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Modelling, Analysis and Control of Counterflow Heat Exchangers:
An Infinite-Dimensional and Thermodynamic Approach

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Resumé:
Dans les applications industrielles où l'énergie est vitale, les échangeurs de chaleur ont acquis une importance économique majeure. En effet, une grande part de l'énergie thermique produite ou collectée en industrie passe au moins une fois par un échangeur de chaleur. Ces dispositifs permettent principalement de transférer de la chaleur d'un fluide chaud vers un fluide froid, et appartiennent à la classe des systèmes thermodynamiques dont la dynamique est décrite par des équations différentielles aux dérivées partielles. Les échangeurs de chaleur sont généralement dimensionnés pour fonctionner en régimes permanents. Malheureusement, dès lors qu'il apparait des perturbations dues notamment à des variations de température ou de débit d'alimentation, l'efficacité du système se dégrade. Alors, est-il possible de maintenir ou d'améliorer les performances en s'intéressant au régime transitoire négligé lors du dimensionnement? Cette problématique est abordée dans cette thèse en trois parties.
La première partie du travail présente la modélisation thermodynamique en dimension infinie d'un échangeur de chaleur à contre-courant, et l'estimation des paramètres du modèle en température (modèle standard en génie des procédés), tout en traitant des questions d'identifiabilité structurelle et pratique.
La seconde partie de la thèse aborde la problématique liée aux propriétés dynamiques de l'échangeur de chaleur. En particulier, les questions d'existence et d'unicité de solutions positives, de commandabilité et d'observabilité, ainsi que celles de passivité et de stabilité au sens thermodynamique sont étudiées.
La dernière partie de la thèse traite premièrement de la commande proportionnelle-intégrale par action frontière tout en cherchant des conditions sur les paramètres du régulateur qui garantissent la stabilité du système commandé. Ensuite, la théorie de commande optimale linéaire quadratique pour les systèmes de dimension infinie est appliquée au modèle d'échangeur de chaleur considéré. Les résultats de simulation montrent que ces deux stratégies de commande permettent de rejeter les perturbations dues aux variations de température et de débits ou vitesses.

Abstract:
In industrial applications where energy is vital, heat exchangers have acquired major economic importance. Indeed, a large part of the thermal energy produced or collected in industry passes at least once through a heat exchanger. These devices mainly make it possible to transfer heat from a hot fluid to a cold fluid, and belong to the class of thermodynamic systems whose dynamics are described by partial differential equations. Heat exchangers are generally sized to operate in steady state. Unfortunately, as soon as disturbances appear due in particular to variations in temperature or supply flow rates, the efficiency of the system deteriorates. So, is it possible to maintain or improve performance by looking at neglected transients during sizing? This issue is addressed in this thesis in three parts.

The first part of the work presents the infinite dimensional thermodynamic modelling of a counterflow heat exchanger, and the estimation of the model parameters in temperature (standard model in process engineering), while dealing with questions of structural and practical identifiability.

The second part of the thesis addresses the issue related to the dynamic properties of the heat exchanger. In particular, the questions of existence and uniqueness of positive solutions, of controllability and observability, as well as those of passivity and stability in the thermodynamic sense are studied.

The last part of the thesis first deals with the proportional-integral control by boundary action by looking for conditions on the parameters of the controller, which guarantee the stability of the closed-loop system. Next, the linear quadratic optimal control theory for the general class of linear infinite-dimensional systems is applied to the heat exchanger model considered. The simulation results show that these two control strategies make it possible to reject the disturbances due to variations in temperature and flow rates or velocities.

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Jacques KADIMA KAZAKU

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List of notations and abbreviations

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t	The time variable
z	The spatial variable
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The set of real vectors of dimension n
$\mathbb{C}$	The set of complex numbers
$\mathbb{C}^n$	The set of complex vectors of dimension n
$\mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$)	Denote the set real (or complex) of $n-$order matrices, respectively
$\Re e(\lambda)$	The real part of the complex number λ
$M^\top$	The transpose of the matrix M
∂_t	The operator of first-order time derivative, i.e. $\frac{\partial}{\partial t}$
∂_z	The operator of first-order space derivative, i.e. $\frac{\partial}{\partial z}$
$\dot{\xi}(t)$	The derivative of $\xi(t)$ with respect to t
I	The identity operator
$L^2(0,1) = L^2\left((0,1);\mathbb{C}\right)$	The Hilbert space of measurable square-integrable function with values in $\mathbb{C}$
$L^\infty(0,1) = L^\infty\left((0,1);\mathbb{C}\right)$	The space of essentially bounded $\mathbb{C}$-valued functions defined on $[0,1]$
$H^1(0,1) = H^1\left((0,1);\mathbb{C}\right)$	The Sobolev space of absolutely continuous $\mathbb{C}-$valued functions whose derivatives are in $L^2(0,1)$
$\langle \xi, \zeta \rangle_{L^2(0,1)}$	The scalar(inner) product between ξ and ζ on $L^2(0,1)$
$\Vert \cdot \Vert_{L^2(0,1)}$	The norm induced by the inner product $\langle \cdot\,,\,\cdot \rangle_{L^2(0,1)}$ on the Hilbert space $L^2(0,1)$
$\mathcal{D}(A)$	The domain of the operator A
$(\mathbb{T}(t))_{t\geq 0}$	C_0-semigroup of bounded linear operators
A^{-1}	The right inverse of the linear operator A, whenever it exists
A^*	The adjoint of the linear operator A
$\rho(A)$	The resolvent set of the linear operator $A : \mathcal{D}(A) \subset X \to X$, i.e. the set of all complex values λ such that $(\lambda I - A)^{-1}$ is a bounded linear operator on X
$R(\lambda, A)$	The resolvent operator of A, given by $(\lambda I - A)^{-1}$ with $\lambda \in \rho(A)$
$\mathrm{Ran}(A)$	The image of the operator A
$\mathrm{Ker}A$	The kernel (or null space) of the operator A
$\mathcal{L}(X,Y)$	The vector space of all bounded linear operators from from X to Y
$\mathcal{L}(X)$	The set of linear and bounded operators on

	the Hilbert space X to X
X_{-1}	The completion of X equipped with the norm $\|g\|_{-1} = \|(\lambda I - A)^{-1} g\|_X$
$\sup_{n\geq 1}\lambda_n$	The supremum of the set of real numbers λ_n, $n \geq 1$ over the natural numbers

List of abbreviations

a.c.	absolutely continuous
a.e.	almost everywhere
C_0-semigroup	Strongly-continuous semigroup
det	determinant of a matrix
diag	diagonal matrix
iff	if and only if
LQ	Linear Quadratic
ODE	Ordinary Differential Equation
ORE	Operator Riccati Equation
PDE	Partial Differential Equation
PI	Proportional Integral
PID	Proportional Integral Derivative
RDE	Riccati Differential Equation
RMSE	Root-Mean-Square Error
VAF	Variance Accounted For
var	Variance

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INTRODUCTION

0.1 General context

INDUSTRIAL processes (chemical, biochemical, metallurgical, etc.) are often characterized by an inhomogeneous environment in which they are implemented. More precisely, their state variables (temperatures, concentrations, biomass, substrate, etc.) are characterized by spatial gradients. This leads to consider partial differential equations (PDEs) to better understand their dynamics; in this case we speak of distributed parameter systems. This is the case of the tubular reactor (plug-flow effect or axial dispersion), ore crushers, heat exchangers, etc. This class of systems has interested researchers in different fields for several decades, particularly for analysis and control purposes.

