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Assessing geometrically the structural safety of masonry arches

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ABSTRACT: Equilibrium, as investigated by many historical theories, is the major concern when studying the behaviour of masonry structures. These existing studies aim at assessing stability and safety through the use of construction rules, which link an arch's radius to its minimum thickness. The first part of the present paper puts forward the main views and methods in this field throughout history. In the second part, a geometrical method is developed in order to assess the structural safety based on graphic statics and admissible geometrical domains. These domains provide graphical information on the minimum and maximum thrusts a given masonry arch can sustain, as well as on its robustness. The different historic rules for masonry design are then reinterpreted and compared taking into account this geometrical approach.

Keywords: Arch Theory, Masonry, Thrust Line, Limit Analysis, Safety Factor, Geometrical Domains

1 INTRODUCTION

When assessing the structural response of an existing building, an engineer may focus on three main requirements: stability, strength, and stiffness (Heyman 2008). The two first requirements are linked to structural safety and the third to the serviceability of the building. Concerning historical masonry structures, as the mean acting stresses are typically relatively low (strength requirement), local crushing of the material is hardly affecting the global integrity (Heyman 1995). In the same vein, working deflections are usually not occurring (stiffness requirement) because the sections are generally large and a rigid material behaviour can be considered. For masonry arches, the focus can then be exclusively put on ultimate limit states related to equilibrium issues (stability requirement). This means avoiding a loss of static equilibrium within the global structure, as well as within any part of it considered as a rigid body.

As it is developed in the Eurocodes (EN 1990), the most common approach for evaluating the structural response is to consider the structure in terms of limit states, intended as states beyond which the structure no longer fulfils the relevant design criteria. This can be formulated as:

$$E_{d,dst} = E_{dst}.\gamma_{dst} \leq E_{d,stb} = \frac{E_{stb}}{\gamma_{stb}} \quad (1)$$

where E_{dst} = characteristic value related to the effects of destabilizing actions; and E_{stb} = characteristic value for stabilizing actions. Design values for these actions (with d indices) are obtained by applying the corresponding safety factors (γ_i).

Amongst these fundamental requirements, it is also mentioned that a structure must be robust, which means this structure has to "withstand events like fire, explosions, impact or consequences of human error, without being damaged to an extent disproportionate to the original cause" (EN 1991–1-7). In practice, there are several types of numerical approaches aiming to assess the level of robustness of existing structures (Starossek 2011).

2 OBJECTIVES AND PLAN

The present paper aims at discussing the theories about the safety of masonry arches. Based on Méry's thrust line approach, it also presents a new way of defining the structural safety based on admissible geometrical domains. These are implemented in a parametric model based on the reciprocal diagrams of graphic statics. Their application to a case study—a semi-circular masonry arch—helps the comparison with the classical geometric safety factor as defined by Heyman (1995).

Section 3 first exposes different approaches that have prevailed in structural theory for masonry arches. These modes of thinking are detailed regarding the historical contributions of various authors. The different attempts for the assessment of the structural behaviour of masonry arches as well as the conditions for their stability are also reviewed.

Section 4 introduces a method for safety assessment, based on graphic statics and admissible geometrical domains. This method is then applied to some canonical arch geometries. The obtained safety indicators are finally compared to the ones given by the classical geometric approach.

3 STRUCTURAL SAFETY THROUGH HISTORY

3.1 *Duality in structural theory*

Historically, structural theory has been composed of two kinds of approaches. Each of them is based on a specific basic principle, as identified by Timoshenko & Young (1945): the principle of virtual works on one side, and the parallelogram principle on the other. The first kind of approaches refers to the so-called kinematic methods based on the first law of thermodynamics (conservation of energy). It leads to the analysis of kinematic collapse mechanisms in instable equilibrium; this is what Karl-Eugen Kurrer (2008) calls the “kinematic view of statics”. He identifies the second type of approaches as the “geometric view of statics”. This approach focuses on the equilibrium conditions of resulting forces acting on elements, considering real rather than potential static conditions. This duality is illustrated in Figure 1 by the two methods that exist to determine the value of the bending moment in the middle span of a statically determinate beam. In the first one (Fig. 1, top), the external work of force F in the cinematically admissible mechanism is set equal to the internal work of the bending moments of the same mechanism.In the second one (Fig. 1, bottom), the equilibrium conditions of the applied force F are expressed through the corresponding force polygon, leading to the same value for the bending moment.

