

The KdV hierarchy: universality and a Painlevé transcendent

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October 20, 2011

Abstract

We study the Cauchy problem for the Korteweg-de Vries (KdV) hierarchy in the small dispersion limit where $\epsilon \rightarrow 0$. For negative analytic initial data with a single negative hump, we prove that for small times, the solution is approximated by the solution to the hyperbolic transport equation which corresponds to $\epsilon = 0$. Near the time of gradient catastrophe for the transport equation, we show that the solution to the KdV hierarchy is approximated by a particular Painlevé transcendent. This supports Dubrovins universality conjecture concerning the critical behavior of Hamiltonian perturbations of hyperbolic equations. We use the Riemann-Hilbert approach to prove our results.

1 Introduction

In this manuscript, we will prove a particular case of a conjecture in [13] about the formation of dispersive shocks [26] in a class of Hamiltonian perturbations of the quasi-linear transport equation

$$u_t + a(u)u_x = 0,$$

where $u = u(x, t)$, $x, u \in \mathbb{R}$, $t \in \mathbb{R}^+$, and where a is an arbitrary regular function of u . We will restrict ourselves to the case of the KdV hierarchy, which, for any $m \in \mathbb{N}$, is a Hamiltonian perturbation of the equation

$$u_{t_m} + C_m u^m u_x = 0 \quad C_m = (-1)^{m+1} \frac{2^m (2m+1)!!}{m!}. \quad (1.1)$$

The equations in the KdV hierarchy can be written in the form

$$u_{t_m} - (-1)^m \partial_x \psi_m(u, \epsilon u_x, \epsilon^2 u_{xx}, \dots, \epsilon^{2m} \partial_x^{2m} u) = 0, \quad t_m \in \mathbb{R}^+, \quad m \in \mathbb{N}, \quad (1.2)$$

where $\epsilon > 0$ and $\partial_x = \frac{\partial}{\partial x}$. The function ψ_m is polynomial in its variables and it is the variational derivative of the Hamiltonian $\mathcal{H}_m$ [20, 22, 40]

$$\psi_m(u, \epsilon u_x, \dots) = \frac{\delta \mathcal{H}_m}{\delta u(x)}, \quad \mathcal{H}_m = \int h_m(u, \epsilon u_x(x), \epsilon^2 u_{xx}(x), \dots) dx,$$

which is defined as

$$\frac{\delta \mathcal{H}_m}{\delta u(x)} = \frac{\partial h_m(u, \epsilon u_x, \dots)}{\partial u} - \frac{d}{dx} \frac{\partial h_m(u, \epsilon u_x, \dots)}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial h_m(u, \epsilon u_x, \dots)}{\partial u_{xx}} - \dots$$

The Hamiltonians $\mathcal{H}_m$ satisfy the Lenard-Magri recurrence relation [33]

$$\begin{aligned} \partial_x \frac{\delta \mathcal{H}_m}{\delta u(x)} &= P \frac{\delta \mathcal{H}_{m-1}}{\delta u(x)}, \quad m \geq 1, & \mathcal{H}_0[u] &= \int \frac{u^2(x)}{2} dx, \\ P &= \epsilon^2 \partial_x^3 + 4u \partial_x + 2u_x. \end{aligned} \quad (1.3)$$

The densities h_m are uniquely determined by (1.3) together with the conditions

$$h_m(u, \epsilon u_x, \epsilon^2 u_{xx}, \dots)|_{\epsilon=0} = \frac{(-1)^{m+1} C_m}{(m+1)(m+2)} u^{m+2}.$$

For $m = 1$ the equation (1.2) coincides with the standard KdV equation

$$u_{t_1} + 6uu_x + \epsilon^2 u_{xxx} = 0, \quad \epsilon > 0, \quad (1.4)$$

and for $m = 2, 3$ one has the equations

$$u_{t_2} - 30u^2 u_x - \epsilon^2 (20u_x u_{xx} + 10uu_{xxx}) - \epsilon^4 u_{xxxxx} = 0, \quad (1.5)$$

$$u_{t_3} + 140u^3 u_x + \epsilon^2 (70u_x^3 + 280uu_x u_{xx} + 70u^2 u_{3x}) + \epsilon^4 (70u_{2x} u_{3x} + 42u_x u_{4x} + 14uu_{5x}) + \epsilon^6 u_{7x} = 0. \quad (1.6)$$

The corresponding Hamiltonians for $m = 1, 2$ are [21, 23, 36]

$$H_1 = \int \left(u^3 - \epsilon^2 \frac{u_x^2}{2} \right) dx, \quad H_2 = \int \left(\frac{5}{2} u^4 - 5\epsilon^2 uu_x^2 + \frac{\epsilon^4}{2} u_{xx}^2 \right) dx.$$

Each equation in the hierarchy can also be written in the Lax form [30]

$$L_{t_m} = [L, A_m], \quad (1.7)$$

where L is the Schrödinger operator

$$L = \epsilon^2 \partial_x^2 + u, \quad (1.8)$$

L_{t_m} is the operator of multiplication by u_{t_m} , and A_m is an antisymmetric higher order operator with leading order $(-1)^{m+1} 4^m \epsilon^{2m} \partial_x^{2m+1}$. The lower order terms are determined by the requirement that $[L, A_m]$ is an operator of multiplication with a function depending on $u, u_x, \dots, u_{(2m+1)x}$. For $m = 1$ and $m = 2$, we have

$$A_1 = 4\epsilon^2 \partial_x^3 + 3(u \partial_x + \partial_x u), \quad (1.9)$$

$$A_2 = -16\epsilon^4 \partial_x^5 - 20\epsilon^2 (\partial_x^3 u + u \partial_x^3) + 5(\partial_x u_{xx} + u_{xx} \partial_x) - 15(\partial_x u^2 + u^2 \partial_x). \quad (1.10)$$

We will study the behavior of solutions to the KdV hierarchy in the small dispersion limit where $\epsilon \rightarrow 0$.

When $\epsilon = 0$ the KdV hierarchy reduces to the transport equation (1.1). Let us assume that we have sufficiently smooth negative initial data $u_0(x)$ with a single local minimum, and which tend to 0 rapidly at $\pm\infty$. The Cauchy problem for (1.1) can then be solved implicitly using the method of characteristics, which leads to

$$u(x, t_m) = u_0(\xi), \quad -x + C_m u_0(\xi)^m t_m + \xi = 0. \quad (1.11)$$

This describes a left-moving solution for any $m \in \mathbb{N}$, but the part of the solution near the minimum moves faster than the less negative parts, so that the slope steepens at the left of the negative hump as t_m increases. The time t_m^c where the slope becomes vertical is called the time of gradient catastrophe, and is given by

$$t_m^c = \frac{1}{m \max_{\xi \in \mathbb{R}} \{-C_m u_0^{m-1}(\xi) u_0'(\xi)\}}. \quad (1.12)$$

After this time, the solution to (1.11) is no longer single-valued.

In order to determine the point of gradient catastrophe x^c and $u^c = u(x^c, t_m^c)$, one has to solve the system of three equations

$$F(u; x, t_m) := -x + C_m u^m t_m + f_L(u) = 0, \quad (1.13)$$

$$F'(u; x, t_m) = m C_m u^{m-1} t_m + f'_L(u) = 0, \quad (1.14)$$

$$F''(u; x, t_m) = m(m-1) C_m u^{m-2} t_m + f''_L(u) = 0, \quad (1.15)$$

for the unknowns u, x, t_m . Here primes denote derivatives with respect to u , and f_L is the inverse of the decreasing part of the initial data u_0 . Among the possibly many solutions of (1.13)-(1.15), the point of gradient catastrophe is the solution $(u^c; x^c, t_m^c)$ with minimal time t_m^c . We say that the gradient catastrophe is generic if

$$k := -F'''(u^c; x^c, t_m^c) = -m(m-1)(m-2)C_m(u^c)^{m-3}t_m^c - f'''_L(u^c) \neq 0. \quad (1.16)$$

In the case $m = 1$, it is well-known that the dispersive term $\epsilon^2 u_{xxx}$ in (1.4) regularizes the gradient catastrophe that occurs for the Hopf equation $u_{t_1} + 6uu_x = 0$: the solution to the KdV equation exists for all $t_1 > 0$ under suitable conditions on the initial data $u_0(x)$. For $t_1 < t_1^c$, the KdV solution is approximated by the Hopf solution for small $\epsilon > 0$, and for $t_1 > t_1^c$, an interval of rapid oscillations is formed where the KdV solution can be modeled using Jacobi elliptic θ -functions [26, 31, 39, 11, 12, 24].

For $m > 1$ we have not been able to find results about global existence in time in the literature. For initial data in weighted Sobolev spaces only local results stating that the solution exists (and stays in the same space) for small times t_m seem to be available, see e.g. [28]. For the class of analytic initial data we will consider, global existence in time should not be an issue for any $m \in \mathbb{N}$. We will comment on this later, see Remark 2.2. However our aim is not to prove an existence result for the KdV hierarchy, rather we want to study the asymptotic behavior for small ϵ of the solution $u(x, t_m, \epsilon)$ to the Cauchy problem for (1.2) under the assumption that it exists. Before gradient catastrophe, one does not expect to see a significant difference between solutions to (1.1) and (1.2) for small ϵ . After the gradient catastrophe, the KdV hierarchy solution will also develop a region of oscillatory behavior for small ϵ .

The purpose of this manuscript is twofold. First, for $t_m < t_m^c$, we will prove that $u(x, t_m, \epsilon) = u(x, t_m) + o(\epsilon)$ as $\epsilon \rightarrow 0$, where $u(x, t_m)$ is the solution to the dispersionless equation. Secondly, for $t_m \approx t_m^c, x \approx x^c$, we will show that the KdV hierarchy solution can be approximated for small ϵ by a special Painlevé transcendent $U = U(X, T)$, which solves the fourth order ODE [27, 2, 13, 29]

$$X = T U - \left[\frac{1}{6} U^3 + \frac{1}{24} (U_X^2 + 2U U_{XX}) + \frac{1}{240} U_{XXX} \right]. \quad (1.17)$$

This ODE is the second member of the Painlevé I hierarchy, and we refer to it as the P_1^2 equation. The relevant solution is real and has the asymptotic behavior

$$U(X, T) = \mp (6|X|)^{1/3} \mp \frac{1}{3} 6^{2/3} T |X|^{-1/3} + \mathcal{O}(|X|^{-1}), \quad \text{as } X \rightarrow \pm\infty, \quad (1.18)$$

for any fixed $T \in \mathbb{R}$, and has no poles for real values of X and T [5, 34, 35]. It is also remarkable that $U(X, T)$ is an exact solution to the KdV equation normalized as

$$U_T + U U_X + \frac{1}{12} U_{XXX} = 0. \quad (1.19)$$

It was conjectured by Dubrovin in [13] that, for any Hamiltonian perturbation of a hyperbolic equation [13, 32], a generic solution $u(x, t, \epsilon)$ has an asymptotic expansion of the form

$$u(x, t, \epsilon) = u^c + a_1 \epsilon^{2/7} U \left(a_2 \epsilon^{-6/7} (x - x^c - a_3(t - t^c)), a_4 \epsilon^{-4/7} (t - t^c) \right) + o \left(\epsilon^{2/7} \right), \quad (1.20)$$

for x, t near the point of gradient catastrophe x^c, t^c of the unperturbed equation, and with constants a_1, a_2, a_3, a_4 depending only on the initial data and on the equation. The expansion should hold in a double scaling limit where $\epsilon \rightarrow 0$, but at the same time x and t should tend to the point x^c and time t^c of gradient catastrophe for the unperturbed equation in such a way that the arguments of U in (1.20) remain bounded. In other words, the limit is such that $\epsilon \rightarrow 0$ and at the same time $\epsilon^{-6/7}(x - x^c - a_3(t - t^c))$ and $\epsilon^{-4/7}(t - t^c)$ remain bounded. The Painlevé transcendent $U(X, T)$ is thus conjectured to describe the behavior of the solution to the perturbed equation near the point of gradient catastrophe for the unperturbed equation, and is expected to be universal in the sense that it is independent of the choice of the equation and independent of the choice of initial data. The only quantities in (1.20) that depend on the initial data and on the equation are the constants a_1, a_2, a_3, a_4 , and the values of x^c, t^c, u^c . The asymptotic formula (1.20) was shown numerically for a certain class of equations including the KdV equation and the second member of the KdV hierarchy [25, 15].

We prove this conjecture in the special case of the KdV hierarchy for a class of analytic initial data with a single negative hump. So far, the conjecture had been proven only for the KdV equation [3]. Similar results appear also in double scaling limits for Hermitian random matrix ensembles [6] and in the semiclassical limit of the focusing nonlinear Schrödinger equation [14].

1.1 Statement of results

We study the Cauchy problem for equation (1.2) with $m \in \mathbb{N}$. Similarly as in [3], we impose the following conditions on the initial data u_0 .

