Weak representability of actions: a study from groups to non-associative algebras

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Actions / Split extensions of groups

- Let B, X be groups. An action of B on X is a map
$$\xi\colon B \times X \to X\colon (b,x) \mapsto b \cdot x$$
such that $(bb') \cdot x = b \cdot (b' \cdot x)$, $1_B \cdot x = x$ and $b \cdot (xx') = (b \cdot x)(b \cdot x')$.
- One can construct the *semi-direct product* $B \ltimes X$ with respect to ξ, that is the group whose underline set is $B \times X$ and the multiplication is determined by
$$(b,x)(b',x') = (bb', x(b \cdot x')),$$
with $1_{B \ltimes X} = (1_B, 1_X)$ and $(b,x)^{-1} = (b^{-1}, b^{-1} \cdot x^{-1})$.
- We can associate with ξ the split extension
$$\begin{tikzcd} 1 \arrow[r] & X \arrow[r, "i_2"] & B \ltimes X \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "i_1"] \arrow[r] & 1. \end{tikzcd}$$

- Conversely, if we start with a split extension of groups

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0
\end{tikzcd}
$$

then we can define an action of B on X by

$$B \times X \to X \colon (b, x) \mapsto \beta(b)k(x)\beta(b)^{-1}$$

and we have a bijection between the set of actions of B on X and the set of isomorphism classes of split extensions of B by X.
- Split extensions of groups are in bijection with the homomorphisms $B \to \mathrm{Aut}(X)$: given a split extension of groups as above, one can define the group homomorphism

$$\varphi \colon B \to \mathrm{Aut}(X) \colon b \mapsto \varphi_b,$$

where $\varphi_b(x) = \beta(b)k(x)\beta(b)^{-1}$.
- Conversely, any homomorphism $\varphi \colon B \to \mathrm{Aut}(X)$ defines a semi-direct product $B \ltimes X$ with multiplication

$$(b, x)(b', x') = (bb', x\varphi_b(x'))$$

and thus a split extension of B by X.

The semi-abelian context

- Semi-abelian categories were introduced in order to provide a categorical setting which would capture algebraic properties of groups, rings and algebras.
- Let $\mathcal{C}$ be a semi-abelian category (i.e. $\mathcal{C}$ is pointed, Barr-exact, Bourn-protomodular with finite coproducts).
- Let X, B be objects of $\mathcal{C}$. A *split extension* of B by X is a diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{C}$ such that $\alpha \circ \beta = 1_B$ and (X, k) is a kernel of α.

- For any object X in $\mathcal{C}$, we consider the functor

$$\mathsf{SplExt}(-, X)\colon \mathcal{C}^{op} \to \mathbf{Set}$$

where, for every object B in $\mathcal{C}$, $\mathsf{SplExt}(B, X)$ is the set of isomorphism classes of split extensions of B by X.

- Recall that we have an an equivalence between split extensions and actions and a canonical functor isomorphism

$$\mathsf{SplExt}(-, X) \cong \mathsf{Act}(-, X).$$

Action representable categories

Definition (F. Borceux, G. Janelidze and G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathsf{SplExt}(-, X)$ is representable. This means that there exists an object $[X]$ of $\mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathsf{SplExt}(-, X) \cong \mathsf{Hom}_{\mathcal{C}}(-, [X]).$$

- The prototype examples are the category **Grp** of groups and the category **Lie** of Lie algebras.
- **Grp** is an action representable category and the actor of a group X is the groups $\mathsf{Aut}(X)$.
- In **Lie**, a split extension of B by X is represented by a homomorphism $B \to \mathsf{Der}(X)$, where $\mathsf{Der}(X)$ is the Lie algebra of derivations of X.

- The notion of action representable category has proven to be quite restrictive.

Theorem (X. García Martínez, M. Tsishyn, T. Van der Linden and C. Vienne, 2021)

Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2$ *and let $\mathcal{V}$ be a non-abelian variety of non-associative algebras over $\mathbb{F}$. If $\mathcal{V}$ is action representable, then $\mathcal{V}$ must be the category* **Lie** *of Lie algebras over $\mathbb{F}$.*

- More recently G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of semi-abelian categories.

Weakly action representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every object X in $\mathcal{C}$, the functor $\mathsf{SplExt}(-,X)$ admits a weak representation. This means that there exist an object T of $\mathcal{C}$ and a monomorphism of functors

$$\tau\colon\ \mathsf{SplExt}(-,X) \rightarrowtail \mathsf{Hom}_{\mathcal{C}}(-,T).$$

An object T as above is called *weak actor* of X, the pair (T,τ) is called *weak representation* of $\mathsf{SplExt}(-,X)$ and $(\varphi\colon B \to T) \in \mathsf{Im}(\tau_B)$ is called *acting morphism*.

Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is a natural isomorphism.

- **G. Janelidze, 2022.** The category **Assoc** of associative algebras over $\mathbb{F}$ is weakly action representable. Given an associative algebra X, a weak actor of X is the associative algebra of *bimultipliers* of X.

$$\mathrm{Bim}(X) = \{(f{*}-, -{*}f) \in \mathrm{End}(X) \times \mathrm{End}(X)^{op} \mid \cdots$$

$$\cdots \mid f{*}(xy) = (f{*}x)y, (xy){*}f = x(y{*}f), x(f{*}y) = (x{*}f)y, \ \forall x, y \in X\}.$$

- **A. S. Cigoli, M. M., G. Metere, 2023.** The category **Leib** of Leibniz algebras over $\mathbb{F}$ is weakly action representable. Given a Leibniz algebra X, a weak actor of X is the Leibniz algebra of *biderivations* of X.

$$\mathrm{Bider}(X) = \{(d, D) \in \mathrm{End}(X)^2 \mid d(xy) = d(x)y + xd(y), \cdots$$

$$\cdots \ D(xy) = D(x)y - D(y)x, \ xd(y) = xD(y), \ \forall x, y \in X\}$$

A counterexample: Jordan algebras

Definition

A *Jordan algebra* over a field $\mathbb{F}$ is a non-associative commutative algebra $(X, \cdot)$ over $\mathbb{F}$ which satisfies the *Jordan identity*

$$(xy)(xx) = x(y(xx)), \ \ \forall x, y \in X.$$

Theorem (G. Janelidze, 2022)

Every weakly action representable category is action accessible.

Remark (A. S. Cigoli and S. Mantovani, 2012)

The category **Jord** of Jordan algebras over a field $\mathbb{F}$ is not action accessible.

Conclusion: Jord is not weakly action representable.

W.R.A. of non-associative algebras

- Let $\mathcal{V}$ be a variety of non-associative algebras over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$;
- We suppose that $\mathcal{V}$ is *operadic*, i.e. all the identities of $\mathcal{V}$ are *multilinear*. This happens, for instance, when $\mathrm{char}(\mathbb{F}) = 0$.
- We also suppose that $\mathcal{V}$ is *action accessible* (or equivalently *algebraically coherent*);
- This is equivalent to saying that the λ/μ *rules* hold, i.e. there exists $\lambda_1, \dots \lambda_8, \mu_1, \dots \mu_8 \in \mathbb{F}$ such that

$$x(yz) = \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) +$$

$$+\lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx),$$

$$(yz)x = \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) +$$

$$+\mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx)$$

are identities in $\mathcal{V}$.

The commutative case

- Let $\mathcal{V}$ be an operadic and action accessible variety of commutative algebras over $\mathbb{F}$. The λ/μ rules reduce to

$$x(yz) = \alpha(xy)z + \beta(xz)y,$$

for some $\alpha, \beta \in \mathbb{F}$.

Proposition

$\mathcal{V}$ is associative or satisfies the Jacobi identity ($x(yz) + y(zx) + z(xy) = 0$).

Corollary

Let $\mathrm{char}(\mathbb{F}) \neq 3$. *If $\mathcal{V}$ is also quadratic, then* $\mathcal{V} = \mathbf{CAssoc}$, $\mathcal{V} = \mathbf{JJord}$ *or* $\mathcal{V} = \mathbf{Nil}_2(\mathrm{Com})$.

Commutative associative algebras

Remark

Let X be a commutative associative algebra. Then

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(U(-), \mathrm{M}(X)),$$

where $\mathrm{M}(X)$ is the associative algebra of *multipliers* of X and $U\colon \mathbf{CAssoc} \to \mathbf{Assoc}$ denotes the forgetful functor.

Theorem (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

Let X be a commutative associative algebra. Then $\mathrm{M}(X)$ is a commutative algebra if and only if the functor $\mathrm{SplExt}(-, X)$ is representable.

Example

If $\mathrm{Ann}(X)$ is trivial or $X^2 = X$, then $\mathrm{M}(X)$ is a commutative associative algebra.

Example

If $X = \mathbb{F}^2$ is the abelian two-dimensional algebra, then $\mathrm{M}(X) = \mathrm{End}(X)$ is not commutative.