3.2 *The theory of masonry arches*

When dealing with masonry arches – ‘still one of the mysteries of architecture’ (Kurrer 2008) – this duality between kinematic and geometric approaches can be recognized amongst the historical development of the related theories. From the first considerations about masonry arches equilibrium, to most recent developments in structural analysis and numerical methods (such as the distinct element modelling), these two approaches have been used alternatively depending on the philosophical or scientific context of the time.

The kinematic approach of statics has prevailed until the eighteenth century due to Aristotle’s tradition of simple machines. Indeed, masonry arches were mainly conceived like kinematic assemblies of rigid bodies in instable equilibrium, for which collapse could occur due to specific displacement conditions, namely at the abutments. For instance, Leonardo da Vinci proposed to consider each single stone of the arch as a wedge (Benvenuto 1991). He then developed a simple measuring arrangement made of ropes and pulleys to characterize the horizontal thrust effect at the springings of the arch (Fig. 2).

Bernardino Baldi (1621) was the first to propose a mechanical approach. He divided the arch into three equal rigid bodies and assumed that collapse would occur for a displacement at the abutments which would transform the central *voussoir* into a three-hinge mechanism (Fig. 3). Authors like Smars (2002) recently reused this theory by developing domains of kinematically admissible displacements.

After Baldi’s approach, Philippe de la Hire proposed what is now referred as a kinematic ultimate load theory, in which he considered the arch and its abutments as a four-hinge mechanism in equilibrium (Kurrer 2008). His first reasoning was based on the wedge and cranked lever mechanical principles, inherited from the five simple machines tradition. He then corrected his theory in 1712, introducing the idea that friction in the joints is such that sliding cannot occur (Heyman 1998). The goal was to understand how to design

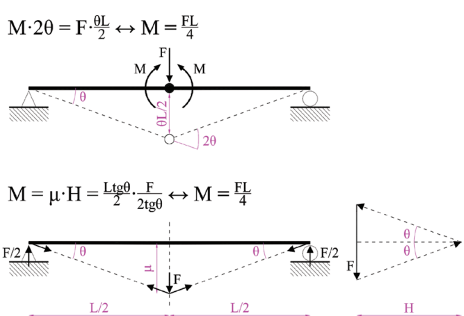

Figure 1. Bending moments calculated using both kinematic and geometric methods (Rondeaux et al. 2018).

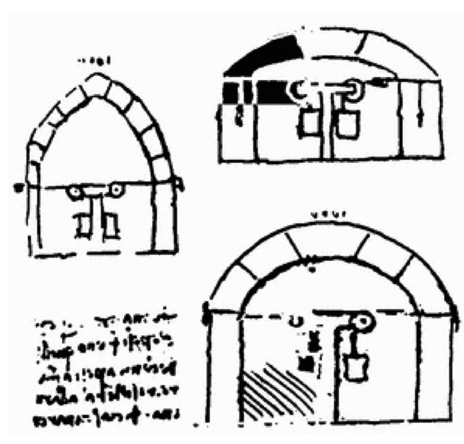

Figure 2. Da Vinci’s intimations for an experimental estimate of the thrust on the abutment piers (Benvenuto 1991).

