

Chapter 5

Management of a pension fund under mortality and financial risks.

Contrary to countries such as USA or UK where the benefits of defined contributions pension plans may entirely depend on the performance of investments, the Belgian regulator (CBFA) imposes an interest rate guarantee on contributions. The purpose of this chapter is precisely to optimize the expected utility of dividend policy and assets allocation, of such pension funds. Our contribution to the literature is to consider this ALM issue in a market with stochastic interest rates and mortality, and to insert an actuarial change of measure in the deflator. As in chapter 4, the optimal dividends and terminal target wealth are determined by martingale method. Whereas the investment strategy hedging at best this target wealth is found by the dynamic programming principle. Finally, we compare solutions for power and exponential utilities. Note that this chapter is adapted from Hainaut & Devolder (2007).

5.1 Introduction.

Owing to the long time horizon of their commitments, pension funds are exposed to important financial risks such as the volatility of markets or a duration gap between assets and liabilities. Furthermore, the modification of the economic environment in past decade is a source of mismatch between the effective investment return and the guarantee granted to affiliates. Another threat facing pension funds is mortality risk. This chapter proposes an ALM framework including those different aspects.

Literature about ALM of pension funds is prolific and from a methodological point of view, two approaches are exploited. The first one is stochastic control, used for the first time by Merton (1969, 1971). Amongst the recent application of this theory to actuarial sciences, we refer to Devolder et al. (2003) who have studied the management of an annuity contract. Menoncin et al. (2004) have done the same exercise for a life annuity. The management of defined benefits plans is addressed in Fonbellida Zapareto (2004). The second method, was developed by Cox and Huang (1989) in the setting of complete markets and relies on the theory of Lagrange multipliers. This approach, also called martingale method, was successfully applied by Boulier et al. (2000), Deelstra et al. (2003, 2004) to study the optimal design and assets allocation of a pension fund, without mortality risk.

The purpose of this work is precisely to incorporate stochastic mortality in a pension fund model and to identify the optimal ALM strategy (dividend policy and assets allocation) in function of the manager's preference. In particular, we consider a portfolio of continuous pension plans with deterministic individual payments, a dividend system and an affiliates' mortality modelled by a Poisson process. The financial market is composed of three assets, cash, a rolling bond and stocks. Interest rates are stochastic and driven by a Vasicek's model. The fund manager aims to maximize the expected utility from dividends and from terminal surplus, defined as the difference between a target asset and an accounting reserve. The optimization is done under the actuarial constraint that the expectation of all intermediate cash flows is at most equal to the current wealth.

The incompleteness of the combination of insurance and financial markets entails that the set of equivalent pricing measures counts more than one element. It is then necessary to assume that the insurer's deflator is well determined, in order to define an unique budget constraint and to apply the martingale method. A second consequence of the incompleteness generated by the mortality risk, is that the optimal target wealth found by the martingale approach is not necessary replicable. Nevertheless, the investment policy hedging at best the optimal target wealth is easily obtained by projection of this target process into the space of self financed wealth processes. The projection method relies on dynamic programming principle.

The outline of the chapter is as follows. Sections 2 and 3 respectively present the actuarial liabilities and the available assets. The next paragraph is a discussion over the deflator. Section 5 introduces the dynamic of the fund and the asset manager's objective. Section 6 develops a general solution applied to CRRA and CARA utility function in the two next paragraphs. Section 9 contains a numerical illustration followed by a conclusion.

5.2 Liabilities.

The fund and the insured agree upon a stream of premiums and a stream of guaranteed benefits. By considering premiums as negative cash flows and benefits as positive cash flows, the payment streams can be merged into a single payment process, L_t , which is the accumulated payment done to one living affiliate at instant t . For the sake of simplicity, there is no death benefit. To avoid any confusion, we insist on the fact that L_t is taken to be predetermined by the policy and is the same for all affiliates. The cumulative payment process has a density noted l_t so that $dL_t = l_t \cdot dt$.

Let $(\Omega^a, \mathcal{F}^a, P^a)$ be the probability space associated to liabilities. As in Møller (1998), we assume that the fund counts initially n_x members of age x and that their remaining lifetimes are independent and identically distributed exponential random variables noted $T_1, T_2, \dots, T_{n_x}$, defined on Ω^a . At time t , the hazard rate of T_i , also called the mortality rate of the affiliates, is a deterministic function, written $\mu(x+t)$. The total number of deaths at instant t is noted N_t and defined by:

$$N_t = \sum_{i=1}^{n_x} I(T_i \leq t)$$

Where $I(\cdot)$ is an indicator variable. The actuarial filtration $\mathcal{F}^a$ is the one generated by N_t : $\mathcal{F}^a = (\mathcal{F}_t^a)_t = \sigma\{N_u : u \leq t\}$. The stochastic intensity of N_t is formally described as follows:

$$\mathbb{E}(dN_t | \mathcal{F}_{t-}^a) = (n_x - N_{t-}) \cdot \mu(x+t) \cdot dt$$

At time t , the compensated process of N_t :

$$M_t = N_t - \int_0^t (n_x - N_{u-}) \cdot \mu(x+u) \cdot du$$

is a martingale under P^a . Notice that, at time s , the total payment done by the fund is equal to $(n_x - N_s) \cdot dL_s$ and the expectation at time $t \leq s$ of this total cash flow is:

$$\begin{aligned} \mathbb{E}((n_x - N_s) \cdot dL_s | \mathcal{F}_t^a) &= \mathbb{E}\left(\sum_{i=1}^{n_x} I(T_i > s) | \mathcal{F}_t^a\right) \cdot dL_s \\ &= \sum_{T_i > t} \mathbb{E}(I(T_i > s) | \mathcal{F}_t^a) \cdot dL_s \\ &= (n_x - N_t) \cdot \underbrace{\exp\left(-\int_t^s \mu(x+u) \cdot du\right)}_{s-t p_{x+t}} \cdot dL_s \end{aligned}$$

$s-t p_{x+t}$ is the real probability that an individual of age $x+t$, survives till age $x+s$.

5.3 Assets.

The financial market is complete and composed of three assets: cash, stocks and a rolling bond. The financial probability space is noted $(\Omega^f, \mathcal{F}^f, P^f)$ and $W_t^{P^f} = (W_t^{r, P^f}, W_t^{S, P^f})$ is a two dimensional Brownian motion generating the filtration

$$\mathcal{F}^f = (\mathcal{F}_t^f)_t = \sigma\left\{\left(W_u^{r, P^f}, W_u^{S, P^f}\right) : u \leq t\right\}$$

A consequence of the completeness of the financial market is the existence of an unique equivalent measure, written Q^f , under which the discounted asset prices are martingales. Let us describe the dynamic of each asset.

The instantaneous risk-free rate r_t is modelled by a Vasicek's model.

$$dr_t = a \cdot (b - r_t) \cdot dt + \sigma_r \cdot dW_t^{r, P^f} \quad (5.3.1)$$

And its dynamic under the risk neutral measure is:

$$dr_t = a \cdot \left(b - \sigma_r \cdot \frac{\lambda_r}{a} - r_t\right) \cdot dt + \sigma_r \cdot \underbrace{\left(dW_t^{r, P^f} + \lambda_r \cdot dt\right)}_{dW_t^{r, Q^f}} \quad (5.3.2)$$

Where W_t^{r, Q^f} is a Brownian motion under Q^f . a , b , σ_r are positive constants whereas λ_r is a negative constant. The rolling bond of maturity K , noted R_t^K , is a zero coupon bond,

continuously rebalanced to keep its maturity constant. R_t^K is described by the following SDE:

$$\begin{aligned}\frac{dR_t^K}{R_t^K} &= r_t \cdot dt - \sigma_r \cdot n(K) \cdot (dW_t^{r,P^f} + \lambda_r \cdot dt) \\ &= r_t \cdot dt - \sigma_r \cdot n(K) \cdot dW_t^{r,Q^f}\end{aligned}$$

$n(K)$ is a function of the maturity of the rolling bond:

$$n(K) = \frac{1}{a} \cdot (1 - e^{-a \cdot K})$$

The risk premium of the rolling bond is constant and noted $\nu_R = -\sigma_r \cdot n(K) \cdot \lambda_r$. Stocks S_t are driven by a geometric Brownian motion:

$$\begin{aligned}\frac{dS_t}{S_t} &= r_t \cdot dt + \sigma_{S_r} \cdot (dW_t^{r,P^f} + \lambda_r \cdot dt) + \sigma_S \cdot (dW_t^{S,P^f} + \lambda_S \cdot dt) \\ &= r_t \cdot dt + \sigma_{S_r} \cdot dW_t^{r,Q^f} + \sigma_S \cdot dW_t^{S,Q^f}\end{aligned}$$

Where σ_{S_r} , σ_S and λ_S are positive constants. For convenience, the stocks risk premium is noted $\nu_S = \sigma_{S_r} \cdot \lambda_r + \sigma_S \cdot \lambda_S$.

5.4 Deflator.

Let $(\Omega, \mathcal{F}, P)$ be the product probability space resulting from the combination of the insurance and financial markets.

$$\Omega = \Omega^a \times \Omega^f \quad \mathcal{F} = \mathcal{F}^a \otimes \mathcal{F}^f \vee \mathcal{N} \quad P = P^a \times P^f$$

Where the sigma algebra $\mathcal{N}$ is generated by all subsets of null sets from $\mathcal{F}^a \otimes \mathcal{F}^f$. The presence of mortality risk, which is not traded, entails that the global market is incomplete and that deflators used to price insurance risk may differ from one insurer to another. The insurer's deflator is composed of one actuarial and one financial part, abusively called financial and actuarial deflators.

5.4.1 Financial deflator.

The financial market is complete and the risk neutral measure Q^f , under which prices of discounted assets are martingales, is defined by the following change of measure:

$$\left(\frac{dQ^f}{dP^f} \right)_t = \exp \left(-\frac{1}{2} \cdot \int_0^t \|\Lambda\|^2 \cdot du - \int_0^t \Lambda \cdot dW_u^{P^f} \right)$$

Where $\Lambda = (\lambda_r, \lambda_S)'$. The unique financial deflator, at instant t , for a payment occurring at time $s \geq t$ is noted $H^f(t, s)$:

$$\begin{aligned}H^f(t, s) &= \frac{\exp \left(-\int_0^s r_u \cdot du \right) \cdot \left(\frac{dQ^f}{dP^f} \right)_s}{\exp \left(-\int_0^t r_u \cdot du \right) \cdot \left(\frac{dQ^f}{dP^f} \right)_t} \\ &= \exp \left(-\int_t^s r_u \cdot du - \frac{1}{2} \cdot \int_t^s \|\Lambda\|^2 \cdot du - \int_t^s \Lambda \cdot dW_u^{P^f} \right)\end{aligned}$$

Notice that the expectation of the deflator $H^f(t, s)$ is equal to the price of a zero coupon bond, noted $B(t, s)$.

$$\begin{aligned} B(t, s) &= \mathbb{E} \left(H^f(t, s) | \mathcal{F}_t \right) \\ &= \mathbb{E}^{Q^f} \left(e^{-\int_t^s r_u \cdot du} | \mathcal{F}_t \right) \end{aligned}$$

The analytic expression of $B(t, s)$ is reminded in appendix A.

