1} = s(w_t h_t, R(k_{t+1}))/(nh_t)$, i.e. does there exist an equilibrium with perfect foresight, and 2) is this solution unique, which allows them to expect $R^e_{t+1} = R(k_{t+1})$?

When the solution is not unique, there is a problem of coordination of expectations (see the discussion and the definition of a rational intertemporal equilibrium in de la Croix-Michel (2002), section 1.5.2). We shall study systematically the conditions of existence and of uniqueness.

2.3.3 Intertemporal equilibrium with perfect foresight

Given the initial capital stock $K_0 > 0$ and the endowments $s_{-1} = K_0/N_{-1}$ of the first old agents, an *intertemporal equilibrium with perfect foresight in the NG-economy* is a sequence of prices $(w_t, R_t)_{t\geq 0}$, of households decisions $(c_t, s_t, d_t)_{t\geq 0}$ and of aggregate variables $(K_t, L_t, Y_t)_{t\geq 0}$ which satisfy all the equilibrium conditions with perfect foresight.

These conditions are the budget constraints of the agents (2.1) and (2.2) with $x_t = 0$, their optimal savings (2.6) with $R^e_{t+1} = R_{t+1}$, and the equilibrium conditions (2.5).

Clearly all the equilibrium variables are simple functions of the sequence of capital-labor ratios $k_t = K_t/L_t$ which satisfies the following dynamic equation

$$nh_t k_{t+1} \;=\; s(w(k_t)h_t, R(k_{t+1})). \tag{2.7}$$

This basic one dimensional dynamics has been studied by many authors in the case $h_t = 1, \forall t$, beginning with Diamond (1965). There are two difficulties: this equation does not give explicitly k_{t+1} as a function of k_t, and the function $s(\cdot,\cdot)$ is defined indirectly by equation (2.6). Moreover, when there is an exogenous technical progress h_t, with $\nu \neq 1$, the dynamics of k_t are time independent only if preferences are homothetic.

2.4 Equilibrium dynamics in the NG-economy with homothetic preferences

2.4.1 Homothetic preferences

Preferences are homothetic when the indifference curves are homothetic with respect to the origin (figure 2.1).

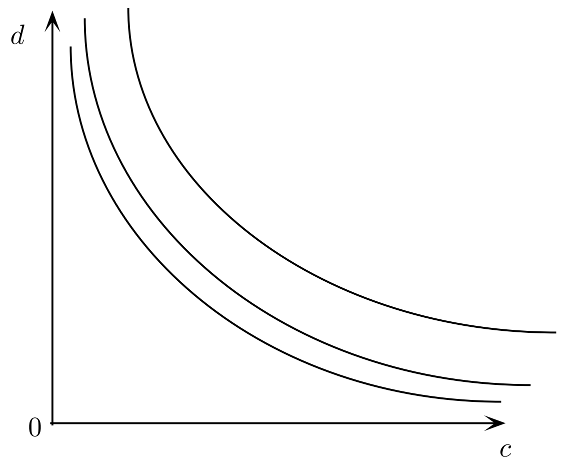

Figure 2.1

Homothetic preferences which are continuous can be represented by an utility function $U(c,d)$ which is homogeneous of degree one. And with such preferences, the savings function $s(\omega, R)$ is linear with respect to income ω, and the saving rate $\tilde{s}(R)$ depends only on the interest factor. Indeed, using the homogeneity property, we have:

$$s(\omega, R) \;=\; \tilde{s}(R)\omega, \text{ with } \tilde{s}(R) = \text{argmax}_s \; U(1-s, Rs).$$

In this case, the dynamical equation (2.7) is equivalent to

$$w(k_t) \;=\; \frac{nk_{t+1}}{\tilde{s}(R(k_{t+1}))} \;\equiv\; \tilde{\varphi}(k_{t+1}). \tag{2.8}$$

The limits of $\tilde{\varphi}(k)$ are: $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}(\infty) = \infty$ (see Appendix 2.A.1 for details). Thus a solution $k_{t+1} > 0$ exists for all $w(k_t) > 0$. It is unique if and only if $\tilde{\varphi}(\cdot)$ is *increasing*. If this assumption is satisfied, then for any $k_0 > 0$, the *intertemporal equilibrium with perfect foresight exists and is unique*. In addition, *the equilibrium dynamics of the capital-labor ratio k_t are monotonic* and *the sequence (k_t) converges*. It converges either to a finite limit $k^* > 0$, and k^* is a steady-state, or it converges to 0, or it converges to ∞ (see the study of monotonic dynamics in Appendix 1.A.3)

These conclusions show three different types of equilibrium dynamics as in the Solow model. But there is one very important difference with the dynamics in the Solow model: *in general, there are several positive steady states* (which of course excludes the global stability property).

Then the long run steady state does depend on the initial capital stock (see below a simple example, section 2.4.3).

There is another important difference: the savings decision is based on the *wage income* and satisfies $s_t \leq w_t h_t$. This implies in the NG-economy without externality, when $R(k) = y'(k)$, that the capital-labor ratio k_t is *bounded* and thus that a growth path at a rate larger than the natural rate is impossible. This property results from the limit property of $w(k)/k$ when k tends to ∞: $w(k)/k = \hat{y}(k) - y'(k)$ tends to 0 (see section 1.1.2, for details see Appendix 2.A.2).

The assumption: $\tilde{\varphi}(\cdot)$ increasing imposes some restrictions which we study in the case of CES-preferences (next section).

2.4.2 Equilibrium in the NG-economy with CES-preferences

We consider the CE utility function (1.7) $u(x) = x^{1-1/\sigma}/(1-1/\sigma)$ for $\sigma > 0$, $\sigma \neq 1$ and $u(x) = \ln x$ if $\sigma = 1$. It satisfies: $u'(x) = x^{-1/\sigma}$ for all $\sigma > 0$. In the NG-economy, we consider the following life-cycle utility function:

$$U(c,d) \;=\; u(c) + \beta u(d), \text{ with } u'(x) = x^{-1/\sigma}, x = c, d.$$

The preferences defined by this utility function are homothetic[19] and we call them CE-preferences in the NG-economy. With these preferences the saving rate $\tilde{s}(R)$ is the solution of the first order condition: $u'(1-\tilde{s}(R)) = \beta R u'(R\tilde{s}(R))$ and thus $\tilde{s}(R) = 1/(1+\beta^{-\sigma}R^{1-\sigma})$. Then we have:

$$\tilde{\varphi}(k) \;=\; nk(1+\beta^{-\sigma}R(k)^{1-\sigma}).$$

- If $\sigma \geq 1$, i.e. the substitution effect is not dominated by the income effect, then $\tilde{\varphi}(\cdot)$ is increasing, since $R(k)^{1-\sigma}$ is non-decreasing.
- In the simple case of a Cobb-Douglas production function, with or without externality (sections 1.5.3 and 1.1.3), we have: $R(k) = A_1 + A_2 k^{\alpha-1}, A_1 > 0, A_2 > 0$ and $0 < \alpha < 1$. Then the derivative of $kR(k)^{1-\sigma}$ is:

$$R(k)^{-\sigma}(R(k) + k(1-\sigma)R'(k)) \;=\; R(k)^{-\sigma}(A_1 + (\alpha + \sigma(1-\alpha))A_2 k^{\alpha-1}) > 0.$$

[19]See Appendix 2.A.3

In this simple case, $\tilde{\varphi}(\cdot)$ is increasing whatever the value of $\sigma > 0$.

- The case of a CES production function (without externality) is studied in the Appendix 2.A.4. In the case of low substitutability of the production factors: $\sigma_F < 1$, there is a condition on $\sigma, \sigma \geq 1 - \sigma_F$, for the function $\tilde{\varphi}(\cdot)$ to be increasing. This example shows that additional assumptions may be necessary to insure the uniqueness of the intertemporal equilibrium in the NG-economy, i.e. the uniqueness of the solution k_{t+1} of equation (2.8) for a given wage $w(k_t)$.

Moreover, there are in general several steady states and the long run dynamics depend on the initial capital-labor ratio k_0. This has been studied in detail by Galor and Ryder (1989). We study a simple example in the next section.

2.4.3 A simple example with several steady states

Consider a loglinear utility function ($\sigma = 1$) which leads to a constant saving rate: $\tilde{s} = 1/(1 + \beta^{-1})$. Assume a CES production function with elasticity $\sigma_F = 1/2$, leading to the product function (without externality and with total depreciation of capital: $\delta = 0$) $y(k) = A(k^{-1}+1)^{-1}$. In this example, the wage rate is: $w(k) = y(k) - ky'(k) = Ak^2/(1+k)^2$, and the equilibrium dynamics are:

$$k_{t+1} = \frac{1}{n(1+\beta^{-1})}w(k_t) = \frac{A}{n(1+\beta^{-1})}\left(\frac{k_t}{1+k_t}\right)^2 \equiv \tilde{f}(k_t).$$

The second order derivative of $w(k), w''(k) = 2A(1-2k)/(1+k)^4$, is positive (negative) for $k < 1/2$ ($k > 1/2$).

Thus $\tilde{f}(\cdot)$ is convex for $k < 1/2$ and concave for $k > 1/2$. The steady states are the solutions of $\tilde{f}(k) = k$. In addition to $k^* = 0$, there are two positive steady states: the roots k_1^* and k_2^* of $n(1+\beta^{-1})(1+k)^2 - Ak = 0$, if A is not too low (i.e. $A > 4n(1+\beta^{-1})$) (see figure 2.2).