The synthesis of control and observation laws of distributed parameter systems has been the subject of intense scientific activity. The complexity of the mathematical processing of PDEs due in particular to the distributed nature of the state variables of the system make this study delicate. For a long time, control law synthesis techniques have been based on an approximate, finite-dimensional model described by ordinary differential equations (ODE). These were obtained by prior reduction of the PDE describing the dynamic behavior of the system using approximation techniques (finite differences, weighted residuals methods, finite elements, finite volumes, method of lines, etc) [Dochain, 1994, Renou, 2000, Burns et al., 2013, Burns and Cliff, 2014a, Burns and Cliff, 2014b]. This strategy has the advantage of accessing all of the theory developed for lumped parameter systems [Bastin and Dochain, 1990, Dochain, 2008, Corriou, 2018]. Nevertheless, to better approximate the behavior of the PDE, the number of ODE equations retained must be sufficiently high. This generally complicates the synthesis step and often leads to a high-dimensional corrector thus constraining the online implementation of the latter.

Significant advances in the field of computer science have pushed the automatic control community to develop control laws from the PDE model. In this approach, the PDE model is represented throughout the synthesis phase in a functional space, and the approximation step only occurs during the simulation and implementation of the control law. Some similarities between the control of lumped and distributed parameter systems have however been highlighted by [El Jai and Pritchard, 1988] and [Amouroux et al., 1994]. In the particular case of linear systems, the theory of infinite dimensional systems has enabled [Curtain and Zwart, 1995] to provide a complete framework for the development of control laws. This framework was used by [Winkin et al., 2000] who applied it to fixed-bed tubular reactors by demonstrating the existence of a mild solution as well as the notions of stability, observability and reachability in the case of plug-flow and axial-dispersion models, with bounded input and output operators.

Note that the modelling of control and observation for certain real systems whose representation is made by PDEs, (generally) involves unbounded control and observation operators. This happens, for example, when we have the boundary control or when the measurements are made at a point or over a limited spatial domain. Unlike the classic case as described by [Curtain and Zwart, 1995] where the operators are bounded, this non-boundedness represents a handicap during the system analysis and control phase. This amounts to the fact that the solution may not exist in the state space, that it does not depend continuously on the control introduced and that the output function may lose its point interpretation beyond a domain of initial states. This problem has been the subject of several studies. A generalization of infinite-dimensional linear systems is the class of well-posed linear systems. This class was introduced by [Salamon, 1987] to formulate and solve in a unified way control problems for systems described by partial differential equations or by functional differential equations, with unbounded input and output operators. An important subclass of well-posed linear systems consists of regular systems, introduced by [Weiss, 1994]. All these theories now constitute an effective toolbox for the analysis (well-posedness, stability, controllability, observability, etc.) of distributed parameter systems (for more information, see the papers [Weiss, 1994, Tucsnak and Weiss, 2014] and the books [Tucsnak and Weiss, 2009, Curtain and Zwart, 2020]).

0.2 Motivation

In this thesis we are interested in the analysis and control of infinite dimensional thermodynamic systems, more specifically to the *heat exchangers*. The choice of this system is based on the fact that in industrial applications where energy is vital, heat exchangers have acquired major economic importance. Almost all the produced or collected thermal energy passes at least once through a heat exchanger. Therefore, it is important to analyze the related energy transfer phenomena, in order to optimize their operation.
Recall that a fundamental notion of Thermodynamics is the concept of energy. It first appears in mechanics where it covers the notion of work capacity, of force in action: it is defined mathematically as the scalar product of a force by its displacement. This notion can be generalized to other areas and we talk about forms of energy related to different phenomena: electrical, magnetic energy etc. Among the many modes of energy conversion, the one that consists in transferring heat from a hot fluid to a cold fluid is carried out in devices called heat exchangers. There are several types of heat exchangers depending on whether their operation is continuous or discontinuous, or even depending on whether the fluids which exchange heat are separated by a wall or, on the contrary, placed in direct contact [Bejan and Kraus, 2003, Chapter 11]. Another classification element relates to the eventual phase change of one of the two fluids: we will talk about

- *heaters* or *coolers* in the absence of phase change (see [Bejan, 2016, Chapter 10]), and
- *evaporators, steam generators, boilers* or *condensers* if the objective pursued is vaporization, boiling or condensation of one of the two fluids (see [Bejan and Kraus, 2003, Chapter 9 and 10]).

Other denominations still highlight the main purpose of energy conversion; this is how, for example, we call *recuperators, preheaters*, or *economizers* devices which make it possible to recover thermal energy in particular for the benefit of preheating. We devote most of this work to the case of devices operating continuously and for which the heat transfer takes place through a wall. It should be noted that depending on the direction of circulation of the two fluids along the exchange wall, a distinction is made between parallel flow exchangers and counterflow exchangers. This last category interests us because it offers better performance, and is the most widespread in the industrial sector. In general, heat exchangers are sized to operate efficiently in steady state. As soon as the temperatures or feed flow rates change, the heat transfer is no longer optimal. It is then obvious to automate the process so as to maintain the performance of the system at an acceptable level.
The control of these devices has attracted the attention of many researchers, and several control approaches have been developed. A large number of contributions, dealing with the control of the temperature of the heat exchangers ranging from classical control techniques (PID) to advanced techniques by extremum seeking, in the context where the dynamics of the system is described by a lumped parameter model, can be found in the literature [Saatci et al., 1981, Padhee et al., 2011, Sahoo et al., 2017, Zitte et al., 2018, Jin et al., 2018, Bastida et al., 2019, Knudsen et al., 2019, Kumar et al., 2021]. A recent thesis gives a general overview of the essentials of these control approaches [Zitte, 2021]. When the dynamics of the heat exchanger are described by PDEs, the review of the literature dealing with this subject shows that the contributions are divided between the pre-approximation strategy (see [Burns et al., 2013, Burns and Cliff, 2014a, Burns and Cliff, 2014b, Sano, 2015, Sano, 2017]) and the post-approximation strategy (see [Derese and Noldus, 1980, Xu and Jerbi, 1995, Grabowski, 2007, Maidi et al., 2008, Maidi et al., 2009, Maidi et al., 2010, Aulisa et al., 2016, Xu and Dubljevic, 2016b, Ozorio Cassol et al., 2019, Sandoval, 2021]). Within the framework of this last strategy, [Grabowski, 2007, Aulisa et al., 2016, Sandoval, 2021] propose external fluid velocity manipulation control approaches to regulate the internal fluid outlet temperature of a counterflow heat exchanger. [Maidi et al., 2009, Maidi et al., 2010] proposes a classic controller (PID) to regulate the outlet temperature of a tubular heat exchanger, by comparing the approaches by velocity manipulation and that by manipulation of the inlet temperature of the external fluid. These control laws are obtained using the relative degrees approach, which consists of differentiating the output until the control variable appears. The conclusion is that the temperature manipulation approach provides better performance than the velocity manipulation approach. Recently advanced control strategies have been extended to the distributed parameter systems. In this context [Xu and