Assumptions 1.1

- (a) $u_0(x)$ is real analytic and has an analytic continuation to the complex plane in the domain

$$\mathcal{S} = \{z \in \mathbb{C} : |\operatorname{Im} z| < \tan \theta |\operatorname{Re} z|\} \cup \{z \in \mathbb{C} : |\operatorname{Im} z| < \sigma\}$$

where $0 < \theta < \pi/2$ and $\sigma > 0$;

- (b) $u_0(x)$ decays as $|x| \rightarrow \infty$ in $\mathcal{S}$ such that

$$u_0(x) = \mathcal{O} \left(\frac{1}{|x|^{3+s}} \right), \quad s > 0, \quad x \in \mathcal{S}, \quad (1.21)$$

- (c) for real x , $u_0(x) < 0$ and u_0 has a single local minimum at a certain point x_M , with

$$u'_0(x_M) = 0, \quad u''_0(x_M) > 0.$$

Without loss of generality, we assume that u_0 is normalized such that $u_0(x_M) = -1$.

We prove the following result.

Theorem 1.2 *Let $u_0(x)$ satisfy the conditions described in Assumptions 1.1, and let $m \in \mathbb{N}$, $x \in \mathbb{R}$ and $t_m < t_m^c$, where t_m^c is the time of gradient catastrophe for (1.1) given by (1.12). If $u(x, t_m, \epsilon)$ solves equation (1.2) with initial condition $u(x, 0, \epsilon) = u_0(x)$, then we have*

$$u(x, t_m, \epsilon) = u(x, t_m) + o(\epsilon), \quad \text{as } \epsilon \rightarrow 0, \quad (1.22)$$

where $u(x, t_m)$ is the solution to the Cauchy problem for (1.1), given by (1.11).

Remark 1.3 We will prove this result in detail for values of x, t_m where $u(x, t_m)$ is decreasing, i.e. for $x < x_M + C_m t_m$ since $x_M + C_m t_m$ is the position of the minimum at time t_m . For $x \geq x_M + C_m t_m$, the proof is similar, but several technical modifications are needed. We will discuss those changes in Remark 4.1.

Our second result describes the behavior of $u(x, t_m, \epsilon)$ when x, t_m approach the point and time of gradient catastrophe at appropriate speeds. We will prove that (1.20) holds, with the values of the constants given by

$$a_1 = \frac{2}{(8k)^{2/7}}, \quad a_2 = \frac{1}{(8k)^{1/7}}, \quad (1.23)$$

$$a_3 = C_m(u^c)^m, \quad a_4 = \frac{2mC_m(u^c)^{m-1}}{(8k)^{3/7}}, \quad (1.24)$$

where k was defined in (1.16). We will give an asymptotic expansion as $\epsilon \rightarrow 0$ for (x, t_m) in a space-time window of size $\mathcal{O}(\epsilon^{4/7})$, and in addition $x - x_c$ must be equal to $C_m(u^c)^m(t - t_m^c)$ plus a correction of size $\mathcal{O}(\epsilon^{6/7})$. In this shrinking (as $\epsilon \rightarrow 0$) region in the (x, t_m) -plane, the transition takes place between asymptotics determined by (1.1) and the oscillatory asymptotics that are expected to be present for $t_m > t_m^c$. The transition is described by the Painlevé transcendent $U(X, T)$. In addition, we will also compute the next term in the asymptotic expansion (1.20), which is of order $\mathcal{O}(\epsilon^{4/7})$. This term is rather complicated but can still be expressed completely in terms of the Painlevé transcendent $U(X, T)$. We have an expansion of the form

$$\begin{aligned} u(x, t_m, \epsilon) = & u_c + a_1 \epsilon^{2/7} U + c_1 \epsilon^{4/7} (QU_X + U_{XX} + 4U^2 - 3c_2 U_T) \\ & + c_3 \epsilon^{4/7} (2U_X Q_T + 4UU_T + \frac{1}{2} U_{XXT}) + \mathcal{O}(\epsilon^{5/7}), \end{aligned} \quad (1.25)$$

where $Q = Q(X, T)$ is given by

$$Q = \frac{1}{240} U_X U_{XXX} - \frac{U_{XX}^2}{480} + XU - \frac{T}{2} U^2 + \frac{U^4}{24} + \frac{1}{24} UU_X^2,$$

and we used the abbreviations

$$U = U\left(a_2 \epsilon^{-6/7}(x - x^c - a_3(t - t^c)), a_4 \epsilon^{-4/7}(t - t^c)\right), \quad (1.26)$$

$$Q = Q\left(a_2 \epsilon^{-6/7}(x - x^c - a_3(t - t^c)), a_4 \epsilon^{-4/7}(t - t^c)\right). \quad (1.27)$$

The values of c_1, c_2, c_3 are

$$c_1 = \frac{32F^{(4)}(u^c)}{63(8k)^{11/7}}, \quad (1.28)$$

$$c_2 = \frac{(x - x^c) - C_m(u^c)^m(t_m - t_m^c)}{(8k)^{1/7} \epsilon^{6/7}}, \quad (1.29)$$

$$c_3 = \frac{mC_m(u^c)^{m-1}(t_m - t_m^c)}{4k\epsilon^{4/7}} \left(\frac{2(m-1)}{5u^c} + \frac{2F^{(4)}(u^c)}{21k} \right), \quad (1.30)$$

with k given by (1.16), and F by (1.13).

Theorem 1.4 *Let $u_0(x)$ satisfy the conditions described in Assumptions 1.1 and assume that the generic condition*

$$k := -F'''(u^c) = -m(m-1)(m-2)(u^c)^{m-3} - f_L'''(u^c) \neq 0 \quad (1.31)$$

holds with $m \in \mathbb{N}$. Write u^c, x^c, t_m^c for the solution to the system (1.13)-(1.14)-(1.15). Let us take a double scaling limit where $\epsilon \rightarrow 0$ and at the same time $x \rightarrow x^c$ and $t_m \rightarrow t_m^c$ in such a way that, for some $X, T \in \mathbb{R}$,

$$\lim \frac{x - x^c - C_m(u^c)^m(t_m - t_m^c)}{(8k)^{\frac{1}{7}}\epsilon^{\frac{6}{7}}} = X, \quad \lim \frac{2mC_m(u^c)^{m-1}(t_m - t_m^c)}{(8k)^{\frac{3}{7}}\epsilon^{\frac{4}{7}}} = T. \quad (1.32)$$

If $u(x, t_m, \epsilon)$ solves equation (1.2) with initial condition $u(x, 0, \epsilon) = u_0(x)$, the asymptotic expansion (1.25) holds in the double scaling limit.

Remark 1.5 The real numbers X, T are seen as parameters: for any choice of the parameters, there is a different double scaling limit. Our result is stated point-wise in X and T but could be strengthened to a uniform statement for $X, T \in [-M, M]$ for any $M > 0$.

Remark 1.6 We believe that our results can be extended, using the same techniques, to linear combinations of different members of the KdV hierarchy of the form

$$L_t = [L, \sum_{j=1}^m a_j A_j], \quad a_j \geq 0, \quad (1.33)$$

where L and A_j are the same operators as the ones used in (1.7). Indeed equations of this form are integrable and the procedure we follow below can be applied, see also Remark 2.1. We thank the referee for pointing this out.

The proofs of our results are based on the direct and inverse scattering transform for the KdV hierarchy. This approach relies on the Lax representation (1.7), and the inverse scattering transform can be formulated as a Riemann-Hilbert problem, where one searches for a function which satisfies a prescribed jump condition, depending on the reflection coefficient for the Schrödinger equation $Lf = \lambda f$. Solving the RH problem asymptotically as $\epsilon \rightarrow 0$ leads to small dispersion asymptotics for $u(x, t_m, \epsilon)$. We will use a Deift/Zhou steepest descent method similar to the one in [11, 12] for the asymptotic analysis of the RH problem. This method consists of a series of transformations $M \mapsto T \mapsto S \mapsto R$ of the RH problem, which results at the end in a RH problem for R which can be solved approximately for small ϵ . The first transformation $M \mapsto T$ involves the construction of a G -function satisfying convenient jump and asymptotic conditions. The second transformation $T \mapsto S$ deforms the jump contour from the real line to a lens-shaped contour. The last transformation $S \mapsto R$ requires the construction of local and global parametrices. A local Airy parametrix will be needed for the proof of Theorem 1.2. For the proof of Theorem 1.4, we will need to build a local parametrix out of a model RH problem related to the P_1^2 equation. The most important new features here compared to [11, 3] are the generalization of the G -function to the case $m > 1$, and the generalized construction of the local P_1^2 parametrix.

2 Riemann-Hilbert problem for the KdV hierarchy

We construct a RH problem using particular solutions to the Schrödinger equation $Lf = \lambda f$, with L given by (1.8) with potential $u = u(x, t_m, \epsilon)$. This construction is well understood [10, 1, 38, 3, 16], but we summarize the main lines here for the convenience of the reader.

For negative u , the Schrödinger operator has no point spectrum. Moreover, if u solves the equation (1.2), as a consequence of the Lax equation (1.7), the eigenvalues of L are independent of t_m . Since our initial data u_0 are negative, it follows that L has no point spectrum at any time $t_m > 0$. If

$$\int_{-\infty}^{+\infty} |u(x, t_m, \epsilon)|(1 + x^2)dx < \infty, \quad (2.1)$$

there exist [10] fundamental Jost solutions $\psi_{\pm} = \psi_{\pm}(\lambda; x, t_m, \epsilon)$ and $\phi_{\pm} = \phi_{\pm}(\lambda; x, t_m, \epsilon)$ to the Schrödinger equation satisfying the asymptotic conditions

$$\lim_{x \rightarrow +\infty} \psi_{\pm}(z; x, t_m, \epsilon) e^{\pm \frac{i}{\epsilon} \sqrt{-\lambda} x} = 1, \quad (2.2)$$

$$\lim_{x \rightarrow -\infty} \phi_{\pm}(z; x, t_m, \epsilon) e^{\mp \frac{i}{\epsilon} \sqrt{-\lambda} x} = 1, \quad (2.3)$$

for $\lambda \in \mathbb{C} \setminus \{0\}$. We fix $\sqrt{-\lambda}$ to be the principal branch of the square root which is analytic for $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$ and positive for $\lambda < 0$. We consider $\psi_{\pm}$ and $\phi_{\pm}$ as functions in the variable λ , whereas x, t_m, ϵ will be parameters. Those solutions can be constructed as a solution to Volterra integral equations as in [10], and the analysis of the integral equations shows that ψ_{-} and ϕ_{-} can be continued analytically for λ in the lower half plane, and ψ_{+} and ϕ_{+} to the upper half plane. Also asymptotics as $\lambda \rightarrow \infty$ for $\phi_{\pm}$ and $\psi_{\pm}$ can be deduced.

The fundamental solutions $\psi_{\pm}$ and $\phi_{\pm}$ are related as follows,

$$\begin{pmatrix} \psi_{+}(\lambda) & \psi_{-}(\lambda) \end{pmatrix} = \begin{pmatrix} \phi_{-}(\lambda) & \phi_{+}(\lambda) \end{pmatrix} \begin{pmatrix} a(\lambda; t_m, \epsilon) & \bar{b}(\lambda; t_m, \epsilon) \\ b(\lambda; t_m, \epsilon) & \bar{a}(\lambda; t_m, \epsilon) \end{pmatrix}, \quad \lambda < 0, \quad (2.4)$$

with

$$|a(\lambda)|^2 - |b(\lambda)|^2 = 1, \quad \text{for } \lambda < 0, \quad (2.5)$$

and where a and b are independent of x . The quantities

$$r(\lambda; t_m, \epsilon) := \frac{b(\lambda; t_m, \epsilon)}{a(\lambda; t_m, \epsilon)}, \quad t(\lambda; t_m, \epsilon) := \frac{1}{a(\lambda; t_m, \epsilon)},$$

are the reflection and transmission coefficients (from the left) for the Schrödinger equation and depend on t_m, ϵ through $u(x, t_m, \epsilon)$. They are continuous for $\lambda \leq 0$. In particular we have

$$\frac{\psi_{+}}{a} \sim \begin{cases} e^{-\frac{i}{\epsilon} \sqrt{-\lambda} x} + \frac{b}{a} e^{\frac{i}{\epsilon} \sqrt{-\lambda} x}, & x \rightarrow -\infty, \\ \frac{1}{a} e^{-\frac{i}{\epsilon} \sqrt{-\lambda} x}, & x \rightarrow +\infty. \end{cases} \quad (2.6)$$

If u solves the higher order KdV equation (1.2), the Gardner-Greene-Kruskal-Miura [22] relations are

$$\frac{da}{dt_m} = 0, \quad \frac{db}{dt_m} = \frac{2i}{\epsilon} 4^m t_m (-\lambda)^{\frac{2m+1}{2}} b. \quad (2.7)$$

Indeed, (1.7) implies that $\frac{d}{dt}\psi_+ + A_m\psi_+$ is also a solution to the Schrödinger equation. The asymptotics as $x \rightarrow +\infty$ then imply that

$$\frac{d}{dt_m}\psi_+ = -A_m\psi_+ + \frac{4^m i}{\epsilon}(-\lambda)^{\frac{2m+1}{2}}\psi_+.$$

Together with (2.6) this implies (2.7) and thus

$$r(\lambda; t_m, \epsilon) = r(\lambda; 0, \epsilon) e^{\frac{8i}{\epsilon} t_m (-\lambda)^{\frac{2m+1}{2}}}.$$

Remark 2.1 For the more general equations of the form (1.33), the time evolution of the reflection coefficient is given by

$$r(\lambda; t, \epsilon) = r(\lambda; 0, \epsilon) e^{\frac{8it}{\epsilon} \sum_{j=1}^m a_j (-\lambda)^{\frac{2j+1}{2}}}.$$

This can be deduced immediately from the Lax form (1.33) of the equations.