The equilibrium dynamics depend on k_0:

- if $k_0 < k_1^*$, the sequence (k_t) is decreasing and tends to 0 (one says that 0 is a poverty trap)

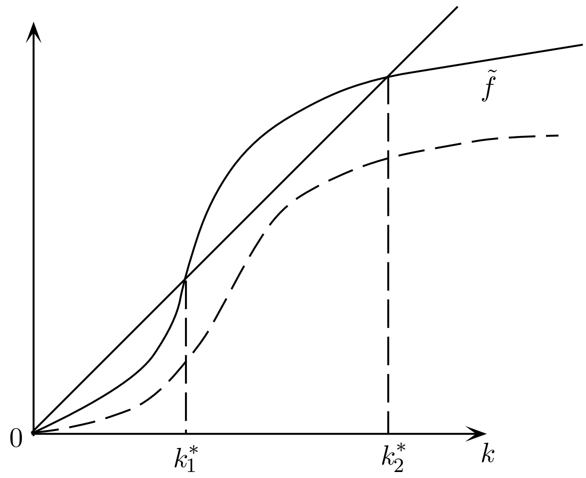

Figure 2.2

- if $k_0 > k_1^*$, the sequence (k_t) is either increasing (if $k_0 < k_2^*$) or decreasing (if $k_0 > k_2^*$) and in both cases, it converges to k_2^*.

For a low value of A, $A < 4n(1 + \beta^{-1})$, 0 is the unique steady state and for any $k_0 > 0$, the sequence (k_t) converges to 0 (broken curve on figure 2.2).

2.5 The economy with gifts in the family (the FG-economy)

In the standard OLG model, there are no gifts. With the assumption of dynastic altruism (Barro 1974) there are bequests linking agents of each generation to the preceding one. Like in the model with an infinite-lived representative consumer, it is then assumed a complete harmony of the sequence of *all* deciders and that *all* future prices are perfectly anticipated. Is this not too much ?

Here we study a simpler model with gifts in the family and only *one decider*, the head of the family. It differs from another simple approach, the joy of giving model[20], in two points: the parent takes into account the wage income of his children and gifts are two-sided, from parent to children or from children to parent.

[20]See e.g. Blinder (1976), Andreoni (1989).

2.5.1 The family organization

Each agent young in period t is the head of family during his life (periods t and $t+1$). In period $t+1$, his ν children belong to this family and each of them creates a new family.

In period t, the head of the family allocates his income $\omega_t = w_t h_t + x_t$ between his consumption c_t and his savings s_t (equation (2.1)). In period $t+1$, there are two sources of income in the family: the gross return on savings $R_{t+1}s_t$ and the wage income of the ν children, $\nu w_{t+1}h_{t+1}$. The family head consumes d_{t+1} and choses the gifts νx_{t+1} (equation (2.2)). These gifts determine the net income of each child: $\omega_{t+1} = w_{t+1}h_{t+1} + x_{t+1}$. There is no restriction on the sign of x_{t+1}. Thus the head of the family allocates the *family income* of period $t+1, \mathcal{W}_{t+1} \equiv R_{t+1}s_t + \nu w_{t+1}h_{t+1}$, between his own consumption d_{t+1} and the net incomes of his ν children, $\nu\omega_{t+1}$:

$$\mathcal{W}_{t+1} \;\equiv\; R_{t+1}s_t + \nu w_{t+1}h_{t+1} \;=\; d_{t+1} + \nu\omega_{t+1}.$$

The gift $x_{t+1} = \omega_{t+1} - w_{t+1}h_{t+1}$ which is positive, negative or nil, results from this choice.

This family organization à la Becker (1991) is represented on figure 2.3

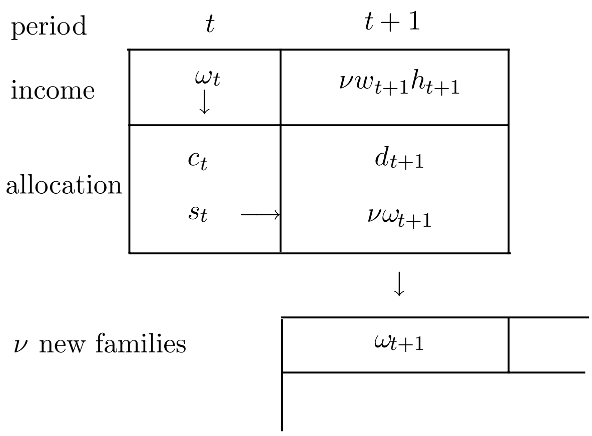

Figure 2.3

An important assumption in this family organization is that there is only *one* decider, the head of the family: children respect his decisions.

2.5.2 Preferences of the head of a family

There are three arguments in the utility function $\mathcal{U}(\cdot)$ of the family head: his life-cycle consumptions c_t, d_{t+1} and the first period income ω_{t+1} of his ν children. In a joy of giving model, the third argument in the utility function is the level of bequest or gift to the children x_{t+1}. Adding another argument, the wage income of children $w_{t+1}h_{t+1}$, we assume perfect substitution between x_{t+1} and $w_{t+1}\ h_{t+1}$ so that the third argument is $\omega_{t+1} = x_{t+1} + w_{t+1}h_{t+1}$. As we shall see, this assumption has important consequences, in particular the neutrality property of a pay-as-go social security system (if there is no redistribution among agents of the same generation).

We shall assume that the preferences of all family heads can be represented with two functions U and V of two arguments which are *homogeneous of degree one*:

$$\mathcal{U}(c_t, d_{t+1}, \omega_{t+1}) \ = \ U(c_t, V(d_{t+1}, \omega_{t+1})).$$

Thus $\mathcal{U}$ is homogeneous of degree one and preferences are homothetic.[21] The head of the family young in period t maximizes his utility $U(c_t, V(d_{t+1}, \omega_{t+1}))$ subject to his life-cycle budget constraints (2.1) and (2.2) and to: $\omega_{t+1} = x_{t+1} + w_{t+1}h_{t+1}$.

In period $t = 0$, the N_{-1} old agents are family heads, live one period, receive $R_0 s_{-1}$, and they maximize their utility $V(d_0, \omega_0)$ subject to: $R_0 s_{-1} = d_0 + \nu x_0$ and $\omega_0 = x_0 + w_0 h_0$.

2.5.3 Intertemporal equilibrium in the FG-economy

Given K_0 and $s_{-1} = K_0/N_{-1}$, an *intertemporal equilibrium with perfect foresight in the FG-economy* is a sequence of prices $(w_t, R_t)_{t\geq 0}$, households decisions $(c_t, s_t, d_t, x_t, \omega_t)_{t\geq 0}$ and aggregate variables $(K_t, L_t, Y_t)_{t\geq 0}$ such that all heads of a family maximize their utilities and the equilibrium conditions (2.5) are satisfied for all $t \geq 0$.

Because *the family income* $\mathcal{W}_t$ is equal to the sum of the gross-return of savings $R_t s_{t-1}$ and the children wages $\nu w_t h_t$, *at equilibrium it is equal to the national product per old agent.* Indeed, at equilibrium, total savings $N_{t-1}s_{t-1}$ is equal to the capital stock $K_t = k_t N_t h_t$ and we have:

$$\mathcal{W}_t \ = \ R_t s_{t-1} + \nu w_t h_t \ = \ \nu(R_t k_t + w_t)h_t \ = \ \nu y(k_t)h_t. \tag{2.9}$$

[21] We shall study later a more general case with homothetic preferences.

Maximizing $V(d_t, \omega_t)$ subject to $d_t + \nu\omega_t = \mathcal{W}_t$ leads to constant proportions of allocation by the head of the family:

$$\omega_t \ = \ ay(k_t)h_t \ \text{ and } \ d_t = (1-a)\nu y(k_t)h_t, \text{ with } 0 < a < 1.$$

We call a the *generosity rate* of the head. Moreover, the optimal consumption c_t of the young head is, under the assumption of perfect foresight, proportional to the next period family income $\mathcal{W}_{t+1} = \nu y(k_{t+1})h_{t+1}$:

$$c_t \ = \ \rho(R_{t+1})\nu y(k_{t+1})h_{t+1}.$$

The proportionality factor $\rho(\cdot)$ is a decreasing function of the interest factor. These properties result from the assumptions made on the functions U and V (for details see Appendix 2.A.5). Thus using (2.1) and (2.5), we obtain:

$$\omega_t = c_t + s_t = \nu\rho(R(k_{t+1}))y(k_{t+1})h_{t+1} + nk_{t+1}h_t \tag{2.10}$$

and with $\omega_t = ay(k_t)h_t$:

$$\frac{y(k_t)}{y(k_{t+1})} \ = \ \frac{n}{a}\rho(R(k_{t+1})) + \frac{nk_{t+1}}{ay(k_{t+1})} \ \equiv \ \psi(k_{t+1}). \tag{2.11}$$

This dynamic equation of the capital-labor ratio k_t fully characterizes an intertemporal equilibrium. All the others equilibrium variables are simple functions of the sequence (k_t).

2.6 Equilibrium dynamics in the FG-economy

Note that $\psi(k) = \frac{n}{a}(\rho(R(k)) + 1/\hat{y}(k))$ is an *increasing* function of k since $\rho(\cdot), R(\cdot)$ and $\hat{y}(\cdot)$ are decreasing functions.

