Ideally Exact Categories, Varieties of Universal Algebras and Multi-Valued Logics

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Definition

An **action** of a group G on a group X is a group homomorphism $\phi\colon G \to \mathrm{Aut}(X)$.

- There is an equivalence between the category of actions of G and the category of split epimorphisms over G.
- The equivalence is established via semidirect products.

In 2005, F. Borceux, G. Janelidze and G. M. Kelly extended the correspondence between **actions** and **split extensions** from the setting of groups to more general **semi-abelian categories**.

Definition (G. Janelidze, L. Márki, W. Tholen, 2002)

A category $\mathcal{C}$ is **semi-abelian** if it is pointed, Barr-exact, Bourn-protomodular with finite coproducts.

- **Examples:** Rng, Assoc, Lie, Hoops, any abelian category,...
- An **internal action** is a $B\flat(-)$-algebra (X, ξ), where $\xi \colon B\flat X \to X$ and

$$B\flat X = \ker(\, B + X \xrightarrow{[1,0]} B \,).$$

Non-pointed categories

Problem: Non-pointed categories *do not* have kernels, but we can see them in a *bigger* category.

Definition (S. Lapenta, G. Metere, L. Spada, 2024)

- A **basic setting** for relative U-ideals is an adjunction $\begin{tikzcd} \mathrm{U} \arrow[r, "U"'] \arrow[r, phantom, "\perp", shift left=2] & \mathrm{V} \arrow[l, bend right, "F"'] \end{tikzcd}$, where V is homological and the functor U is conservative and faithful.
- A morphism $k\colon X \to U(A)$ in V is a **relative U-ideal** of an object A in U if

$$\exists \begin{array}{c} A \\ \downarrow f \\ B \end{array} \text{ in U such that } \begin{tikzcd} X \arrow[r, "k"] \arrow[d] & U(A) \arrow[d, "U(f)"] \\ 0 \arrow[r] & U(B) \end{tikzcd} \text{ is a pullback.}$$

- **The varietal case:** $\mathrm{V} = (\mathrm{V}, \Sigma_{\mathrm{V}}, Z_{\mathrm{V}})$, $\mathrm{U} = (\mathrm{U}, \Sigma_{\mathrm{U}}, Z_{\mathrm{U}})$ algebraic varieties, V homological (and hence semi-abelian), $\Sigma_{\mathrm{V}} \subseteq \Sigma_{\mathrm{U}}$, $Z_{\mathrm{V}} \subseteq Z_{\mathrm{U}}$ and $U\colon \mathrm{U} \to \mathrm{V}$ forgetful functor.

Definition (G. Janelidze, 2024)

A category $\mathbb{U}$ is **ideally exact** if it is Barr exact, protomodular, has finite coproducts and the unique morphism $0 \to 1$ is a regular epimorphism.

Theorem (G. Janelidze, 2024)

$\mathbb{U}$ is ideally exact if and only if it is Barr exact, has finite coproducts and there is a monadic functor $U\colon \mathbb{U} \to \mathbb{V}$, with $\mathbb{V}$ semi-abelian.

- If $\mathbb{U}$ is ideally exact, the pullback functor $\mathbb{U} \to (\mathbb{U} \downarrow 0)$ along $0 \to 1$ is monadic.
- A monadic adjunction $\begin{tikzcd} \mathbb{U} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbb{V} \arrow[l, bend right, "F"'] \end{tikzcd}$, with $\mathbb{U}$ ideally exact and $\mathbb{V}$ semi-abelian, is associated to $0 \to 1$ if and only if the unit of the adjunction is cartesian.
- A non-trivial algebraic variety is ideally exact if and only if it is protomodular.

▷ **Ideally exact context**: a monadic adjunction with cartesian unit $F \dashv U$, with $\mathbb{U}$ ideally exact and $\mathbb{V}$ semi-abelian.

Wajsberg hoops and MV-algebras

A **hoop** is an algebra $(A, \cdot, \to, 1)$ such that

- $x \to x = 1$;
- $x \cdot (x \to y) = y \cdot (y \to x)$;
- $x \cdot y \to z = x \to (y \to z)$.

A **Wajsberg hoop** is a hoop such that

- $(x \to y) \to y = (y \to x) \to x$.

A **bounded hoop** is a hoop with a constant 0 such that

- $0 \to x = 1$.