The transmission coefficient is analytic for λ in the upper half plane, and the reflection coefficient is analytic in a region of the form $\{\pi - \theta_0 < \arg \lambda < \pi\}$, with $\theta_0 > 0$. If the potential u is smooth in x , the reflection coefficient decays rapidly as $\lambda \rightarrow -\infty$.

We will now construct the solution to a RH problem using the Jost solutions $\psi_{\pm}$ and $\phi_{\pm}$: write $M = M(\lambda; x, t_m, \epsilon)$ by

$$M(\lambda; x, t_m, \epsilon) = \begin{cases} \begin{pmatrix} \phi_+ & \frac{1}{a}\psi_+ \\ \epsilon \frac{d}{dx}\phi_+ & \frac{\epsilon}{a} \frac{d}{dx}\psi_+ \end{pmatrix} e^{\frac{-i}{\epsilon}\alpha(\lambda; x, t_m)\sigma_3}, & \text{as } \lambda \in \mathbb{C}^+, \\ \begin{pmatrix} \frac{1}{a^*}\psi_- & \phi_- \\ \frac{\epsilon}{a^*} \frac{d}{dx}\psi_- & \epsilon \frac{d}{dx}\phi_- \end{pmatrix} e^{\frac{-i}{\epsilon}\alpha(\lambda; x, t_m)\sigma_3}, & \text{as } \lambda \in \mathbb{C}^-, \end{cases} \quad (2.8)$$

where

$$\phi_{\pm} = \phi_{\pm}(\lambda; x, t_m, \epsilon), \quad \phi_{\pm} = \phi_{\pm}(\lambda; x, t_m, \epsilon), \quad (2.9)$$

$$a = a(\lambda; \epsilon), \quad a^* = \overline{a(\bar{\lambda}; \epsilon)}, \quad (2.10)$$

and

$$\alpha(\lambda; x, t_m) = x(-\lambda)^{\frac{1}{2}} + 4^m t_m (-\lambda)^{m+\frac{1}{2}}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.11)$$

Using (2.4) and the asymptotics for the Jost solutions as $\lambda \rightarrow \infty$, one shows that M solves a RH problem, see [1, 38, 3]:

RH problem for M

- (a) $M(\lambda; x, t_m, \epsilon)$ is analytic for $\lambda \in \mathbb{C} \setminus \mathbb{R}$,
- (b) M has continuous boundary conditions $M_{\pm}(\lambda)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ that satisfy the jump conditions

$$M_+(\lambda) = M_-(\lambda) \begin{pmatrix} 1 & r_0(\lambda; \epsilon) e^{2i\alpha(\lambda; x, t_m)/\epsilon} \\ -\bar{r}_0(\lambda; \epsilon) e^{-2i\alpha(\lambda; x, t_m)/\epsilon} & 1 - |r_0(\lambda; \epsilon)|^2 \end{pmatrix} \quad \text{for } \lambda < 0, \quad (2.12)$$

$$M_+(\lambda) = M_-(\lambda) \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } \lambda > 0, \quad (2.13)$$

with $r_0(\lambda; \epsilon) := r(\lambda; 0, \epsilon)$.

(c) We have

$$M(\lambda; x, t_m, \epsilon) = \begin{pmatrix} 1 & 1 \\ i\sqrt{-\lambda} & -i\sqrt{-\lambda} \end{pmatrix} \left(I - \frac{q}{2i\epsilon\sqrt{-\lambda}} \sigma_3 + \mathcal{O}(\lambda^{-1}) \right), \text{ for } \lambda \rightarrow \infty, \quad (2.14)$$

with $q = q(x, t_m, \epsilon)$ independent of λ , and $u = q_x$ is the solution to equation (1.2) with initial data u_0 .

In other words, the solution to the KdV hierarchy with initial data u_0 can be recovered from the solution to the RH problem by the formula

$$u(x, t_m, \epsilon) = -2i\epsilon \partial_x M_{1,11}(x, t_m, \epsilon), \quad \partial_x = \frac{\partial}{\partial x}, \quad (2.15)$$

where $M_{1,11}$ denotes the $(1,1)$ -entry of the matrix M_1 defined as follows in terms of the $(1,1)$ -entry of M ,

$$M_{11}(\lambda; x, t_m, \epsilon) = 1 + \frac{M_{1,11}(x, t_m, \epsilon)}{\sqrt{-\lambda}} + \mathcal{O}(1/\lambda), \quad \text{as } \lambda \rightarrow \infty.$$

Remark 2.2 Using the vanishing lemma approach developed in [19, 18, 17], one can show that the RH problem for M is solvable for any value of $x, t \in \mathbb{R}$, $\epsilon > 0$ (not necessarily small) if r_0 has sufficient regularity and sufficient decay at $-\infty$ and if $|r_0(\lambda)| < 1$ for $\lambda < 0$. The solvability of the RH problem can be used to prove that the Cauchy problem for equation (1.2) is solvable for initial data in a suitable space following the proofs in [41].

3 Asymptotic analysis of the RH problem as $\epsilon \rightarrow 0$

This section contains the asymptotic analysis of the RH problem for M as $\epsilon \rightarrow 0$. We will study the RH problem for fixed x, t_m (for the proof of Theorem 1.2), and in a double scaling limit for $x \approx x^c$, $t_m \approx t_m^c$ (for the proof of Theorem 1.4). The construction of the G -function in Section 3.1 is almost the same in both cases, and also for the transformations $M \mapsto T \mapsto S$ in Sections 3.2 and 3.3 and the construction of the outside parametrix in Section 3.4, we do not need to distinguish between the regular and the critical case. It is only when we construct a local parametrix that there is an essential difference. For the regular case, we assume that x and t_m are such that $u_x(x, t_m) < 0$, i.e. we consider only the decreasing part where $x < x_M + C_m t_m$, the position of the minimum at time t_m . For the increasing part, several changes have to be made, see Remark 4.1.

3.1 The G -function

In this section, we will define a G -function, which will be needed to modify the jumps of the RH problem in a suitable way. Let us first write the Abel transform

$$F_A(\lambda; x, t_m) = \frac{1}{2} \int_{\lambda}^0 \frac{F(\xi; x, t_m) d\xi}{\sqrt{\xi - \lambda}}, \quad \text{for } \lambda \in [-1, 0),$$

where F is defined by (1.13). We define the G -function $G = G(\lambda; x, t_m; u)$ as

$$G(\lambda; x, t_m; u) = \frac{\sqrt{u - \lambda}}{\pi} \int_u^0 \frac{F_A(\eta; x, t_m)}{(\eta - \lambda)\sqrt{\eta - u}} d\eta. \quad (3.1)$$

We will have to make two different choices for $u \in (-1, 0)$. For the proof of Theorem 1.2, we need to choose $u = u(x, t_m)$ to be the solution to equation (1.1): this solution is given by (1.11) or, equivalently, by (1.13). For the proof of Theorem 1.4 on the other hand, we fix u to be $u = u^c = u(x^c, t_m^c)$. Whenever the choice of u , x , and t_m is unimportant below, we will simply write $G_m(\lambda)$ for $G_m(\lambda; x, t_m; u)$.

G is analytic for $\lambda \in \mathbb{C} \setminus [u, +\infty)$. As $\lambda \rightarrow \infty$, we have $G(\lambda) = \mathcal{O}(\lambda^{-1/2})$, and writing

$$G_1(x, t_m; u) := \lim_{\lambda \rightarrow \infty} (-\lambda)^{1/2} G(\lambda; x, t_m, u), \quad (3.2)$$

one verifies that, for both choices of u made above, we have the identity

$$\partial_x G_1(x, t_m) = \frac{u}{2}. \quad (3.3)$$

For λ on the branch cut, G satisfies the jump properties

$$G_+(\lambda) + G_-(\lambda) = 0, \quad \text{for } \lambda \in (0, +\infty), \quad (3.4)$$

$$G_+(\lambda) + G_-(\lambda) - 2\rho(\lambda) + 2\alpha(\lambda) = 0, \quad \text{for } \lambda \in (u, 0), \quad (3.5)$$

with ρ given by

$$\rho(\lambda) = \frac{1}{2} \int_{\lambda}^0 \frac{f_L(\xi)}{\sqrt{\xi - \lambda}} d\xi. \quad (3.6)$$

The function $\rho(\lambda)$ is clearly well-defined for $-1 < \lambda < 0$, but we can extend it to an analytic function. Because of the analyticity of the initial data u_0 , ρ is analytic in a neighborhood of $(-1, 0)$. It cannot be extended to an analytic function in a full neighborhood of -1 because the inverse f_L behaves like a square root near -1 , but we can extend it to an analytic function in a neighborhood of $[-1, 0)$, except for a branch cut which we choose along $\lambda < -1$. We write $\Omega = \Omega^+ \cup \Omega^- \cup (-1, 0)$ for such a region where ρ is analytic, with $\Omega^+ \subset \mathbb{C}^+$ and $\Omega^- = \overline{\Omega^+}$. For later convenience, we choose Ω^+ sufficiently small so that it is contained in the region $\{\pi - \theta_0 < \arg \lambda < \pi\}$ where $r_0(\lambda)$ is analytic (see Fig. 1)

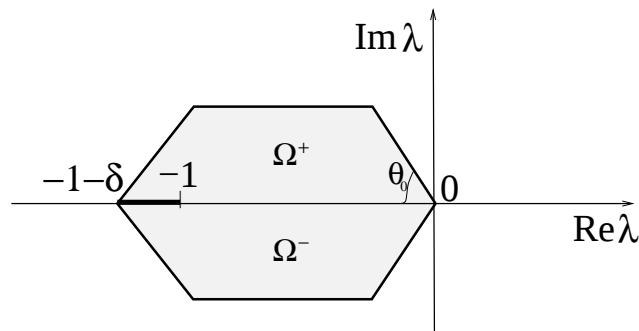


Figure 1: The regions Ω^+ and Ω^- (note that $(-1 - \delta, -1]$ is not part of Ω).

Let us define an auxiliary function ϕ in $\Omega^+ \cup \Omega^-$ by

$$\phi(\lambda; x, t_m) = G(\lambda; x, t_m) - \rho(\lambda) + \alpha(\lambda; x, t_m), \quad (3.7)$$

so that ϕ is analytic across $(-1, u)$, but not on $(-1 - \delta, -1)$, because this is a part of the branch cut for ρ , and not on $(u, 0)$, because this is part of the branch cut for G . It is an analytic function for $\lambda \in \Omega \setminus [u, 0)$. By (3.5), we have

$$2\phi_+(\lambda; x, t_m) = G_+(\lambda; x, t_m) - G_-(\lambda; x, t_m), \quad \text{for } \lambda \in (u, 0). \quad (3.8)$$

After a straightforward integral calculation as in [3], one observes that ϕ can be written as

$$\begin{aligned}\phi(\lambda; x, t_m) &= \frac{1}{2} \int_u^\lambda \frac{F(\xi; x, t_m)}{\sqrt{\xi - \lambda}} d\xi \\ &= -\sqrt{u - \lambda} F(u; x, t_m) + \frac{2}{3} (u - \lambda)^{\frac{3}{2}} F'(u; x, t_m) \\ &\quad - \frac{4}{15} (u - \lambda)^{\frac{5}{2}} F''(u; x, t_m) - \frac{4}{15} \int_u^\lambda F'''(\xi; x, t_m) (\xi - \lambda)^{\frac{5}{2}} d\xi.\end{aligned}\quad (3.9)$$

3.2 First transformation $M \mapsto T$

We are now ready to perform a first transformation of the RH problem. This will lead to more convenient jump matrices for the asymptotic analysis as $\epsilon \rightarrow 0$. A crucial role is played by the G -function and its properties discussed before.