2.6.1 Existence and uniqueness of the intertemporal equilibrium

The equilibrium dynamics are given by

$$y(k_t) \ = \ \psi(k_{t+1})y(k_{t+1}) = \frac{n}{a}k_{t+1} + \frac{n}{a}\rho(R(k_{t+1}))y(k_{t+1}) \ \equiv \ \zeta(k_{t+1})$$

$\zeta(\cdot)$ is increasing and $\zeta(\infty) = \infty$ (since $\zeta(k) > nk/a$). Given k_t, the solution k_{t+1} of $\zeta(k_{t+1}) = y(k_t)$ exists and is positive if and only if $y(k_t) > \zeta(0) = \frac{n}{a}\rho(R(0))y(0)$. When this condition is not satisfied, we set: $k_{t+1} = 0$, which is the equilibrium solution when imposing a non-negativity constraint on savings (see Appendix 2.A.6 for details in this special case).

Thus, for all $k_0 > 0$, there exists a unique intertemporal equilibrium with initial capital-labor ratio k_0, and the dynamics of (k_t) are monotonic.

2.6.2 The three different possibilities of equilibrium dynamics

Because $\psi(\cdot)$ is increasing, there are three different possibilities of the dynamics.

1. *Convergence to a long run growth path at the natural rate.* If $\psi(0) < 1 < \psi(\infty)$, then there exists a unique steady-state $k^* > 0$ for the dynamics (2.11) and this steady state is *globally stable.* k^* is the solution of $\psi(k^*) = 1$. Indeed, if $k_t < k^*$, then we have: $\psi(k_t) < \psi(k^*) = 1$ and thus:

$$\zeta(k_t) = \psi(k_t)y(k_t) < y(k_t) < y(k^*) = \zeta(k^*).$$

   From the intermediary value theorem, $\zeta(k_{t+1}) = y(k_t)$ admits an unique solution k_{t+1} such that: $k_t < k_{t+1} < k^*$. Similarly, if $k_t > k^*$, k_{t+1} satisfies $k^* < k_{t+1} < k_t$. The monotonic sequence (k_t) is bounded and necessarily converges to k^* since there is no other steady state.

2. *unlimited growth of k_t.* If $\psi(\infty) \leq 1$, then for all $k_t > 0$, we have: $y(k_{t+1}) > y(k_{t+1})\psi(k_{t+1})$ $= y(k_t)$, and thus: $k_{t+1} > k_t$. The increasing sequence (k_t) tends to ∞, since there is no steady state. The long run growth factor of $y(k_t)$ is equal to $1/\psi(\infty)$. In the long run, the growth rate of the economy is larger than the natural rate if $\psi(\infty) < 1$.

3. *decrease of k_t tending to* 0. If $\psi(0) \geq 1$, then the sequence (k_t) is decreasing (since $y(k_{t+1}) < y(k_{t+1})\psi(k_{t+1}) = y(k_t)$) and it tends to 0.

The Inada conditions on $R(\cdot) : R(0) = \infty$ and $R(\infty) = 0$ imply: $\psi(0) = 0$ and $\psi(\infty) = \infty$. Thus these conditions imply the existence of a steady state $k^* > 0$ which is globally stable (more details on the limits $\psi(0)$ and $\psi(\infty)$ are in Appendix 2.A.7).

2.6.3 Dynamics in the FG-economy with CE-preferences

Consider the CE-utility function u defined in (1.7) satisfying $u'(x) = x^{-1/\sigma}, \sigma > 0$ and assume the following utility function of the family heads:

$$\mathcal{U}(c_t, d_{t+1}, \omega_{t+1}) \;=\; u(c_t) + \beta[u(d_{t+1}) + \mu\nu u(\omega_{t+1})], \text{ where } u'(x) = x^{-1/\sigma}, \sigma > 0. \tag{2.12}$$

$\beta > 0$ is the discount factor and $\mu > 0$ is the weighting on ν times the utility of each child's income. It can be considered as the reduced form of the second period utility function $u(d_{t+1}) + \mu \sum_{i=1}^{i=\nu} u(\omega_{t+1}^i)$ when children in the family have different wage incomes $w_{t+1}h_{t+1}^i$. Then the optimal choice of transfers x_{t+1}^i lead to the same value of $\omega_{t+1} = x_{t+1}^i + w_{t+1}h_{t+1}^i$ for all children. Assuming that the mean h_{t+1} of the efficient labor endowments is the same in all families, we obtain the equivalent reduced form (2.12).

The first order conditions of the problem of the head of family young in t are: $u'(c_t) = R_{t+1}\beta u'(d_{t+1})$ and $u'(d_{t+1}) = \mu u'(\omega_{t+1})$. With $d_{t+1} = \mu^{-\sigma}\omega_{t+1}$, we obtain the generosity rate $a = 1/(1 + \mu^{-\sigma}\nu^{-1})$. With $c_t = R_{t+1}^{-\sigma}\beta^{-\sigma}d_{t+1}$, we obtain (2.10) with $\rho(R) = (1-a)\beta^{-\sigma}R^{-\sigma}$.

With the CE-preferences given by (2.12), the equilibrium dynamics of the capital-labor ratio (2.11) are:

$$\frac{y(k_t)}{y(k_{t+1})} \;=\; \frac{n(1-a)}{a}\beta^{-\sigma}R(k_{t+1})^{-\sigma} + \frac{nk_{t+1}}{ay(k_{t+1})} \;\equiv\; \tilde{\psi}(k_{t+1}, a).$$

Here we have explicitly the effect of the generosity rate a on the equilibrium dynamics, with $\tilde{\psi}(k, a)$ increasing in k and decreasing in a. Then the three possibilities for the dynamics of k_t depend on the value of a. Let us define the solution a_0 (a_1) of $\tilde{\psi}(0, a) = 1$ ($\tilde{\psi}(\infty, a) = 1$) if it exists, and $a_0 = 0 (a_1 = 1)$ if not. From the study of section 2.6.2, it results:

1. if $a_0 < a < a_1$, the sequence (k_t) converges to $k^*(a) > 0$, the unique positive steady state solution of $\tilde{\psi}(k^*(a), a) = 1$ and $k^*(a)$ is globally stable;

2. if $a \geq a_1, (k_t)$ is increasing and tends to ∞,

3. if $a \leq a_0, (k_t)$ is decreasing and tends to 0.

The second possibility occurs (with some $a < 1$) only if $a_1 < 1$ and the third one only if $a_0 > 0$. These conditions are explicitly studied in Appendix 2.A.8.
The *Golden Rule capital stock* k_{GR} exists if and only if $y'(\infty) < n < y'(0)$ (section 1.3.1). Then we have: $\tilde{\psi}(k_{GR}, 0) = \infty$ and $\tilde{\psi}(k_{GR}, 1) = n/\hat{y}(k_{GR}) < 1$ since $\hat{y}(k_{GR}) > y'(k_{GR}) = n$. This implies that there exists a unique value a_{GR} of the generosity rate, $0 < a_{GR} < 1$, such that

$\tilde{\psi}(k_{GR}, a_{GR}) = 0$. Thus for $a = a_{GR}$, the steady state $k^*(a_{GR})$ is the Golden Rule capital stock. Moreover, for $a_0 < a < a_1$, the steady state $k^*(a)$ is in under-accumulation if $a < a_{GR}$ and in over-accumulation of capital if $a > a_{GR}$.

2.6.4 Examples

In the *simple example* with $\sigma = 1$ (loglinear utility), $y(k) = Ak^\alpha$, $(0 < \alpha < 1, A > 0)$ and $R(k) = y'(k) = \alpha A k^{\alpha-1}$, the equilibrium dynamics are $nk_{t+1} = sAk_t^\alpha$, with a constant savings rate $s = a/(1 + \frac{1-a}{\alpha}\beta^{-1})$. This savings rate is increasing with a and with β. For any $a, 0 < a < 1, (k_t)$ converges to $k^*(a) = (sA/n)^{1/(1-\alpha)}$. This steady state is the Golden Rule capital stock for $a_{GR} = \frac{1+\alpha\beta}{1+\beta}$.

Another simple example with $\sigma = 1$ and the CES product function $y(k) = A(k^{-1} + 1)^{-1}$ has been studied in the NG-economy (section 2.4.3). In the FG-economy, we have for this example:

$$\tilde{\psi}(k, a) \;=\; \frac{n(1-a)}{a\beta A}(1+k)^2 + \frac{n}{aA}(1+k).$$

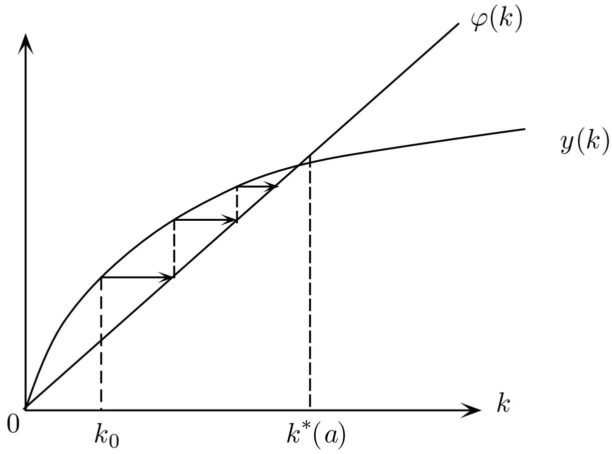

Figure 2.4

The solution a_0 of $\tilde{\psi}(0, a_0) = 1$ is: $a_0 = n(1+\beta)/(n+\beta A)$ and it satisfies $a_0 < 1$ if $A > n$. The dynamics of (k_t):

$$y(k_t) \equiv \frac{Ak_t}{1+k_t} = \frac{n(1-a)}{\beta a}k_{t+1}(1+k_{t+1}) + \frac{n}{a}k_{t+1} \equiv \zeta(k_{t+1})$$

are represented on figure 2.4, when $a_0 < a < 1$.