Dubljevic, 2016a, Xu and Dubljevic, 2016b] have proposed (state-feedback) boundary control strategies by internal model, based on the manipulation of the inlet temperature of the external fluid. In general, these control laws are synthesized on the assumption that the state of the system is measured at each point in space. This assumption is purely theoretical. In practice, it is rare or even impossible in the case of distributed parameter systems to access the complete state of the system. Indeed, the dimension of the state space of these systems is infinite, while that of the observations is finite (in other words, the measurement is only accessible on certain subsets of the domain of the system). Hence the need to develop software sensors (state observers) that can produce an estimate of the variables necessary for the synthesis of control laws [Zobiri et al., 2017, Ozorio Cassol et al., 2019, Ghousein et al., 2020].

The main objective of this thesis project is to propose effective approaches for controlling counterflow exchangers in a context where the dynamics are described by PDEs, so as to take into account the distributed character of the state variables. Firstly, the synthesis approaches by Lyapunov and semi-group methods will be discussed, in a context of boundary output feedback. Next, the optimal observer-based control will be discussed. Such objectives can only be achieved by having a good model, i.e. describing the dynamics of the system fairly faithfully. Added to this is a good knowledge of the dynamic properties of the system. Thus, well before addressing the issue relating to control, we will address the issues of modelling, parameter estimation, as well as analysis in an infinite dimensional context.
Throughout this thesis, we will restrict ourselves to the study of single-phase liquid-liquid heat exchangers, for which the fluids are considered incompressible. Indeed, gas is a more unstable phase characterized by a high sensitivity of its density with respect to thermodynamic conditions (temperature, pressure).

0.3 Structure of the thesis

This study is organized into three main parts: modelling, analysis, and finally the control of a counterflow heat exchanger.
The **first part** deals with the modelling and the identification of the parameters of a heat exchanger, and consists of two chapters. Chapter 1 begins with a reminder of the fundamental properties of thermodynamics on which the modeling of heat exchangers is based. These properties will also be used in Chapter 4. Next, we present the energetic and entropic representations of the counterflow heat exchanger model. We end the chapter by proposing a model based only on fluids temperatures, which we derive from the entropic representation of the heat exchanger. An important stage of modeling consists not only of choosing a model suitable and appropriate for describing the heat exchanger dynamics studied but also of calibrating the parameters of this

model. This stage is far from being understood, given the complexity of models. Chapter 2 will attempt to introduce the problem of identification of the parameters of the model of the counterflow heat exchanger, by dealing with questions of structural and practical identifiability as well as the suitable methods to carry out this identification. For this chapter in particular, the structural identifiability study will be completely done on the PDEs model while that of practical identifiability on the ODEs model obtained by the method of lines.

The **second part** deals with the analysis of the dynamic properties of the counterflow heat exchanger, and consists of two chapters. In chapter 3, we are first interested in the existence and uniqueness of positive solution for the uncontrolled system. By a pseudo-spectral analysis, it is shown numerically that the stability is guaranteed despite the fact that the semigroup is non-normal. Next, we show that the abstract problem constitutes a well-posed boundary control-observation system, with unbounded control and observation operators this in the case where the control and the observation are applied at the boundary. Finally, the properties of controllability and exact observability are addressed. In Chapter 4, we perform an analysis of the equilibrium profiles. In this analysis we show the importance of the thermal pinch as an energy efficiency factor. Finally we study the passivity and the asymptotic stability of the considered heat exchanger essentially basing ourselves on the entropy as a storage function, entropy production as a dissipation function, and the thermodynamic availability function as a Lyapunov function. In this part, the discretization of the PDE model will only intervene to simulate the obtained results.

The **third part** concerns control issues and also consists of two chapters. Chapter 5 will focus on developing the basic concepts of boundary control of a heat exchanger, in which we are choosing a proportional-integral (PI)-controller implemented on an output feedback from a boundary condition. In particular, we are looking for conditions on controller gains using Lyapunov's method, as well as some results based on operator perturbation theory. The Chapter 6 will concentrate on the development of more sophisticated control methods with the objective of guaranteeing the best possible counterflow heat exchanger operation. Emphasis will be placed on optimal control method. In this part, the discretization of the PDEs model will only occur for the implementation of the control and observation laws.

0.4 Contributions by part

- *First part*: The infinite-dimensional thermodynamic representation of a counterflow heat exchanger, estimation and validation of the model parameters in temperature (standard model in process engineering), while dealing with questions of structural and practical identifiability, whose objective is to verify the possibility of determining the parameters of the system in a unique way from the available data.

- *Second part*: First, the consolidation of existing results relating to the analysis of counterflow heat exchangers in an appropriate Hilbert space by providing a condition on the model parameters (and giving a thermodynamic interpretation to this condition) that guarantees the existence and uniqueness of positive solutions. This gives meaning to the model. Next, the study of passivity and stability in the thermodynamic sense, precisely using the entropy as a storage function, the entropy production as a dissipation function for to proof the passivity, and the thermodynamic availability function for asymptotic stability.
- *Third part*: The use of PI regulation in a boundary control context, while searching the conditions on the controller parameters by Lyapunov method as well as by semigroup techniques, that guarantee the stability of the closed-loop system. The application of LQ-optimal control theory for infinite dimensional systems.