Define

$$T(\lambda; x, t_m, \epsilon) = \begin{pmatrix} 1 & 0 \\ \frac{q}{2} & 1 \end{pmatrix} M(\lambda; x, t_m, \epsilon) e^{-\frac{i}{\epsilon} G(\lambda; x, t_m) \sigma_3}. \quad (3.11)$$

We then have

RH problem for T

(a) T is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(b) $T_+(\lambda) = T_-(\lambda) v_T(\lambda)$, as $\lambda \in \mathbb{R}$, with

$$v_T(\lambda) = e^{\frac{i}{\epsilon} \mathcal{G}_-(\lambda) \sigma_3} v_M(\lambda) e^{-\frac{i}{\epsilon} \mathcal{G}_+(\lambda) \sigma_3}, \quad (3.12)$$

and v_M is the jump matrix in the RH problem for M given by

$$v_M(\lambda) = \begin{cases} \begin{pmatrix} 1 & r_0(\lambda; \epsilon) e^{2i\alpha(\lambda; x, t_m)/\epsilon} \\ -\bar{r}_0(\lambda; \epsilon) e^{-2i\alpha(\lambda; x, t_m)/\epsilon} & 1 - |r_0(\lambda; \epsilon)|^2 \end{pmatrix}, & \text{for } \lambda < 0, \\ \sigma_1, & \text{for } \lambda > 0. \end{cases} \quad (3.13)$$

(c) As $\lambda \rightarrow \infty$,

$$T(\lambda) = (I + \mathcal{O}(\lambda^{-1})) \begin{pmatrix} 1 & 1 \\ i\sqrt{-\lambda} & -i\sqrt{-\lambda} \end{pmatrix}. \quad (3.14)$$

The left multiplication with the triangular matrix in (3.11) was needed to transform (2.14) in (3.14), but has no effect on the jumps for T . The jump matrix (3.12) can be simplified using the properties (3.4)-(3.5) of the G -function and the definition (3.7) of ϕ . We write the jump matrix in a different form depending on the value of $\lambda \in \mathbb{R}$. For $\lambda > 0$, by (3.4) we have

$$v_T(\lambda) = \sigma_1. \quad (3.15)$$

For a sufficiently small choice of $\delta_1 > 0$, we can write the jump matrix in terms of the previously defined function ϕ on the interval $(-1 - \delta_1, 0)$. For $\lambda \in (u, 0)$, we use (3.5) and (3.8) to conclude that

$$v_T(\lambda) = \begin{pmatrix} e^{-\frac{2i}{\epsilon} \phi_+(\lambda)} & i\kappa_+(\lambda) \\ i\kappa_-^*(\lambda) & (1 - |r_0(\lambda)|^2) e^{\frac{2i}{\epsilon} \phi_+(\lambda)} \end{pmatrix}, \quad \text{as } \lambda \in (u, 0), \quad (3.16)$$

where we have written κ for

$$\kappa(\lambda; \epsilon) = -ir_0(\lambda; \epsilon)e^{\frac{2i}{\epsilon}\rho(\lambda)}, \quad \text{for } \lambda \in \Omega^+, \quad (3.17)$$

with boundary values on $\mathbb{R}$ denoted by $\kappa_+(\lambda)$ and $\kappa_-^*(\lambda) = \bar{\kappa}_+(\lambda)$. For $\lambda \in (-1 - \delta_1, u)$, by (3.5) and (3.7),

$$v_T(\lambda) = \begin{pmatrix} 1 & i\kappa_+(\lambda)e^{\frac{2i}{\epsilon}\phi_+(\lambda)} \\ i\kappa_-^*(\lambda)e^{-\frac{2i}{\epsilon}\phi_-(\lambda)} & 1 - |r_0(\lambda)|^2 \end{pmatrix}. \quad (3.18)$$

Here, the boundary values $\phi_\pm$ are needed only on $(-1 - \delta_1, -1]$, on $(-1, u)$, we have $\phi = \phi_\pm$. Finally, on $(-\infty, -1 - \delta_1)$, we have

$$v_T(\lambda) = \begin{pmatrix} 1 & r_0(\lambda)e^{\frac{2i}{\epsilon}(\mathcal{G}(\lambda) + \alpha(\lambda))} \\ -\bar{r}_0(\lambda)e^{-\frac{2i}{\epsilon}(\mathcal{G}(\lambda) + \alpha(\lambda))} & 1 - |r_0(\lambda)|^2 \end{pmatrix}, \quad \text{as } \lambda < -1 - \delta_1. \quad (3.19)$$

Using (3.3), (3.11), and (2.15), we recover the solution of the higher order KdV equation by

$$\begin{aligned} u(x, t_m, \epsilon) &= 2\partial_x \mathcal{G}_1(x, t_m; u) - 2i\epsilon \partial_x T_{11}^1(x, t_m, \epsilon) \\ &= u - 2i\epsilon \partial_x T_{11}^1(x, t_m, \epsilon), \end{aligned} \quad (3.20)$$

where T_{11}^1 is given by

$$T_{11}(\lambda; x, t_m, \epsilon) = 1 + \frac{T_{11}^1(x, t_m, \epsilon)}{\sqrt{-\lambda}} + O(\lambda^{-1}), \quad \text{as } \lambda \rightarrow \infty.$$

The aim of this RH analysis is to end up with jump matrices that decay to the identity matrix when $\epsilon \rightarrow 0$. To get a feeling for the small ϵ behavior of the jump matrix v_T , we need to have information about the reflection coefficient $r_0(\lambda; \epsilon)$ for small values of ϵ . We have the following results, see [3, 4] in combination with [37], for any choice of $\delta_1 > 0$.

(i) For $\lambda < -1 - \delta_1$,

$$r_0(\lambda; \epsilon) = \mathcal{O}(e^{-c\sqrt{-\lambda}/\epsilon}), \quad \text{as } \epsilon \rightarrow 0, \quad (3.21)$$

with $c > 0$.

(ii) for λ lying in a region Ω_+ as defined in Section 3.1, but λ bounded away from -1 , say $|\lambda + 1| > \frac{\delta_1}{2}$, we have

$$\kappa(\lambda; \epsilon) = 1 + \mathcal{O}(\epsilon), \quad \text{as } \epsilon \rightarrow 0, \quad (3.22)$$

(iii) for $\lambda \in (u + \delta_1, 0)$, we have

$$(1 - |r_0(\lambda)|^2)e^{\frac{2i}{\epsilon}\phi_+(\lambda)} = \mathcal{O}(e^{-c/\epsilon}), \quad c > 0, \quad \text{as } \epsilon \rightarrow 0. \quad (3.23)$$

Statements (i) and (ii) were shown in [3], and (iii) was shown for $m = 1$ and $u = u^c$, but exactly the same argument as in [3, Proposition 3.3] applies to the case $m > 1$ and $u = u(x, t_m)$.

This implies that the jump matrix $v_T(\lambda)$ tends to $i\sigma_1$ for $\lambda \in (u, 0)$, and that it tends to I for $\lambda < -1 - \delta_1$. On $(-1 - \delta_1, u)$, the jump matrix is oscillatory for small ϵ . In the next section, we will deform the contour in such a way that the oscillatory behavior turns into exponential decay.

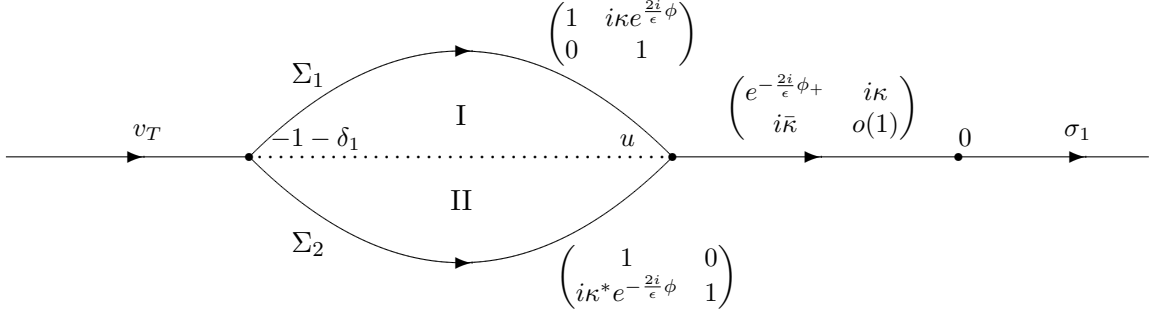


Figure 2: The jump contour Σ_S and the jumps for S

3.3 Opening of the lens $T \mapsto S$

The jump matrix $v_T(\lambda)$ can be written in the following factorized form for $-1 - \delta_1 < \lambda < u$,

$$v_T(\lambda) = \begin{pmatrix} 1 & 0 \\ i\kappa_-^*(\lambda)e^{-\frac{2i}{\epsilon}\phi_-(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} 1 & i\kappa_+(\lambda)e^{\frac{2i}{\epsilon}\phi_+(\lambda)} \\ 0 & 1 \end{pmatrix}. \quad (3.24)$$

Because the first factor is analytic in a complex region $\overline{\Omega_+}$ and the second in Ω_+ , this factorization can be used to deform the jump contour: the interval $(-1 - \delta_1, u)$ can be deformed to a lens-shaped contour as shown in Figure 2.

Define S as follows,

$$S(\lambda) = \begin{cases} T(\lambda) \begin{pmatrix} 1 & -i\kappa(\lambda)e^{\frac{2i}{\epsilon}\phi(\lambda)} \\ 0 & 1 \end{pmatrix}, & \text{in region I,} \\ T(\lambda) \begin{pmatrix} 1 & 0 \\ i\kappa^*(\lambda)e^{-\frac{2i}{\epsilon}\phi(\lambda)} & 1 \end{pmatrix}, & \text{in region II,} \\ T(\lambda), & \text{elsewhere,} \end{cases} \quad (3.25)$$

with $\kappa^*(\lambda) = \bar{\kappa}(\bar{\lambda})$. Since κ (resp. κ^*) is analytic in Ω_+ (resp. $\overline{\Omega_+}$), S is analytic in each of the regions in Figure 2 if we choose the lens sufficiently close to the real line so that it lies in Ω .

These are the RH conditions for S .

RH problem for S

(a) S is analytic in $\mathbb{C} \setminus \Sigma_S$,

(b) $S_+(\lambda) = S_-(\lambda)v_S$ for $\lambda \in \Sigma_S$, with

$$v_S(\lambda) = \begin{cases} \begin{pmatrix} 1 & i\kappa(\lambda)e^{\frac{2i}{\epsilon}\phi(\lambda)} \\ 0 & 1 \end{pmatrix}, & \text{on } \Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ i\kappa^*(\lambda)e^{-\frac{2i}{\epsilon}\phi(\lambda)} & 1 \end{pmatrix}, & \text{on } \Sigma_2, \\ \begin{pmatrix} e^{-\frac{2i}{\epsilon}\phi_+(\lambda)} & i\kappa(\lambda) \\ i\bar{\kappa}(\lambda) & (1 - |r_0(\lambda)|^2)e^{\frac{2i}{\epsilon}\phi_+(\lambda)} \end{pmatrix}, & \text{as } \lambda \in (u, 0), \\ v_T(\lambda), & \text{as } \lambda \in (-\infty, -1 - \delta_1) \cup (0, +\infty). \end{cases} \quad (3.26)$$

$$(c) \ S(\lambda) = (I + \mathcal{O}(\lambda^{-1})) \begin{pmatrix} 1 & 1 \\ i\sqrt{-\lambda} & -i\sqrt{-\lambda} \end{pmatrix} \text{ as } \lambda \rightarrow \infty.$$

For large λ , $S(\lambda) = T(\lambda)$, and by (3.20) we have

$$u(x, t, \epsilon) = u - 2i\epsilon \partial_x S_{11}^1(x, t, \epsilon), \quad (3.27)$$

where

$$S_{11}(\lambda; x, t, \epsilon) = 1 + \frac{S_{11}^1(x, t, \epsilon)}{\sqrt{-\lambda}} + \mathcal{O}(\lambda^{-1}), \quad \text{as } \lambda \rightarrow \infty. \quad (3.28)$$

For $u = u(x, t_m)$ in the definition of the G -function (3.1), we have $F(u; x, t_m) = 0$ if x belongs to the decreasing part of $u(x, t_m)$. For $t_m < t_m^c$ we also have $F'(\xi; x, t_m) < 0$ for $\xi \in (-1, 0)$. If the contours Σ_1 and Σ_2 are chosen sufficiently close to the real line, one uses (3.9) to verify that, for any neighborhood $\mathcal{U}$ of u , there is a $c > 0$ such that

$$\operatorname{Im} \phi(\lambda; x, t_m; u) > c, \quad \text{for } \lambda \in \Sigma_1 \setminus \mathcal{U}, \quad (3.29)$$

$$\operatorname{Im} \phi(\lambda; x, t_m; u) < -c, \quad \lambda \in \Sigma_2 \setminus \mathcal{U}, \quad (3.30)$$

$$\operatorname{Im} \phi_+(\lambda; x, t_m; u) < -c, \quad \text{for } \lambda \in (u, 0) \setminus \mathcal{U}. \quad (3.31)$$

For $u = u^c = u(x^c, t_m^c)$, we have $F(u; x^c, t_m^c) = 0$ and $F'(\xi; x^c, t_m^c) < 0$ for $\xi \in (-1, 0) \setminus \{u\}$, and one again uses (3.10) to check that the inequalities (3.29)-(3.31) hold. Since ϕ is continuous in x and t_m , there must be a $\delta > 0$ such that (3.29)-(3.31) hold also for $u = u^c$, and for $|x - x^c| < \delta$, $|t_m - t_m^c| < \delta$ (possibly for a smaller $c > 0$). The inequalities imply together with (3.21), (3.22), and (3.23) that the jump matrix v_S for S is uniformly close to constant matrices on $\Sigma_S \setminus \mathcal{U}$ as $\epsilon \rightarrow 0$: we have exponential decay to I for $\lambda < -1 - \delta_1$ and on $(\Sigma_1 \cup \Sigma_2) \setminus \mathcal{U}$, and decay to $i\sigma_1$ on $(u, 0) \setminus \mathcal{U}$, with an error of order $\mathcal{O}(\epsilon)$.