The behavior assumptions in the FG-economy give nice micro fundations to the Solow model, much better than the OLG model does because savings are based on the national product instead of the wage income. Moreover, these savings depend on the generosity rate of the head of the family. Last, but not least, the long run growth path does not depend on the initial capital-labor ratio.

2.7 The economy with parents' gifts (PG-economy)

It is often assumed that parents make gifts (or bequests) to their children, but that children make no gift to their parents. This assumption applies to the study of dynastic altruism (e.g. Weil (1987)), to studies of joy of giving, of education (e.g. Glomm and Ravikumar (1992)), etc... In his study of familial altruism, Lambrecht (2000) considers gifts in the family which are only from parents to children, like we do now.[22]

2.7.1 The economy with a non-negative constraint on gifts

In the economy with gifts in the family, the additional constraint: $x_t \geq 0$ for all t, means that children refuse to make a gift to the head of the family. The authority of the head is restricted, children also take a decision which is to leave the family if they should make a gift to the head and to create new families with their wage income. Children are assumed to be identical and the family organization is modified as in figure 2.5.

In this economy, only parents make gifts (to their children). We call it the PG-economy, and the agent young will be called the *parent*. Each parent young in period t maximizes his utility which is unchanged:

$$U(c_t, V(d_{t+1}, x_{t+1} + w_{t+1}h_{t+1}))$$

[22] See also Lambrecht, Michel and Thibault (2000) and Lambrecht, Michel and Vidal (2001). Ascending altruism and two-sided altruism have been studied less (see the survey by Michel, Thibault and Vidal (2001)).

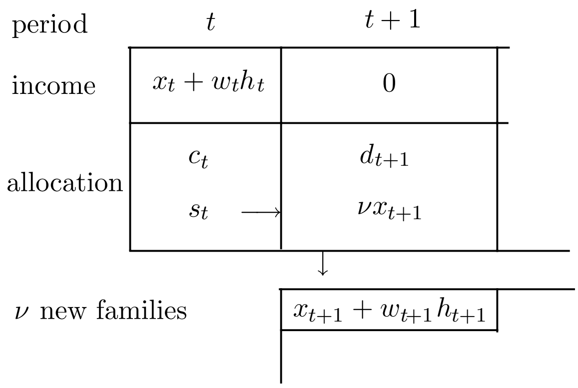

Figure 2.5

subject to his two budgets constraints: (2.1), (2.2) *and* the constraint: $x_{t+1} \geq 0$. Also the first old agents are submitted to the constraint of non-negative gifts. They maximize $V(d_0, x_0 + w_0 h_0)$ subject to the constraints: $R_0 s_{-1} = d_0 + \nu x_0$ and $x_0 \geq 0$.

In this PG-economy, the definition of an *intertemporal equilibrium with perfect foresight* is quite similar to the definition in the FG-economy. The only change is in the optimal problem of parents the additional constraint of non-negative gifts. But this of course may modify their decisions of consumption and savings.

The F.O.C. are obtained by maximizing $U(\omega_t - s_t, V(R_{t+1}s_t - \nu x_{t+1}, x_{t+1} + w_{t+1}h_{t+1}))$ with respect to s_t and to x_{t+1} subject to the constraint: $x_{t+1} \geq 0$. The maximum with respect to s_t leads to: $U_1' = U_2' V_1' R_{t+1}$. The maximum with respect to x_{t+1} leads to *two possible regimes*:

1. the parent is *not constrained*: his optimal choice ignoring the constraint satisfies it: $x_{t+1} \geq 0$. Then we have: $\nu V_1' = V_2'$;

2. the optimal choice (with the constraint) is a *zero gift*: $x_{t+1} = 0$ and then we have: $-\nu V_1' + V_2' \leq 0$ (the marginal utility of a zero gift is non-positive).

Note that the two regimes apply simultaneously when zero gifts are the optimal choice without the constraint of non-negative gifts: $x_{t+1} = 0$ and $\nu V_1' = V_2'$.

2.7.2 Study of the two regimes: unconstrained and zero gift

1) In the *unconstrained regime*, the F.O.C. are the same as in the FG-economy. Then, at equilibrium, the relation (2.10) applies and we have

$$\frac{\omega_t}{h_t} \;=\; n\rho(R(k_{t+1}))y(k_{t+1}) + nk_{t+1} \;\equiv\; \varphi_1(k_{t+1}). \tag{2.13}$$

$\varphi_1(k) = a\zeta(k)$ is an increasing function of k (section 2.6.1).

For the unconstrained regime to prevail, it is necessary and sufficient that the corresponding gift is non-negative, i.e.

$$x_{t+1} = \omega_{t+1} - w_{t+1}h_{t+1} = ay(k_{t+1})h_{t+1} - w(k_{t+1})h_{t+1} \geq 0.$$

This condition is: $ay(k_{t+1}) \geq w(k_{t+1})$.

2) In the *zero gift regime*, i.e. $x_{t+1} = 0$, we have at equilibrium

$$d_{t+1} = R_{t+1}s_t = R(k_{t+1})\nu k_{t+1}h_{t+1} \;\text{ and }\; \omega_{t+1} = w(k_{t+1})h_{t+1}.$$

Using the homogeneity properties of V (degree 1) and V_1' (degree 0), we have then

$$\begin{aligned} &V(d_{t+1}, \omega_{t+1}) = v(k_{t+1})h_{t+1}, \text{ where } v(k_{t+1}) \equiv V(\nu R(k_{t+1})k_{t+1}, w(k_{t+1})) \\ &V_1'(d_{t+1}, \omega_{t+1}) = V_1'\Big(\frac{d_{t+1}}{\omega_{t+1}}, 1\Big) \;=\; V_1'\Big(\frac{\nu R(k_{t+1})k_{t+1}}{w(k_{t+1})}, 1\Big) \;\equiv\; v_1(k_{t+1}). \end{aligned}$$

The F.O.C.: $U_1' = U_2'V_1'R_{t+1}$ is equivalent to: c_t is proportional to $V(d_{t+1}, \omega_{t+1})$ and the coefficient of proportionality ρ_0 depends on $V_1'R_{t+1}$. Thus at equilibrium with the zero gift regime in $t+1$, we have $c_t = \rho_0(k_{t+1})v(k_{t+1})h_{t+1}$ and, using $\omega_t = c_t + s_t = c_t + nk_{t+1}h_t$:

$$\frac{\omega_t}{h_t} \;=\; \eta\rho_0(k_{t+1})v(k_{t+1}) + nk_{t+1} \;\equiv\; \varphi_0(k_{t+1}). \tag{2.14}$$

The other F.O.C. is the inequality: $V_2' \leq \nu V_1$, and it is equivalent to: $ay(k_{t+1}) \leq w(k_{t+1})$, i.e. the wage income of children is larger than or equal to the wealth that the parent would like for them.

We have also the following *property* of φ_0 an φ_1:

$$\text{for any } k>0 \begin{cases} \text{the equality } ay(k)=w(k) \text{ implies } \varphi_0(k)=\varphi_1(k) \\ \text{the inequality } ay(k)<w(k) \text{ implies } \varphi_0(k)<\varphi_1(k). \end{cases} \tag{2.15}$$

The equality $\varphi_0(k) = \varphi_1(k)$ holds when the two regimes apply. The inequality $\varphi_0(k) < \varphi_1(k)$ results from $\eta\rho_0(k)v(k) < n\rho(R(k))y(k)$, consumption being lower when it is constrained, i.e. if $w(k) > ay(k)$. The detailed proofs are in the Appendix 2.A.9.

The equality property in (2.15) permits to gather the two equations (2.13) and (2.14) of the two regimes in a *unique equation*: $w_t/h_t = \varphi(k_{t+1})$, where $\varphi(\cdot)$ is defined by

$$\varphi(k) = \varphi_0(k) \text{ if } ay(k) \leq w(k) \text{ and } \varphi(k) = \varphi_1(k) \text{ if } ay(k) \geq w(k). \tag{2.16}$$

The function $\varphi(\cdot)$ is continuous and $\varphi(\infty) = \infty$ (since $\varphi(k) > nk$). The prevailing regime in period $t+1$ is unconstrained if $ay(k_{t+1}) \geq w(k_{t+1})$ and with zero gift if $ay(k_{t+1}) \leq w(k_{t+1})$. In both cases, the corresponding initial wealth of the young in $t+1$ is:

$$\omega_{t+1} \;=\; \tilde{\omega}(k_{t+1})h_{t+1}, \text{ where } \tilde{\omega}(k_{t+1}) = \max\{ay(k_{t+1}), w(k_{t+1})\}.$$

For the first old agents, s_{-1} is given. They simply choose d_0 and x_0 subject to the constraint $x_0 \geq 0$. The unconstrained regime prevails if $ay(k_0) \geq w(k_0)$ and then: $\omega_0 = ay(k_0)h_0$. The zero gift regime prevails if $ay(k_0) \leq w(k_0)$ and then $\omega_0 = w(k_0)h_0$. In both cases, we have: $\omega_0 = \tilde{\omega}(k_0)h_0$, where $\tilde{\omega}(k) = \max\{ay(k), w(k)\}$.

2.7.3 Existence and uniqueness of the intertemporal equilibrium in the PG-economy

The sequence of capital-labor ratios (k_t) satisfies at equilibrium for all $t \geq 0$:

$$\frac{\omega_t}{h_t} \;=\; \max\{ay(k_t), w(k_t)\} \;\equiv\; \tilde{\omega}(k_t) \;\text{ and }\; \tilde{\omega}(k_t) = \varphi(k_{t+1}). \tag{2.17}$$

Wealth of the young is equal to the maximum of their wage and of the income that their parent would like for them. Dynamics satisfy (2.13) if parents are not constrained and (2.14) if their constrained choice is a zero gift. All other variables are simple functions of the sequence (k_t).