5.4.2 Actuarial deflator.

On the liability side, as the mortality is not a treadable asset, the pricing measure of an insurance cash flow may differ from the real measure and varies from one company to another. More precisely, for any $\mathcal{F}^a$ -predictable process h_s , such that $h_s > -1$, an equivalent actuarial measure $Q^{a,h}$ is defined by the random variable solution of the SDE:

$$\begin{aligned} d \left(\frac{dQ^{a,h}}{dP^a} \right)_t &= \left(\frac{dQ^{a,h}}{dP^a} \right)_t \cdot h_t \cdot d \left(N_t - \int_0^t (n_x - N_{u-}) \cdot \mu(x+u) \cdot du \right) \\ &= \left(\frac{dQ^{a,h}}{dP^a} \right)_t \cdot h_t \cdot dM_t \end{aligned} \quad (5.4.1)$$

And we have the property that the process $M_t^{a,h}$ defined by

$$M_t^{a,h} = N_t - \int_0^t (n_x - N_{u-}) \cdot \mu(x+u) \cdot (1 + h_u) \cdot du$$

is a martingale under $Q^{a,h}$. We adopt the notation $\lambda_{N,u} = (n_x - N_{u-}) \cdot \mu(x+u)$ for the intensity of jumps. The solution of the SDE (5.4.1) is (for details, see Duffie (2001), appendix I on counting processes or Biffis et al. (2005)):

$$\begin{aligned} \left(\frac{dQ^{a,h}}{dP^a} \right)_t &= \prod_{T_i \leq t} (1 + h_{T_i}) \cdot \exp \left(- \int_0^t h_u \cdot \lambda_{N,u} \cdot du \right) \\ &= \exp \left(\int_0^t \ln(1 + h_u) \cdot dN_u - \int_0^t h_u \cdot \lambda_{N,u} \cdot du \right) \end{aligned}$$

And we note $H^a(t, s)$, the deflator at instant t , for a payment occurring at time $s \geq t$, defined by:

$$H^a(t, s) = \frac{\left(\frac{dQ^{a,h}}{dP^a} \right)_s}{\left(\frac{dQ^{a,h}}{dP^a} \right)_t} = \exp \left(\int_t^s \ln(1 + h_u) \cdot dN_u - \int_t^s h_u \cdot \lambda_{N,u} \cdot du \right) \quad (5.4.2)$$

Remark that the expectation under $Q^{a,h}$, of a liability cash flow, at time s is:

$$\mathbb{E}^{Q^{a,h}} \left((n_x - N_s) \cdot dL_s | \mathcal{F}_t^a \right) = (n_x - N_t) \cdot \underbrace{\mathbb{E} \left(\exp \left(- \int_t^s \mu(x+u) \cdot (1 + h_u) \cdot du \right) | \mathcal{F}_t^a \right)}_{s-tP_{x+t}^h} \cdot l_s \cdot ds$$

$s-tP_{x+t}^h$ may be interpreted as a modified survival probability, taking into account the pricing preferences of the insurer. In the sequel of this work, we restrict our field of research to a constant process $h_u = h$. The reason motivating this choice is that, in this particular case, some interesting analytic results can be presented. If h is negative, it can be seen as a security margin against mortality risk.

5.4.3 Combined deflator.

As the financial and the actuarial sources of risk are independent, the deflator of the combined market is simply the product of the financial deflator and of the actuarial change of measure:

$$H(t, s) = \frac{\exp\left(-\int_0^s r_u \cdot du\right) \cdot \left(\frac{dQ^f}{dP^f}\right)_s \cdot \left(\frac{dQ^{a,h}}{dP^a}\right)_s}{\exp\left(-\int_0^t r_u \cdot du\right) \cdot \left(\frac{dQ^f}{dP^f}\right)_t \cdot \left(\frac{dQ^{a,h}}{dP^a}\right)_t} \quad (5.4.3)$$

5.5 The dynamic of the fund and the optimization problem.

We address the case of a fund manager who optimizes the dividend and investment policies so as to maximize the utility U_1 of dividends, noted D_t , and the utility U_2 of terminal surplus. This surplus, at the end of the optimization period (time T), is defined as the difference between the total target asset $\tilde{X}_T$ and the number of alive affiliates times the individual accounting reserve, noted IR_T . (Note that the real wealth process, defined hereafter will be noted X_t). Let T_m , r^* and ${}_t p_x^*$ be respectively the maturity of liabilities, the technical discount rate (constant) and the survival probability of first order. The individual reserve is then defined by:

$$IR_T = \int_T^{T_m} e^{-r^* \cdot (s-T)} \cdot {}_{s-T} p_{x+T}^* \cdot l_s \cdot ds \quad (5.5.1)$$

The value function $V(t, x, n)$ at time t , for a wealth x and for a total number of deceases n is defined by:

$$V(t, x, n) = \sup_{D_t, \tilde{X}_T \in \mathcal{A}_t(x)} \mathbb{E} \left(\int_t^T U_1(D_s) \cdot ds + U_2(\tilde{X}_T - (n_x - N_T) \cdot IR_T) \mid \underbrace{\tilde{X}_t = x, N_t = n}_{\mathcal{F}_t} \right) \quad (5.5.2)$$

The utility functions $U_1(\cdot)$ $U_2(\cdot)$ are strictly increasing and concave. The controls are chosen in a set $\mathcal{A}_t(x)$ which is delimited by a constraint stipulating that the expected sum of all future deflated cash flows and of the deflated target terminal wealth, is lower than the current wealth x .

$$\mathcal{A}_t(x) = \left\{ \left((D_s)_{s \in [t, T]}, \tilde{X}_T \right) \text{ such that } \mathbb{E} \left(\int_t^T H(t, s) \cdot (D_s + (n_x - N_s) \cdot l_s) \cdot ds + H(t, T) \cdot \tilde{X}_T \mid \mathcal{F}_t \right) \leq x \right\}$$

In the sequel, this constraint is called the budget constraint. It guarantees the actuarial equivalence between the current wealth of the company and expected future benefits. Clearly, the set $\mathcal{A}_t(x)$ is function of the deflator and in particular of $\frac{dQ^{a,h}}{dP^a}$. And a direct consequence of the actuarial incompleteness is that the belonging of controls to $\mathcal{A}_t(x)$ doesn't guarantee that the target wealth process is replicable by an adapted investment policy.

Under the assumption that the fund is closed (no cash in or cash out excepted dividends and liabilities cash flows), the assets allocation replicating at best the optimal target wealth is

obtained by projection of this target process in the space of attainable wealth, noted $\mathcal{A}_t^\pi(x)$. If X_t , π_t^S and π_t^R point out respectively the attainable wealth, the fraction of the total asset invested in stocks and in rolling bonds, $\mathcal{A}_t^\pi(x)$ is defined by:

$$\begin{aligned} \mathcal{A}_t^\pi(x) = & \left\{ \left((D_s)_{s \in [t, T]}, X_T \right) \mid \exists (\pi_t^S)_t (\pi_t^R)_t \text{ } F_t - \text{adapted} : \right. \\ & e^{-\int_t^T r_s ds} \cdot X_T = x - \int_t^T e^{-\int_t^s r_u du} \cdot (D_s + (n_x - N_s) \cdot l_s) \cdot ds \\ & \left. + \int_t^T \frac{\pi_s^S \cdot X_s}{S_s} \cdot d \left(e^{-\int_t^s r_u du} \cdot S_s \right) + \int_t^T \frac{\pi_s^R \cdot X_s}{R_s} \cdot d \left(e^{-\int_t^s r_u du} \cdot R_s^K \right) \right\} \end{aligned}$$

By definition of $\mathcal{A}_t^\pi(x)$, a self financed strategy, $((D_t)_t, X_T)$ obeys to the following SDE:

$$\begin{aligned} dX_t = & \left((r_t + \pi_t^S \cdot \nu_S + \pi_t^R \cdot \nu_R) \cdot X_t - D_t - (n_x - N_t) \cdot l_t \right) \cdot dt \\ & + \pi_t^S \cdot \sigma_S \cdot X_t \cdot dW_t^{S, P^f} + (\pi_t^S \cdot \sigma_{Sr} - \pi_t^R \cdot \sigma_r \cdot n(K)) \cdot X_t \cdot dW_t^{r, P^f} \end{aligned}$$

The projection method is detailed in the next section. We draw the attention of the reader on the fact that the problem (5.5.2) is badly stated for the particular class of power utility if we restrict the domain of resolution to the set of replicable one, $\mathcal{A}_t^\pi(x)$. Intuitively, if the available asset is insufficient to cover an important adverse deviation of mortality (that's generally the case of many life insurers), the probability of having a negative terminal surplus or a negative dividends is not null and their utilities are no more defined. Enlarging the set of admissible controls to $\mathcal{A}_t(x)$ allows us to avoid this drawback.

5.6 A general solution.

This paragraph presents the optimal dividend and investment strategies without specifying the form of utility functions. Those general results are applied in the following sections to the particular cases of power (CRRA) and exponential (CARA) utility functions.