3.4 Outside parametrix

We now deal with the jumps that do not tend to the identity matrix as $\epsilon \rightarrow 0$. Therefore we ignore for a moment all jumps that tend to I as $\epsilon \rightarrow 0$, and a small neighborhood of u . We are then left with a jump $i\sigma_1$ on $(u, 0)$, and a jump σ_1 on $(0, +\infty)$. We can explicitly construct a function with those jump properties, and with the same asymptotic behavior as S . Indeed, if we define

$$P^{(\infty)}(\lambda) = (-\lambda)^{1/4} (u - \lambda)^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad (3.32)$$

we have

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus [u, +\infty) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic,

(b) $P^{(\infty)}$ satisfies the following jump conditions on $(u, +\infty)$,

$$P_+^{(\infty)} = P_-^{(\infty)} \sigma_1, \quad \text{as } \lambda \in (0, +\infty), \quad (3.33)$$

$$P_+^{(\infty)} = iP_-^{(\infty)} \sigma_1, \quad \text{as } \lambda \in (u, 0), \quad (3.34)$$

(c) $P^{(\infty)}$ has the asymptotic behavior

$$P^{(\infty)}(\lambda) = \left(I + \frac{u}{4\lambda} \sigma_3 + \mathcal{O}(\lambda^{-2}) \right) \begin{pmatrix} 1 & 1 \\ i(-\lambda)^{1/2} & -i(-\lambda)^{1/2} \end{pmatrix}, \quad \text{as } \lambda \rightarrow \infty. \quad (3.35)$$

This outside parametrix will determine the leading order asymptotic behavior of S as $\epsilon \rightarrow 0$, but to prove this, we need to control the jump matrices also in the vicinity of u . We will do this by constructing a local parametrix near u .

3.5 Local parametrix near u

So far, there was no need to distinguish between the regular case, where x, t_m are fixed and where $u = u(x, t_m)$, and the singular case, where x, t_m are involved in a double scaling limit together with ϵ and where $u = u^c = u(x^c, t_m^c)$. The choices of u will however be crucial for the construction of the local parametrix. In either case, we want to construct a local parametrix in a neighborhood $\mathcal{U}$ of u in such a way that

RH problem for P

- (a) $P : \overline{\mathcal{U}} \setminus \Sigma_S \rightarrow \mathbb{C}^{2 \times 2}$ is analytic,
- (b) P satisfies the following jump condition on $\mathcal{U} \cap \Sigma_S$,

$$P_+(\lambda) = P_-(\lambda)v_P(\lambda), \quad (3.36)$$

with v_P given by

$$v_P(\lambda) = \begin{cases} \begin{pmatrix} 1 & ie^{\frac{2i}{\epsilon}\phi(\lambda; x, t_m)} \\ 0 & 1 \end{pmatrix}, & \text{as } \lambda \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ ie^{-\frac{2i}{\epsilon}\phi(\lambda; x, t_m)} & 1 \end{pmatrix}, & \text{as } \lambda \in \Sigma_2, \\ \begin{pmatrix} e^{-\frac{2i}{\epsilon}\phi_+(\lambda; x, t_m)} & i \\ i & 0 \end{pmatrix}, & \text{as } \lambda \in (u^c, 0), \end{cases} \quad (3.37)$$

- (c) As $\epsilon \rightarrow 0$, we have the matching condition

$$P(\lambda) = (I + \mathcal{O}(\epsilon^\gamma))P^{(\infty)}(\lambda), \quad \text{for } \lambda \in \partial\mathcal{U}, \quad \gamma > 0, \quad (3.38)$$

between the local parametrix and the outside parametrix.

Such a parametrix has, for $\epsilon \rightarrow 0$, approximately the same jumps as S has on $\mathcal{U} \cap \Sigma_S$. The idea is that P will approximate S near u for small ϵ , whereas $P^{(\infty)}$ will be a good approximation elsewhere.

Note that by (3.10), $\phi(u; x, t_m)$ behaves like $c(u - \lambda)^{3/2}$ as $\lambda \rightarrow u$ for $t_m < t_m^c$, but $\phi(u^c, x^c, t_m^c)$ behaves like $c(u^c - \lambda)^{7/2}$ as $\lambda \rightarrow u^c$, so the jump matrix v_P behaves differently in the regular case than in the singular case in the vicinity of u . This is the reason why we need to build different local parametrices in both cases. In the regular case, we will be able to construct a local parametrix with the value of γ in (3.38) equal to 1, in the singular case we will have a weaker matching with $\gamma = 1/7$.

3.5.1 The regular case: $t_m < t_m^c$

Here we let $u = u(x, t_m)$ be defined by the equation (1.11), i.e. $u(x, t_m)$ is the solution to the unperturbed equation (1.1). Then we will construct the local parametrix near u using the following model RH problem.

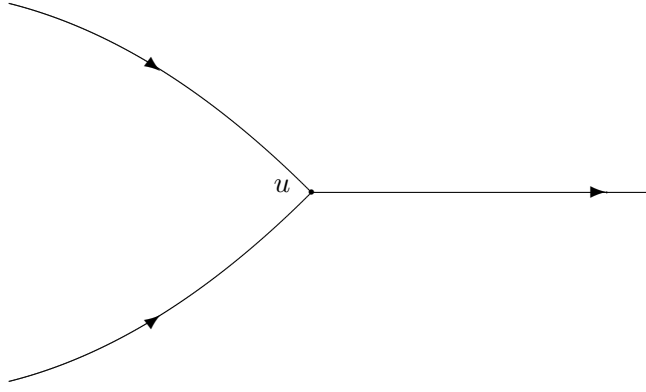


Figure 3: The jump contour for P

RH problem for Φ

- (a) Φ is analytic for $\zeta \in \mathbb{C} \setminus \Gamma$, with $\Gamma = \{\zeta \in \mathbb{C} : \arg \zeta = 0, \arg \zeta = \frac{2\pi}{3}, \text{ or } \arg \zeta = -\frac{2\pi}{3}\}$.
- (b) Φ satisfies the jump relations

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} e^{-2\theta(\zeta)} & i \\ i & 0 \end{pmatrix}, \quad \text{for } \arg \zeta = 0, \quad (3.39)$$

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & ie^{2\theta(\zeta)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \arg \zeta = \frac{2\pi}{3}. \quad (3.40)$$

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & 0 \\ ie^{2\theta(\zeta)} & 1 \end{pmatrix}, \quad \text{for } \arg \zeta = -\frac{2\pi}{3}, \quad (3.41)$$

where $\theta(\zeta) = \frac{2}{3}\zeta^{3/2}$, with branch cut on $(-\infty, 0)$.

- (c) Φ has the following behavior as $\zeta \rightarrow \infty$,

$$\Phi(\zeta) = \frac{1}{\sqrt{2}}(-\zeta)^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left(I + \mathcal{O}(\zeta^{-3/2}) \right), \quad (3.42)$$

with the branch cuts of $(-\zeta)^{\pm\frac{1}{4}}$ along $(0, +\infty)$.

Several RH problems equivalent to this one have appeared in the literature, see e.g. [8, 9, 7, 11], and it is well-known that this problem can be solved explicitly in terms of the Airy function. The unique solution to this precise problem has been constructed in [4]. In the singular case later on, we will use the same model RH problem, but with a different value for θ . We only need the existence of a 2×2 matrix-valued function Φ satisfying the above RH conditions, the precise construction in terms of the Airy function is irrelevant.

We define the parametrix $P = P(\lambda; x, t_m, \epsilon)$ by

$$P(\lambda) = E(\lambda; \epsilon) \Phi(\epsilon^{-2/3} f(\lambda)), \quad (3.43)$$

where E is an analytic function in $\mathcal{U}$ and f is a conformal map from $\mathcal{U}$ to a neighborhood of 0. Let us first define f by the requirement that

$$\theta(f(\lambda)) = \frac{2}{3}f(\lambda)^{3/2} = \widehat{\phi}(\lambda), \quad (3.44)$$

and

$$\widehat{\phi}(\lambda) = \pm i\phi(\lambda), \quad \text{for } \pm \text{Im } \lambda > 0, \quad (3.45)$$

so that the branch cut of $\widehat{\phi}$ coincides with the one of $f^{3/2}$ along $\lambda < u$. By (3.10), equation (3.44) only defines f analytically if $F(u; x, t_m) = 0$, which is true because of our choice of u . Then we have

$$f(u) = 0, \quad f'(u) = F'(u; x, t_m)^{2/3} > 0. \quad (3.46)$$

Now we can choose the lens Σ_S in $\mathcal{U}$ in such a way that f maps $\Sigma_S \cap \mathcal{U}$ to the jump contour Γ for Φ . Then P has jumps on $\Sigma_S \cap \mathcal{U}$ which are given by (3.37).

Now define E by

$$E(\lambda; \epsilon) = \frac{1}{\sqrt{2}} P^{(\infty)}(\lambda) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} (-\epsilon^{-2/3} f(\lambda))^{\frac{\sigma_3}{4}}. \quad (3.47)$$

Then it is easily verified that E is analytic in $\mathcal{U}$, and using (3.42), we obtain the matching

$$P(\lambda) P^{(\infty)}(\lambda)^{-1} = P^{(\infty)}(\lambda) (I + \mathcal{O}(\epsilon)) P^{(\infty)}(\lambda)^{-1} = I + \mathcal{O}(\epsilon), \quad (3.48)$$

for $\lambda \in \partial \mathcal{U}$ as $\epsilon \rightarrow 0$. When x, t_m approach x^c, t_m^c , the uniform convergence breaks down because $f'(u)$ approaches zero. Therefore we will now construct a different local parametrix in the singular case.

3.5.2 The singular case: double scaling limit $\epsilon \rightarrow 0, x \rightarrow x^c, t_m \rightarrow t_m^c$

In this case we need a slightly modified model RH problem, which is obtained by replacing θ in the RH problem for Φ by

$$\theta^c(\zeta; X, T) = \frac{1}{105} \zeta^{7/2} - \frac{T}{3} \zeta^{3/2} + X \zeta^{1/2}. \quad (3.49)$$

We then obtain the RH problem

RH problem for Φ^c

(a) Φ^c is analytic for $\zeta \in \mathbb{C} \setminus \Gamma^c$, with $\Gamma^c = \{\zeta \in \mathbb{C} : \arg \zeta = 0, \arg \zeta = \frac{6\pi}{7}, \text{ or } \arg \zeta = -\frac{6\pi}{7}\}$.