Existence. Consider $k_0 > 0$ and the intertemporal equilibrium (k_t^1) in the FG-economy starting at $k_0^1 = k_0$. We show (by induction) that there exists an intertemporal equilibrium (k_t) in the

PG-economy which satisfies: $\forall t \geq 0, k_t \geq k_t^1$. We assume:[23] $k_t \geq k_t^1$ and $ay(k_t^1) = \varphi_1(k_{t+1}^1)$. This implies: $\varphi_1(k_{t+1}^1) \leq \tilde{\omega}(k_t^1) \leq \tilde{\omega}(k_t)$ and, since $\varphi_1(\infty) = \infty$, there exists $k_{t+1}^2 \geq k_{t+1}^1$ satisfying $\varphi_1(k_{t+1}^2) = \tilde{\omega}(k_t)$. Then we have:

- either $ay(k_{t+1}^2) \geq w(k_{t+1}^2)$ and $k_{t+1} = k_{t+1}^2$ satisfies: $\varphi(k_{t+1}) = \varphi_1(k_{t+1}^2) = \tilde{\omega}(k_t)$
- or $ay(k_{t+1}^2) < w(k_{t+1}^2)$ and $\varphi(k_{t+1}^2) = \varphi_0(k_{t+1}^2) < \varphi_1(k_{t+1}^2) = \tilde{\omega}(k_t)$ (property (2.15)); we also have: $\varphi(\infty) = \infty$ and there exists $k_{t+1} > k_{t+1}^2$ satisfying $\varphi(k_{t+1}) = \tilde{\omega}(k_t)$.

In both cases, there exists $k_{t+1} \geq k_{t+1}^1$ solution of $\varphi(k_{t+1}) = \tilde{\omega}(k_t)$. *The non-negativity constraint on gifts can only increase the capital accumulation* as it can only increase the income of the young.

Uniqueness. The intertemporal equilibrium in the PG-economy is unique if and only if $\varphi(\cdot)$ is increasing. Since $\varphi_1(\cdot)$ is increasing, it is sufficient that $\varphi_0(\cdot)$ is increasing. This condition is similar to the monotony condition in the NG-economy, and with CE-preferences it is exactly the same ($\varphi_0 = \tilde{\varphi}$ see section 2.7.5 below).

2.7.4 Equilibrium dynamics in the PG-economy

We assume that $\varphi(\cdot)$ is increasing. Then for any $k_0 > 0$, there exists a unique intertemporal equilibrium in the PG-economy. The equilibrium dynamics of (k_t) are monotonic and converge either to a steady state or to ∞ or to 0. They satisfy: $k_t \geq k_t^1$ for all $t \geq 0$, where (k_t^1) is the equilibrium dynamics in the FG-economy starting at $k_0^1 = k_0$. There are three possibilities (section 2.6.2).

1. the sequence (k_t^1) converges to the unique globally stable steady state k^*, solution of $\psi(k^*) = 1$, if $\psi(0) < 1 < \psi(\infty)$. Consider $k_0 < k^*$. Then the equilibrium dynamics (k_t) in the PG-economy is increasing and it converges to $k^{**} \geq k^*$, such that:
   - either $k^{**} = k^*$: if $ay(k^*) \geq w(k^*)$, then: $\varphi(k^*) = \tilde{\omega}(k^*)$ and k^* is a steady state of the PG-dynamics;

[23] The dynamics in the FG-economy are: $a(k_t^1) = a\zeta(k_{t+1}^1) = \varphi_1(k_{t+1}^1)$ if $k_{t+1}^1 > 0$. But we may also have $k_{t+1}^1 = 0$ when $\varphi_1(0) \geq ay(k_t^1)$, see Appendix 2.A.6. Then we have $k_{t+1} > k_{t+1}^1 = 0$ (see Appendix 2.A.10).

- or $k^{**} > k^*$: if $ay(k^*) < w(k^*)$, and the lowest solution $k^{**} > k^*$ of $\varphi_0(k) = w(k)$ is finite, then (k_t) converges to k^{**};
- or $k^{**} = \infty$ if $\varphi_0(k) < w(k)$ for all $k > k^*$. This case is excluded in a pure competitive economy without externality. It is possible only if the distribution of factor incomes is enough in favor of wages (e.g. by external effects, section 1.5, or by collective negociation, section 1.8.1). For details, see Appendix 2.A.11.

2. the sequence (k_t^1) increases and tends to ∞, if $\psi(\infty) \leq 1$. Then for any $k_0 > 0$, the equilibrium dynamics (k_t) in the PG-economy increases and tends to ∞.

3. the sequence (k_t^1) decreases and tends to 0, if $\psi(0) \geq 1$. Then either (k_t) decreases and tends to 0, or the trap of zero capital-labor ratio is avoided by the non-negativity constraint on gifts.

As in the NG-economy, there may be several steady states in the zero gift regime (except in case 2).

The prevailing regime. The zero gift condition in period t: $ay(k_t) \leq w(k_t)$ is equivalent to $k_t R(k_t)/y(k_t) \leq 1 - a$ (since $w(k_t) = y(k_t) - k_t R(k_t)$).

$\varepsilon(k) = kR(k)/y(k)$ is the fraction of the capital income in the national income, and in the pure competitive economy without externality, it is equal to the elasticity $ky'(k)/y(k)$ of the product function.

Examples. In the simple Cobb-Douglas case where $y(k) = Ak^\alpha$ and $R(k) = \alpha Ak^{\alpha-1}$, $\varepsilon(k) = \alpha$ is constant and there is *no change of regime*. The prevailing regime is zero gift if $a \leq 1 - \alpha$, or unconstrained if $a \geq 1 - \alpha$. In the example with a CES production function, total depreciation of capital ($\delta = 0$) and no externality ($R(k) = y'(k)$), there is no more than *one change of regime* (see Appendix 2.A.12).

2.7.5 Equilibrium in the PG-economy with CE-preferences

CE-preferences are defined by the utility function (2.12) which is additively separable. Then in the zero gift regime, savings are the same as in the NG-economy and we have (section 2.4.2):

$$\varphi_0(k) = \tilde{\varphi}(k) \;=\; nk(1 + \beta^{-\sigma} R(k)^{1-\sigma}).$$

The assumption of uniqueness of the equilibrium in the NG-economy, i.e. $\tilde{\varphi}(\cdot)$ is increasing, implies the uniqueness of the equilibrium in the PG-economy. Moreover, the unconstrained regime leads to (section 2.6.3):

$$\varphi_1(k) \; = \; \tilde{\psi}(k,a)ay(k) = n(1-a)\beta^{-\sigma}R(k)^{-\sigma}y(k) + nk.$$

Then the difference $\varphi_1(k) - \varphi_0(k) = n\beta^{-\sigma}R(k)^{-\sigma}y(k)(1-a-\varepsilon(k))$ is non-negative in the zero gift regime ($\varepsilon(k) = kR(k)/y(k) \leq 1-a$) and it is non-positive in the unconstrained regime ($\varepsilon(k) \geq 1-a$). In both regimes, we have:

$$\varphi(k) \; = \; \min\left\{\varphi_0(k), \varphi_1(k)\right\}.$$

We assume $\varphi_0(\cdot)$ increasing which means uniqueness of the equilibrium in the NG-economy. In the PG-dynamics, the capital-labor ratio k_t is larger than or equal to what it would be both in the FG-economy k_t^1 and in the NG-economy k_t^0. Indeed for (k_t^0) the equilibrium dynamics in the NG-economy starting at $k_0^0 = k_0$, we have by induction: if $k_t \geq k_t^0$, then $\varphi(k_{t+1}) = \tilde{\omega}(k_t) \geq w(k_t^0) = \varphi_0(k_{t+1}^0) \geq \varphi(k_{t+1}^0)$ and $k_{t+1} \geq k_{t+1}^0$. The sequence (k_t) is increasing if at least one of the two sequences (k_t^0) and (k_t^1) is increasing. Using the threshold values a_0 and a_1 of the generosity rate (section 2.6.3), the dynamics in the PG-economy with CE-preferences satisfy:

- unlimited increase of (k_t) occurs in the PG-economy if $a \geq a_1$ and also if $a < a_1$ when (k_t^0) increases and tends to ∞;
- convergence to a steady state $k^{**} > 0$ occurs if $a_0 < a < a_1$ and (k_t^0) is bounded, and if $a \leq a_0$ and (k_t^0) does not tend to 0;
- convergence to 0 only occurs when both (k_t^1) and (k_t^0) tend to 0.

Example. The example of section 2.4.3: $\sigma = 1, y(k) = A(k^{-1}+1)^{-1}, R(k) = y'(k) = A(1+k)^{-2}$ is studied in detail in the Appendix 2.A.13. The zero gift regime prevails if $k \geq a/(1-a)$ and the unconstrained regime prevails if $k \leq a/(1-a)$.