To construct the solution, we use the method of Lagrange multipliers and refer to Karatzas and Shreve 1998, for details on this approach. Let $y_t \in \mathbb{R}^+$ be the Lagrange multiplier associated to the budget constraint at instant t . The Lagrangian is defined by:

$$\begin{aligned} \mathcal{L} \left(t, x, n, (D_s)_s, \tilde{X}_T, y_t \right) = & \mathbb{E} \left(\int_t^T U_1(D_s) \cdot ds + U_2(\tilde{X}_T - (n_x - N_T) \cdot IR_T) \mid \mathcal{F}_t \right) + \\ & y_t \cdot \left(x - \mathbb{E} \left(\int_t^T H(t, s) \cdot (D_s + (n_x - N_s) \cdot l_s) \cdot ds + H(t, T) \cdot \tilde{X}_T \mid \mathcal{F}_t \right) \right) \end{aligned} \quad (5.6.1)$$

A sufficient condition to obtain an optimal dividend strategy $(D_s^*)_{s \in [t, T]}$ and an optimal target wealth $\tilde{X}_T^*$, is the existence of an optimal Lagrange multiplier $y_t^* > 0$ such that the couple $((D_s^*)_{s \in [t, T]}, \tilde{X}_T^*)$ is a saddle point of the Lagrangian. The value function may therefore be

reformulated as:

$$\begin{aligned} V(t, x, n) &= \inf_{y_t \in \mathbb{R}^+} \left(\sup_{(D_s)_s, \tilde{X}_T} \mathcal{L} \left(t, x, n, (D_s)_s, \tilde{X}_T, y_t \right) \right) \\ &= \inf_{y_t \in \mathbb{R}^+} \tilde{V}(t, x, n, y_t) \end{aligned} \quad (5.6.2)$$

And

$$V(t, x, n) = \tilde{V}(t, x, n, y_t^*)$$

Under the assumptions that utilities U_1 and U_2 are strictly concave, increasing $\mathbb{C}^1$ functions, satisfying $\lim_{x \rightarrow +\infty} U'_{i=1,2}(x) = 0$, their derivatives admit continuous inverse functions $I_{i=1,2}(\cdot)$:

$$U'_1(I_1(x)) = x \quad U'_2(I_2(x)) = x$$

It can be proved (formally, it is sufficient to derive equation (5.6.1) with respect to D_s and X_T) that the optimal dividend and terminal wealth are:

$$D_s^* = I_1(y_t^* \cdot H(t, s)) \quad (5.6.3)$$

$$\tilde{X}_T^* = I_2(y_t^* \cdot H(t, T)) + (n_x - N_T) \cdot IR_T \quad (5.6.4)$$

And the optimal Lagrange multiplier y_t^* saturates the budget constraint i.e.

$$\begin{aligned} x &= \mathbb{E} \left(\int_t^T H(t, s) \cdot (I_1(y_t^* \cdot H(t, s)) + (n_x - N_s) \cdot l_s) \cdot ds | \mathcal{F}_t \right) \\ &\quad + \mathbb{E} (H(t, T) \cdot (I_2(y_t^* \cdot H(t, T)) + (n_x - N_T) \cdot IR_T) | \mathcal{F}_t) \end{aligned}$$

Once the optimal Lagrange multiplier determined, the value function is also calculable:

$$V(t, x, n) = \mathbb{E} \left(\int_t^T U_1(I_1(y_t^* \cdot H(t, s))) \cdot ds + U_2(I_2(y_t^* \cdot H(t, T))) | \mathcal{F}_t \right) \quad (5.6.5)$$

It remains to determine the strategy of investment replicating at best the target terminal wealth $\tilde{X}_T^*$. As this process, depends on mortality, it cannot be perfectly replicated. However, two ways are conceivable to establish the best investment policy. The first one consists to split $(\tilde{X}_t^*)_t$ into a sum of an adapted process, of a Brownian integral and of a zero mean martingale, orthogonal to the Brownian integral. The investment strategy is then obtained by comparison of the real wealth process, $(X_t)_t$, and this decomposition. The second possibility, that we have adopted, relies on dynamic programming (for an introduction, see Fleming and Rishel 1975). We refer to chapter 4 for a comparison of those two methods in the simplified setting of the management of pure endowments. For a small step of time, Δt , the dynamic programming principle states that

$$V(t, x, n) = \mathbb{E} \left(\int_t^{t+\Delta t} U_1(D_s^*) ds + V(t + \Delta t, \tilde{X}_{t+\Delta t}^*, N_{t+\Delta t}) | \mathcal{F}_t \right)$$

Given that $(\tilde{X}_t^*)_t$ is the process maximizing the value function, any other processes $(X_t)_t \neq (\tilde{X}_t^*)_t$ belonging to the set of replicable processes $\mathcal{A}_t^\pi(x) \subset \mathcal{A}_t(x)$, satisfy the inequality:

$$V(t, x, n) \geq \mathbb{E} \left(\int_t^{t+\Delta t} U_1(D_s^*) ds + V(t + \Delta t, X_{t+\Delta t}, N_{t+\Delta t}) | \mathcal{F}_t \right) \quad (5.6.6)$$

And the closest process to $(\tilde{X}_t^*)_t$ is determined by an investment strategy maximizing the right hand term of (5.6.6).

By application of the Itô's lemma for jump processes (see Øksendal and Sulem 2004, chapter one), the expectation of the value function at instant $t + \Delta t$ is given by:

$$\begin{aligned} \mathbb{E}(V(t + \Delta t, X_{t+\Delta t}, N_{t+\Delta t} | \mathcal{F}_t) = \\ V(t, x, n) + \mathbb{E}\left(\int_t^{t+\Delta t} G^\pi(s, X_s, N_s).ds | \mathcal{F}_t\right) + \\ \mathbb{E}\left(\int_t^{t+\Delta t} (V(s, X_s, N_s) - V(s, X_s, N_{s-})).dN_s | \mathcal{F}_t\right) \end{aligned} \quad (5.6.7)$$

Where $G^\pi(s, X_s, N_s)$ is the generator of the value function:

$$\begin{aligned} G^\pi(s, X_s, N_s) = & V_s + a.(b - r_s).V_r + \frac{1}{2}.\sigma_r^2.V_{rr} + \\ & ((r_s + \pi_s^S.\nu_S + \pi_s^R.\nu_R).X_s - D_s^* - l_s.(n_x - N_s)).V_X + \\ & \frac{1}{2}.X_s^2.((\pi_s^S.\sigma_S)^2 + (\pi_s^S.\sigma_{Sr} - \pi_s^R.\sigma_r.n(K))^2).V_{XX} + \\ & X_s.\sigma_r.(\pi_s^S.\sigma_{Sr} - \pi_s^R.\sigma_r.n(K)).V_{Xr} \end{aligned}$$

$V_s, V_X, V_r, V_{XX}, V_{Xr}, V_{rr}$ are partial derivatives of first and second orders with respect to time, fund and interest rate. When Δt tends to zero, the optimal investment strategy maximizing the right hand term of (5.6.6) is the one maximizing the generator G^π . Deriving G^π with respect to π_t^R and π_t^S , give us the optimal investment policy in function of V_X, V_{XX}, V_{Xr} . In particular, the optimal percentage of the fund invested in stocks is:

$$\pi_t^{S*} = \left(-\frac{\nu_R.\sigma_{Sr}}{\sigma_S^2.\sigma_r.n(K)} - \frac{\nu_S}{\sigma_S^2}\right) \cdot \frac{V_X}{V_{XX}} \cdot \frac{1}{X_t} \quad (5.6.8)$$

Whereas the fraction of bonds is:

$$\begin{aligned} \pi_t^{R*} = & \left(-\frac{\nu_S.\sigma_{Sr}}{\sigma_S^2.\sigma_r.n(K)} - \frac{\nu_R}{\sigma_r^2.n(K)^2} \cdot \left(1 + \frac{\sigma_{Sr}^2}{\sigma_S^2}\right)\right) \cdot \frac{V_X}{V_{XX}} \cdot \frac{1}{X_t} \\ & + \frac{1}{n(K)} \cdot \frac{V_{Xr}}{V_{XX}} \cdot \frac{1}{X_t} \end{aligned} \quad (5.6.9)$$

When the form of utility functions is know, partial derivatives V_X, V_{XX}, V_{Xr} can be inferred from equation (5.6.5). Korn and Kraft (2001), have established similar results for the wealth optimization.

5.7 CRRA utility.

In this paragraph, results of the previous section are applied to utility functions belonging to the CRRA family (constant relative risk aversion) with a risk aversion parameter noted γ

($-1 \leq \gamma < 1$). The value function at time t becomes:

$$V(t, x, n) = \sup_{D_t, \tilde{X}_T \in \mathcal{A}_t(x)} \mathbb{E} \left(\int_t^T u_1 \cdot \frac{D_s^\gamma}{\gamma} \cdot ds + u_2 \cdot \frac{(\tilde{X}_T - (n_x - N_T) \cdot IR_T)^\gamma}{\gamma} \middle| \mathcal{F}_t \right)$$

Where u_1 and u_2 are the weights respectively given to the maximization of the utility from dividends and from the terminal surplus. Working with CRRA utilities entails that the problem is not defined for negative terminal surplus and negative dividends. Formulae (5.6.3) and (5.6.4) directly leads to the optimal dividend and terminal wealth.

$$D_s^* = u_1^{-\frac{1}{\gamma-1}} \cdot (y_t^* \cdot H(t, s))^{\frac{1}{\gamma-1}}$$

$$\tilde{X}_T^* = u_2^{-\frac{1}{\gamma-1}} \cdot (y_t^* \cdot H(t, T))^{\frac{1}{\gamma-1}} + (n_x - N_T) \cdot IR_T$$

Where y_t^* is the optimal Lagrange multiplier such that the budget constraint is saturated:

$$\begin{aligned} x = & \mathbb{E} \left(\int_t^T \left(u_1^{\frac{1}{1-\gamma}} \cdot y_t^{*\frac{1}{\gamma-1}} \cdot H(t, s)^{\frac{\gamma}{\gamma-1}} + l_s \cdot (n_x - N_s) \cdot H(t, s) \right) \cdot ds \middle| \mathcal{F}_t \right) \\ & + \mathbb{E} \left(u_2^{\frac{1}{1-\gamma}} \cdot y_t^{*\frac{1}{\gamma-1}} \cdot H(t, T)^{\frac{\gamma}{\gamma-1}} + IR_T \cdot (n_x - N_T) \cdot H(t, T) \middle| \mathcal{F}_t \right) \end{aligned} \quad (5.7.1)$$

Some notations are now developed to enhance the readability of future calculations. Firstly, remark that:

$$\begin{aligned} & \mathbb{E} \left(\int_t^T l_s \cdot (n_x - N_s) \cdot H(t, s) \cdot ds \middle| \mathcal{F}_t \right) \\ &= (n_x - N_t) \cdot \int_t^T l_s \cdot \underbrace{e^{-\int_t^s \mu(x+u) \cdot (1+h_u) \cdot du}}_{s-t p_{x+t}^h} \cdot \mathbb{E}^Q \left(e^{-\int_t^s r_u \cdot du} \middle| \mathcal{F}_t \right) \cdot ds \\ &= (n_x - N_t) \cdot \int_t^T l_s \cdot \bar{E}_{t,s} \cdot ds \end{aligned} \quad (5.7.2)$$

Where $\bar{E}_{t,s}$ is the market price of a pure endowment subscribed by an individual of age $x+t$, delivering 1 unit at age $x+s$, conditionally to the survival of the agent. See appendix A for the analytic formula of $\bar{E}_{t,s}$. In a similar way, we have that:

$$\begin{aligned} \mathbb{E} (IR_T \cdot (n_x - N_T) \cdot H(t, T) \middle| \mathcal{F}_t) &= (n_x - N_t) \cdot IR_T \cdot T-t p_{x+t}^h \cdot \mathbb{E}^Q \left(e^{-\int_t^T r_u \cdot du} \middle| \mathcal{F}_t \right) \\ & \quad (n_x - N_t) \cdot IR_T \cdot \bar{E}_{t,T} \end{aligned} \quad (5.7.3)$$

The second notation adopted is:

$$\tilde{E}_{t,s} = \mathbb{E} \left(H(t, s)^{\frac{\gamma}{\gamma-1}} \middle| \mathcal{F}_t \right) \quad (5.7.4)$$

And $\tilde{E}_{t,s}$ is defined by the next proposition.