Corollary (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

The variety **CAssoc** *of commutative associative algebras over* $\mathbb{F}$ *is not action representable.*

Jacobi-Jordan algebras

Definition

A commutative algebra X is called a *Jacobi-Jordan* algebra if it satisfies the *Jacobi identity*

$$x(yz) + y(zx) + z(xy) = 0, \quad \forall x, y, z \in X.$$

Definition

An *anti-derivation* is a linear map $d\colon X \to X$ such that

$$d(xy) = -d(x)y - d(y)x, \quad \forall x, y \in X.$$

Example

The (left) multiplications $L_x = x \cdot -$, $x \in X$, are particular anti-derivations, called *inner anti-derivations*.

Split extensions

- Let

$$0 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$

be a split extension of Jacobi-Jordan algebras.

- We have that

$$A \cong B \ltimes X = (B \oplus X, \cdot)$$

and the Jacobi-Jordan algebra structure is given by

$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + b' * x).$$

- The action of B on X is defined by

$$b * x = \beta(b) \cdot_A k(x), \quad \forall b \in B, \ \forall x \in X.$$

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ using the formula above, we have the following characterization.

Proposition

$(B \oplus X, \cdot)$ is a Jacobi-Jordan algebra if and only if

1. $b * (xx') = -(b * x)x' - (b * x) * x'$;
2. $(bb') * x = -b * (b' * x) - b' * (b' * x)$;

for every $b, b' \in B$ and $x, x' \in X$.

In an equivalent way, a split extension of B by X (or an action of B on X) is a linear map

$$B \to \mathrm{ADer}(X) \colon b \mapsto b * -$$

such that

$$(bb') * - = \{b * -, b' * -\}, \quad \forall b, b' \in B,$$

where $\{f, f'\} = -f \circ f' - f' \circ f$ denotes the *anti-commutator* between two linear endomorphisms of X.

Remark

The space $\mathrm{ADer}(X)$ endowed with the anti-commutator $\{-,-\}$ is not in general a (Jacobi-Jordan) algebra.

Example

Let $X = \mathbb{F}$ be the abelian one-dimensional algebra. Then $\mathrm{ADer}(X) = \mathrm{End}(X) \cong \mathbb{F}$ does not satisfy the Jacoby identity.

Definition

A *partial algebra* is a vector space X endowed with a *bilinear partial operation*

$$\Omega \subseteq X \times X \to X.$$

A homomorphism of partial algebras is a linear map $f\colon X \to Y$ such that $f(xy) = f(x)f(y)$, for every $x, y \in X$ such that xy and $f(x)f(y)$ are defined.

Definition

We denote by **PAlg** the category whose objects are the partial algebras over $\mathbb{F}$ and morphisms are the homomorphisms of partial algebras.

Theorem (X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2023)

Let X be a Jacobi-Jordan algebra. Then

(i) $\mathsf{SplExt}(-, X) \cong \mathsf{Hom}_{\mathbf{PAlg}}(U(-), \mathsf{ADer}(X))$, *where* $U \colon \mathbf{JJord} \to \mathbf{PAlg}$ *denotes the forgetful functor.*

(ii) *if* $\mathsf{ADer}(X)$ *is a Jacobi-Jordan algebra, then it is the actor of* X.

Two-step nilpotent commutative algebras

Theorem (X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2023)

- *The variety* $\mathbf{Nil}_2(\mathrm{Com})$ *is weakly action representable;*
- *A weak actor of an object* X *of* $\mathbf{Nil}_2(\mathrm{Com})$ *is the abelian algebra*

$$[X]_2 = \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y = 0, \ \forall x \in X\}.$$

- *An arrow* $B \to [X]_2$*, defined by* $b \mapsto b * -$*, is an acting morphism if and only if*

$$b * (b' * x) = 0, \quad \forall b, b' \in B, \ \forall x \in X.$$

The non-commutative case

- Let $\mathcal{V}$ be an operadic and action accessible variety of non-associative algebras over $\mathbb{F}$ and let

$$\Phi_{k,i}(x_1, \dots, x_k) = 0, \quad i = 1, \dots, n,$$

where k is the degree of Φ_k, be the multilinear identities which define $\mathcal{V}$.

- We look for a vector space $\mathcal{E}(X)$ such that

$$\mathrm{Inn}(X) = \{(L_x, R_x) \mid x \in X\} \leq \mathcal{E}(X) \leq \mathrm{End}(X)^2$$

and we want to endow it with a bilinear partial operation

$$\langle -, - \rangle \colon \Omega \subseteq X \times X \to X,$$

such that we can associate in a natural way a morphism of partial algebras $B \to \mathcal{E}(X)$, with every object B in $\mathcal{V}$ and every split extension of B by X.