For intermediary values of $a, b_1 < a < b_2$[24], we have $k^*(a) > a/(1-a)$ and the equilibrium dynamics converge to k_2^*, the largest steady state of the NG-dynamics (see section 2.4.3 and figure 2.6). Both for low values of a ($a_0 < a < b_1$) and for large values of a ($b_2 < a < 1$), $k^*(a)$ is a

[24] b_1 and b_2 are the roots of $a(1-a) = B$, where $B = n(\beta^{-1}+1)/A$ is assumed smaller than 1/4 (see Appendix 2.A.13).

steady state of the PG-dynamics (the unconstrained regime applies in $k^*(a)$) and if $k_0 < k^*(a)$, the sequence (k_t) converges to $k^*(a)$. But for low values of a, there are three steady states: $k^*(a) < k_1^* < k_2^*$, and if $k_0 > k_1^*$, the sequence (k_t) converges to k_2^* (figure 2.8). The two curves defining the dynamics: $\tilde{\omega}(k_t) = \varphi(k_{t+1})$ are:

$$\text{if } k \leq \frac{a}{1-a},\ \tilde{\omega}(k) = \frac{aAk}{1+k} \text{ and } \varphi(k) = nk + n\beta^{-1}(1-a)k(1+k)$$
$$\text{if } k \geq \frac{a}{1-a},\ \tilde{\omega}(k) = \frac{Ak^2}{(1+k)^2} \text{ and } \varphi(k) = n(1+\beta^{-1})k$$

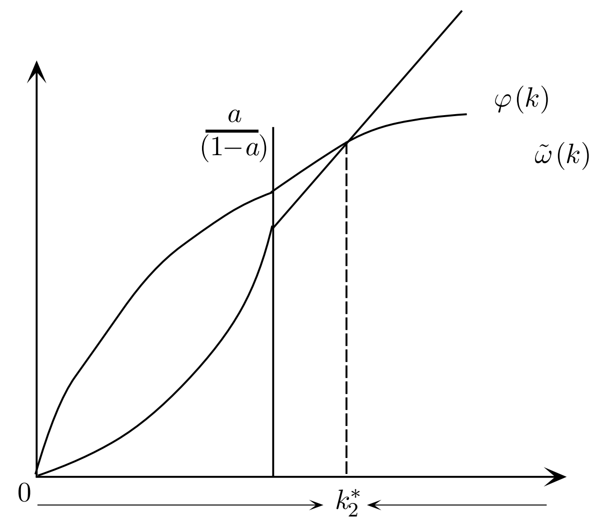

Figure 2.6 ($b_1 < a < b_2$)

In this example, if the generosity rate a is large enough, $a > b_1$, there exists a unique steady state of the dynamics in the PG-economy, and this steady state is globally stable (figures 2.6 and 2.7). But there exist three steady states if the generosity rate is low.

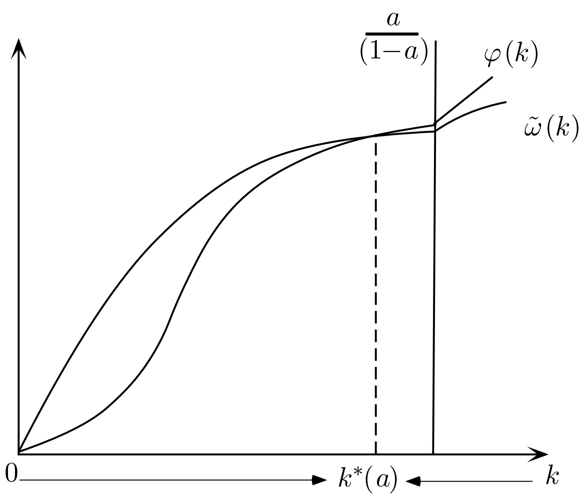

Figure 2.7 ($b_2 < a < 1$)

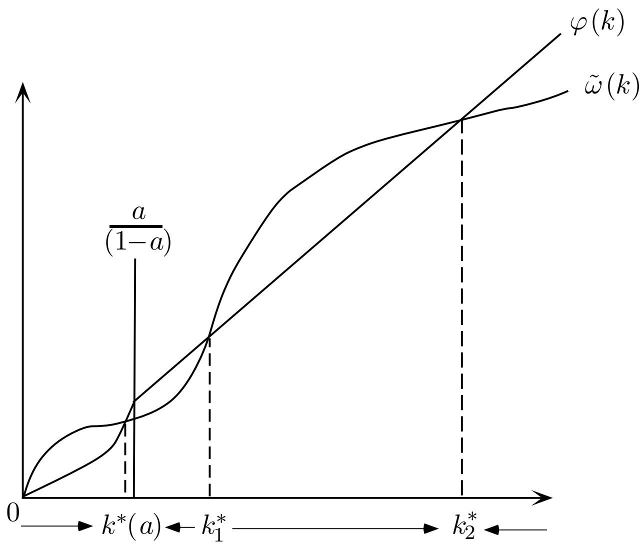

Figure 2.8 ($a_0 < a < b_1$)

APPENDIX of Section 2

2.A.1 The limits of $\tilde{\varphi}(k) = nk/\tilde{s}(R(k))$

- When k tends to ∞. By definition of $\tilde{s}(R) = \text{argmax}\ U(1-s, Rs)$, we have $\tilde{s}(R) \leq 1$, and thus $\tilde{\varphi}(k) \geq nk$. This implies $\tilde{\varphi}(\infty) = \infty$.

- When k tends to 0, we distinguish to possibilities.

  a) if $R(0)$ is finite, then $\tilde{s}(R(0))$ is positive and $\tilde{\varphi}(0) = 0$

  b) if $R(0) = \infty$, we have $\tilde{\varphi}(k) = \frac{nkR(k)}{R(k)\tilde{s}(R(k))} \leq \frac{ny(k)}{\tilde{d}(R(k))}$, and $\tilde{d}(R) = R\tilde{s}(R) = \text{argmax}\ U(1 - d/R, d)$. Assume that the second period consumption good d is non-Giffen good (its demand does not increase with its price $1/R$). Then $\tilde{d}(R)$ is non decreasing and its limit $\tilde{d}(\infty)$ is necessarily equal to ∞ (a finite limit $\overline{d}$ does not maximize the limit of the utility $U(1, \overline{d})$). This implies $\tilde{\varphi}(0) = \frac{ny(0)}{\tilde{d}(\infty)} = 0$.

2.A.2 Unlimited growth of k_t is excluded in the NG-economy when $R(k) = y'(k)$

(and also when $y'(k) \leq R(k)$ as in section 1.8.2). If $y'(k) \leq R(k)$, we have $w(k)/k = \hat{y}(k) - R(k) \leq \hat{y}(k) - y'(k)$. As seen in section 1.1.2 (and Appendix A.1.1), we also have $\hat{y}(\infty) = y'(\infty)$, and thus the limit of $w(k)/k$ when k tends to ∞ is equal to 0. This implies that for k large enough, say $k > \overline{k}$, we have: $w(k)/k < n$ and thus, if $k_t > \overline{k}$:

$$\frac{k_{t+1}}{k_t} \;=\; \frac{s_t}{nh_tk_t} \leq \frac{w(k_t)}{nk_t} < 1.$$

If for some t, $k_t > \overline{k}$, the monotonic sequence (k_t) is decreasing. In all cases this sequence is bounded by $\max\{k_0, \overline{k}\}$.

2.A.3 CE-preferences in the NG-economy

Consider $\sigma > 0, \sigma \neq 1$, and let $\lambda = 1 - 1/\sigma$. The following utility function

$$\tilde{U}(c,d) \;=\; (c^{\lambda} + \beta d^{\lambda})^{1/\lambda} \;=\; [\lambda(u(c) + \beta u(d))]^{1/\lambda}$$

where $u(c) = c^{\lambda}/\lambda$, defines the same preferences as $U(c,d) = u(c) + \beta u(d)$ since $(\lambda x)^{1/\lambda}$ is an increasing function of x. The utility function $\tilde{U}(c,d)$ is homogeneous of degree one and it defines homothetic preferences.

For $\sigma = 1$, with $u(c) = \ln c$, we have:

$$\tilde{U}(c, d) = c^{\alpha} d^{1-\alpha} = \exp[\alpha(\ln c + \beta \ln d)]$$

with $\alpha = 1/(1+\beta) > 0$ and the same conclusion holds.

Remark. With CE-preferences, we have $\tilde{s}(R) = 1/(1 + \beta^{-\sigma} R^{1-\sigma})$. Corresponding to a wage income ω and an interest factor R, consumptions are $c = (1 - \tilde{s}(R))\omega$ and $d = R\tilde{s}(R)\omega$, and they satisfy $d/c = \beta^{\sigma} R^{\sigma}$. The intertemporal elasticity of substitution, i.e. the elasticity of d/c with respect to R, is constant and equal to σ.

2.A.4 Study of the monotony of $\tilde{\varphi}(\cdot)$ in the NG-economy with CES-preferences and a CES production function.

With a CES production function and no externality, we have (Appendix 1.A.2):

$$R(k) = y'(k) = \alpha A(\alpha + (1-\alpha)k^{-\lambda})^{1/\lambda - 1} + \delta,$$

$\lambda < 1, \lambda \neq 0, A > 0, 0 < \alpha < 1$. The derivative of $kR(k)^{1-\sigma}$ has the same sign as

$$R(k) + k(1-\sigma)R'(k) = \delta + \alpha A(\alpha + (1-\alpha)k^{-1})^{1/\lambda - 2} B(k)$$

where

$$B(k) = \alpha + (1-\alpha)k^{-\lambda}(1 - (1-\sigma)(1-\lambda)).$$

We have $B(k) \geq 0$ for all $k > 0$ if and only if $(1-\sigma)(1-\lambda) \leq 1$, and this condition imposes a lower bound on $\sigma : \sigma \geq 1 - \frac{1}{1-\lambda} = 1 - \sigma_F$, when $\lambda < 0$ (the case of low substitutability $\sigma_F < 1$).