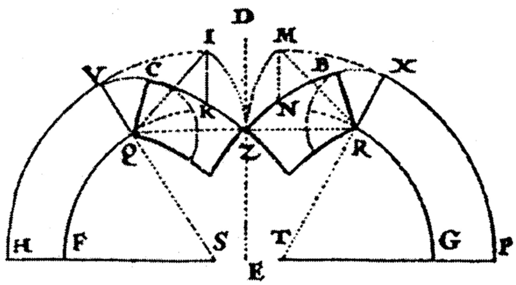

Figure 3. Baldi's mechanical treatment of arch stability (1621).

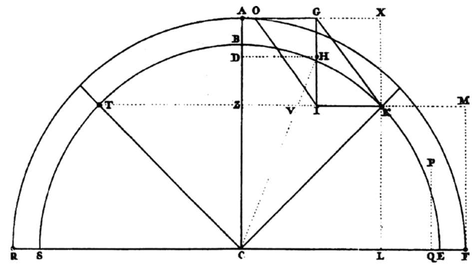

Figure 4. Bélidor's model for assessing arch stability (1729).

the abutments in order to ensure the stable behaviour of the arch. Bélidor (1729) reused this theory, putting forward three simplifying assumptions for practical purposes (Fig. 4). First, he proposed to consider series of adjacent blocks as a sole rigid stone. He then assumed that the arch failure would occur at a 45° angle, dividing the arch in four equal *voussoirs*. He finally supposed that the crown *voussoirs* are acting as wedges on the two others, which are considered being monolithically bound with the piers. Bélidor recognised that this last hypothesis puts his theory 'on the safe side', since he considered no friction between *voussoirs*.

Couplet (1731) further developed Bélidor's and de la Hire's kinematic methods. He thought of the same collapse mechanism, in which the arch is divided into four equal rigid bodies. But he considered instead that the failure mechanism occurs with a rotation of *voussoirs* around specific points of the intrados and extrados (Fig. 5). He was the first to articulate explicitly the three basic assumption for masonry structures: infinite friction between *voussoirs*, infinite compressive strength and no tensile strength for the material.

The kinematic approach was almost exclusively used until Perronet's (1810) observations on case studies led to a new—and more correct—theory for vaulted structures, based on a geometric approach of statics. It was then spread by nineteenth century handbooks for engineers, for instance by Chaix (1890) who explicited various geometric procedures for masonry bridge design including abutments, vaults, etc (Fig. 6).

This geometric point of view led to the so-called "classical approach" (Roca et al. 2010). As it only deals with equilibrium considerations, the stability of vaulted structures and masonry arch bridges is ensured by the possibility to find at least one thrust line—defined as the locus of points where the resultant force passes through the joints between the *voussoirs* (Ochsendorf 2002) – lying entirely inside the masonry envelope (Huerta 2001). The thrust line theory is attributed to Gerstner in German literature, to Méry in French literature

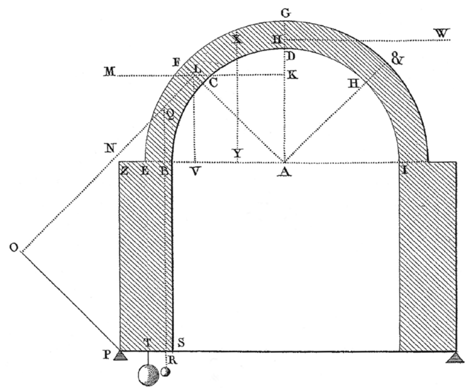

Figure 5. Couplet's model for assessing arch stability (1731).