Proposition 5.7.1. *Under the assumptions that interest rates follow (5.3.1), that the deflator is defined by (5.4.3), and that the process defining the actuarial measure $Q^{a,h}$ is constant, $h_t = h$ with $h > \frac{1-\gamma}{\gamma}$, we have:*

$$\begin{aligned} \mathbb{E} \left(H(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right) = & \exp \left(-\frac{1}{2} \cdot \frac{\gamma}{\gamma-1} \cdot \int_t^s \|\Lambda\|^2 \cdot du + \frac{1}{2} \cdot \int_t^s \left\| \frac{\gamma}{\gamma-1} \cdot \Lambda \right\|^2 \cdot du \right) \cdot \\ & \exp \left(-\beta^{\tilde{P}} \cdot (s-t) + n(s-t) \cdot \left(\beta^{\tilde{P}} - \frac{\gamma}{\gamma-1} \cdot r_t \right) - \left(\frac{\gamma}{\gamma-1} \right)^2 \cdot \frac{\sigma_r^2}{4 \cdot a} \cdot n(s-t)^2 \right) \cdot \\ & \sum_{n=1}^{n_x - N_t} \frac{(n_x - N_t)!}{(n_x - N_t - n)! \cdot n!} \left(k^n \cdot \left({}_{s-t}p_{x+t}^{\frac{\gamma}{\gamma-1}h} \right)^{n_x - N_t - n} \cdot \left(1 - {}_{s-t}p_{x+t}^{\frac{\gamma}{\gamma-1}h} \right)^n \right) \end{aligned}$$

Where $\beta^{\tilde{P}}$ and k are constant and defined by:

$$\beta^{\tilde{P}} = \frac{\gamma}{\gamma-1} \cdot b - \left(\frac{\gamma}{\gamma-1} \right)^2 \cdot \frac{\sigma_r \cdot \lambda_r}{a} - \left(\frac{\gamma}{\gamma-1} \right)^2 \cdot \frac{\sigma_r^2}{2 \cdot a^2}$$

$$k = \frac{(1+h)^{\frac{\gamma}{\gamma-1}}}{\left(1 + \frac{\gamma}{\gamma-1} \cdot h \right)}$$

${}_{s-t}p_{x+t}^{\frac{\gamma}{\gamma-1}h}$ is a probability of survival under a modified measure of probability:

$${}_{s-t}p_{x+t}^{\frac{\gamma}{\gamma-1}h} = \exp \left(- \int_t^s \mu(x+u) \cdot \left(1 + \frac{\gamma}{\gamma-1} \cdot h \right) \cdot du \right)$$

And $n(s-t)$ is a positive decreasing function, null when $s=t$,

$$n(s-t) = \frac{1 - e^{-a(s-t)}}{a}$$

The proof is detailed in appendix B. Regrouping (5.7.1) (5.7.2) (5.7.3) and (5.7.4) leads to the optimal Lagrange multiplier:

$$y_t^{*\frac{1}{\gamma-1}} = \frac{x - (n_x - N_t) \cdot \left(\int_t^T l_s \cdot \bar{E}_{t,s} \cdot ds + IR_T \cdot \bar{E}_{t,T} \right)}{u_1^{\frac{1}{1-\gamma}} \cdot \int_t^T \tilde{E}_{t,s} \cdot ds + u_2^{\frac{1}{1-\gamma}} \cdot \tilde{E}_{t,T}}$$

Remark that y_t^* is function of $x - (n_x - N_t) \cdot \left(\int_t^T l_s \cdot \bar{E}_{t,s} \cdot ds + IR_T \cdot \bar{E}_{t,T} \right)$, the spread between the asset value and the market value of fund manager's future commitments. This quantity may be seen as the equity of the pension fund and plays a crucial role in the optimal assets allocation.

Once the multiplier known, the optimal dividend and terminal wealth are calculable and the corresponding value function is:

$$\begin{aligned} V(t, x, n) &= \frac{1}{\gamma} \cdot y_t^{*\frac{\gamma}{\gamma-1}} \cdot \mathbb{E} \left(u_1^{\frac{1}{1-\gamma}} \cdot \int_t^T H(t, s)^{\frac{\gamma}{\gamma-1}} ds + u_2^{\frac{1}{1-\gamma}} \cdot H(t, T)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right) \\ &= \frac{1}{\gamma} \cdot \frac{\left(x - (n_x - n) \cdot \left(\int_t^T l_s \cdot \bar{E}_{t,s} \cdot ds + IR_T \cdot \bar{E}_{t,T} \right) \right)^\gamma}{\left(u_1^{\frac{1}{1-\gamma}} \cdot \int_t^T \tilde{E}_{t,s} \cdot ds + u_2^{\frac{1}{1-\gamma}} \cdot \tilde{E}_{t,T} \right)^{\gamma-1}} \end{aligned}$$

Remark that the value function is not defined for a negative equity. Next, it suffices to calculate the partial derivatives V_X , V_{XX} and V_{Xr} to obtain the optimal assets allocation by formulas (5.6.8) and (5.6.9). The fraction of the fund invested in stocks is a constant percentage of the equity:

$$\begin{aligned} \pi_t^{S*} \cdot X_t &= \underbrace{\left(-\frac{\nu_R \cdot \sigma_{Sr}}{\sigma_S^2 \cdot \sigma_r \cdot n(K)} - \frac{\nu_S}{\sigma_S^2} \right)}_{constant} \cdot \frac{1}{\gamma - 1} \cdot \\ &\quad \underbrace{\left(x - (n_x - n) \cdot \left(\int_t^T l_s \cdot \bar{E}_{t,s} \cdot ds + IR_T \cdot \bar{E}_{t,T} \right) \right)}_{equity} \end{aligned} \quad (5.7.5)$$

Whereas the optimal part of the fund invested in bonds is the sum of two components. One is a constant percentage of the equity and the other one is a correction term.

$$\begin{aligned} \pi_t^{R*} \cdot X_t &= \underbrace{\left(-\frac{\nu_S \cdot \sigma_{Sr}}{\sigma_S^2 \cdot \sigma_r \cdot n(K)} - \frac{\nu_R}{\sigma_r^2 \cdot n(K)^2} \cdot \left(1 + \frac{\sigma_{Sr}^2}{\sigma_S^2} \right) \right)}_{constant} \cdot \frac{1}{\gamma - 1} \cdot \\ &\quad \underbrace{\left(x - (n_x - n) \cdot \left(\int_t^T l_s \cdot \bar{E}_{t,s} \cdot du + IR_T \cdot \bar{E}_{t,T} \right) \right)}_{equity} \\ &\quad + \underbrace{\frac{1}{n(K)} \cdot \frac{V_{xr}}{V_{xx}}}_{correction} \end{aligned} \quad (5.7.6)$$

From appendix A and B, we have that

$$\frac{\partial \bar{E}_{t,s}}{\partial r_t} = -n(s-t) \cdot \bar{E}_{t,s} \quad \frac{\partial \tilde{E}_{t,s}}{\partial r_t} = -\frac{\gamma}{\gamma-1} \cdot n(s-t) \cdot \tilde{E}_{t,s}$$

The correction term is then totally calculable:

$$\begin{aligned} \frac{1}{n(K)} \frac{V_{xr}}{V_{xx}} &= \frac{1}{n(K)} \left[(n_x - n) \cdot \left(\int_t^T n(s-t) \cdot l_s \cdot \bar{E}_{t,s} \cdot ds + IR_T \cdot n(T-t) \cdot \bar{E}_{t,T} \right) \right. \\ &\quad + \frac{x - (n_x - n) \cdot \left(\int_t^T l_s \cdot \bar{E}_{t,s} \cdot ds + IR_T \cdot \bar{E}_{t,T} \right)}{u_1^{\frac{1}{1-\gamma}} \cdot \int_t^T \tilde{E}_{t,s} \cdot ds + u_2^{\frac{1}{1-\gamma}} \cdot \tilde{E}_{t,T}} \cdot \frac{\gamma}{\gamma-1} \cdot \\ &\quad \left. \left(u_1^{\frac{1}{1-\gamma}} \cdot \int_t^T n(s-t) \cdot \tilde{E}_{t,s} \cdot ds + u_2^{\frac{1}{1-\gamma}} \cdot n(T-t) \cdot \tilde{E}_{t,T} \right) \right] \end{aligned} \quad (5.7.7)$$

As all terms of (5.7.7) are dependent on $n(s-t)$, the correction term is a function tending to zero when $t \rightarrow T$. Note that, in the example detailed in section 5.9, integrals containing $\tilde{E}_{t,s}$ and $\bar{E}_{t,s}$ are computed numerically.

5.8 C.A.R.A. utility.

In this paragraph, results of section 5.6 are applied to utility functions from the CARA family (constant absolute risk aversion) with a risk aversion parameter noted α . The value function at time t is then rewritten:

$$V(t, x, n) = \sup_{D_t, \tilde{X}_T \in \mathcal{A}_t(x)} \mathbb{E} \left(- \int_t^T u_1 \cdot \frac{e^{-\alpha \cdot D_s}}{\alpha} \cdot ds - u_2 \cdot \frac{e^{-\alpha \cdot (\tilde{X}_T - (n_x - N_T) \cdot IR_T)}}{\alpha} \middle| \mathcal{F}_t \right)$$

Where u_1 and u_2 are again weights respectively given to the maximization of the utility from dividends and from the terminal surplus. The optimal dividend and terminal wealth are:

$$D_s^* = -\frac{1}{\alpha} \cdot \ln \left(\frac{1}{u_1} \cdot y^* \cdot H(t, s) \right)$$

$$\tilde{X}_T^* = -\frac{1}{\alpha} \cdot \ln \left(\frac{1}{u_2} \cdot y^* \cdot H(t, T) \right) + (n_x - N_T) \cdot IR_T$$

Where y_t^* is the optimal Lagrange multiplier such that the budget constraint is saturated:

$$\begin{aligned} x = & \mathbb{E} \left(\int_t^T H(t, s) \cdot \left(-\frac{1}{\alpha} \cdot \ln \left(\frac{1}{u_1} \cdot y^* \cdot H(t, s) \right) + l_s \cdot (n_x - N_s) \right) \cdot ds \middle| \mathcal{F}_t \right) \\ & + \mathbb{E} \left(H(t, T) \cdot \left(-\frac{1}{\alpha} \cdot \ln \left(\frac{1}{u_2} \cdot y^* \cdot H(t, T) \right) + IR_T \cdot (n_x - N_T) \right) \middle| \mathcal{F}_t \right) \end{aligned} \quad (5.8.1)$$