(b) Φ^c satisfies the jump relations

$$\Phi_+^c(\zeta) = \Phi_-^c(\zeta) \begin{pmatrix} e^{-2\theta^c(\zeta)} & i \\ i & 0 \end{pmatrix}, \quad \text{for } \arg \zeta = 0, \quad (3.50)$$

$$\Phi_+^c(\zeta) = \Phi_-^c(\zeta) \begin{pmatrix} 1 & ie^{2\theta^c(\zeta)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \arg \zeta = \frac{6\pi}{7}. \quad (3.51)$$

$$\Phi_+^c(\zeta) = \Phi_-^c(\zeta) \begin{pmatrix} 1 & 0 \\ ie^{2\theta^c(\zeta)} & 1 \end{pmatrix}, \quad \text{for } \arg \zeta = -\frac{6\pi}{7}. \quad (3.52)$$

(c) Φ^c has the following behavior as $\zeta \rightarrow \infty$,

$$\begin{aligned} \Phi^c(\zeta) = \frac{1}{\sqrt{2}} (-\zeta)^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left(I + iQ\sigma_3(-\zeta)^{-1/2} - \frac{1}{2} \begin{pmatrix} Q^2 & U \\ U & Q^2 \end{pmatrix} (-\zeta)^{-1} \right. \\ \left. - \frac{i}{2} (-\zeta)^{-3/2} \begin{pmatrix} W & V \\ -V & -W \end{pmatrix} + \mathcal{O}(\zeta^{-2}) \right). \end{aligned} \quad (3.53)$$

In [5], existence of a solution for a slightly different but equivalent RH problem has been proved, and the transformation to the RH problem for Φ^c has been given in [3]. We write $\Phi^c = \Phi^c(\zeta; X, T)$ for the solution to this RH problem. In the regular case, the constants Q, U vanished, but this is no longer true in the singular case: they now depend in a transcendental way on the parameters X, T . We have $Q = Q(X, T)$, $U = U(X, T)$, where U solves the P_I^2 equation with asymptotics given by (1.18). Furthermore we have the relations

$$Q_X = U, \quad (3.54)$$

$$W_X = Q^2 U + U^2, \quad (3.55)$$

$$V = -QU - \frac{1}{2}U_X. \quad (3.56)$$

Those identities follow from the fact that $(\Phi^c e^{-\theta^c \sigma_3})_X^c (\Phi^c e^{-\theta^c \sigma_3})^{-1}$ is a polynomial in ζ . Substituting (3.53) gives a function with terms proportional to $\zeta^{-1}, \zeta^{-2}, \dots$ as $\zeta \rightarrow \infty$. The requirement that those terms vanish, leads to (3.54)-(3.56). Using the equation (1.17) and the asymptotics for Q obtained in [5], it is straightforward to derive the identity

$$Q = \frac{1}{240}U_X U_{XX} - \frac{U_{XX}^2}{480} + XU - \frac{T}{2}U^2 + \frac{U^4}{24} + \frac{1}{24}UU_X^2. \quad (3.57)$$

For any $X_0, T_0 \in \mathbb{R}$, there are complex neighborhoods of X_0 and T_0 such that the RH problem is solvable for X, T in those neighborhoods, and such that (3.42) holds uniformly in X, T .

We want to construct a parametrix of the form

$$P(\lambda) = E(\lambda; \epsilon) \Phi^c \left(\epsilon^{-2/7} f(\lambda); \epsilon^{-6/7} g_1(\lambda; x, t_m), \epsilon^{-4/7} g_2(\lambda; t_m) \right). \quad (3.58)$$

where f, g_1, g_2, E are analytic functions which are to be determined. If we want P to satisfy the jump conditions given in (3.36), we need to construct f, g_1, g_2 in such a way that

$$\theta^c \left(\epsilon^{-2/7} f(\lambda); \epsilon^{-6/7} g_1(\lambda; x, t_m), \epsilon^{-4/7} g_2(\lambda; t_m) \right) = \frac{1}{\epsilon} \widehat{\phi}(\lambda), \quad (3.59)$$

or equivalently

$$\theta^c (f(\lambda); g_1(\lambda; x, t_m), g_2(\lambda; t_m)) = \widehat{\phi}(\lambda). \quad (3.60)$$

Moreover we want that $\epsilon^{-6/7} g_1(\lambda; x, t_m), \epsilon^{-4/7} g_2(\lambda; t_m)$ remain bounded in the double scaling limit where $\epsilon \rightarrow 0$, $x \rightarrow x^c$, $t_m \rightarrow t_m^c$ for fixed $\epsilon > 0$. This is the reason why we need to restrict to the scalings (1.32).

If we take $u = u^c = u(x^c, t_m^c)$, we have

$$F(u^c; x^c, t_m^c) = F'(u^c; x^c, t_m^c) = F''(u^c; x^c, t_m^c) = 0,$$

and (3.10) implies

$$\begin{aligned} \widehat{\phi}(\lambda; x, t_m, \epsilon) &= -\sqrt{\lambda - u^c} (F(u^c; x, t_m) - F(u^c; x^c, t_m^c)) \\ &\quad - \frac{2}{3}(\lambda - u^c)^{\frac{3}{2}} (F'(u^c; x, t_m) - F'(u^c; x^c, t_m^c)) - \frac{4}{15}(\lambda - u^c)^{\frac{5}{2}} (F''(u^c; x, t_m) - F''(u^c; x^c, t_m^c)) \\ &\quad - \frac{4}{15} \int_{u^c}^{\lambda} F'''(\xi; x, t_m) (\lambda - \xi)^{\frac{5}{2}} d\xi. \end{aligned} \quad (3.61)$$

Then it is clear that the first line in (3.61) vanishes like a square root as $\lambda \rightarrow u^c$, the second line vanishes like $c(\lambda - u^c)^{3/2}$, and the third line behaves like $c'(\lambda - u^c)^{7/2}$. Therefore we define f, g_1, g_2 by the equations

$$\frac{1}{105}f(\lambda)^{7/2} = -\frac{4}{15} \int_{u^c}^{\lambda} F'''(\xi; x^c, t_m^c)(\lambda - \xi)^{\frac{5}{2}} d\xi \quad (3.62)$$

$$g_1(\lambda; x, t_m)f(\lambda)^{1/2} = -\sqrt{\lambda - u^c}(F(u^c; x, t_m) - F(u^c; x^c, t_m^c)), \quad (3.63)$$

$$\begin{aligned} -\frac{g_2(\lambda; t_m)}{3}f(\lambda)^{3/2} &= -\frac{2}{3}(\lambda - u^c)^{\frac{3}{2}}(F'(u^c; x, t_m) - F'(u^c; x^c, t_m^c)) \\ &\quad -\frac{4}{15}(\lambda - u^c)^{\frac{5}{2}}(F''(u^c; x, t_m) - F''(u^c; x^c, t_m^c)) \\ &\quad -\frac{4}{15} \int_{u^c}^{\lambda} (F'''(\xi; x, t_m) - F'''(\xi; x^c, t_m^c))(\lambda - \xi)^{\frac{5}{2}} d\xi. \end{aligned} \quad (3.64)$$

One verifies that this defines f, g_1, g_2 analytically in $\mathcal{U}$ and moreover in such a way that (3.60) holds. A straightforward calculation yields

$$f(u^c) = 0, \quad f'(u^c) = (8k)^{\frac{2}{7}}, \quad f''(u^c) = -\frac{64}{63} \frac{F^{(4)}(u^c)}{(8k)^{5/7}}, \quad (3.65)$$

$$g_1(u^c; x, t_m) = \frac{\tilde{x} - \tilde{x}^c}{(8k)^{1/7}}, \quad g_1'(u^c, x, t_m) = -\frac{f''(u^c)}{4(8k)^{2/7}} g_1(u^c, x, t_m), \quad (3.66)$$

$$g_2(u^c; t_m) = \frac{2mC_m(u^c)^{m-1}(t_m - t_m^c)}{(8k)^{3/7}}, \quad (3.67)$$

$$g_2'(u^c; t_m) = \left(\frac{2(m-1)}{5u^c} - \frac{3f''(u^c)}{4f'(u^c)} \right) g_2(u^c; t_m), \quad (3.68)$$

where

$$\tilde{x} = x - C_m(u^c)^m t_m, \quad k = -F'''(u^c) > 0. \quad (3.69)$$

Similarly as in the regular case, we can choose the lens in such a way that $f(\Sigma_S \cap \mathcal{U}) \subset \Gamma$. Then P satisfies the jump condition (3.36) for $\lambda \in \Sigma_S \cap \mathcal{U}$.

In order to have a good matching between P and $P^{(\infty)}$ on $\partial\mathcal{U}$, we define the analytic pre-factor E by

$$E(\lambda; \epsilon) = \frac{1}{\sqrt{2}} P^{(\infty)}(\lambda) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} (-\epsilon^{-2/7} f(\lambda))^{\frac{\sigma_3}{4}}. \quad (3.70)$$

Using the definition (3.32) of $P^{(\infty)}$, one checks directly that E is analytic in $\mathcal{U}$. For $\lambda \in \partial\mathcal{U}$ and $\epsilon^{-4/7} g_1(\lambda; x, t_m), \epsilon^{-6/7} g_2(\lambda; t_m)$ in small complex neighborhoods of $X, T \in \mathbb{R}$, we have the matching

$$\begin{aligned} P(\lambda)P^{(\infty)}(\lambda)^{-1} &= P^\infty(\lambda) \left(I + iQ\sigma_3(-f(\lambda))^{-1/2}\epsilon^{1/7} \right. \\ &\quad \left. -\frac{1}{2} \begin{pmatrix} Q^2 & U \\ U & Q^2 \end{pmatrix} (-f(\lambda))^{-1}\epsilon^{2/7} - \frac{i}{2} \begin{pmatrix} W & V \\ -V & -W \end{pmatrix} (-f(\lambda))^{-3/2}\epsilon^{3/7} + \mathcal{O}(\epsilon^{4/7}) \right) P^\infty(\lambda)^{-1}, \end{aligned} \quad (3.71)$$

as $\epsilon \rightarrow 0$, with

$$U = U(\epsilon^{-6/7} g_1(\lambda; x, t_m), \epsilon^{-4/7} g_2(\lambda; t_m)), \quad (3.72)$$

$$Q = Q(\epsilon^{-6/7} g_1(\lambda; x, t_m), \epsilon^{-4/7} g_2(\lambda; t_m)), \quad (3.73)$$

$$W = W(\epsilon^{-6/7} g_1(\lambda; x, t_m), \epsilon^{-4/7} g_2(\lambda; t_m)), \quad (3.74)$$

$$V = V(\epsilon^{-6/7} g_1(\lambda; x, t_m), \epsilon^{-4/7} g_2(\lambda; t_m)). \quad (3.75)$$

In the double scaling limit given by (1.32), we have by (3.66)-(3.67) that

$$\epsilon^{-4/7}g_1(u^c; x, t_m) \rightarrow T, \quad \epsilon^{-6/7}g_2(u^c; t_m) \rightarrow X,$$

and from the definitions of g_1 and g_2 , it follows that $\epsilon^{-4/7}g_1(\lambda; x, t_m), \epsilon^{-6/7}g_2(\lambda; t_m)$ lie in a small neighborhood of X, T if λ is sufficiently close to u^c . Summarizing, the matching (3.71) holds in the double scaling limit if we have chosen $\mathcal{U}$ small enough.

3.6 Final transformation $S \mapsto R$

We define

$$R(\lambda) = \begin{cases} S(\lambda)P^{(\infty)}(\lambda)^{-1}, & \text{as } \lambda \in \mathbb{C} \setminus \mathcal{U}, \\ S(\lambda)P(\lambda)^{-1}, & \text{as } \lambda \in \mathcal{U}. \end{cases} \quad (3.76)$$

The outside parametrix has been constructed in such a way that the jumps of R are close to I as $\epsilon \rightarrow 0$. On $(0, +\infty)$, the jump for S cancels out exactly with the jump of $P^{(\infty)}$. On the other parts of $\Sigma_S \setminus \overline{\mathcal{U}}$, we have

$$R_-^{-1}(\lambda)R_+(\lambda) = P_-^{(\infty)}(\lambda)v_S(\lambda)v_{P^{(\infty)}}^{-1}(\lambda)P_-^{(\infty)}(\lambda)^{-1} \quad (3.77)$$

$$= I + \mathcal{O}(\epsilon), \quad \text{as } \epsilon \rightarrow 0. \quad (3.78)$$

For $\lambda \in \mathcal{U} \cap \Sigma_S$, we have

$$R_-^{-1}(\lambda)R_+(\lambda) = P_-(\lambda)v_S(\lambda)v_P^{-1}(\lambda)P_-^{-1}(\lambda). \quad (3.79)$$

On one hand it follows from the construction of the parametrix that $P_-(\lambda)$ is uniformly bounded by $c\epsilon^{-\alpha}$ for $\lambda \in \mathcal{U} \cap \Sigma_S$, where $\alpha = 1/6$ in the regular case and $\alpha = 1/14$ in the critical case. On the other hand

$$v_S v_P^{-1} = \begin{cases} \begin{pmatrix} 1 & i(\kappa - 1)e^{\frac{2i}{\epsilon}\phi} \\ 0 & 1 \end{pmatrix}, & \text{on } \Sigma_1 \cap \mathcal{U}, \\ \begin{pmatrix} 1 & 0 \\ i(\kappa^* - 1)e^{-\frac{2i}{\epsilon}\phi} & 1 \end{pmatrix}, & \text{on } \Sigma_2 \cap \mathcal{U}, \\ \begin{pmatrix} \kappa & i(\kappa - 1)e^{-\frac{2i}{\epsilon}\phi_+} \\ -i(1 - |r|^2)e^{\frac{2i}{\epsilon}\phi_+} & \kappa^* + (1 - |r|^2) \end{pmatrix}, & \text{on } (u^c, 0) \cap \mathcal{U}. \end{cases} \quad (3.80)$$

Except for the 21-entry on $(u^c, 0)$, the exponentials in the above matrices are uniformly bounded on the jump contours inside $\mathcal{U}$. The asymptotics (3.22) and (3.23) for κ and r ensure that $v_S v_P^{-1} = I + \mathcal{O}(\epsilon)$. We obtain the following RH problem for R .