Split extensions

- Let

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

  be a split extension in $\mathcal{V}$.
- We have that

$$A \cong B \ltimes X = (B \oplus X, \cdot)$$

  with

$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + x * b').$$

- The action of B on X is defined by the pair of bilinear maps

$$l \colon B \times X \to X, \quad r \colon X \times B \to X,$$

  with $b * x' = l(b, x') = \beta(b) \cdot_A k(x')$, $x * b' = r(x, b') = k(x) \cdot_A \beta(b')$.

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ as before, we have:

Proposition

Let X and B be two algebras in $\mathcal{V}$. Given a pair of bilinear maps

$$l\colon B \times X \to X, \quad r\colon X \times B \to X,$$

we construct $(B \oplus X, \cdot)$ as before. Then $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$ if and only if

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad \forall i = 1, \dots, n,$$

where at least one of the α_1, ..., α_k is an element of of the form $(0, x)$, with $x \in X$, and the others are of the form $(b, 0)$, with $b \in B$. The resulting algebra is the semi-direct product of B and X, denoted by $B \ltimes X$.

- We write $b * x$ (resp. $x * b$) for the multiplication $(b, 0) \cdot (0, x)$ (resp. $(0, x) \cdot (b, 0)$) in $B \ltimes X$.

The partial algebra $\mathcal{E}(X)$

We define $\mathcal{E}(X)$ as the subspace of all pairs $(f * -, - * f) \in \mathrm{End}(X)^2$ satisfying

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad \forall i = 1, \dots, n,$$

for each choice of $\alpha_j = f$ and $\alpha_t \in X$, where $t \neq j \in \{1, \dots, k\}$ and $fx := f * x$, $xf := x * f$.
We endow it with the bilinear map $\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (h * -, - * h),$$

where

$$\begin{aligned} x * h = &\lambda_1 (x * f) * g + \lambda_2 (f * x) * g + \lambda_3 g * (x * f) + \lambda_4 g * (f * x) \\ &+ \lambda_5 (x * g) * f + \lambda_6 (g * x) * f + \lambda_7 f * (x * g) + \lambda_8 f * (g * x) \end{aligned}$$

and

$$\begin{aligned} h * x = &\mu_1 (x * f) * g + \mu_2 (f * x) * g + \mu_3 g * (x * f) + \mu_4 g * (f * x) \\ &+ \mu_5 (x * g) * f + \mu_6 (g * x) * f + \mu_7 f * (x * g) + \mu_8 f * (g * x). \end{aligned}$$

Example

If $\mathcal{V} = \textbf{Assoc}$, then $\mathcal{E}(X) = \mathrm{Bim}(X)$. If $\mathcal{V} = \textbf{Leib}$, then $\mathcal{E}(X) \cong \mathrm{Bider}(X)$.

Remark

A split extension of B by X in $\mathcal{V}$ is the same as a linear map

$$B \to \mathcal{E}(X) \colon b \mapsto (b * -, - * b),$$

such that $((bb') * -, - * (bb')) = \langle (b * -, - * b), (b' * -, - * b') \rangle$ and

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad i = 1, \dots, n,$$

where α_1, ..., α_k are as above.

Remark

The bilinear map

$$\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$$

defines a partial operation $\langle -, - \rangle \colon \Omega \to \mathcal{E}(X)$, where Ω is the preimage

$$\langle -, - \rangle^{-1}(\mathcal{E}(X))$$

of the inclusion $\mathcal{E}(X) \hookrightarrow \mathrm{End}(X)^2$.

Alternative algebras

Let $\mathcal{V} = \textbf{Alt}$ be the variety of alternative algebras over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ ($(yx)x - yx^2 = x(xy) - x^2y = 0$). Then $\mathcal{E}(X)$ consists of the pairs $(f * -, - * f) \in \mathrm{End}(X)^2$ satisfying

$$f * (xy) = (x * f)y + (f * x)y - x(f * y),$$

$$(xy) * f = x(f * y) + x(y * f) - (x * f)y,$$

$$x(y * f) = (yx) * f + (xy) * f - y(x * f)$$

$$(f * x)y = f * (yx) + f * (xy) - (f * y)x$$

for any $x, y \in X$, and the bilinear map

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (h * -, - * h)$$

is given by

$$h * x = -(f * x) * g + f * (g * x) + f * (x * g)$$

$$x * h = (x * f) * g + (f * x) * g - f * (x * g).$$

One can check that $\langle -, - \rangle$ does not define an algebra structure.