For readability purposes, the following notation is adopted :

$$\hat{E}_{t,s} = \mathbb{E} (H(t, s) \cdot \ln H(t, s) | \mathcal{F}_t) \quad (5.8.2)$$

The analytic expression of $\hat{E}_{t,s}$ is given by the next proposition:

Proposition 5.8.1. *Under the assumptions that interest rates follow (5.3.1), that the deflator is defined by (5.4.3), and that the process defining the actuarial measure $Q^{a,h}$ is constant, $h_t = h$, we have:*

$$\begin{aligned} \mathbb{E} (H(t, s) \cdot \ln H(t, s) | \mathcal{F}_t) = & B(t, s) \cdot \left(\frac{1}{2} \cdot \left(\lambda_r^2 + \lambda_s^2 + \frac{\lambda_r \cdot \sigma_r}{a} \right) \cdot (s-t) - \frac{\lambda_r \cdot \sigma_r}{a} \cdot n(s-t) - \mathbb{E}^{F(t,s)} \left(\int_t^s r_u du \middle| \mathcal{F}_t \right) \right) \\ & + B(t, s) \cdot (n_x - N_t) \cdot \left(\ln(1+h) \cdot \left(1 - {}_{s-t}p_{x+t}^h \right) - h \cdot \int_t^s {}_{u-t}p_{x+t}^h \cdot \mu(x+u) \cdot du \right) \end{aligned}$$

Where

$$\begin{aligned} \mathbb{E}^{F(t,s)} \left(\int_t^s r_u du \middle| \mathcal{F}_t \right) = & \left(b - \frac{\lambda_r \cdot \sigma_r}{a} - \frac{\sigma_r^2}{a^2} \right) \cdot (s-t) + \\ & \left(r_t - b + \frac{\lambda_r \cdot \sigma_r}{a} + \left(\frac{\sigma_r^2}{a^2} + \frac{1}{2} \cdot \frac{\sigma_r^2}{a} \cdot n(s-t) \right) \right) \cdot n(s-t) \end{aligned} \quad (5.8.3)$$

And

$$n(s-t) = \frac{1}{a} \cdot \left(1 - e^{-a \cdot (s-t)}\right)$$

The proof of proposition 5.8.1 is established in appendix C. Recall that $\bar{E}_{t,s}$ is the actuarial discount factor and $B(t,s)$ the financial discount factor. The optimal Lagrange multiplier, inferred from equation (5.8.1), is then:

$$\begin{aligned} -\ln y_t^* = & \frac{1}{\int_t^T B(t,s).ds + B(t,T)} \cdot \left[\alpha \cdot \left(x - (n_x - N_t) \cdot \left(\int_t^T l_s \cdot \bar{E}_{t,s} ds + IR_T \cdot \bar{E}_{t,T} \right) \right) \right. \\ & \left. + \int_t^T \hat{E}_{t,s} ds + \hat{E}_{t,T} - \ln(u_1) \cdot \int_t^T B(t,s).ds - \ln(u_2) \cdot B(t,T) \right] \end{aligned}$$

The value function is next easily expressed in term of y_t^* :

$$V(t, x, n) = -\frac{1}{\alpha} \cdot y_t^* \cdot \left(\int_t^T B(t,s).ds + B(t,T) \right)$$

The optimal assets allocation is finally obtained by formulas (5.6.8) and (5.6.9). The fraction of the fund invested in stocks is a constant percentage of the sum of a financial annuity and of a discount factor, divided by the aversion parameter α . It means that the assets allocation is therefore completely independent of the size of the fund!

$$\pi_t^{S*} \cdot X_t = \underbrace{\left(\frac{\nu_R \cdot \sigma_{Sr}}{\sigma_S^2 \cdot \sigma_r \cdot n(K)} + \frac{\nu_S}{\sigma_S^2} \right)}_{constant} \cdot \underbrace{\frac{\int_t^T B(t,s).ds + B(t,T)}{\alpha}}_{financial\ quantity} \quad (5.8.4)$$

The optimal part of the fund invested in bonds is the sum of two components. One is a constant percentage of a financial quantity, independent of the fund, and the other one is a correction term.

$$\begin{aligned} \pi_t^{R*} \cdot X_t = & \underbrace{\left(\frac{\nu_S \cdot \sigma_{Sr}}{\sigma_S^2 \cdot \sigma_r \cdot n(K)} + \frac{\nu_r}{\sigma_r^2 \cdot n(K)^2} \cdot \left(1 + \frac{\sigma_{Sr}^2}{\sigma_S^2} \right) \right)}_{constant} \cdot \underbrace{\frac{\int_t^T B(t,s).ds + B(t,T)}{\alpha}}_{financial\ quantity} \\ & + \underbrace{\frac{1}{n(K)} \cdot \frac{V_{xr}}{V_{xx}}}_{correction} \end{aligned} \quad (5.8.5)$$

From appendix C, we know that

$$\frac{\partial \hat{E}_{t,s}}{\partial r_t} = -n(s-t) \cdot \left(\hat{E}_{t,s} + B(t,s) \right)$$

The correction term is:

$$\begin{aligned}
\frac{1}{n(K)} \cdot \frac{V_{xr}}{V_{xx}} = & -\frac{1}{\alpha \cdot n(K)} \cdot \ln y_t^* \cdot \left(\int_t^T n(s-t) \cdot B(t, s) \cdot ds + n(T-t) \cdot B(t, T) \right) \\
& -\frac{1}{\alpha \cdot n(K)} \cdot \left[-\alpha \cdot (n_x - n) \cdot \left(\int_t^T n(s-t) \cdot \bar{E}_{t,s} ds + n(T-t) \cdot IR_T \cdot \bar{E}_{t,T} \right) \right. \\
& + \int_t^T n(s-t) \cdot \left(\hat{E}_{t,s} + B(t, s) \right) \cdot ds + n(T-t) \cdot \left(\hat{E}_{t,T} + B(t, T) \right) \\
& \left. - \ln u_1 \cdot \int_t^T n(s-t) \cdot B(t, s) \cdot ds - \ln u_2 \cdot B(t, T) \cdot n(T-t) \right] \quad (5.8.6)
\end{aligned}$$

As in the CRRA case, all terms of (5.8.6) are dependent on $n(s-t)$ and the correction term tends to zero when $t \rightarrow T$. Remark that, in the example detailed in section 5.9, integrals including $\hat{E}_{t,s}$ and $B(t, s)$ are computed numerically.

5.9 Numerical illustration.

We consider a pension fund counting 100 male members, aged of 60. Each individual pays in a premium of 1 till his retirement at the age of 65 years, and receives next, conditionally to his survival, a continuous annuity rate of 2.45. This annuity rate corresponds to 25 years of contributions and is established with the first order bases presented in table 5.9.1. Those bases are also used to calculate the individual mathematical reserve IR_T , defined by equation 5.5.1. We assume that the initial market value of assets is equal to 1.05 times the total accounting reserve, $X_{t=0} = 1.05 \cdot IR_{t=0}$. The asset manager's time horizon, T , is 10 years.

Table 5.9.1: First order basis.

r^*	2%
T_m	110 years
${}_t p_x^*$	equal to ${}_t p_x$, see appendix D

The constant h defining the actuarial deflator is set to -1%. Under $Q^{a,h}$, mortality rates $\mu(x+t)$ are hence multiplied by 99%. Table 5.9.3 presents the parameters of the Vasicek's model and of the rolling bond. Those are inferred from the Belgian bonds market (data from January 2000 to December 2005). The stocks parameters, in table 5.9.5, are calibrated on the return of the equity index MSCI Europe from January 1994 to December 2005.

Table 5.9.3: Vasicek's parameters.

a	12.72%
b	3.88%
σ_r	1.75%
λ_r	2.36%
$r_{t=0}$	2%
K	15 years
ν_R	2.77%

Table 5.9.5: Stocks parameters.

λ_S	34.94%
σ_S	15.24%
σ_{SR}	-0.1%
ν_S	5.35%

The sequel of the analysis focuses on one scenario in which the returns of assets are constant and the observed mortality is equal to the average mortality. We assume that cash, stocks and rolling bond have respectively a constant return of $r_t = 2\%$, $r_t + \frac{\nu_S}{2} = 4.67\%$ and $r_t + \frac{\nu_R}{2} = 3.38\%$.

C.R.R.A. utility.

The risk aversion parameter γ , is set to 5% and the weights given to the utility of dividends and of terminal surplus are respectively $u_1 = 1$, $u_2 = 1$. Figure 5.9.1 depicts the evolution of the fund, of the equity (such as defined in equation (5.7.5)) and of the total accounting reserve, $(100 - N_t).IR_t$. This evolution is split into an accumulation and a decumulation phase. In the selected scenario, the equity decreases due to the distribution of a dividend (figure 5.9.2). During the first five years, the dividend percentage is stable, round about 2% of the fund, and it next gradually increases to 3%.

Figure 5.9.1: Fund, reserve, equity evolution.