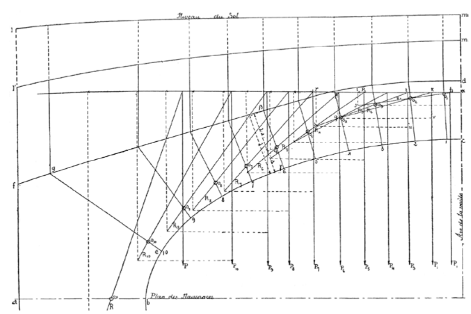

Figure 6. Geometric procedure following Méry's method for masonry arch design (Chaix 1890).

and to Moseley in the English one. These two last authors correlate the geometry of the thrust line with collapse mechanisms: collapse indeed occurs when the thrust line reaches the masonry envelope in such an amount of points that a mechanism is formed. The question of finding the 'true' line of thrust—intended as the one actually defining the equilibrium conditions—has then led to various interpretations from various authors (Scheffer, Culmann or Durand-Claye) until the elastic methods prevailed. The correct analytical description of the thrust line was given by Milankovitch (1907), for a circular arch made of radial *voussoirs*. He found a ratio of 0.1075 between the minimum arch thickness and its average radius. Makris has then discussed this value when considering a different shape for the *voussoirs* (Makris & Alexakis 2013).

Leonardo da Vinci's proposal that "the arch will not break if the outer arc chord does not touch the inner arc" can actually—and with some anachronism—be considered as a first attempt to define a thrust line lying inside the masonry envelope (Benvenuto 1991). Fabri's mechanical model follows the same assumption (Kurrer 2008). He proposed a graphical methodology to determine the minimum thickness of an arch's abutments—and consequently of the arch itself (see section 4.3). In his configuration, the thrust line is reduced to straight lines intersecting the extrados at the crown and at the abutments.

The geometric point of view of statics is also the one that Heyman (1995), a trailblazer in the field of plastic theory, proposed for evaluating the structural safety of masonry arches. He integrated the thrust line theory within the framework of plastic theory, by assuming three fundamental hypotheses for masonry: infinite compressive strength, zero tensile strength, and no failure due to *voussoirs* sliding. Following these, masonry can be considered as a rigid-plastic material. Note that these hypotheses are also the ones that Méry (1840) assumed—and discussed—in his *Mémoire.* Various authors have then developed this method for design and analysis purposes. O'Dwyer (1999) proposed a new technique for the limit state analysis of masonry vaults, using a discrete network of forces at equilibrium inside the masonry envelope. Ochsendorf (2002; 2004) developed a general method to assess the stability of masonry buttresses against overturning or failure, taking into account the effects of leaning. Block, Ciblac and Ochsendorf (2006) further developed applications for real-time limit analysis of arches and vaults.

Within the geometric approach of statics, the safety of masonry arches is logically ensured by the increase of the dimensions of the arch's cross section with respect to the minimal one as defined by Milankovitch. This method, turned into a more general definition, is discussed in the following section.

4 STRUCTURAL SAFETY USING ADMISSIBLE GEOMETRICAL DOMAINS

4.1 *Description of the method*

For masonry arches, the question of measuring the structural safety as in Equation (1) is not straightforward. The general expression of the safety factor as a ratio between stabilizing and destabilizing actions induces a difficulty for the choice of the appropriate parameter (especially for arches under self-weight only). Heyman (1995) proposed instead a geometric safety factor, defined as the ratio between the actual thickness of the *voussoirs* and the minimum one, i.e. the one for which the limit state of equilibrium is reached. This way of defining the structural safety is interesting, because it only deals with the geometrical characteristics of the arch rather than with material strength. Assuming for instance a geometrical safety factor of three would mean that it would be possible to find a thrust line contained in the middle third of the arch. This corresponds to the classical design criteria for masonry developed in Eurocode 6, for blocky elements without known/specific performances (Hurez et al. 2009). However, despite this remarkable coherence, the geometric approach may be uncomfortable for practical use in case of irregular geometries. Indeed, it would first require the identification of the critical section to which apply the safety factor, which is not always easy to determine. Another limitation of this method is that it cannot easily be connected to the magnitudes of the external loads acting on the structure. This is not an issue for existing structures under self-weight only, but it becomes for instance a clear drawback when evaluating the response of an existing structure under a modified external loading. Furthermore, this approach cannot easily be linked with the safety factor formulations commonly used by structural engineers, which are expressed in terms of ratios between characteristic and design stresses or deformations.