- Bounded Wajsberg hoops (i.e., Wajsberg algebras) are term equivalent to **MV-algebras**.
- In WAlg, the initial object is $L_2 = \{0, 1\}$, and the terminal object is $\{1\}$.

- The varieties Hoops and WHoops are semi-abelian categories.
- The variety WAlg $\simeq$ MVAlg is an ideally exact category.

Consider the adjunction $\begin{tikzcd} \mathrm{MVAlg} \arrow[r, "U"', ""{name=D}] & \mathrm{WHoops} \arrow[l, bend right, "M"', ""{name=C}] \arrow[from=C, to=D, phantom, "\perp"] \end{tikzcd}$, where:

- U is the functor that forgets the constant 0.
- M is the **MV-closure**, i.e., it maps every Wajsberg hoop W to the MV-algebra

$$M(W) = (W \times \{0,1\}, \cdot, \to, 0, 1),$$

where $0 := (1,0)$, $1 := (1,1)$,

$$(a,i)\cdot(b,j) = \begin{cases} (a\cdot b, 1), & \text{if } i=j=1,\\ (a\to b, 0), & \text{if } i=1,\ j=0,\\ (b\to a, 0), & \text{if } i=0,\ j=1,\\ ((a\to(a\cdot b))\to b, 0), & \text{if } i=j=0. \end{cases} \qquad (a,i)\to(b,j) = \begin{cases} (a\to b, 1), & \text{if } i=j=1,\\ (a\cdot b, 0), & \text{if } i=1,\ j=0,\\ ((a\to(a\cdot b))\to b, 1), & \text{if } i=0,\ j=1,\\ (b\to a, 1), & \text{if } i=j=0. \end{cases}$$

▻ The unit $\eta\colon 1_{\mathrm{WHoops}} \Rightarrow UM$ is cartesian, and hence the adjunction is associated with the unique map $L_2 \to \{1\}$.

▻ This gives an ideally exact context.

Coherent and ideal actions

Definition (M. M., G. Metere, F. Piazza, 2026)

A split epimorphism $A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$ in V is **ideal** if there exists a split epimorphism $A' \underset{s'}{\overset{p'}{\rightleftarrows}} B$ in U, and an isomorphism $\sigma\colon U(A') \to A$ that induces an isomorphism of points.

Definition (M. M., G. Metere, F. Piazza, 2026)

Let $\xi\colon U(B)\flat X \to X$ be the internal action associated with a split epimorphism $A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$, and let $\xi_0\colon UF(0)\flat X \to X$ be the internal action associated with the canonical split epimorphism $UF(X) \underset{UF(\iota)}{\overset{UF(\tau)}{\rightleftarrows}} UF(0)$. We say that:

- ξ is a **coherent action** if $\xi \circ (U(\iota_B)\flat\, \mathrm{id}_X) = \xi_0$.
- ξ is an **ideal action** if the corresponding split epimorphism is ideal.

Lemma (M. M., G. Metere, F. Piazza, 2026)

An internal action ξ is coherent if and only if $\exists\, f : UF(X) \to A$ in V such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, equal] & UF(X) \arrow[r, shift left, "UF(\tau_X)"] \arrow[d, "f"'] & UF(0) \arrow[l, shift left, "UF(\iota_X)"] \arrow[d, "U(\iota_B)"] \\
X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & U(B). \arrow[l, shift left, "s"]
\end{tikzcd}$$

is a morphism of split extensions in V. Moreover, the square on the right is a split pullback.

Theorem (M. M., G. Metere, F. Piazza, 2026)

In any ideally exact context, all ideal actions are coherent.

Definition (M. M., G. Metere, F. Piazza, 2026)

An ideally exact context has a **good theory of actions** (or it is **BAT**) if all coherent actions are ideal, and all morphisms of such actions are ideal.

Theorem (M. M., G. Metere, F. Piazza, 2026)

The following ideally exact contexts are BAT:

- $\begin{tikzcd} \mathrm{MVAlg} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathrm{WHoops} \arrow[l, bend right=40, "M"'] \end{tikzcd}$;
- $\begin{tikzcd} \mathrm{PAlg} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathrm{PHoops} \arrow[l, bend right=40, "K"'] \end{tikzcd}$;
- $\begin{tikzcd} \mathrm{Ring} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathrm{Rng} \arrow[l, bend right=40, "F"'] \end{tikzcd}$;
- $\begin{tikzcd} \mathrm{V}_1 \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathrm{V} \arrow[l, bend right=40, "F"'] \end{tikzcd}$, where V is a unit-closed variety of non-associative algebras over a field.