Most of the research presented in this manuscript has been published in international conference proceedings (with peer review process), submitted, or in preparation for submission for publication in scientific journals or has been the object of a communication as a talk or a poster presentation, at scientific and engineering conferences and workshops.
The most important publications and communications are listed below:

International conference proceedings with oral communication

- J. K. Kazaku, D. Dochain, M. M. Kahilu, and J. K. K. Kasongo, "*Optimal Boundary State Feedback Control by Triangularization of the Counterflow Heat Exchanger Model*", 2023, submitted to the 62nd IEEE Conference on Decision and Control;
- J. K. Kazaku, D. Dochain, M. M. Kahilu, and J. K. K. Kasongo," *Parameter Estimation of Hyperbolic Model of Counterflow Heat Exchanger*," 22th IFAC World Congress 2023, (July 09-14, Yokohama, Japan), *to appear in IFAC-POL (IFAC PapersOnLine)*;
- J. K. Kazaku, D. Dochain, M. M. Kahilu, and J. K. K. Kasongo," *Thermodynamics Analysis of the Distributed Parameter Model of Counterflow Heat Exchanger*," IFAC World Congress 2023, (July 09-14, Yokohama, Japan), *to appear in IFAC-POL (IFAC PapersOnLine)*;
- J. K. Kazaku, D. Dochain, M. M. Kahilu, and J. K. K. Kasongo, "*P-I Boundary Control of Countercurrent Heat Exchanger*," *IFAC-PapersOnLine*, vol. 55, no. 40, pages. 49–54, 2022, 1st IFAC Workshop on Control of Complex Systems COSY 2022, (November 24-25, Bologna, Italy);
- J. K. Kazaku, D. Dochain, M. M. Kahilu, and J. K. K. Kasongo, "*Optimal Observer-Based LQ-Feedback Regulation for Hyperbolic Model of a Countercurrent Heat Exchanger*", 2022 American Control Conference (ACC), (June 8-10, Atlanta, USA), pages 1679–1684;

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Part I

MODELLING

1

MODELLING OF COUNTERFLOW HEAT EXCHANGER

Contents

1.1 Introduction

THE notion of dynamic model plays a central role in system theory and control. It is indeed on the basis of the knowledge of the evolution over time of the system that all the analysis, design and implementation of monitoring and control methods are carried out. In the context of heat exchangers, the most natural way to find models that characterize the dynamics of the energy transfer is to consider the energy balances of each fluid. Under some assumptions, this can lead to lumped-parameter models (see [Jonsson et al., 1992], [Mutambara and Al-Haik, 1999]), as well as parabolic (see [Burns et al., 2013], [Aulisa et al., 2016]) or hyperbolic (see [Maidi et al., 2009], [Burns and Cliff, 2014a]) distributed parameter models, depending on whether or not heat diffusion phenomenon is considered.
In this chapter we propose to provide different thermodynamic representations of the heat exchanger, from energy and entropy balance.

The chapter is organized as follows. In section 1.2, we recall the thermodynamic foundations of heat exchangers (the thermodynamics laws, then the local equilibrium assumption based on the Gibbs equation for non-equilibrium systems). In section 1.3 we give the assumptions and the constitutive laws of the considered phenomena. Next, we present different models resulting from energy and entropy balances, from which we deduce the standard model (whose dynamics is defined from the temperatures of each fluid) generally used in process engineering.

1.2 Theoretical overview of thermodynamics

We devote this part to a brief reminder of the thermodynamics notions, that will allow to model the heat exchanger while integrating the transport phenomena specific to it. As any scientific discipline, thermodynamics is based on some precise definitions. Before going any further, let us start by recalling the essentials of these definitions.

1.2.0.1 Intensive and extensive variables

An **extensive variable**, associated with a system, is a quantity proportional to the quantity of matter contained in this system, i.e. additive and with positive value. A quantity is additive if its value in the system $\Sigma \cup \Sigma'$ is equal to the sum of its values in the system Σ and Σ' provided that Σ and Σ' are disjoint and initially in equilibrium.
For example, mass, volume, enthalpy,... are extensive quantities or variables.
An **intensive variable** is an independent variable of the quantity of matter considered. For example, pressure, velocity, temperature,... are intensive variables. They are defined punctually.

1.2.0.2 State variables and state functions

The **state variables** are extensive, independent quantities, knowledge of which is sufficient to define the macroscopic state of the system. The state quantities not chosen as state variables constitute the **state functions**; they are deduced from the state variables by state equations.
For any isolated system, we can define a function of state variables, extensive, called **energy** E, which is conservative, i.e. which remains constant, in all circumstances, when the system does not exchange energy with the external environment.
The following definition highlights the concept of *internal energy* as defined and used in thermodynamics. For that, one proposes to restrict the total energy to the mechanical and material domain.

Definition 1.2.1 (Internal energy)**.** *We define the internal energy by the quantity U, intrinsically associated with the considered system such that:*

$$U = E - (E_p + E_c), \tag{1.2.1}$$

where E_p is the potential energy associated with external forces and E_c the macroscopic kinetic energy. For systems macroscopically at rest ($E_c = 0$) and not subject to an external field ($E_p = 0$), the total energy E is deduced from the internal energy U.

Definition 1.2.2 (Entropy)**.** *For any isolated system, the entropy S is a non-conservative state function which expresses the degradation of energy, and which satisfies the following relation:*

$$S = S(U, V, N), \tag{1.2.2}$$

where U is internal energy, V is the volume and N the number of moles.

1.2.1 Equilibrium Thermodynamics

The concept of thermodynamic equilibrium characterizes the tendency of an isolated system to evolve towards an state independent of its past, and this state is characterized by a finite number of variables. The equilibrium states of a system are completely characterized macroscopically by their internal energy U, their volume V, and the numbers of moles N_j or the mass m_j for j particles. This characterizes the extensive variables on which balance sheets are written and at the same time defines the state of the system [Callen, 2006].
The *first law* of thermodynamics is a *conservation* principle. It stipulates that the variation of the internal energy during an infinitesimal transformation of an isolated system is equal to the energy exchanged with the external environment. However, this principle imposes no constraint on the direction of the energy exchange between the system and the external environment. Yet evolution always happens spontaneously. This meaning is specified by the *second law* of thermodynamics which is indeed a principle of *evolution*. This principle introduces the notion of irreversibility of physical phenomena. This irreversibility is formalized by the entropy function S, a non-conservative extensive state function.
The thermodynamic equilibrium of an isolated system (which exchanges neither work, nor heat, nor matter with the exterior) is characterized by a maximum of entropy and a minimum of internal energy [de Groot et al., 1984, Chapter 5]. The equilibrium of an isolated system (which does not exchange matter with the outside) under constraint is characterized by the minimum of a specific thermodynamic potential, which is also a state function, which corresponds to a maximum of the overall entropy of the system and its exterior. The entropy and the thermodynamic potentials consequently possess characteristics of concavity or convexity which determine some properties of the intensive variables.

From the second law, we give the following postulates (for more details see the books [Callen, 2006] and [de Groot et al., 1984]):

Postulate 1.2.1. *The entropy S has the following properties:*

- *In an isolated system, the values taken by the extensive quantities are those which maximize the entropy over all the equilibrium states.*
- *The entropy S is a function of the extensive variables U, V, N_j.*
- *S is homogeneous function of degree 1 in these variables[a] :*

$$S(\lambda U, \lambda V, \lambda N_j) = \lambda S(U, V, N_j), \quad \forall \lambda \in \mathbb{R}. \tag{1.2.3}$$

[a] A consequence of this point is that we can invert S with respect to U. This implies that U is a definite, continuous, differentiable function of S, V, N_j.

Postulate 1.2.2. *According to the second law, for an isolated system (which does not exchange matter or energy in any form whatsoever with the exterior) S cannot decrease.*

$$\frac{\partial S}{\partial U} \geq 0. \tag{1.2.4}$$

In the case of a reversible transformation, it is clearly seen that this postulate attributes a null origin to the entropy.