2.A.5 Solution of the optimization problem of the head in the FG-economy

Maximizing $U(\omega_t - s_t, V(R_{t+1}s_t - \nu x_{t+1}, w_{t+1}h_{t+1} + x_{t+1}))$ with respect to s_t and x_{t+1} leads to the F.O.C.: $U_1' = U_2' R_{t+1} V_1'$ and $\nu V_1' = V_2'$. The partial derivatives U_i' and V_i' are homogeneous of degree 0 and thus only depend on the ratio of their arguments.

We assume that the marginal rates for substitution $U_1'(x,1)/U_2'(x,1)$ and $V_1'(x,1)/V_2'(x,1)$ are decreasing functions of x, with limits ∞ (0) when x tends to 0 (∞). We denote by $\mu_U(\cdot)$ and $\mu_V(\cdot)$ the corresponding inverse functions.

Note that strict concavity of U with respect to each argument and homogeneity of degree one implies that $U_1'(x,1)$ is decreasing and $U_2'(x,1) = U_2'(1,1/x)$ is increasing. The limit properties result from the Inada conditions.

The F.O.C. $\nu V_1'(d,\omega) = V_2'(d,\omega)$ is equivalent to: $d/\omega = \mu_V(1/\nu)$, and with the family budget constraint $d + \nu\omega = \mathcal{W}$, we obtain constant proportions $d = (1-a)\mathcal{W}$ and $\nu\omega = a\mathcal{W}$, where $a = \nu/(\nu + \mu_V(1/\nu))$.

In addition, we have: $V_1'(d,\omega) = V_1'(1-a, a/\nu) = b_1$ (since V_1' is homogeneous of degree 0), and $V(d,\omega) = b_0\mathcal{W}$ with $b_0 = V(1-a, a/\nu)$, since V is homogeneous of degree 1. These conditions apply to each period $t \geq 0$, with constants b_1 and b_0. Then the F.O.C. $U_1' = U_2' R_{t+1} V_1' = b_1 R_{t+1} U_2'$ is equivalent to

$$c_t = \mu_U(b_1 R_{t+1}) V(d_{t+1}, \omega_{t+1}) = \rho(R_{t+1})\mathcal{W}_{t+1},$$

$\rho(R_{t+1}) = b_0 \mu_U(b_1 R_{t+1})$ is a decreasing function of R_{t+1} with limits $\rho(0) = \infty$ and $\rho(\infty) = 0$.

2.A.6 Economic growth with zero capital in the FG-economy (when $y(0) > 0$)

Only in the special case where $y(0) > 0$ and $a < n\rho(R(0))$, there exists $k > 0$ such that $y(k) < \zeta(0) = \frac{n}{a}\rho(R(0))y(0)$. This implies a finite value for $R(0)$, since $\rho(\infty) = 0$, and then $w(0) = y(0)$. Positive production with zero capital: $y(0) > 0$, finite interest factor $R(0)$ and an enough small generosity rate a may lead to the desired choice by the head of the family of *negative savings*. Indeed, we have if $y(k_t) < \zeta(0)$, for any $k_{t+1} \geq 0$:

$$s_t = \omega_t - c_t = [ay(k_t) - n\rho(R(k_{t+1}))y(k_{t+1})]h_t < 0.$$

Since $s_t < 0$ is impossible, we have to add the constraint $s_t \geq 0$, and then optimal savings is $s_t = 0$ and $k_{t+1} = 0$. The intuition of this possibility is clear: with positive wages $w(0) = y(0) > 0$, finite $R(0)$ and low a, the head of the family want to consume more than $ay(k_t)$ in his first period of life. The constraint $s_t \geq 0$ then implies $c_t = ay(k_t)$ and $k_{t+1} = 0$.

Then the dynamics of k_t in the FG-economy are:

$$y(k_t) = \zeta(k_{t+1}) \text{ if } y(k_t) > \zeta(0) \text{ and } k_{t+1} = 0 \text{ if } y(k_t) \leq \zeta(0).$$

Note that, when k_t takes the value 0 in finite time, the economy continues to grow at the natural rate with

$$c_t = \omega_t = ay(0)h_t, s_t = 0, d_t = (1-a)\nu y(0)h_t, \text{ and } \frac{Y_{t+1}}{Y_t} = \frac{L_{t+1}y(0)}{L_t y(0)} = n$$

because production is possible with zero capital, $y(0) > 0$.

2.A.7 Study of the limits of $\psi(k) = \frac{n}{a}(\rho(R(k)) + 1/\hat{y}(k))$

We have $\rho(0) = \infty$ and $\rho(\infty) = 0$ (Appendix 2.A.5). We also have:

$$\psi(0) = \frac{n}{a}(\rho(R(0)) + \frac{1}{\hat{y}(0)}) \text{ and } \psi(\infty) = \frac{n}{a}(\rho(R(\infty)) + \frac{1}{\hat{y}(\infty)}).$$

The limit $\psi(0)$ is equal to 0 if and only if $R(0) = \infty$. Assume $\psi(0) = 0$, then necessarily both $\rho(R(0))$ and $1/\hat{y}(0)$ are equal to 0, $\rho(R(0)) = 0$ implies $R(0) = \infty$. Assume $R(0) = \infty$, then $\rho(R(0)) = 0$ and, since $\hat{y}(k) > R(k)$ for all $k > 0$, we also have $\hat{y}(0) = \infty$ and thus $\psi(0) = 0$. The limit $\psi(\infty)$ is equal to ∞ if and only if $R(\infty) = 0$. Assume $R(\infty) = 0$, then $\rho(R(\infty)) = \infty$ and $\psi(\infty) = \infty$. Assume $\psi(\infty) = \infty$, then either $\rho(R(\infty)) = \infty$ and $R(\infty) = 0$, or $\hat{y}(\infty) = 0$ and also $R(\infty) = 0$ since $R(k) < \hat{y}(k)$ for all $k > 0$.

2.A.8 The effects of the generosity rate on the dynamics in the FG-economy with CES-preferences

The limit $\tilde{\psi}(0, a)$ is equal to $\frac{n(1-a)}{a}\beta^{-\sigma}R(0)^{-\sigma} + \frac{n}{a}\hat{y}(0)^{-1}$. This limit is positive if and only if $R(0)$ is finite (we have: $\hat{y}(0) \geq R(0)$), and then, the solution a_0 of $\tilde{\psi}(0, a_0) = 1$ is equal to $(n\beta^{-\sigma}R(0)^{-\sigma} + n\hat{y}(0)^{-1})/(1 + n\beta^{-\sigma}R(0)^{-\sigma})$. If $R(0) = \infty$, we set $a_0 = 0$.

The limit $\tilde{\psi}(\infty, a)$ is equal to $\frac{n(1-a)}{a}\beta^{-\sigma}R(\infty)^{-\sigma} + \frac{n}{a}\hat{y}(\infty)^{-1}$. This limit is finite if and only if $R(\infty) > 0$ (since $\hat{y}(\infty) \geq R(\infty)$) and then the solution a_1 of $\tilde{\psi}(\infty, a_1) = 1$ is equal to $(n\beta^{-\sigma}R(\infty)^{-\sigma} + n\hat{y}(\infty)^{-1})/(1 + n\beta^{-\sigma}R(\infty)^{-\sigma})$. If $R(\infty) = 0$, we set $a_1 = 1$.

The condition: $a_1 < 1$ is equivalent to $R(\infty) > 0$ and $\hat{y}(\infty) > n$. Note that, if $R(\infty) = y'(\infty) = \hat{y}(\infty)$, this condition is also equivalent to the condition of unlimited growth of (k_t) in the Solow dynamics ($s\hat{y}(\infty) > n$ for some $s < 1$).

2.A.9 Study of the zero gift regime in the PG-economy

We use the notations and the results of Appendix 2.A.5. The F.O.C. $U_1' = U_2' V_1' R_{t+1}$ is equivalent to $c_t = \mu_U(V_1' R_{t+1}) V(d_{t+1}, \omega_{t+1})$, and we have:

$$\begin{aligned} V_1'(d_{t+1}, \omega_{t+1}) &= V_1'(\nu R(k_{t+1}) k_{t+1}/w(k_{t+1}), 1) \equiv v_1(k_{t+1}) \\ V(d_{t+1}, \omega_{t+1}) &= V(\nu R(k_{t+1}) k_{t+1}, w(k_{t+1})) h_{t+1} \equiv v(k_{t+1}) h_{t+1} \end{aligned}$$

Thus $c_t = \mu_U(v_1(k_{t+1}) R(k_{t+1})) v(k_{t+1}) h_{t+1} \equiv \rho_0(k_{t+1}) v(k_{t+1}) h_{t+1}$. The other F.O.C.: $V_2' \leq \nu V_1'$ is equivalent to $d_{t+1}/\omega_{t+1} \leq \mu_V(1/\nu)$ because $\mu_V(\cdot)$, the inverse function of $V_1'(x,1)/V_2'(x,1)$, is decreasing. By definition of $a = \nu/(\nu + \mu_V(1/\nu))$, we have: $\mu_V(1/\nu) = \nu(1-a)/a$. With zero gift, we have at equilibrium:

$$\frac{d_{t+1}}{\omega_{t+1}} = \frac{\nu R(k_{t+1}) k_{t+1}}{w(k_{t+1})} = \nu \Big(\frac{y(k_{t+1}) - w(k_{t+1})}{w(k_{t+1})} \Big).$$

Thus the F.O.C.: $V_2' \leq \nu V_1'$ is equivalent to $a y(k_{t+1}) \leq w(k_{t+1})$.