Unitary alternative algebras

If X is a *unitary* alternative algebra, such as the algebra of octonions $\mathbb{O}$, and $e = 1_X$, then the elements of $\mathcal{E}(X)$ satisfy the following set of equations

$$\begin{aligned} f * x =& (x * f)e + (f * x)e - x(f * e), \\ x * f =& e(f * x) + e(x * f) - (e * f)x, \\ x * f =& x * f + x * f - x(e * f), \\ f * x =& f * x + f * x - (f * e)x, \end{aligned}$$

for any $x \in X$. It turns out that an element of $\mathcal{E}(X)$ is uniquely determined by $\alpha = f * e = e * f$ of X, i.e.

$$\mathcal{E}(X) \cong \{(\alpha, \alpha) \mid \alpha \in X\} \cong X$$

is an object of **Alt**.

Theorem (X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2023)

Let $\mathcal{V}$ be an action accessible, operadic variety of non-associative algebras over a field $\mathbb{F}$ and let $U\colon \mathcal{V} \to \mathbf{PAlg}$ denote the forgetful functor.

- *Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors*

$$\tau\colon \operatorname{SplExt}(-,X) \rightarrowtail \operatorname{Hom}_{\mathbf{PAlg}}(U(-),\mathcal{E}(X))$$

and, for every object B of $\mathcal{V}$, τ_B is the injection which sends an element of $\operatorname{SplExt}(B,X)$ to the corresponding partial algebra homomorphism

$$B \to \mathcal{E}(X)\colon b \mapsto (b*-,-*b).$$

- *Let B, X be objects of $\mathcal{V}$. The homomorphism of partial algebras*

$$B \to \mathcal{E}(X)\colon b \mapsto (b*-,-*b)$$

belongs to $\operatorname{Im}(\tau_B)$ if and only if $\Phi_{k,i}(\alpha_1,\ldots,\alpha_k)=0$, as before.

Theorem

- *If* $(\mathcal{E}(X), \langle -, - \rangle)$ *is an object of* $\mathcal{V}$*, then* $(\mathcal{E}(X), \tau)$ *is a weak representation of* $\mathrm{SplExt}(-, X)$*.*
- *If* $\mathcal{V}$ *is a variety of commutative or anti-commutative algebras, then* $\mathcal{E}(X)$ *is isomorphic to the partial algebra*

$$\{f \in \mathrm{End}(X) \mid \Phi_{k,i}(f, x_2, \ldots, x_k) = 0, \ \forall x_2, \ldots, x_k \in X\}$$

endowed with the bilinear partial operation $\langle f, g \rangle = \alpha(f \circ g) + \beta(g \circ f)$*, where* α*,* $\beta \in \mathbb{F}$ *are given by the* λ/μ *rules.*

Definition

The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X. The pair $(\mathcal{E}(X), \tau)$ is called *external weak representation* of the functor $\mathrm{SplExt}(-, X)$. When τ is a natural isomorphism, we say that $\mathcal{E}(X)$ is an *external actor* of X.

Example

We may check that, with the *obvious* choices of the λ/μ rules:

1. if $\mathcal{V} = \mathbf{AbAlg}$, then $\mathcal{E}(X) = 0$ is the actor of X;
2. if $\mathcal{V} = \mathbf{CAssoc}$, then $\mathcal{E}(X) \cong \mathrm{M}(X)$ is an external actor of X;
3. if $\mathcal{V} = \mathbf{JJord}$, then the external actor is isomorphic to the partial algebra $\mathrm{ADer}(X)$ of anti-derivations of X;
4. if $\mathcal{V} = \mathbf{Lie}$, then $\mathcal{E}(X) \cong \mathrm{Der}(X)$ is the actor of X;
5. if $\mathcal{V} = \mathbf{Nil}_2(\mathbf{CAlg})$, then $\mathcal{E}(X) \cong [X]_2$ is a weak actor of X;
6. if $\mathcal{V} = \mathbf{Alt}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and X is unitary, then $\mathcal{E}(X) \cong X$ is an alternative algebra and we have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Alt}}(-, X)$$

i.e. X is the actor of itself. In particular, the algebra of octonions $\mathbb{O}$ has representable actions.

Open problems and future directions

- Does the converse of the implication

  weakly action representable category $\Rightarrow$ *action accessible category*

  hold in the context of varieties of algebras over a field?
- Is it possible to generalize the costruction of the external weak actor to any operadic and action accessible variety of algebras with two not necessarily associative bilinear operations? We already have positive answers for the categories **Pois** and **CPois** of (commutative) Poisson algebras.
- We can use the construction of the external weak actor to study the representability of actions of varieties of unitary algebras, which are *ideally exact categories* in the sense of G. Janelidze. For instance, we have that the category of unitary associative algebras and that of unitary alternative algebras are action representable.

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