Based on the postulate 1.2.2, it is necessary to write:

$$U = U(S, V, N_j), \tag{1.2.5}$$

which can be put in the differential form as follows:

$$dU = TdS - pdV + \sum_{j=1}^{n} \mu_j dN_j, \tag{1.2.6}$$

where:

- $T = \left(\frac{\partial U}{\partial S}\right)$, represents the temperature,
- $p = -\left(\frac{\partial U}{\partial V}\right)$, represents the pressure,
- $\mu_j = \left(\frac{\partial U}{\partial N_j}\right)$, represents the chemical potential of component j.

The quantities T, p and μ_j are *intensive* variables defined on the partial derivatives of the internal energy. Equation (1.2.6) is the *fundamental Gibbs' equation* from equilibrium thermodynamics, and is applicable at all times. It can be rewritten in the following compact form:

$$dU = w_u^\top dZ, \tag{1.2.7}$$

with $Z = (S \quad V \quad N_j)^\top$, the vector of extensive variables and $w_u^\top = \left(\frac{\partial U}{\partial Z}\right)$ the vector of intensive variables. The Euler relation, which follows from the homogeneous first-order property of internal energy U, is ([Callen, 2006, page 284]):

$$U = w_u^\top Z, \tag{1.2.8}$$

and the Gibbs-Duhem relation is

$$dw_u^\top Z = 0. \tag{1.2.9}$$

From the Gibbs' relation given by equation (1.2.6) we can deduce the entropic representation given by:

$$dS = \frac{1}{T}dU + \frac{p}{T}dV - \sum_{j=1}^{n} \frac{\mu_j}{T} dN_j = w_s^\top dZ, \tag{1.2.10}$$

with $w_s^\top = \left(\frac{1}{T}, \quad \frac{p}{T}, \quad \frac{-\mu_j}{T}\right)$ the vector of intensive variables, and $Z = (U, V, N_j)$ the vector of extensive variables. S is a homogeneous function of degree with respect to the extensive variables Z. We can then rewrite S explicitly by Euler's theorem: $S = w_s^\top Z$. Equivalently we have:

$$dw_s^\top Z = 0, \tag{1.2.11}$$

that represents the Gibbs-Duhem relation in entropy representation.

1.2.1.1 Concavity of the entropy

Consider any two isolated thermodynamic systems, denoted α and β. Each of these two systems is at thermodynamic equilibrium independently of the other. Their entropy respectively is $S^\alpha = S(U^\alpha, V^\alpha, N^\alpha)$ and $S^\beta = S(U^\beta, V^\beta, N^\alpha)$. Let γ be any number with $0 < \gamma < 1$. Imagine, then, to form a single isolated system by putting together a fraction of system α and a fraction $1 - \gamma$ of system β. The entropy variables being extensive, these system fractions $\gamma\alpha$ and $[1 - \gamma]\,\beta$ are defined by the fractions of the initial variables, respectively. Entropy being itself extensive at equilibrium, the entropies of these two fractions of systems correspond to the fractions of the entropies of the initial α and β systems, and are respectively equal to [Prestipino and Giaquinta, 2003]:

$$\begin{aligned}
S^{\gamma\alpha} &= S(\gamma U^\alpha, \gamma V^\alpha, \gamma N^\alpha) = \gamma S(U^\alpha, V^\alpha, N^\alpha) = \gamma S^\alpha, \\
S^{\gamma\beta} &= S((1-\gamma)U^\beta, (1-\gamma)V^\beta, (1-\gamma)N^\beta) \\
&= (1-\gamma)S(U^\beta, V^\beta, N^\beta) \\
&= (1-\gamma)S^\beta,
\end{aligned} \tag{1.2.12}$$

which explains the third assertion of Postulate 1.2.1. These two fractions form a new isolated system $\gamma\alpha + [1-\gamma]\,\beta$; and since the entropy variables are all extensive, they add up. In particular, the new system being isolated, then by the first law of thermodynamics, its internal energy cannot evolve and can only be equal to $\gamma\, U^{\alpha} + (1-\gamma)\, U^{\beta}$. Therefore, the new system is defined by the sums of the variables of the two fractions, and its entropy is a such that:

$$\begin{aligned} S^{\gamma\alpha+(1-\gamma)\beta} = S\Big(&\gamma U^{\alpha} + (1-\gamma)U^{\beta}, \gamma V^{\alpha} + (1-\gamma)V^{\beta}, \\ &\gamma N^{\alpha} + (1-\gamma)N^{\beta}\Big). \end{aligned} \tag{1.2.13}$$

If the two fractions are not in equilibrium, then the new system is the seat of a thermodynamic transformation. According to the second law of thermodynamics, the entropy of the new system can only increase during this transformation and, when it has reached equilibrium, can only be greater than the sum of the entropies of the two fractions. Therefore, at equilibrium, entropy S of new system $\gamma\alpha + (1-\gamma)\beta$ is a concave function of (U, V, N) [Callen, 2006, Page 207-208]:

$$S^{\gamma\alpha+(1-\gamma)\beta} \geq \gamma S^{\alpha} + (1-\gamma)S^{\beta}. \tag{1.2.14}$$

1.2.1.2 Convexity of internal energy

The convexity of the internal energy $U(S, V, N)$ can be deduced from the concavity of the entropy $S(U, V, N)$. Let us consider the differential form of the internal energy (1.2.6). Its second derivative is written:

$$d^2U = dTdS - dpdV + \sum_{j=1}^{n} d\mu_j dN_j. \tag{1.2.15}$$

Substituting it in the expression for the second derivative of entropy, we get:

$$\begin{aligned} d^2S &= d\left(\frac{1}{T}\right)dU + d\left(\frac{p}{T}\right)dV - \sum_{j=1}^{n} d\left(\frac{\mu_j}{T}\right)dN_j \\ &= -\frac{1}{T^2}dTdU - \frac{p}{T^2}dTdV + \frac{1}{T}dpdV - \sum_{j=1}^{n}\frac{1}{T}d\mu_j dN_j \\ &\quad + \sum_{j=1}^{n}\frac{\mu_j}{T^2}dTdN_j. \end{aligned}$$

Then, substituting (1.2.6) in this expression we get:

$$\begin{aligned} d^2S &= -\frac{1}{T}dTdS + \frac{p}{T^2}dTdV - \sum_{j=1}^{n}\frac{\mu_j}{T^2}dTdN_j - \frac{p}{T^2}dTdV \\ &\quad + \frac{1}{T}dpdV - \sum_{j=1}^{n}\frac{1}{T}d\mu_j dN_j + \sum_{j=1}^{n}\frac{\mu_j}{T^2}dTdN_j \\ &= -\frac{1}{T}\left(dTdS - dpdV + \sum_{j=1}^{n}d\mu_j dN_j\right) \\ &\overset{\text{Eq.(1.2.15)}}{=} -\frac{1}{T}d^2U \Longrightarrow d^2U = -Td^2S. \end{aligned} \tag{1.2.16}$$

Since the temperature T can only be positive, then d^2S and d^2U have opposite signs. The entropy being a concave function, the internal energy is a convex function with respect to its variables (S, V, N_j) [Callen, 2006, Page 207-208].