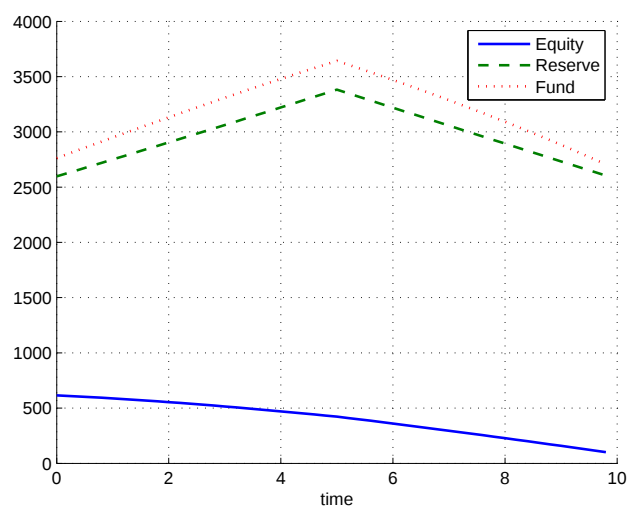
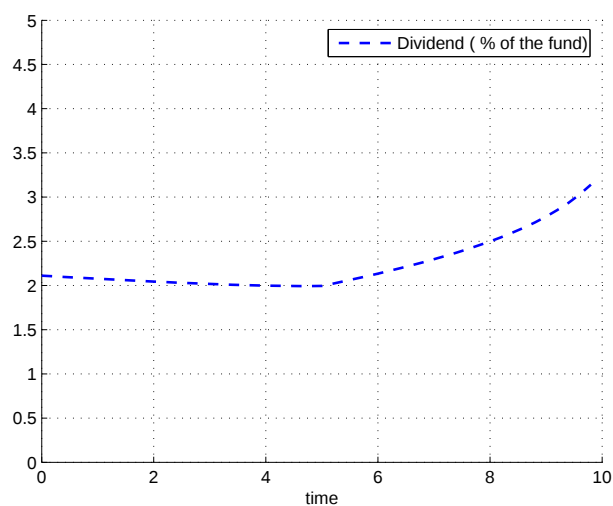
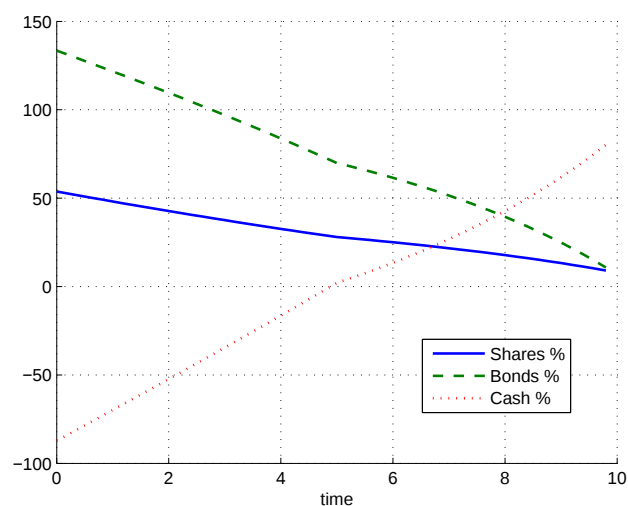


Figure 5.9.2: Dividend.



The figure 5.9.3 emphasizes the dependence of the investment strategy on the time remaining before T . During the first years, cash is borrowed (around 85% of the fund) in order to buy risky assets and in particular rolling bonds. Positions in risky assets are next gradually reduced with time and replaced by cash which finally represents more than 80% of the portfolio. The stocks purchased are equal to 2.41 times the equity and decreases in this particular scenario, from 53% to 9% of the fund. During the first 2 years, the amount invested in bonds exceeds the total value of the fund and is equal to 6 times the equity.

Figure 5.9.3: assets allocation.



C.A.R.A. utility.

The parameters defining the exponential utility functions are $\alpha = 5\%$, $u_1 = 1$ and $u_2 = 1$. The figure 5.9.4 presents the evolution of the fund, of the equity and of the reserve. After 4 years, the equity becomes negative whereas a positive dividend of 2.52% (see figure 5.9.5) is still distributed. After 7 years, shareholders have to pay a contribution which finally worths 3.29% of the fund.

Figure 5.9.4: Fund, reserve, equity evolution.

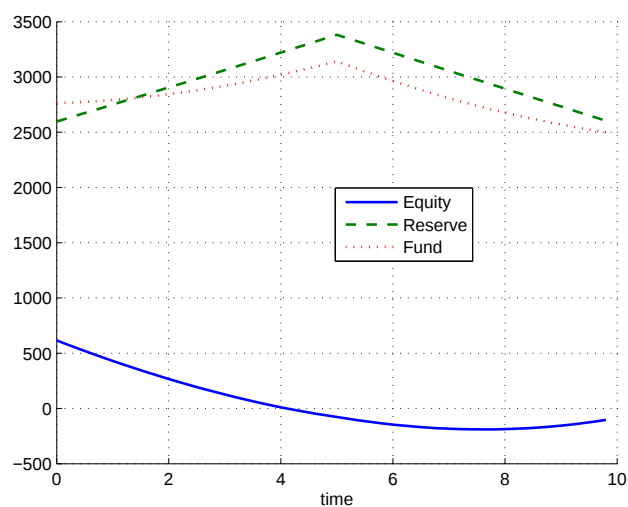
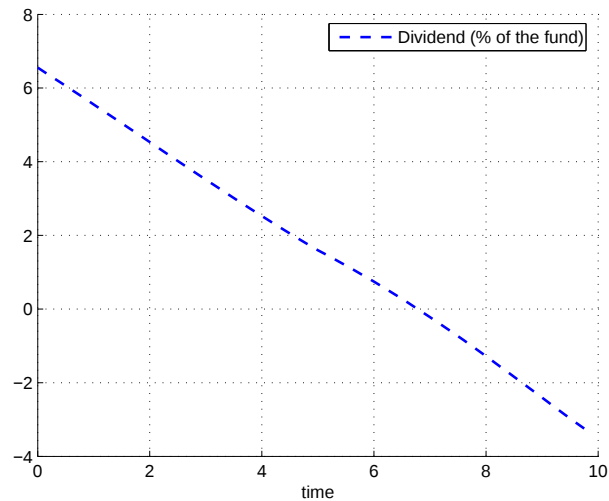
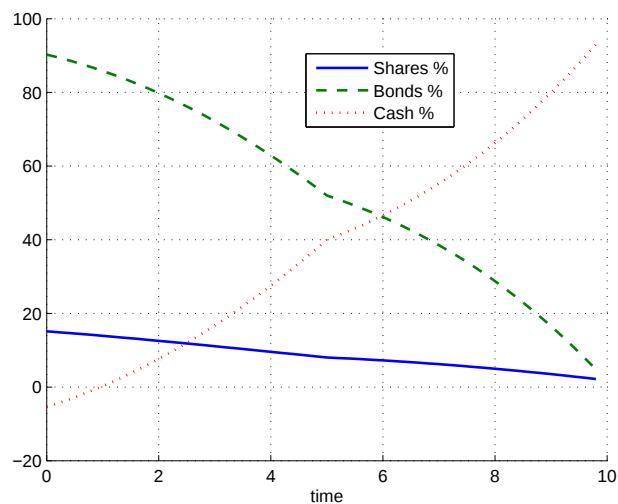


Figure 5.9.5: Dividend.



The figure 5.9.6 presents the evolution of the assets allocation. As for CRRA utilities, the positions in risky assets are reduced with time and replaced by cash. However, we insist on the fact that the assets allocation only depends on the financial quantity $(\int_t^T B(t, s).ds + B(t, T))$ which is totally independent of the situation of the fund. In particular, the position in stocks is equal to 45.8 times this quantity.

Figure 5.9.6: Assets allocation.



5.10 Conclusion.

The contribution of this chapter is to solve by martingale approach, the problem of the management of a pension fund under mortality and financial risks. In particular, we consider the case of a fund manager who optimizes the expected utility of dividends and of terminal surplus under a budget constraint guaranteeing that the expectation of deflated intermediate cash flows is at most equal to the current wealth. Owing to the presence of mortality risk, the deflator is not unique. This non uniqueness has two drawbacks.

Firstly, we need to fix the deflator in order to apply the Cox and Huang's method. This assumption is however not really impeding and is widely spread in actuarial practice. Actuaries relies indeed on diversification to hedge the mortality risk. In this work, the insurer's deflator is equal to the product of the financial deflator and of the actuarial change of measure. In particular, we focus on actuarial change of measure multiplying the real mortality rate by a constant factor $(1+h)$ under the pricing measure. The factor h may be seen as a tariff loading.

Secondly, the optimal target wealth solution found by the martingale approach, is not necessary replicable by an adapted investment strategy. The budget constraint delimits indeed a domain of wealth processes wider than the set of replicable ones. However, dynamic programming allows us to find the optimal investment strategy replicating at best the unattainable target wealth, by maximization of the value function generator. As mentioned in section 5.7, restricting the domain to replicable wealth processes entails that the problem is badly stated for power utility function. This point is particularly annoying in reason of the well established properties of CRRA utilities in complete markets. The method developed in this chapter precisely circumvents this difficulty.

Applying the martingale method to power and exponential utilities reveals that the choice of the utility has a huge impact on the asset liability management policy. In the CRRA case, dividends, value function and optimal investment policy are function of the fund equity, which is defined as the difference between the available asset and the deflated value of the manager's future commitments. The higher is the equity, the higher are dividends and positions in risky assets. For CARA utility functions, the optimal assets allocation is totally independent from liabilities. It is clear that this characteristic is not adapted to the concerns of a fund manager.

This work is concluded by a numerical comparison of CRRA and CARA optimal policies in the same scenario. For each utility function, positions in risky assets are reduced with time. Dividends are always positive for CRRA functions whereas a contribution can be required from shareholders for CARA utilities. This point disqualifies exponential utility for ALM purposes.

Appendix A.

$\bar{E}_{t,s}$ is the market price of a pure endowment subscribed by an individual of age $x+t$, delivering 1 unit at age $x+s$, conditionally to the survival of the agent. It is the product of the pricing

survival probability times the price of a zero coupon.

$$\bar{E}_{t,s} = \underbrace{e^{-\int_t^s \mu(x+u).(1+h_u).du}}_{s-tP_{x+t}^h} \cdot \underbrace{\mathbb{E}^Q \left(e^{-\int_t^s r_u.du} | \mathcal{F}_t \right)}_{B(t,s)}$$

And in the Vasicek's model (for details on this model, we refer to Cairns 2004) , the price of a zero coupon bond is given by

$$B(t, s) = \exp \left(-\beta.(s-t) + n(s-t).(\beta - r_t) - \frac{\sigma_r^2}{4.a}.n(s-t)^2 \right) \quad (5.10.1)$$

Where

$$\beta = b^Q - \frac{\sigma_r^2}{2.a^2} = b - \sigma_r \cdot \frac{\lambda_r}{a} - \frac{\sigma_r^2}{2.a^2}$$

and $n(s-t)$ is a positive decreasing function, null when $s = t$:

$$n(s-t) = \frac{1}{a} \cdot \left(1 - e^{-a.(s-t)} \right)$$

The derivative of the pure endowment with respect to r_t , used in paragraph 5.7 to obtain the optimal bonds strategy, is:

$$\begin{aligned} \frac{\partial \bar{E}_{t,s}}{\partial r_t} &= s-tP_{x+t}^h \cdot \frac{\partial B(t, s)}{\partial r_t} \\ &= -n(s-t) \cdot \bar{E}_{t,s} \end{aligned}$$

Appendix B.

This appendix presents the proof of the proposition (5.7.1). The calculation of $\tilde{E}_{t,s}$ defined as,

$$\tilde{E}_{t,s} = \mathbb{E}^P \left(H(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right)$$

requires two changes of measure: one on the asset side, and one on the liability side. At first, the independence between insurance and financial markets, allows us to split the deflator into actuarial and financial components, which are next calculated separately.

$$\mathbb{E}^P \left(H(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right) = \underbrace{\mathbb{E}^{P^f} \left(H^f(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right)}_{Financial} \cdot \underbrace{\mathbb{E}^{P^a} \left(H^a(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right)}_{actuarial} \quad (5.10.2)$$

Calculation of the financial component.