To overcome these limitations, the authors suggest working with the form and force diagrams of graphic statics. The form diagram represents the geometry of the considered structure and the path forces will follow inside it, which is represented by a thrust line formasonry arches (Fig. 7, right). The force diagram then gives the magnitudes of these through the lengths of the different bars that compose it, each segment in this diagram being related to one sole segment parallel to it in the other diagram (Fig. 7, left). Graphic statics then presents the advantage of being a very simple and visual tool to control force equilibrium (Zalewski et al. 1998). The concept of admissible geometrical domains as developed in Fivet's framework of Constraint Based Graphic Statics (2013) has been used in that context. The aforementioned domains are intended as the locus of statically admissible positions for the different points of the force diagram. For masonry arches, this is equal to the sets of poles leading to thrust lines that lie entirely within the masonry envelope, depicting all the possible equilibrium configurations. Points on

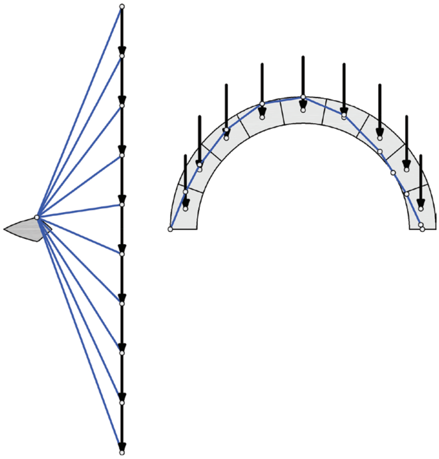

Figure 7. Graphic statics' force and form diagrams (Rondeaux et al. 2018).

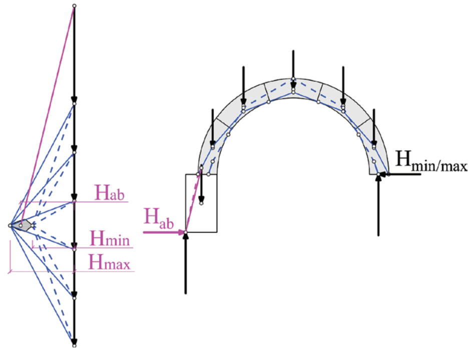

Figure 8. Safety assessment using admissible geometrical domains (Rondeaux et al. 2018).

the vertices of the domain (Fig. 7, left) are then related to possible collapse mechanisms (i.e. corresponding to limit states), as the corresponding thrust line reaches the masonry envelope in an amount of points in such a way that it produces a mechanism (Fig. 7, right) (Rondeaux 2017). Based on the safe theorem of plasticity, it means that the failure of an arch can only occur if its admissible geometrical domain is an empty set of points. Consequently, the structural safety of an arch can be deduced from the analysis of its admissible geometrical domain and the three different types of information it can provide, detailed in the following section.

4.2 *Measuring the structural safety*

First, for a masonry arch to be able to stand, the forces applied to it (or its self-weight) need to be balanced by the thrust forces at its abutments. Let H_{min} be the horizontal distance in the force diagram between the rightmost point of the domain and the force polygon describing the equilibrium of external forces (Fig. 8). This distance corresponds to the minimum horizontal force that has to be mobilised at the arch's supports in order to find a thrust line entirely lying inside the ma-sonry envelope. In order to evaluate the arch's stability, this value can be compared with the maximum thrust H_{ab} the abutments are able to support. In Eurocodes formulation, a first safety indicator can be given by:

$$\lambda_1 = \frac{H_{ab}}{H_{\min}} = \gamma_{ab} \cdot \gamma_{\min} \tag{2}$$

where $H_{d,min} = H_{min.}\gamma_{min}$ = design value related to the minimum horizontal force acting at the arch's supports; and $H_{d,ab} = H_{ab}/\gamma_{ab}$ = design value related to the stabilizing forces given by the abutments.