1.2.2 Non-equilibrium thermodynamics

Classical thermodynamics makes it possible to describe thermodynamic systems which evolve through a series of equilibrium states (reversible transformation); which seems contradictory to real processes, which generally exchange energy or matter with their environment. However, the objective is to have a system in equilibrium at all times. This difficulty of the irreversibility of real processes can be circumvented by considering the local equilibrium assumption which states that [de Groot et al., 1984, Chapter 3.2]: "the local and instantaneous relations of a irreversible system are the same as those of a system at thermodynamic equilibrium". Once admitted, this assumption implies that the thermodynamic relations (1.2.4) and (1.2.6) remain valid for irreversible systems ([Hoang and Dochain, 2013], [Zhou et al., 2015]). Thus, the differential form of the Gibbs equation (1.2.6) is written:

$$\frac{dU}{dt} = T\frac{dS}{dt} - p\frac{dV}{dt} + \sum_{j=1}^{n}\mu_j\frac{dN_j}{dt}. \tag{1.2.17}$$

We can also write (1.2.10) in entropic version as follows:

$$\frac{dS}{dt} = \frac{1}{T}\frac{dU}{dt} + \frac{p}{T}\frac{dV}{dt} - \sum_{j=1}^{n}\frac{\mu_j}{T}\frac{dN_j}{dt}, \tag{1.2.18}$$

where the state variables are the internal energy U, the total volume V and the number of moles per component N_j.
In the rest of the document, we will more readily use the specific mass m_j

instead of the number of moles N_j. Therefore the relations (1.2.17) and (1.2.18) become respectively:

$$\frac{dU}{dt} = T\frac{dS}{dt} - p\frac{dV}{dt} + \sum_{j=1}^{n} \mu_j^m \frac{dm_j}{dt}, \tag{1.2.19}$$

$$\frac{dS}{dt} = \frac{1}{T}\frac{dU}{dt} + \frac{p}{T}\frac{dV}{dt} - \sum_{j=1}^{n} \frac{\mu_j^m}{T}\frac{dm_j}{dt}, \tag{1.2.20}$$

where μ_j^n represents the mass chemical potential.

1.2.3 Extension in infinite dimension

Most industrial chemical and biochemical engineering processes are the site of transport phenomena (diffusion, convection) characterized by a spatial gradient of the state variables, leading to a model governed by partial differential equations [Zhou et al., 2015], [Dochain, 1994]. This kind of systems or processes belongs to a large class of systems which are not in thermodynamic equilibrium, and whose intensive quantities such as temperature, concentration, pressure remain locally well-defined concepts. Extensive variables such as entropy and internal energy should be replaced by the corresponding densities. The thermodynamic variables are thus functions of the position z and the time t:

$$T = T(z,t), \quad p = p(z,t), \quad m_j = m_j(z,t)$$

This is the meaning of the local equilibrium hypothesis [de Groot et al., 1984]. Therefore, by considering the relationship between the elementary mass and volume variations $dm = \rho dV$ (ρ density), the overall energy of the system for a sufficiently small volume V is written:

$$U = u(z,t)V = Ts(z,t)V - p(z,t)V + \sum_{j=1}^{n} \mu_j^m m_j(z,t)V, \tag{1.2.21}$$

In infinite dimension, Gibbs's equation retains its meaning if it follows matter in its mean motion (defined by the convection velocity v). In the one-dimensional case, the differential operator of this equation takes on the meaning of a material derivative [Byron et al., 1976]:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial z}, \tag{1.2.22}$$

with z the spatial coordinate.
By considering equations (1.2.19) and (1.2.20), the Gibbs' equation in energy

representation can be written after some developments as follows:

$$\frac{Du(z,t)}{Dt} = w^\top \frac{Dx(z,t)}{Dt}, \tag{1.2.23}$$

where $w = (T \quad \mu_1 \quad \cdots \quad \mu_n)^\top$ and $x(z,t) = (s(z,t)\; m_1(z,t) \;\cdots\; m_n(z,t))^\top$
In entropic representation we will have:

$$\frac{Ds(z,t)}{Dt} = w^\top \frac{Dx(z,t)}{Dt} \tag{1.2.24}$$

with $w = \left(\frac{1}{T} \quad -\frac{\mu_1}{T} \quad -\frac{\mu_n}{T}\right)^\top$.
Note, however, that when the pressure p is assumed to be constant, another state function called enthalpy denoted H can be introduced. As any state function, it only depends on system state variables. Its analytical expression is given by:

$$H = h(z,t)V = U + pV = (u+p)V. \tag{1.2.25}$$

Thus the energy form of the Gibbs' equation can be rewritten as a function of the enthalpy $h(z,t)$:

$$\begin{aligned} Dh(z,t) &= TDs(z,t) + \sum_{j=1}^{n} \mu_j^m Dm_j(z,t), \\ &= w^\top Dx(z,t), \end{aligned} \tag{1.2.26}$$

with $w = (T\; \mu_1^m \;\cdots\; \mu_j^m)^\top$ and $x(z,t) = (s(z,t)\; m_1(z,t) \;\cdots\; m_j(z,t))^\top$.
The material derivative is expressed by:

$$\frac{Dh}{Dt} = w^\top \frac{Dx}{Dt}. \tag{1.2.27}$$

In entropic representation we obtain equivalently:

$$\begin{aligned} Ds(z,t) &= \frac{1}{T} Dh(z,t) - \sum_{j=1}^{n} \frac{\mu_j^m}{T} Dm_j(z,t) \\ &= w^\top Dx(z,t), \end{aligned} \tag{1.2.28}$$

where $w = \left(\frac{1}{T} \;-\frac{\mu_1^m}{T} \;\cdots\; -\frac{\mu_j^m}{T}\right)^\top$, and
$x(z,t) = (h(z,t)\; m_1(z,t) \;\cdots\; m_j(z,t))^\top$. We still get:

$$\frac{Ds}{Dt} = w^\top \frac{Dx}{Dt}. \tag{1.2.29}$$

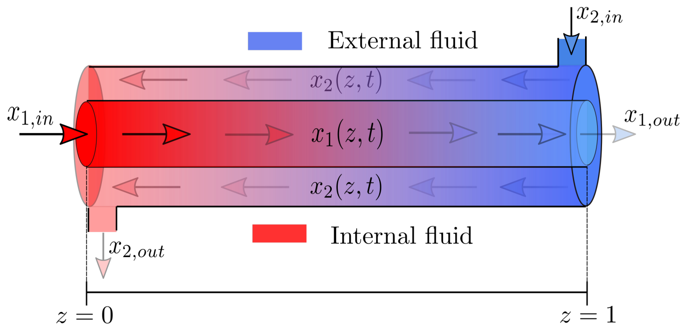


Figure 1.1: A counterflow heat exchanger

1.3 Dynamical modelling of heat exchanger

Let us derive the dynamical equations of the counterflow heat exchanger shown in Figure 1.1. The meaning and the units of the parameters are summarized in Table 1.1. The derivation of the model considers various dynamical phenomena due to the thermal exchanges. To do this, it is necessary first to admit a certain number of simplifying assumptions, the main ones of which are:

- The outer tube sufficiently insulates the exchanger, so that heat losses are not taken into account.
- Axial heat diffusion is negligible in fluids and in walls.
- The transport velocities v_j (with $j = 1, 2$) and the global heat exchange coefficient U_g are assumed to be constant and uniform along the exchanger.
- Fluids are incompressible, so that viscous dissipation phenomena can be neglected.
- The pressure is assumed to be constant and uniform.