RH problem for R

- (a) R is analytic in $\mathbb{C} \setminus \Sigma_R$, with $\Sigma_R = (\Sigma_S \cup \partial\mathcal{U}) \setminus (0, +\infty)$ as shown in Figure 4,
- (b) $R_+(\lambda) = R_-(\lambda)v_R(\lambda)$, where the jump matrix v_R is given by

$$v_R(\lambda) = \begin{cases} P(\lambda)P^{(\infty)}(\lambda)^{-1}, & \text{as } \lambda \in \partial\mathcal{U}, \\ I + \mathcal{O}(\epsilon^{1-2\alpha}), & \text{as } \lambda \in \Sigma_R \setminus \partial\mathcal{U} \text{ as } \epsilon \rightarrow 0, \end{cases} \quad (3.81)$$

with α as above.

- (c) $R(\lambda) = I + \mathcal{O}(\lambda^{-1})$ as $\lambda \rightarrow \infty$.

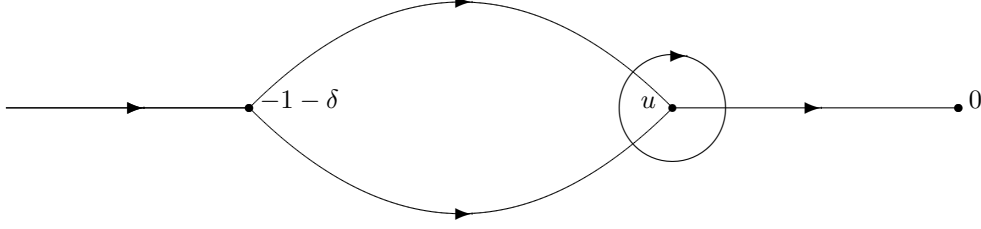


Figure 4: The contour Σ_R after the final transformation $S \mapsto R$.

From the matching (3.48) and (3.81) in the regular case where $t_m < t_m^c$, it follows that

$$v_R(\lambda) = I + \mathcal{O}(\epsilon^{2/3}), \quad \text{as } \epsilon \rightarrow 0, \quad (3.82)$$

uniformly for $\lambda \in \Sigma_R$, since $P^{(\infty)}$ is bounded on $\partial\mathcal{U}$. In the singular case, we use (3.71) to conclude that

$$v_R(\lambda) = I + \mathcal{O}(\epsilon^{1/7}), \quad (3.83)$$

uniformly for $\lambda \in \Sigma_R$, in the double scaling limit where $\epsilon \rightarrow 0$, $x \rightarrow x^c$, and $t_m \rightarrow t_m^c$ in such a way that (1.32) holds.

Following the general theory for small-norm RH problems [9], one shows that

$$R(\lambda) = I + \frac{1}{2\pi i} \int_{\Sigma_R} R_-(s)(v_R(s) - I) \frac{ds}{s - \lambda}, \quad \text{for } \lambda \in \mathbb{C} \setminus R, \quad (3.84)$$

and that

$$\|R_- - I\|_{L^2(\Sigma_R)} = \mathcal{O}(\epsilon^\gamma), \quad (3.85)$$

where $\gamma = 2/3$ in the regular case as $\epsilon \rightarrow 0$, and $\gamma = 1/7$ in the double scaling limit in the singular case.

4 Proof of the results

4.1 Proof of Theorem 1.2

Note that (3.84) implies that

$$R(\lambda) = I + \frac{R_1}{\lambda} + \mathcal{O}(\lambda^{-2}), \quad \text{as } \lambda \rightarrow \infty, \quad (4.1)$$

with

$$R_1 = R_1(x, t_m, \epsilon) = -\frac{1}{2\pi i} \int_{\Sigma_R} R_-(s)(v(s) - I) ds. \quad (4.2)$$

This gives

$$R_1 = -\frac{1}{2\pi i} \int_{\Sigma_R} (R_-(s) - I)(v(s) - I) ds - \frac{1}{2\pi i} \int_{\Sigma_R} (v_R(s) - I) ds = \mathcal{O}(\epsilon^{2/3}), \quad (4.3)$$

as $\epsilon \rightarrow 0$ by the Cauchy-Schwarz inequality. Differentiating the integral equation (3.84) and (4.3) in x , one similarly obtains the estimates

$$\|R_{x,-}\|_{L^2(\Sigma_R)} = \mathcal{O}(\epsilon^{2/3}), \quad \partial_x R_1 = \mathcal{O}(\epsilon^{2/3}), \quad (4.4)$$

using the fact that $\|v_{R,x}\|_{L^2(\Sigma_R)} = \mathcal{O}(\epsilon^{2/3})$. Now by (3.76), we have that $S(\lambda) = R(\lambda)P^{(\infty)}(\lambda)$ for large λ , which implies by (3.35) that

$$S_{11}(\lambda) = 1 - i \frac{R_{1,12}}{(-\lambda)^{1/2}} + \mathcal{O}(\lambda^{-1}), \quad \text{as } \lambda \rightarrow \infty. \quad (4.5)$$

By (3.27), we obtain

$$u(x, t_m, \epsilon) = u(x, t_m) - 2\epsilon \partial_x R_{1,12}(\tilde{x}, t_m, \epsilon) = u(x, t_m) + o(\epsilon), \quad (4.6)$$

which proves Theorem 1.2 in the case where $x < x_M + C_m t_m$.

Remark 4.1 If $x = x_M + C_m t_m$, the RH analysis remains the same, except for the fact that $u = -1$ and $F'(u; x, t_m) = +\infty$. This implies that the jump matrix for S decays to I also near u , and there is no need to construct a local parametrix. If $x > x_M + C_m t_m$, we have that $-x + C_m u^{m-1} t_m + f_R(u) = 0$, with f_R the inverse of the increasing part of u_0 , but $F(u; x, t_m) = -x + C_m u^{m-1} t_m + f_L(u)$ does not vanish (in general). Because of this, we need to modify the G -function:

$$G_m(\lambda; x, t_m; u) = \frac{\sqrt{u - \lambda}}{\pi} \left[\int_u^0 \frac{F_A(\eta; x, t_m)}{(\eta - \lambda)\sqrt{\eta - u}} d\eta - \int_{-1}^u \frac{\tau(\eta)}{(\eta - \lambda)\sqrt{u - \eta}} d\eta \right],$$

with

$$\tau(\lambda) = \int_{f_L(\lambda)}^{f_R(\lambda)} \sqrt{\lambda - u_0(x)} dx, \quad \text{for } -1 < \lambda < 0.$$

This gives the additional jump condition

$$G_{m,+}(\lambda) - G_{m,-}(\lambda) = -2i\tau(\lambda), \quad \text{for } \lambda \in (-1, u).$$

Then ϕ takes the form

$$\begin{aligned} \phi(\lambda; x, t_m) = & -i\tau(\lambda) - \sqrt{u - \lambda} F(u; x, t_m) + \frac{2}{3}(u - \lambda)^{\frac{3}{2}} F'(u; x, t_m) \\ & - \frac{4}{15}(u - \lambda)^{\frac{5}{2}} F''(u; x, t_m) - \frac{4}{15} \int_u^\lambda F'''(\xi; x, t_m) (\xi - \lambda)^{\frac{5}{2}} d\xi. \end{aligned}$$

With those modifications, the RH analysis can be carried on similarly as before, see also [11].

4.2 Proof of Theorem 1.4

Using (3.71), we can expand the jump matrix v_R in fractional powers of ϵ in the double scaling limit,

$$v_R(\lambda) = I + \epsilon^{1/7} \Delta^{(1)}(\lambda) + \epsilon^{2/7} \Delta^{(2)}(\lambda) + \epsilon^{3/7} \Delta^{(3)}(\lambda) + \mathcal{O}(\epsilon^{4/7}), \quad (4.7)$$

with

$$\Delta^{(1)}(\lambda) = iQ \cdot (-f(\lambda))^{-1/2} P^{(\infty)}(\lambda) \sigma_3 P^{(\infty)}(\lambda)^{-1}, \quad (4.8)$$

$$\Delta^{(2)}(\lambda) = -\frac{1}{2}(-f(\lambda))^{-1} P^{(\infty)}(\lambda) \begin{pmatrix} Q^2 & U \\ U & Q^2 \end{pmatrix} P^{(\infty)}(\lambda)^{-1}, \quad (4.9)$$

$$\Delta^{(3)}(\lambda) = -\frac{i}{2}(-f(\lambda))^{-3/2} P^{(\infty)}(\lambda) \begin{pmatrix} W & V \\ -V & -W \end{pmatrix} P^{(\infty)}(\lambda)^{-1}, \quad (4.10)$$

for $\lambda \in \partial\mathcal{U}$, and

$$\Delta^{(1)}(\lambda) = \Delta^{(2)}(\lambda) = \Delta^{(3)}(\lambda) = 0, \quad \text{for } \lambda \in \Sigma_R \setminus \partial\mathcal{U}, \quad (4.11)$$

since the jump matrices are equal to I up to an error of $\mathcal{O}(\epsilon)$ on the other parts of the contour. Note that the functions $\Delta^{(1)}$ and $\Delta^{(2)}$ are meromorphic functions in $\mathcal{U}$ with simple poles at u^c , and that $\Delta^{(3)}$ is meromorphic in $\mathcal{U}$ with a double pole at u^c . Observe that the x -derivative of v_R is not close to I on $\partial\mathcal{U}$ as $\epsilon \rightarrow 0$, since the x -derivatives of U and Q cause multiplication with $\epsilon^{-6/7}$ by (3.72)-(3.73).

Substituting (4.7) in (3.84) yields an asymptotic expansion for the RH solution R of the form

$$R(\lambda) = I + \epsilon^{1/7} R^{(1)}(\lambda) + \epsilon^{2/7} R^{(2)}(\lambda) + \epsilon^{3/7} R^{(3)}(\lambda) + \mathcal{O}(\epsilon^{4/7}). \quad (4.12)$$

Combining (4.7) with (4.12) and the jump relation $R_+(\lambda) = R_-(\lambda)v_R(\lambda)$ gives the following relations for $\lambda \in \partial\mathcal{U}$,

$$R_+^{(1)}(\lambda) = R_-^{(1)}(\lambda) + \Delta^{(1)}(\lambda), \quad (4.13)$$

$$R_+^{(2)}(\lambda) = R_-^{(2)}(\lambda) + R_-^{(1)}(\lambda)\Delta^{(1)}(\lambda) + \Delta^{(2)}(\lambda) \quad (4.14)$$

$$R_+^{(3)}(\lambda) = R_-^{(3)}(\lambda) + R_-^{(1)}(\lambda)\Delta^{(2)}(\lambda) + R_-^{(2)}(\lambda)\Delta^{(1)}(\lambda) + \Delta^{(3)}(\lambda). \quad (4.15)$$

In addition we know that $R(\lambda) \rightarrow I$ as $\lambda \rightarrow \infty$, and consequently $R^{(j)}(\lambda) \rightarrow 0$ for $j = 1, 2, 3$. So we have additive jump relations and asymptotic conditions for $R^{(1)}$, $R^{(2)}$, and $R^{(3)}$, and it is easily verified that those conditions determine $R^{(1)}$, $R^{(2)}$, and $R^{(3)}$ uniquely. We obtain

$$R^{(1)}(\lambda) = \begin{cases} \frac{1}{\lambda - u^c} \text{Res}(\Delta^{(1)}; u^c), & \text{as } \lambda \in \mathbb{C} \setminus \overline{\mathcal{U}} \\ \frac{1}{\lambda - u^c} \text{Res}(\Delta^{(1)}; u^c) - \Delta^{(1)}(\lambda), & \text{as } \lambda \in \mathcal{U}, \end{cases} \quad (4.16)$$

$$R^{(2)}(\lambda) = \begin{cases} \frac{1}{\lambda - u^c} \text{Res}(R^{(1)}\Delta^{(1)} + \Delta^{(2)}; u^c), & \text{as } \lambda \in \mathbb{C} \setminus \overline{\mathcal{U}}, \\ \frac{1}{\lambda - u^c} \text{Res}(R^{(1)}\Delta^{(1)} + \Delta^{(2)}; u^c) - R^{(1)}\Delta^{(1)}(\lambda) - \Delta^{(2)}(\lambda), & \text{as } \lambda \in \mathcal{U}. \end{cases} \quad (4.17)$$

After a straightforward calculation we find using (4.8), (4.9), and (3.32) that, for $\lambda \in \mathbb{C} \setminus \overline{\mathcal{U}}$,

$$R^{(1)}(\lambda) = -Qf'(u^c)^{-1/2} \frac{1}{\lambda - u^c} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (4.18)$$

$$R^{(2)}(\lambda) = \frac{1}{2f'(u^c)(\lambda - u^c)} \begin{pmatrix} U + Q^2 & 0 \\ 0 & -U - Q^2 \end{pmatrix}, \quad (4.19)$$

where

$$\begin{aligned} Q &= Q(\epsilon^{-6/7}g_1(u^c; x, t_m), \epsilon^{-4/7}g_2(u^c; t_m)), \\ U &= U(\epsilon^{-6/7}g_1(u^c; x, t_m), \epsilon^{-4/7}g_2(u^c; t_m)). \end{aligned}$$