To obtain the second information, the overall ability of the arch to undertake horizontal thrusts has to be evaluated. In the force diagram, this is represented by the horizontal range of the admissible geometrical domain (Fig. 8). In terms of safety indicator, this gives:

$$\lambda_2 = \frac{H_{\max}}{H_{\min}} = \gamma_{\max} \cdot \gamma_{\min} \tag{3}$$

where $H_{d,max} = H_{max}/\gamma_{max}$ = design value related to the maximum admissible thrust inside the arch, linked with the leftmost point of the domain; and H_{min} is obtained as in Equation (2).

Finally, the domains' area gives a measure of the arch's ability for plastic force redistributions, corresponding to the set of possible thrust lines lying inside the masonry envelope. This area can be taken as a geometrical criterion to assess the arch's robustness (Zastavni et al. 2016). Indeed, in case of a local failure, a larger domain will indicate a larger structural capacity to equilibrate the applied forces by finding another path for the thrust line. Taking inspiration from the robustness deterministic index developed by Frangopol & Curley (1987), a robustness indicator can be given by:

$$\lambda_3 = \frac{A_{\text{int}}}{A_{\text{int}} - A_{dam}} \tag{4}$$

where A_{int} = area of the admissible geometrical domain for an intact arch geometry; and A_{dam} = remaining domain area after the same arch suffered damage.

For example, if movements of the abutments lead to a crack which position can be clearly identified, this means that a plastic hinge has formed in this particular section and that consequently the thrust line must pass by this well-defined point. The ratio between domain areas before and after that crack occurred then represents the ability of the arch to survive to this damage.

4.3 *Interpretation of the historical theories using admissible geometrical domains*

Some of the historical theories mentioned in Section 3.2 are now reinterpreted in terms of admissible geometrical domains. For this purpose, an arch discretized in 21 radial *voussoirs* has been considered.

First, this arch was given a minimal thickness $d_F = 0.343\ R_e$ as proposed by Fabri, with R_e the external radius of the arch. Following Fabri's construction rule, the thrust line is reduced to two straight lines lying entirely inside the masonry envelope, leading to the domain on the left of Figure 9. The domain corresponding to the full 21-*voussoirs* thrust line is the one depicted in grey.

The arch was then given a minimal thickness $d_V = 0.293\ R_e$ as proposed by da Vinci. According to Benvenuto (1991), da Vinci's proposal that "the arch will not break if the outer arc chord does not touch the inner arc" probably comes from the assimilation of the arch to a system of two rigid bars. This means that, assuming an arch is symmetric, its weight is transmitted from the crown down to the supports along two straight lines that intersect the extrados at the crown and at the abutments, and that lie entirely within the masonry envelope. The orientation of the chords (Fig. 10) is then uniquely defined, giving value d_V for minimal thickness. The domain corresponding to the full 21-*voussoirs* thrust line is the one depicted in grey.

A third arch's thickness $d_B = 0.261\ R_e$ is deduced from Bélidor's theory. The internal resulting forces applied on the *voussoirs* are determined in terms of direction (perpendicular to the joints) and position (middle point of the 45° cross sections). A single corresponding funicular polygon can then be drawn, as depicted in Figure 11. This gives value d_B for minimal thickness. The pole of this polygon lies within the admissible geometrical domain that can be obtained for the full 21-*voussoirs* arch geometry (in grey).

It was then considered that $d_A = 0{,}1\ D_i = 0.167\ R_e$ as proposed by Alberti, with D_i the internal diameter of the arch (Proske, 2009). The corresponding admissible geometrical domain is depicted in grey in Figure 12.