- Let U (resp. V) be a non-trivial variety of universal algebras with signature Σ_{U} (resp. Σ_{V}) defined by a set Z_{U} (resp. Z_{V}) of identities.
- V is semi-abelian with a unique constant 0.
- $\Sigma_{\mathsf{V}} \subseteq \Sigma_{\mathsf{U}}$, $Z_{\mathsf{V}} \subseteq Z_{\mathsf{U}}$ and $\Sigma_{\mathsf{U}} \backslash \Sigma_{\mathsf{V}}$ is a set of nullary operations.
- $\Gamma := I_{\mathsf{U}} \backslash 0_{\mathsf{V}} = I_{\mathsf{U}} \backslash T_{\mathsf{U}}$ is the set of constants in U which do not belong to V.
- $U\colon \mathsf{U} \to \mathsf{V}$ is the functor which forgets all the constants in Γ as well as all the identities involving such constants.
- There is a free-forgetful adjunction

$$\begin{tikzcd} \mathsf{U} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{V} \arrow[l, bend right, "F"'] \end{tikzcd} \tag{1}$$

which defines an ideally exact context if and only if the unit $\eta\colon 1_{\mathsf{V}} \Rightarrow UF$ is cartesian.

Theorem (M. M., F. Piazza, 2026)

Let (1) be an ideally exact context, and let

$$\xi\colon U(B)\flat X \to X$$

be a coherent relative U-action with associated split epimorphism

$$A \underset{s}{\overset{p}{\rightleftarrows}} U(B).$$

Then, ξ is an ideal action if and only if

$$\omega_A = \omega(\omega_A, \dots, \omega_A, s(c_1), \dots, s(c_q)), \tag{2}$$

for every identity $\omega = 0$ in $Z_{\mathrm{U}} \backslash Z_{\mathrm{V}}$, and for every $a_1, \dots, a_p \in A$, where

$$\omega_A = \omega(a_1, \dots, a_p, s(c_1), \dots, s(c_q)) \in A.$$

Main result

Theorem (M. M., F. Piazza, 2026)

The ideally exact context

$$\begin{tikzcd} \mathsf{U} \arrow[r, "U"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathsf{V} \arrow[l, bend right, "F"'] \end{tikzcd}$$

is BAT if and only if condition (2) holds for any coherent internal action and for any identity $\omega = 0$ in $Z_{\mathsf{U}} \backslash Z_{\mathsf{V}}$.

Example

Let U = MVAlg and let V = WHoops. The unique identity which involves the constant 0 is $\omega(a, 0) = 0 \to a = 1$. Then, condition (2) reduces to

$$s(0) \to (s(0) \to a) = s(0) \to a,$$

which holds since $a \to (b \to c) = (a \cdot b) \to c$.

Example

Let U = Assoc_1 and let V = Assoc. The only identities which involve the constant 1 are $\omega_1(a, 1) = a - a \cdot 1 = 0$ and $\omega_2(a, 1) = a - 1 \cdot a = 0$. Then, one has

$$\omega_i(\omega_i(a, s(1)), s(1)) = \omega_i(a, s(1)), \quad \forall i = 1, 2.$$

Open Problem

It is not known whether the units of the adjunctions

$$\begin{tikzcd} \mathsf{GAlg} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathsf{GHoops} \arrow[l, bend right=40, "G"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{BLAlg} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathsf{BHoops} \arrow[l, bend right=40, "B"'] \end{tikzcd}$$

are cartesian.

It remains open whether these adjunctions define ideally exact contexts.

If they do, then they are BAT by the characterization above.

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- M. Mancini and F. Piazza. *Ideally Exact Categories, Varieties of Universal Algebras and Multi-Valued Logics*. In: B. Vantaggi et al. (eds), Information Processing and Management of Uncertainty in Knowledge-Based Systems, volume 2. **IPMU 2026**. Communications in Computer and Information Science **3020** (2026), no. 19. Springer, Cham.