Consider now $k > 0$ such that $a y(k) < w(k)$. We have:

$$\frac{\nu R(k) k}{w(k)} = \frac{\nu(y(k) - w(k))}{w(k)} < \frac{\nu(1-a)}{a} \text{ and } v_1(k) > V_1'(\frac{\nu(1-a)}{a}, 1) = b_1,$$

since $V_1'(x,1)$ is decreasing and homogeneous of degree 0. Thus: $\rho_0(k) < \mu_U(b_1 R(k))$ since μ_U is decreasing.

We also have $v(k) = V(\nu R(k) k, w(k)) \leq V(1-a)\nu y(k), a y(k)) = b_0 \nu y(k)$, because this last term is the maximum of $V(d, \omega)$ subject to: $d + \nu\omega = \nu y(k)$. These properties imply: $\rho_0(k) v(k) < b_0 \nu y(k) \mu_U(b_1 R(k)) \equiv \nu y(k) \rho(k)$, and

$$\varphi_0(k) = \eta \rho_0(k) v(k) + nk < n\rho(k) y(k) + nk = \varphi_1(k).$$

Assume now: $a y(k) = w(k)$, then all the preceding inequalities are satisfied as equalities and we have: $\varphi_0(k) = \varphi_1(k)$.

2.A.10 Equilibrium with zero capital-labor ratio in the PG-economy

Let $(k_t^1)_{t\geq 0}$ be the equilibrium dynamics in the FG-economy starting at $k_0^1 = k_0 > 0$. We study the case in which $ay(k_t^1) \leq \varphi_1(0)$ which implies $k_{t+1}^1 = 0$ (assuming a non-negativity constraint on savings, Appendix 2.A.6). This may happen only if $y(0) > 0$ and $R(0)$ is finite which implies $w(0) = y(0)$. Now consider the equilibrium in the PG-economy and assume $k_t \geq k_t^1$.

Since $w(0) = y(0) > ay(0)$, the zero gift regime prevails and we have: $\varphi(0) = \varphi_0(0) = 0$. Indeed, the conclusion of Appendix 2.A.1 applies to the utility function $U(c, V(d, w(0)))$ in the case $R(0)$ finite. This implies $\varphi(0) = 0 < \tilde{\omega}(k_t)$ and there exists $k_{t+1} > 0$ solution of $\varphi(k_{t+1} = \tilde{\omega}(k_t)$.

2.A.11 Equilibrium dynamics in the PG-economy

Assuming that $\varphi(\cdot)$ is increasing, these dynamics satisfying: $\tilde{\omega}(k_t) = \varphi(k_{t+1})$ are monotonic, and converge either to a steady state k^{**} or to ∞ or to 0.

When the FG-dynamics converge to $k^* > 0$ solution of $ay(k^*) = \varphi_1(k^*)$ and $k_0 < k^*$, then k_t is necessarily increasing (since $k_t \geq k_t^1, \forall t$). Either it converges to the smallest steady state $k^{**} \geq k^*$ or it tends to ∞ (Appendix 1.A.3). We study the dynamics for $k_t > k^*$.

- Consider $k_t > k^*$ and assume that the unconstrained regime prevails in period t, i.e. $\tilde{\omega}(k_t) = ay(k_t)$ and $\varphi(k_t) = \varphi_1(k_t)$, then we have: $k_{t+1} < k_t$. Indeed, k_{t+1}^2 solution of $\varphi_1(k_{t+1}^2) = ay(k_t)$ satisfies: $k_{t+1}^2 < k_t$ (the FG-dynamics starting at $k_t > k^*$ decreases). Then we have: $\varphi(k_{t+1}) = \tilde{\omega}(k_t) = ay(k_t) = \varphi_1(k_{t+1}^2) < \varphi_1(k_t) = \varphi(k_t)$, and $k_{t+1} < k_t$.

  As a consequence, if (k_t) increases and $k_{t_0} > k^*$, then necessarily the *zero-gift regime applies in all periods* $t \geq t_0$ and the PG-dynamics satisfy: $w(k_t) = \varphi_0(k_{t+1})$ for all $t \geq t_0$.

- In general, when k_t becomes larger than k^* (i.e. $ay(k^*) < w(k^*)$), the sequence (k_t) converge to k^{**} which is the smallest steady state greater than k^* of the dynamics with zero gift: $w(k_t) = \varphi_0(k_{t+1})$. This is necessarily the case in a competitive economy without externality in which $w(k)/\varphi_0(k) < w(k)/(nk)$ tends to 0 when k tends to ∞ (section 1.4.1).

2.A.12 Study of the regimes in the PG-economy with a CES production function

We consider $y(k) = A(\alpha k^\lambda + 1 - \alpha)^{1/\lambda}, \lambda < 1, \lambda \neq 0$. The elasticity of $y(\cdot)$ is:

$$\varepsilon(k) \;=\; \frac{ky'(k)}{y(k)} \;=\; \frac{\alpha k^\lambda}{\alpha k^\lambda + 1 - \alpha}.$$

If $\lambda > 0$ (high substitution: $\sigma_F = \frac{1}{1-\lambda} > 1), \varepsilon(k)$ increases from 0 to 1 when k increases from 0 to ∞. Then the regime with zero gift, $\varepsilon(k) \le 1 - a$, holds if and only if $k \le \hat{k}_a$, where $\hat{k}_a$ satisfies $\varepsilon(\hat{k}_a) = 1 - a$.

If $\lambda < 0$ (low substitution: $\sigma_F < 1$), $\varepsilon(k)$ decreases from 1 to 0 when k increases from 0 to ∞. Then the regime with zero gift holds if and only if $k \ge \hat{k}_a$.

In both cases, along a monotonic equilibrium dynamics (k_t), there is no more than one change of regime.

When the depreciation of capital is not total: $\delta > 0$, there may be more than one change of regime. Consider $y(k) = A(k^{-1} + 1)^{-1} + \delta k$. Then we have:

$$\varepsilon(k) \;=\; \frac{A + \delta(1+k)^2}{A(1+k) + \delta(1+k)^2} \quad \text{and} \quad \varepsilon(0) = 1 = \varepsilon(\infty).$$

The derivative $\varepsilon'(k)$ has the same sign as $\delta k^2 - \delta - A$ which is negative for $k < \overline{k}$ and positive for $k > \overline{k}$, where $\overline{k} = (1 + A/\delta)^{1/2}$. The minimum of $\varepsilon(k)$ is $\varepsilon(\overline{k}) = 2/(1+\overline{k})$. If $\varepsilon(\overline{k}) \ge 1 - a$, the unconstrained regime prevails and there is non change of regime. If $\varepsilon(\overline{k}) < 1 - a$, there is an interval on which the zero gift regime prevails (for $\varepsilon(k) \le 1 - a$) and there are two changes of regime.

2.A.13 A simple example of dynamics in the PG-economy

The example: $\sigma = 1, y(k) = Ak/(1+k), R(k) = y'(k) = A/(1+k)^2$ has been studied in the NG-economy (section 2.4.3). If $B = n(1+\beta^{-1})/A < 1/4$, then there are two positive steady states: $k_1^* < k_2^*$, the two roots of $B(1+k)^2 - k = 0$. In the FG-economy, we have for this example

$$\tilde{\psi}(k, a) \;=\; \frac{n(1-a)}{a\beta A}(1+k)^2 + \frac{n}{aA}(1+k).$$

The solution a_0 of $\tilde{\psi}(0, a) = 1$ is: $a_0 = n(1+\beta)/(n + \beta A)$ and it satisfies: $a_0 < B$. We have $a_1 = 1$ since $\tilde{\psi}(\infty, a) = \infty$. For $a_0 < a < 1$, the FG-dynamics converge monotonically to the steady state $k^*(a)$ solution of $\tilde{\psi}(k^*(a), a) = 1$ (figure 2.4 in section 2.6.3).

In the PG-economy, the condition of zero gift: $ky'(k)/y(k) \leq 1-a$ is equivalent to $k \geq a/(1-a)$. For $k = k^*(a)$ this condition is also equivalent to $\tilde{\psi}(a/(1-a), a) \leq 1$, since $\tilde{\psi}(k,a)$ is increasing in k. By substitution, we obtain:

$$k^*(a) \geq \frac{a}{1-a} \;\Leftrightarrow\; a(1-a) \geq \frac{n(1+\beta^{-1})}{A} = B.$$

Assuming $B < 1/4$, this condition is equivalent to $a \in [b_1, b_2]$, where $b_1 = 1/2 - (1/4 - B)^{1/2}$ and $b_2 = 1/2 + (1/4 - B)^{1/2}$ are the 2 roots of $a(1-a) = B$. Note that $B < b_1$. In addition, we have: $k^*(b_i) = b_i/(1-b_i)$ and $B(1+k^*(b_i))^2 - k^*(b_i) = 0$, for $i = 1, 2$. Thus $k^*(b_1) = k_1^*$ and $k^*(b_2) = k_2^*$ are the steady states of the NG-dynamics.

In the PG-economy, the unconstrained regime prevails for $k \leq a/(1-a)$ and the zero gift prevails for $k \geq a/(1-a)$. We have in the PG-economy:

- if $a_0 < a < b_1$, there are three steady states: $k^*(a) < a/(1-a) < k_1^* < k_2^*$
- if $b_1 < a < b_2$, there is only one steady state k_2^*, since $k^*(a) > \frac{a}{1-a}$ and $k_1^* < \frac{a}{1-a}$ are excluded
- if $b_2 < a < 1$, there is only one steady state $k^*(a) < \frac{a}{1-a}$ since $k_1^* < k_2^* < \frac{a}{1-a}$.

The three type of dynamics are represented on figures 2.6, 2.7, 2.8.

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