Milankovitch's theory was then expressed in terms of the ratio between the arch's minimal thickness and its external radius, giving $d_M = 0.102\ R_e$. As his formulation leads to the correct analytical

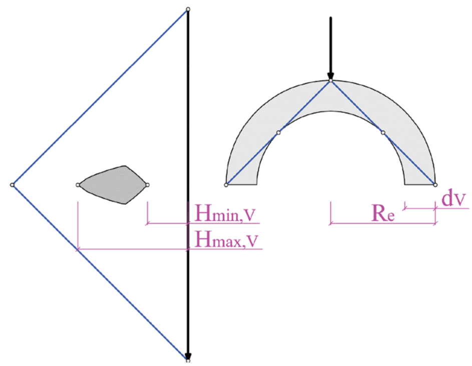

Figure 10. Geometrical construction after da Vinci's theory and its admissible geometrical domain (grey) according to thrust line theory (Rondeaux et al. 2018).

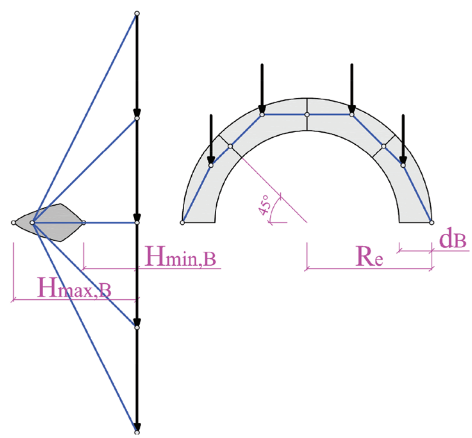

Figure 11. Geometrical construction after Bélidor's theory and its admissible geometrical domain (grey) according to thrust line theory (Rondeaux et al. 2018).

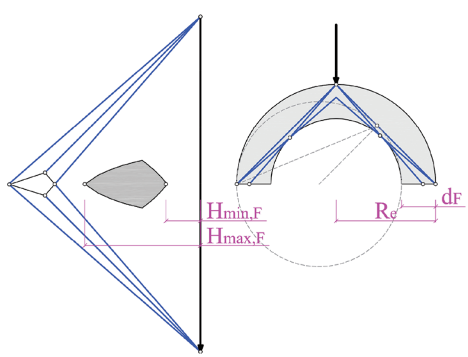

Figure 9. Geometrical construction of a semi-circular arch after Fabry's theory and its admissible geometrical domain (grey) according to thrust line theory (Rondeaux et al. 2018).

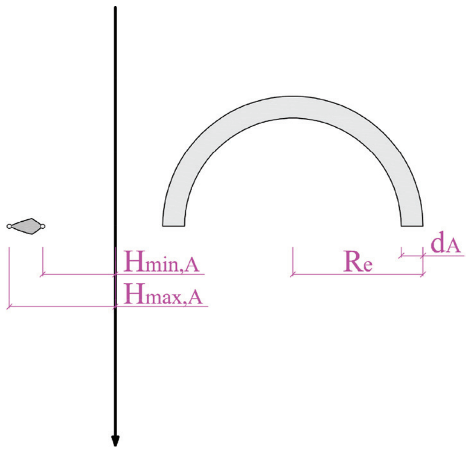

Figure 12. Admissible geometrical domain corresponding to Alberti's theory (Rondeaux et al. 2018).

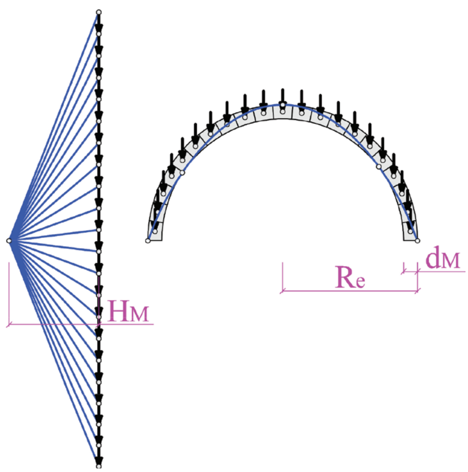

Figure 13. Admissible geometrical domain corresponding to Milankovitch's theory (Rondeaux et al. 2018).

description of the thrust line, the corresponding admissible geometrical domain is reduced to one sole point (Fig. 13).