Let $\tilde{P}$ be an equivalent measure to P^f defined by:

$$\left(\frac{d\tilde{P}}{dP^f} \right)_t = \exp \left(- \int_0^t \frac{\gamma}{\gamma-1} \cdot \Lambda \cdot dW_u^{P^f} - \frac{1}{2} \cdot \int_0^t \left\| \frac{\gamma}{\gamma-1} \cdot \Lambda \right\|^2 \cdot du \right)$$

Under $\tilde{P}$, the following elements are Brownian motions:

$$d\tilde{W}_u^{r, \tilde{P}} = dW_u^{r, P^f} + \frac{\gamma}{\gamma-1} \cdot \lambda_r \cdot du$$

$$d\tilde{W}_u^{S,\tilde{P}} = dW_u^{S,P^f} + \frac{\gamma}{\gamma-1} \cdot \lambda_S \cdot du$$

And the financial component of (5.10.2) is:

$$\begin{aligned} & \mathbb{E}^{P^f} \left(H^f(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right) \\ &= \mathbb{E}^{P^f} \left(\exp \left(-\frac{\gamma}{\gamma-1} \cdot \left(\int_t^s r_u \cdot du + \int_t^s \Lambda \cdot dW_u^{P^f} + \frac{1}{2} \cdot \int_t^s \|\Lambda\|^2 \cdot du \right) \right) | \mathcal{F}_t \right) \\ &= \exp \left(-\frac{1}{2} \cdot \frac{\gamma}{\gamma-1} \cdot \int_t^s \|\Lambda\|^2 \cdot du + \frac{1}{2} \cdot \int_t^s \left\| \frac{\gamma}{\gamma-1} \cdot \Lambda \right\|^2 \cdot du \right) \cdot \mathbb{E}^{\tilde{P}} \left(e^{-\int_t^s \frac{\gamma}{\gamma-1} \cdot r_u \cdot du} | \mathcal{F}_t \right) \end{aligned}$$

As $\frac{\gamma}{\gamma-1} \cdot r_u$ has a mean reverting dynamic under $\tilde{P}$,

$$\begin{aligned} d \left(\frac{\gamma}{\gamma-1} \cdot r_u \right) &= a \cdot \left(\frac{\gamma}{\gamma-1} \cdot b - \left(\frac{\gamma}{\gamma-1} \right)^2 \cdot \frac{\sigma_r \cdot \lambda_r}{a} - \frac{\gamma}{\gamma-1} \cdot r_u \right) \cdot dt \\ &\quad + \frac{\gamma}{\gamma-1} \cdot \sigma_r \cdot d\tilde{W}_u^{r,\tilde{P}} \end{aligned}$$

it suffices therefore to apply the Vasicek's formula to obtain that:

$$\begin{aligned} & \mathbb{E}^{P^f} \left(H^f(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right) = \\ & \exp \left(-\frac{1}{2} \cdot \frac{\gamma}{\gamma-1} \cdot \int_t^s \|\Lambda\|^2 \cdot du + \frac{1}{2} \cdot \int_t^s \left\| \frac{\gamma}{\gamma-1} \cdot \Lambda \right\|^2 \cdot du \right) \cdot \\ & \exp \left(-\beta^{\tilde{P}} \cdot (s-t) + n(s-t) \cdot \left(\beta^{\tilde{P}} - \frac{\gamma}{\gamma-1} \cdot r_t \right) - \left(\frac{\gamma}{\gamma-1} \right)^2 \cdot \frac{\sigma_r^2}{4a} \cdot n(s-t)^2 \right) \end{aligned}$$

Where

$$\beta^{\tilde{P}} = \frac{\gamma}{\gamma-1} \cdot b - \left(\frac{\gamma}{\gamma-1} \right)^2 \cdot \frac{\sigma_r \cdot \lambda_r}{a} - \left(\frac{\gamma}{\gamma-1} \right)^2 \cdot \frac{\sigma_r^2}{2a^2}$$

And

$$n(s-t) = \frac{1}{a} \cdot \left(1 - e^{-a \cdot (s-t)} \right)$$

At this point, we can already calculate the derivative $\frac{\partial \tilde{E}_{t,s}}{\partial r_t}$, used in paragraph 5.7 :

$$\frac{\partial \tilde{E}_{t,s}}{\partial r_t} = -\frac{\gamma}{\gamma-1} \cdot n(s-t) \cdot \tilde{E}_{t,s}$$

Calculation of the actuarial component.

$$\begin{aligned} & \mathbb{E}^{P^a} \left(H^a(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right) \\ &= \mathbb{E}^{P^a} \left(\exp \left(\int_t^s \ln \left((1+h_u)^{\frac{\gamma}{\gamma-1}} \right) \cdot dN_u - \int_t^s \frac{\gamma}{\gamma-1} \cdot h_u \cdot \lambda_{N,u} \cdot du \right) | \mathcal{F}_t \right) \end{aligned} \quad (5.10.3)$$

In the particular case of a constant process $h_t = h$, this expectation has an analytic solution. Assume that $h > \frac{1-\gamma}{\gamma}$, it is therefore possible to define a positive constant k :

$$k = \frac{(1+h)^{\frac{\gamma}{\gamma-1}}}{\left(1 + \frac{\gamma}{\gamma-1}.h\right)}$$

Such that equation (5.10.3) can be rewritten:

$$\begin{aligned} \mathbb{E}^{P^a} \left(H^a(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right) = & \mathbb{E}^{P^a} \left(\exp \left(\int_t^s \ln(k).dN_u \right) \cdot \right. \\ & \left. \underbrace{\exp \left(\int_t^s \ln \left(1 + \frac{\gamma}{\gamma-1}.h \right) .dN_u - \int_t^s \frac{\gamma}{\gamma-1}.h.\lambda_{N,u}.du \right)}_{\frac{dQ^{a, \frac{\gamma}{\gamma-1}.h}}{dP^a}} \middle| \mathcal{F}_t \right) \end{aligned} \quad (5.10.4)$$

The term $\frac{dQ^{a, \frac{\gamma}{\gamma-1}.h}}{dP^a}$ defines a new actuarial measure $Q^{a, \frac{\gamma}{\gamma-1}.h}$, under which the following centered process

$$M_t^{a, \frac{\gamma}{\gamma-1}.h} = N_t - \int_0^t (n_x - N_{u-}) . \mu(x+u) . \left(1 + \frac{\gamma}{\gamma-1}.h\right) . du$$

is a martingale. And the expected number of survivors at time s , conditionally to instant t is:

$$\mathbb{E}^{Q^{a, \frac{\gamma}{\gamma-1}.h}} ((n_x - N_s) | \mathcal{F}_t^a) = (n_x - N_t) . \underbrace{\exp \left(- \int_t^s \mu(x+u) . \left(1 + \frac{\gamma}{\gamma-1}.h\right) . du \right)}_{s-t p_{x+t}^{\frac{\gamma}{\gamma-1}.h}}$$

Equation (5.10.4) is finally rewritten as the expectation under $Q^{a, \frac{\gamma}{\gamma-1}.h}$ of a constant k , exponent the number of deceases.

$$\mathbb{E}^{P^a} \left(H^a(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right) = \mathbb{E}^{Q^{a, \frac{\gamma}{\gamma-1}.h}} \left(k^{N_s - N_t} | \mathcal{F}_t \right)$$

Under $Q^{a, \frac{\gamma}{\gamma-1}.h}$, the probability of observing n deceases in the interval of time (t, s) is a binomial variable of parameters $(n_x - N_t, 1 - s-t p_{x+t}^{\frac{\gamma}{\gamma-1}.h})$. The expected value of $k^{N_s - N_t}$ is then computable by the following formula:

$$\begin{aligned} \mathbb{E}^{P^a} \left(H^a(t, s)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right) &= \mathbb{E}^{Q^{a, \frac{\gamma}{\gamma-1}.h}} \left(k^{N_s - N_t} | \mathcal{F}_t \right) \\ &= \sum_{n=1}^{n_x - N_t} \frac{(n_x - N_t)!}{(n_x - N_t - n)! n!} \left(k^n \cdot \left(s-t p_{x+t}^{\frac{\gamma}{\gamma-1}.h} \right)^{n_x - N_t - n} \cdot \left(1 - s-t p_{x+t}^{\frac{\gamma}{\gamma-1}.h} \right)^n \right) \end{aligned}$$

Appendix C.

In section 5.8, we have defined $\hat{E}_{t,s}$ as:

$$\hat{E}_{t,s} = \mathbb{E}(H(t,s) \cdot \ln H(t,s) | \mathcal{F}_t)$$

Due to the independence of the insurance and financial markets, this last expression can be split in a product of a financial term times an actuarial term, which are calculated separately in the sequel of this paragraph.

$$\begin{aligned} \mathbb{E}(H(t,s) \cdot \ln H(t,s) | \mathcal{F}_t) &= \underbrace{\mathbb{E}^{Q^f} \left(e^{-\int_t^s r_u \cdot du} \cdot \ln \left(H^f(t,s) \right) | \mathcal{F}_t \right)}_{\text{financial}} + \\ &\quad \underbrace{\mathbb{E}^{Q^f} \left(e^{-\int_t^s r_u \cdot du} | \mathcal{F}_t \right) \cdot \mathbb{E}^{Q^{a,h}} (\ln H^a(t,s) | \mathcal{F}_t)}_{\text{actuarial}} \end{aligned} \quad (5.10.5)$$

Calculation of the financial component.