For the matrix $R^{(3)}$, using (4.15) and (4.16)-(4.17), one obtains

$$R^{(3)}(\lambda) = \frac{R^{(1)}(u^c)}{\lambda - u^c} \text{Res}(\Delta^{(2)}(\lambda); u^c) + \frac{R^{(2)}(u^c)}{\lambda - u^c} \text{Res}(\Delta^{(1)}(\lambda); u^c) + \frac{1}{\lambda - u^c} \text{Res}(\Delta^{(3)}(\lambda); u^c) \\ + \frac{1}{(\lambda - u^c)^2} \text{Res}((\lambda - u^c)\Delta^{(3)}(\lambda); u^c), \quad \text{for } \lambda \in \mathbb{C} \setminus \overline{\mathcal{U}}.$$

We are interested only in the 12-entry of R_3 , so it is sufficient to compute the entries in the first line of $R^{(1)}(u^c)$ and $R^{(2)}(u^c)$:

$$R_{11}^{(1)}(u^c) = 0, \\ R_{12}^{(1)}(u^c) = \left(\frac{\epsilon^{-6/7} g_1'(u^c) Q_X + \epsilon^{-4/7} g_2'(u^c) Q_T}{\sqrt{f'(u^c)}} - \frac{Q f''(u^c)}{4 f'(u^c)^{3/2}} \right), \\ R_{11}^{(2)}(u^c) = \frac{f''(u^c)}{4 f'(u^c)^2} U - \frac{\epsilon^{-6/7} g_1'(u^c)}{2 f'(u^c)} U_X - \frac{\epsilon^{-4/7} g_2'(u^c)}{2 f'(u^c)} U_T, \\ R_{12}^{(2)}(u^c) = 0.$$

Consequently,

$$R_{12}^{(3)}(\lambda) = \frac{1}{\lambda - u^c} Z + \frac{1}{(\lambda - u^c)^2} \frac{V - W}{2 f'(u^c)^{3/2}}, \quad \text{for } \lambda \in \mathbb{C} \setminus \overline{\mathcal{U}}, \quad (4.20)$$

where

$$Z = \frac{\epsilon^{-6/7} g_1'(u^c)}{2 f'(u^c)^{3/2}} (Q^2 Q_X + U_X Q - U Q_X + (V - W)_X) \\ + \frac{\epsilon^{-4/7} g_2'(u^c)}{2 f'(u^c)^{3/2}} (Q^2 Q_T + U_T Q - U Q_T + (V - W)_T) \\ - \frac{f''(u^c)}{8 f'(u^c)^{5/2}} (Q^3 + U Q + 3(V - W)), \quad (4.21)$$

and

$$V = V(\epsilon^{-6/7} g_1(u^c; x, t_m), \epsilon^{-4/7} g_2(u^c; t_m)), \\ W = W(\epsilon^{-6/7} g_1(u^c; x, t_m), \epsilon^{-4/7} g_2(u^c; t_m)).$$

Compatibility of the small ϵ -expansion (4.12) with the large λ -expansion (4.1) learns us that

$$R_{1,12}(x, t, \epsilon) = -\epsilon^{1/7} Q f'(u^c)^{-1/2} + \epsilon^{3/7} Z + \mathcal{O}(\epsilon^{4/7}), \quad (4.22)$$

in the double scaling limit. Taking x -derivatives in (4.3), one justifies that we can formally differentiate (4.22) to obtain

$$-2\epsilon \partial_x R_{1,12}(x, t, \epsilon) = 2\epsilon^{2/7} \frac{\partial_x g_1(u^c)}{f'(u^c)^{1/2}} U - 2\epsilon^{4/7} \partial_x g_1(u^c) Z_X \\ - 2\epsilon^{4/7} \frac{\partial_x g_1'(u^c)}{2 f'(u^c)^{3/2}} (Q^2 Q_X + U_X Q - U Q_X + V_X - W_X) + \mathcal{O}(\epsilon^{5/7}), \quad (4.23)$$

since $Q_X = U$. Now we obtain by (3.65)-(3.68), the first equality in (4.6), and (4.21) that

$$\begin{aligned} u(x, t_m, \epsilon) &= u_c + \left(\frac{2\epsilon^2}{k^2} \right)^{1/7} U \\ &\quad - \epsilon^{4/7} \frac{f''(u^c)}{2(8k)^{6/7}} \left(QU_X + U_{XX} + 4U^2 - 3U_T[\epsilon^{-6/7}g_1(u^c)] \right) \\ &\quad + \frac{g_2(u^c)}{(8k)^{4/7}} \left(\frac{2(m-1)}{5u^c} - \frac{3}{4} \frac{f''}{f'} \right) (2U_X Q_T + 4UU_T + \frac{1}{2}U_{XXT}) + \mathcal{O}(\epsilon^{5/7}). \end{aligned}$$

In the derivation of this expansion we have used (3.54)-(3.56). For the third term, we also used the identity (1.19). The term in the third line is also of order $\mathcal{O}(\epsilon^{4/7})$ because $g_2(u^c)$ is of order $\mathcal{O}(\epsilon^{4/7})$. The expansion for $u(x, t_m, \epsilon)$ can be written in terms of U exclusively by substituting the formula (3.54) for Q and $Q_T = -U^2/2 - U_{XX}/12$. Substituting the values for $f''(u^c)$, $g_1(u^c)$, and $g_2(u^c)$ given in (3.65), (3.66), and (3.67), we find (1.25), with the constants c_1, c_2, c_3 given by (1.28)-(1.30). This proves Theorem 1.4.

Acknowledgements

The authors are grateful to P. Miller for useful comments. This work has been supported by the project FroM-PDE funded by the European Research Council through the Advanced Investigator Grant Scheme. TC acknowledges support by the Belgian Interuniversity Attraction Pole P06/02. TG acknowledges support by Italian MUR Research Project Geometric Methods in the Theory of Nonlinear Waves and Their Applications.

References

- [1] R. Beals, P. Deift, and C. Tomei, Direct and inverse scattering on the line, *Mathematical Surveys and Monographs* **28** (1988), American Mathematical Society, Providence, RI.
- [2] E. Brézin, E. Marinari, and G. Parisi, A nonperturbative ambiguity free solution of a string model, *Phys. Lett. B* **242** (1990), no. 1, 35–38.
- [3] T. Claeys and T. Grava, Universality of the break-up profile for the KdV equation in the small dispersion limit using the Riemann-Hilbert approach, *Comm. Math. Phys.* **286** (2009), 979–1009.
- [4] T. Claeys and T. Grava, Painlevé II asymptotics near the leading edge of the oscillatory zone for the Korteweg-de Vries equation in the small dispersion limit, *Comm. Pure Appl. Math.* **63** (2010), 203–232.
- [5] T. Claeys and M. Vanlessen, The existence of a real pole-free solution of the fourth order analogue of the Painlevé I equation, *Nonlinearity* **20** (2007), 1163–1184.
- [6] T. Claeys and M. Vanlessen, Universality of a double scaling limit near singular edge points in random matrix models, *Comm. Math. Phys.* **273** (2007), 499–532.
- [7] P. Deift, “Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach”, Courant Lecture Notes 3, New York University 1999.

- [8] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides, and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* **52** (1999), 1335–1425.
- [9] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides, and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* **52** (1999), 1491–1552.
- [10] P. Deift and E. Trubowitz, Inverse scattering on the line, *Comm. Pure Appl. Math.* **32** (1979), 121–251.
- [11] P. Deift, S. Venakides, and X. Zhou, New result in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems, *Internat. Math. Res. Notices* **6** (1997), 285–299.
- [12] P. Deift, S. Venakides, and X. Zhou, An extension of the steepest descent method for Riemann-Hilbert problems: the small dispersion limit of the Korteweg-de Vries equation, *Proc. Natl. Acad. Sci. USA* **95** (1998), no. 2, 450–454.
- [13] B. Dubrovin, On Hamiltonian perturbations of hyperbolic systems of conservation laws, II: universality of critical behaviour, *Comm. Math. Phys.* **267** (2006), 117–139.
- [14] B.Dubrovin, T.Grava, C.Klein, On universality of critical behaviour in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the *tritronquée* solution to the Painlevé-I equation, *J. Nonlinear Sci.* **19** (2009), no. 1, 57–94.
- [15] B.Dubrovin, T.Grava, C.Klein, Numerical study of break up in generalized Korteweg de Vries equation and Kawahara equation, preprint arxiv=1101.0268v1.
- [16] L.D. Faddeev, The inverse problem in the quantum theory of scattering. II, *Current problems in mathematics* **3** (1974), 93–180.
- [17] A.S. Fokas, A.R. Its, A.A. Kapaev, and V.Yu. Novokshenov, “Painlevé transcendents: the Riemann-Hilbert approach”, AMS Mathematical Surveys and Monographs **128** (2006).
- [18] A.S. Fokas, U. Mugan, and X. Zhou, On the solvability of Painlevé I, III and V, *Inverse Problems* **8** (1992), no. 5, 757–785.
- [19] A.S. Fokas and X. Zhou, On the solvability of Painlevé II and IV, *Comm. Math. Phys.* **144** (1992), no. 3, 601–622.
- [20] S.C. Gardner, Korteweg-de Vries equation and generalizations. IV. The Korteweg-de Vries equation as a Hamiltonian system, *J. Math. Phys.* **12** (1971), 1548–1551.
- [21] C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura, *Phys. Rev. Letters* **19** (1967), 1095–1097.
- [22] S.C. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura, Korteweg-de Vries equation and generalizations. VI. Methods for exact solution, *Comm. Pure Appl. Math.* **27** (1974), 97–133.
- [23] C.S. Gardner, M.D. Kruskal and R.M. Miura, Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion, *J. Math. Phys.* **9** (1968), 1204–1209.

- [24] T. Grava and C. Klein, Numerical solution of the small dispersion limit of Korteweg de Vries and Whitham equations, *Comm. Pure Appl. Math.* **60** (2007) 1623-1664.
- [25] T. Grava, C. Klein, Numerical study of a multiscale expansion of Korteweg-de Vries and Camassa-Holm equation, Integrable systems and random matrices, *Contemp. Math.* **458**, Amer. Math. Soc., Providence, RI, 2008, 81-98.
- [26] A.G. Gurevich and L.P. Pitaevskii, Non stationary structure of a collisionless shock waves, *JEPT Letters* **17** (1973), 193-195.
- [27] A.A. Kapaev, Weakly nonlinear solutions of equation P_I^2 , *J. Math. Sc.* **73** (1995), no. 4, 468-481.
- [28] C. E. Kenig, G. Ponce, and L. Vega, Higher-order nonlinear dispersive equations, *Proc. Amer. Math. Soc.* **122** (1994), 157-166.
- [29] V. Kudashev, B. Suleimanov, A soft mechanism for the generation of dissipationless shock waves, *Phys. Lett. A* **221** (1996), 204-208.
- [30] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968), 467-490.
- [31] P.D. Lax and C.D. Levermore, The small dispersion limit of the Korteweg de Vries equation, I,II,III, *Comm. Pure Appl. Math.* **36** (1983), 253-290, 571-593, 809-830.
- [32] P. Lorenzoni, Deformations of bi-Hamiltonian structures of hydrodynamic type, *J. Geom. Phys.* **44** (2002), no. 2-3, 331-375.
- [33] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* **19** (1978), no. 5, 1156-1162.
- [34] A. Menikoff, The existence of unbounded solutions of the Korteweg-de Vries equation, *Comm. Pure Appl. Math.* **25** (1972), 407-432.
- [35] G. Moore, Geometry of the string equations, *Comm. Math. Phys.* **133** (1990), no. 2, 261-304.
- [36] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of solitons. The inverse scattering method. Translated from the Russian. Contemporary Soviet Mathematics. New York, 1984. xi+276 pp. ISBN: 0-306-10977-8, 35Q20 (58F07 76B25)
- [37] T. Ramond, Semiclassical study of quantum scattering on the line, *Comm. Math. Phys.* **177** (1996), no. 1, 221-254.
- [38] A.B. Shabat, "One dimensional perturbations of a differential operator and the inverse scattering problem" in Problems in Mechanics and Mathematical Physics, Nauka, Moscow, 1976.
- [39] S. Venakides, The Korteweg de Vries equations with small dispersion: higher order Lax-Levermore theory, *Comm. Pure Appl. Math.* **43** (1990), 335-361.
- [40] V.E. Zaharov and L.D. Faddeev, The Korteweg-de Vries equation is a fully integrable Hamiltonian system, *Functional Anal. Appl.* **5** (1971), 280-287.
- [41] X. Zhou, L2-Sobolev space bijectivity of the scattering and inverse scattering transforms, *Comm. Pure Appl. Math.* **51** (1998), no. 7, 697731,

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