Finally, the arch was modelled with a thickness of $d_C = 0.096\ R_e$ as proposed by Couplet. Since this value is not conservative, no thrust line lying entirely in the arch's geometry can be found, and the corresponding admissible geometrical domain is null.

Table 1 summarises the values obtained in each case for the minimum thickness ratios d_i/R_e, the domain areas $A_{dom,I}$, as well as the maximum and minimum thrusts $H_{max,i}$ and $H_{min,i}$ for the 21-*voussoirs* arch geometry. Note that the force diagrams depicted in Figures 9 to 13 are represented at different scales, in order to have a normalized weight for the *voussoirs*. Indeed, the thicker the arch is, the larger the force diagram becomes.

Table 1. Results.

	d_i/R_e	$A_{dom,i}$	$H_{max,i}$	$H_{min,i}$	$\lambda_{2,i}$	d_i/d_M
Authors	/	kN²	kN	kN	/	/
Fabri	0.343	498.2	53.81	16.04	3.36	3.36
Da Vinci	0.293	113.5	29.39	10.82	2.72	2.87
Bélidor	0.261	38.99	19.48	8.42	2.31	2.55
Alberti	0.167	0.607	3.32	4.83	1.46	1.64
Milankovitch	0.102	0	1.26	1.26	1	1
Couplet	0.096	/	/	/	/	/

Safety indicators $\lambda_{2,i}$ have then been calculated as in Equation (3), using the results obtained via the proposed domain approach. These values are then compared to the geometrical safety indicator as intended by Heyman (1995), obtained by dividing each arch's thickness by the absolute minimum thickness d_M as given by Milankovitch. Both these values are given in the last two columns of Table 1. It can be observed that they are very close to each other, since only slight differences (less than 3%) can be noted. The results also confirm that Couplet's approach is unsafe. This means that, although it is computed using a discrete analysis, the proposed approach based on admissible geometrical domains provides a very good estimation of the structural safety of masonry arches. Moreover, the approach gives in each case a safe estimation of structural safety, as the values obtained for $\lambda_{2,i}$ are always smaller or equal to those of d_i/d_M.

5 CONCLUSION

In section 3, some key contributions concerning the historical development of several masonry arches theories have been reviewed. Because of the low-stress states characterising historical masonry structures, authors have always considered strength requirements as a secondary issue compared to overall stability. Therefore they have linked the structural safety of masonry arches to their geometry, proposing different ways to define minimum values for the arch's thickness. These minimum values have here been expressed as a ratio between the *voussoirs*' thickness (d_i) and the extrados' radius (R_e), defining a geometrical structural safety factor (Heyman, 1995). This approach can be better related to the effective arch's mechanical behaviour than a strength-based safety factor.

The approach based on admissible geometrical domains has then been presented, introducing a way to assess the structural safety of masonry arches using a stability indicator λ_2. This method, based on the classical thrust line approach from which the values of λ_2 are extracted, enables to evaluate the historical construction rules and theories for masonry arches proposed by Fabri, da Vinci, Bélidor, Couplet, etc. It appears that the results of such an analysis show a strong correlation with Heyman's geometrical safety factor. Nevertheless, the proposed indicator λ_2 presents the advantage of being easily objectified and extendable to any structural shape and loading conditions. It can also be reinterpreted in terms of design codes' formulation.

This analysis brings forward some important conclusions about the respective reliability of classical masonry arch theories. It also provides a significant contribution to construction history by completing the theories with a more actual formulation of structural safety, which nowadays hardly include the issue of masonry arches. Interesting perspectives of development can be expected, as the proposed methodology can easily be extended to cases with more complex geometries: masonry vaults, buttressed arches, unreinforced concrete shells; as well as to the analysis of some aspects of robustness of these various types of structures.

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