We associate to these assumptions the following constitutive laws:

- The heat exchange with the double jacket is characterized by the following equation: $q_j(z,t) = A_t U_g(x_k(z,t) - x_j(z,t))$, where $x_k(z,t)$ (with $k = 2, 1$) is the distributed temperature of the jacket.
- The convective enthalpy flow of the fluid j noted $\Phi^{h,j}_{conv}$, is given by $\Phi^{h,j}_{conv} = v_j h_j$.
- The convective entropy flow of the fluid j noted $\Phi^{s,j}_{conv}$, is given by $\Phi^{s,j}_{conv} = v_j s_j$.

Table 1.1: System parameters

Constant	Unit	Description
L	m	Length of the heat exchanger
A_t	m^2	Total surface area for heat transfer
S_1	m^2	Internal Cross section
S_2	m^2	External cross section
V_j	m^3	Total volume of component j
A_s	m	Heat transfer surface area per unit length
v_j	m/s	Velocity of component j
$\dot{\mathrm{m}}_j$	L/min	Mass flow rate of component j
ρ_j	$\mathrm{Kg/m}^3$	Density of component j
c_{p_j}	$\mathrm{J\ (Kg \cdot {}^\circ C)^{-1}}$	Heat capacity of component j
U_g	$\mathrm{W/(m^2.{}^\circ C)}$	Overall heat transfer coefficient
q_{c_j}	W/°C	thermal capacity flow rate of component j
$x_{j,ref}$	°C	Reference temperature of fluid j
$h_{j,ref}$	Kj/Kg	Reference enthalpy of fluid j
$\Delta\mathrm{T}_{\mathrm{min}}$		Minimum temperature differences between the hot and cold fluids in the heat exchanger
$x_{1,int}$	°C	Inlet temperature of fluid 1
$x_{2,int}$	°C	Inlet temperature of fluid 2
$x_{1,out}$	°C	Outlet temperature of fluid 1
$x_{2,out}$	°C	Outlet temperature of fluid 2

Without loss of generality, the distributed temperature of the hot and cold fluid are represented in this work by $x_1(z,t)$ and $x_2(z,t)$, respectively (see Figure 1.1).

1.3.1 Balance equations

In this part we derive the balance equations by by considering the works of [Callen, 2006], [de Groot et al., 1984], [Byron et al., 1976], and also the specific work on the thermodynamic representation of chemical reactors [Ngoc Ha, 2009] [Bahroum et al., 2011], [Zhou et al., 2013], and taking into account the above assumptions.

1.3.1.1 Energy balance

The total energy balance is established on the basis of the first law of thermodynamics. According to this law, the sources of energy are: the power of forces outside the system, the power of forces outside the system, the calorific power brought into play inside the system and the heat flow.
Given the modeling assumptions, the total energy corresponds to the internal energy. Moreover, as the pressure is assumed to be constant, this balance is equal to the enthalpy balance and is written:

$$\frac{\partial h_j}{\partial t} = -\frac{\partial}{\partial z}\Phi^h_{conv} + q_j(z,t). \tag{1.3.1}$$

1.3.1.2 Entropic balance

The local entropy balance requires that at all time and for any component j, there exists the following relation:

$$\frac{\partial s_j}{\partial t} = -\frac{\partial}{\partial z}\Phi^s_{conv} + \frac{q_j(z,t)}{x_{ref}(z,t)} + \sigma_s(z,t), \tag{1.3.2}$$

with $x_{ref}(z,t)$ the reference temperature, and where $\sigma_s(z,t)$ represents the irreversible entropy production. This quantity is a tool for diagnosis and measurement of irreversibilities or energy inefficiency. It is generally used as an evolution criterion [i] and as an optimization criterion [Bejan and Errera, 1998], [Gupta et al., 2007], [Sarangi and Chowdhury, 1982], [Bejan, 1980], [Bejan, 1987]. In [Favache and Dochain, 2009] for example, it is used as a Lyapunov function for stability analysis of the Continuous Stirred Tank Reactor (CSTR) model.

1.3.2 Thermodynamic representation of the heat exchanger

In this subsection, we write the different representations or thermodynamics models of the heat exchanger, using the balance equations obtained previously. Next, from one of the representations we will deduce the classical model generally used process engineering.

1.3.2.1 Energy representation

The energy approach uses internal energy as an explicit quantity and the rest of quantities as state variable. In our case, this representation corresponds to the

[i] According to the second law of thermodynamics, the entropy production is net and positive (see Postulate 1.2.2).

entropic balance given by equation (1.3.2).
For each component $j \to 1, 2$ it is written:

$$\begin{cases} \dfrac{\partial s_1}{\partial t} = -v_1 \dfrac{\partial s_1}{\partial z} - \dfrac{q(z,t)}{x_1(z,t)} \\ \dfrac{\partial s_2}{\partial t} = v_2 \dfrac{\partial s_2}{\partial z} + \dfrac{q(z,t)}{x_2(z,t)}, \end{cases} \tag{1.3.3}$$

where $x_j(z,t)$ represents the temperature of fluid j, $q(z,t) = A_s U_g (x_1(z,t) - x_2(z,t))$. The total entropy per unit of length define as $s(z,t) = s_1(z,t) + s_2(z,t)$ has the following dynamical behaviour:

$$\frac{Ds}{Dt} = q\left(\frac{1}{x_2} - \frac{1}{x_1}\right) = A_s U_g \frac{(x_1 - x_2)^2}{x_1 x_2} := \sigma(z,t). \tag{1.3.4}$$