It requires the use of a forward measure. Under the risk neutral measure, the dynamic of the zero coupon bond $B(t,s)$ is such that :

$$\frac{dB(t,s)}{B(t,s)} = r \cdot dt - \sigma_r \cdot n(s-t) \cdot dW_t^{r,Q^f}$$

And the solution of this SDE is:

$$\frac{B(t,s)}{B(0,s)} = \exp \left(\int_0^t r_u \cdot du - \frac{1}{2} \int_0^t \sigma_r^2 \cdot n(s-u)^2 \cdot du - \int_0^t \sigma_r \cdot n(s-u) \cdot dW_u^{Q^f} \right)$$

A change of probability toward $F^{(t,s)}$, the forward measure related to the numeraire $B(t,s)$, is defined by the next variable:

$$\begin{aligned} \frac{dF^{(t,s)}}{dQ^f} &= \frac{B(t,s)}{B(0,s)} \cdot \exp \left(- \int_0^t r_u \cdot du \right) \\ &= \exp \left(- \frac{1}{2} \int_0^t \sigma_r^2 \cdot n(s-u)^2 \cdot du - \int_0^t \sigma_r \cdot n(s-u) \cdot dW_u^{Q^f} \right) \end{aligned}$$

And under the forward measure, we have the interesting properties that:

$$\begin{aligned} \mathbb{E}^{F^{(t,s)}} \left(\ln \left(H^f(t,s) \right) | \mathcal{F}_t \right) &= \frac{\mathbb{E}^{Q^f} \left(\frac{B(s,s) \cdot \exp(-\int_0^s r_u \cdot du)}{B(0,s)} \cdot \ln \left(H^f(t,s) \right) | \mathcal{F}_t \right)}{\mathbb{E}^{Q^f} \left(\frac{B(t,s) \cdot \exp(-\int_0^t r_u \cdot du)}{B(0,s)} | \mathcal{F}_t \right)} \\ &= \frac{1}{B(t,s)} \mathbb{E}^{Q^f} \left(e^{-\int_t^s r_u \cdot du} \cdot \ln \left(H^f(t,s) \right) | \mathcal{F}_t \right) \end{aligned}$$

The financial component of (5.10.5) is therefore reformulated as follows:

$$\mathbb{E}^{Q^f} \left(e^{-\int_t^s r_u \cdot du} \cdot \ln \left(H^f(t,s) \right) | \mathcal{F}_t \right) = B(t,s) \cdot \mathbb{E}^{F^{(t,s)}} \left(\ln \left(H^f(t,s) \right) | \mathcal{F}_t \right)$$

The zero coupon bond is easily calculated by the Vasicek's formula (5.10.1). Whereas the expectation under $F^{(t,s)}$ of the logarithm of the deflator requires additional calculations.

$$\begin{aligned} \mathbb{E}^{F^{(t,s)}} \left(\ln \left(H^f(t, s) \right) | \mathcal{F}_t \right) = \\ \mathbb{E}^{F^{(t,s)}} \left(- \int_t^s r_u . du - \int_t^s \Lambda . dW_u^{Pf} - \frac{1}{2} \cdot \int_t^s \|\Lambda\|^2 . du | \mathcal{F}_t \right) \end{aligned} \quad (5.10.6)$$

As under $F^{(t,s)}$, the following elements are Brownian motions

$$\begin{aligned} dW_u^{r,F^{(t,s)}} &= dW_u^{r,Pf} + \lambda_r . du + \sigma_r . n(s-u) . du \\ dW_u^{s,F^{(t,s)}} &= dW_u^{s,Pf} + \lambda_s . du \end{aligned} \quad (5.10.7)$$

Equation (5.10.6) is rewritten as:

$$\begin{aligned} \mathbb{E}^{F^{(t,s)}} \left(\ln \left(H^f(t, s) \right) | \mathcal{F}_t \right) = \\ - \mathbb{E}^{F^{(t,s)}} \left(\int_t^s r_u du | \mathcal{F}_t \right) - \mathbb{E}^{F^{(t,s)}} \left(\int_t^s \Lambda . dW_u^{F^{(t,s)}} | \mathcal{F}_t \right) \\ + \int_t^s \Lambda . \begin{pmatrix} \lambda_r + \sigma_r . n(s-u) \\ \lambda_s \end{pmatrix} . du - \frac{1}{2} \cdot \int_t^s \|\Lambda\|^2 . du \end{aligned} \quad (5.10.8)$$

The expectation of the Brownian stochastic integral is null and the following proposition gives the value of $\mathbb{E}^{F^{(t,s)}} \left(\int_t^s r_u du | \mathcal{F}_t \right)$.

Proposition 5.10.1. *Under the forward measure $F^{(t,s)}$, we have that*

$$\begin{aligned} \mathbb{E}^{F^{(t,s)}} \left(\int_t^s r_u du | \mathcal{F}_t \right) = & \left(b - \frac{\lambda_r . \sigma_r}{a} - \frac{\sigma_r^2}{a^2} \right) . (s-t) + \\ & \left(r_t - b + \frac{\lambda_r . \sigma_r}{a} + \left(\frac{\sigma_r^2}{a^2} + \frac{1}{2} \cdot \frac{\sigma_r^2}{a} n(s-t) \right) \right) n(s-t) \end{aligned} \quad (5.10.9)$$

Where

$$n(s-t) = \frac{1}{a} \cdot \left(1 - e^{-a \cdot (s-t)} \right)$$

Proof. At first, we calculate r_u under $F^{(t,s)}$. It results from (5.3.1) and (5.10.7) that the dynamic of the instantaneous risk free rate is given by:

$$dr_u = a \cdot \left(\underbrace{b - \frac{\lambda_r . \sigma_r}{a} - \frac{\sigma_r^2 . n(s-u)}{a}}_{b(s-u)} - r_u \right) . du + \sigma_r . dW_u^{r,F^{(t,s)}} \quad (5.10.10)$$

Consider a process X_u defined by:

$$X_u = e^{a \cdot u} \cdot (b(s-u) - r_u) \quad (5.10.11)$$

Taking into account (5.10.10), the differential of X_u is so that:

$$\begin{aligned} dX_u &= a \cdot e^{a \cdot u} \cdot (b(s-u) - r_u) . du + e^{a \cdot u} \cdot \frac{\partial b(s-u)}{\partial u} . du - e^{a \cdot u} . dr_u \\ &= e^{a \cdot u} \cdot \frac{\partial b(s-u)}{\partial u} . du - e^{a \cdot u} \cdot \sigma_r . dW_u^{r,F^{(t,s)}} \end{aligned}$$

And then the process X_u may be rewritten as the sum of integrals:

$$X_u = X_t + \int_t^u e^{a.z} \cdot \frac{\partial b(s-z)}{\partial z} \cdot dz - \int_t^u e^{a.z} \cdot \sigma_r \cdot dW_z^{r, F(t,s)} \quad (5.10.12)$$

From relation (5.10.11), we know that

$$r_u = b(s-u) - e^{-a.u} \cdot X_u \quad X_t = e^{a.t} \cdot (b(s-t) - r_t) \quad (5.10.13)$$

It suffices therefore to combine (5.10.12) (5.10.13) and to differentiate $b(s-z)$ to get that

$$\begin{aligned} r_u &= b(s-u) - e^{-a.(u-t)} \cdot b(s-t) + e^{-a.(u-t)} \cdot r_t \\ &\quad - \int_t^u \frac{\sigma_r^2}{a} \cdot e^{-a.(u+s-2.z)} \cdot dz + \int_t^u e^{-a.(u-z)} \cdot \sigma_r \cdot dW_z^{r, F(t,s)} \end{aligned} \quad (5.10.14)$$

The short term rate r_u is hence Gaussian under $F^{(t,s)}$. Expression (5.10.9) is finally calculated by taking the expectation of the integral of (5.10.14). $\square$

To summarize, the expectation of the logarithm of the financial deflator is equal to

$$\begin{aligned} \mathbb{E}^{F(t,s)} \left(\ln \left(H^f(t,s) \right) | \mathcal{F}_t \right) &= \\ &\quad \frac{1}{2} \cdot \lambda_r^2 \cdot (s-t) + \frac{1}{2} \cdot \lambda_s^2 \cdot (s-t) + \frac{\lambda_r \cdot \sigma_r}{a} \cdot ((s-t) - n(s-t)) \\ &\quad - \mathbb{E}^{F(t,s)} \left(\int_t^s r_u du | \mathcal{F}_t \right) \end{aligned} \quad (5.10.15)$$

Where the expectation of the integral of r_u is given by formula (5.10.9). At this point, we may already calculate the derivative $\frac{\partial \hat{E}_{t,s}}{\partial r_t}$, used in paragraph 5.8 :

$$\frac{\partial \hat{E}_{t,s}}{\partial r_t} = -n(s-t) \cdot \hat{E}_{t,s} - n(s-t) \cdot B(t,s)$$

Calculation of the actuarial component.

The actuarial component of $\hat{E}_{t,s}$ is equal to the product of a zero coupon bond (formula (5.10.1)) times the expectation, under the actuarial measure, of the logarithm of the deflator.

$$\mathbb{E}^{Q^f} \left(e^{-\int_t^s r_u \cdot du} | \mathcal{F}_t \right) \cdot \mathbb{E}^{Q^{a,h}} (\ln H^a(t,s) | \mathcal{F}_t) = B(t,s) \cdot \mathbb{E}^{Q^{a,h}} (\ln H^a(t,s) | \mathcal{F}_t)$$

Remember that

$$H^a(t,s) = \exp \left(\int_t^s \ln(1+h_u) \cdot dN_u - \int_t^s h_u \cdot \lambda_{N,u} \cdot du \right)$$

In the particular case of a constant process $h_u = h$, we have that:

$$\mathbb{E}^{Q^{a,h}} (\ln H^a(t,s) | \mathcal{F}_t) = \ln(1+h) \cdot \mathbb{E}^{Q^{a,h}} (N_s - N_t | \mathcal{F}_t) - h \cdot \int_t^s \mathbb{E}^{Q^{a,h}} (\lambda_{N,u} | \mathcal{F}_t) \cdot du$$

Where $\mathbb{E}^{Q^{a,h}}(N_s - N_t | \mathcal{F}_t)$ is the expected number of deceases, under the insurer's pricing probability:

$$\mathbb{E}^{Q^{a,h}}(N_s - N_t | \mathcal{F}_t) = (n_x - N_t) \cdot (1 - {}_{s-t}p_{x+t}^h)$$

Whereas,

$$\int_t^s \mathbb{E}^{Q^{a,h}}(\lambda_{N,u} | \mathcal{F}_t) \cdot du = \int_t^s (n_x - N_t) \cdot {}_{u-t}p_{x+t}^h \cdot \mu(x+u) \cdot du$$

To summarize, the expectation of the logarithm of the actuarial deflator is equal to

$$\begin{aligned} \mathbb{E}^{Q^{a,h}}(\ln H^a(t, s) | \mathcal{F}_t) &= \ln(1+h) \cdot (n_x - N_t) \cdot (1 - {}_{s-t}p_{x+t}^h) \\ &\quad - h \cdot (n_x - N_t) \cdot \int_t^s {}_{u-t}p_{x+t}^h \cdot \mu(x+u) \cdot du \end{aligned} \quad (5.10.16)$$

Appendix D.

In the examples presented in this chapter, real mortality rates and accounting mortality rates obey to a Gompertz-Makeham distribution. The parameters are those defined by the Belgian regulator for the pricing of a life insurance purchased by a man. For an individual of age x , the mortality rate is :

$$\mu(x) = a_\mu + b_\mu \cdot c^x \quad a_\mu = -\ln(s_\mu) \quad b_\mu = \ln(g_\mu) \cdot \ln(c_\mu)$$

Where the parameters s_μ , g_μ , c_μ take the values showed in the table 6.8.1.

Table 5.10.1: Belgian legal mortality, for life insurance products, and for a male population.

$s_\mu :$	0.999441703848
$g_\mu :$	0.999733441115
$c_\mu :$	1.116792453830