In particular, when x_1 and x_2 are square-integrable functions, then for $z \in [0, L]$, $x_1 x_2 = x_m = \int_0^L x_1 x_2 dz$ is the average temperature between x_1 and x_2. $\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + v_1 \frac{\partial s_1}{\partial z} - v_2 \frac{\partial s_2}{\partial z}$ is the material derivative of entropy, and $\sigma(z,t)$ the density of entropy production per unit of length inside the heat exchanger due to heat transfer between the two fluids [Kjelstrup et al., 2010, Chapter 9, page 165-166]. It is obtained by identification with the dynamic entropy equations (1.3.3). Therefore, the total entropy production in the heat exchanger is [Sandoval, 2021]:

$$\Sigma_s(t) = \int_0^L \sigma(z,t) dz. \tag{1.3.5}$$

Note that the entropy production due to the heat flux $q(z,t)$ is a quadratic form of the temperature difference $\Delta \mathrm{T}(z) \equiv (x_1(z) - x_2(z))$. In the equilibrium state $x_1(z) = x_2(z)$, and the entropy production takes its minimum value, namely zero. For a non-equilibrium state (where $x_1(z) \neq x_2(z)$), it evolves towards its minimum value (therefore zero) with a continuous increase in entropy. Furthermore, it is important to specify "where the produced entropy goes". In our case, by considering the temperature x_1 as reference temperature (i.e $x_{ref} = x_1$) we check that $1/x_2 > 1/x_1$. This implies that the entropy q/x_2 received by the cold fluid is greater than the entropy q/x_1 yielded by the hot fluid, and that the heat flow goes irreversibly from hot (at temperature x_1) to cold (at temperature x_2). So the whole of the entropy produced $\sigma(z,t)$ is transferred to the cold fluid.

Another component of irreversible entropy production is that due to the interface model, based on the continuity of the transferred energy flow. This model allows to calculate the component of the entropy production due to exchanges of the system with its external environment, at the boundary of the heat exchanger.
Assuming zero work received by the system, the entropy balance at the interface

of the heat exchanger is written:

$$\Phi_s^+ + \Sigma_I = \Phi_s^-, \tag{1.3.6}$$

where $\Phi_s^+ = q^+ \left(\frac{1}{x_2^+} - \frac{1}{x_1^+}\right)$ and $\Phi_s^- = q^- \left(\frac{1}{x_2^-} - \frac{1}{x_1^-}\right)$ represent the inlet and outlet entropy flow of the heat exchanger, respectively. We deduce that the entropy production at the interface is worth:

$$\Sigma_I = q^- \left(\frac{1}{x_2^-} - \frac{1}{x_1^-}\right) - q^+ \left(\frac{1}{x_2^+} - \frac{1}{x_1^+}\right). \tag{1.3.7}$$

1.3.2.2 Entropic representation

In this representation, entropy is considered as an implicit function, and all other quantities as state variables. In our case, this representation corresponds to the energy balance given by equation (1.3.1).
For $j \to 1, 2$ fluid it is written:

$$\begin{cases} \frac{\partial h_1}{\partial t} = -v_1 \frac{\partial h_1}{\partial z} + A_s U_g \left(x_2(z,t) - x_1(z,t)\right) \\ \frac{\partial h_2}{\partial t} = v_2 \frac{\partial h_2}{\partial z} + A_s U_g \left(x_1(z,t) - x_2(z,t)\right). \end{cases} \tag{1.3.8}$$

1.3.3 Standard model in process engineering

In general, it is impossible to measure a thermodynamic potential (such as enthalpy or entropy in this case) from the instrumentation (sensor). The numerical values of these state functions are generally obtained experimentally, or from the relations which link them to intensive quantities [Coogan et al., 2016]. For this, it is preferable to express the balance equations as a function of intensive variables (temperature, pressure, concentration, etc) [Bastin and Dochain, 1990], [Dochain, 2008], [Corriou, 2018]. In the following we show how to obtain from the state equations of the heat exchanger, the classical infinite-dimensional model generally used in process engineering [Burns and Cliff, 2014a], [Maidi et al., 2009].

The entropic model (1.3.8) of heat exchanger written as a function of the enthalpies can be expressed as a function of the temperatures using the expressions of the partial mass enthalpies for an ideal mixture: $h_j = \rho_j V_j c_{p_j}(x_j - x_{ref}) + h_{j_{ref}}$. Thus, from (1.3.8) we arrive at the partial differential equations on the temperatures of each fluid, widely used in the literature for the simulation, analysis and design of the control laws ([Xu and Sallet., 1993], [Maidi et al., 2009], [Burns and Cliff, 2014a], [Grabowski, 2007]):

$$\frac{\partial x}{\partial t} = \underbrace{\begin{pmatrix} -v_1 & 0 \\ 0 & v_2 \end{pmatrix}}_{\Lambda} \frac{\partial x}{\partial z} + \underbrace{\begin{pmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{pmatrix}}_{M} x(z,t), \tag{1.3.9}$$

with $x(z,t) = (x_1(z,t)\; x_2(z,t))^\top$ the distributed state vector, and $\alpha_j = \frac{A_t U_g}{\rho_j V_j c_{p_j}} [s^{-1}]$.
To solve model (1.3.9), it must be associated with initial and boundary conditions. Therefore we consider that the temperatures at the boundary are continuous and equal to the inlet temperatures:

$$x_1(0,t) = x_{1,in}(t), \qquad x_2(L,t) = x_{2,in}(t). \tag{1.3.10}$$

At $t = 0$, we assume that the temperature profiles exist. Hence the initial conditions:

$$x_1(z,0) = x_{10}(z), \qquad x_2(z,0) = x_{20}(z). \tag{1.3.11}$$

We will discuss the numerical resolution of problem (1.3.9)-(1.3.11) in the next chapter.

Remark 1.3.1. *It is important to note that the parameter $\alpha_j[s^{-1}]$ represents the inverse of the thermal time constant $\tau_j[s]$ of the component j. Indeed, the time constant τ of a thermal system is linked to the product $\rho V c_p$ as well as to the thermal resistance $1/U_g$ of the thermal exchange medium by the relation: $\tau = \frac{\rho V c_p}{A_t U_g}$ [Robertson and Smith, 1981, Hedbrant, 2001, Levermore, 2020].*

1.4 Conclusion

The aim of this chapter was to present the thermodynamics modelling of a counterflow heat exchanger. This system belongs to the class of distributed parameter systems, for which the theories of thermodynamics of lumped parameter systems can only be applied after some further considerations. After a brief overview of the thermodynamics and considering a number of assumptions, we derived the models in entropic and energetic representations. Next, from these we deduced the classical distributed parameter model generally used in process engineering for analysis and control.
An important step in modelling consists not only in choosing the appropriate model able of describing the dynamics of the studied system, but also in calibrating the parameters of this model. In the next chapter, we will study the calibration of the standard process engineering model.

2

PARAMETER IDENTIFICATION AND GLOBAL SIMULATION

Contents

2.1 Introduction

The derivation of of a mathematical model consists in representing a system by a mathematical model that best describes its dynamical behavior. The development of such a model can be carried out either by considering exclusively physical principles governing the behavior of the process (white-box model), by considering approximations on the time response to a test signal (black-box model), or by combining the two approaches (grey-box model) [Ljung and Söderström, 1983].
In this chapter we are interested in the identification of the counterflow heat exchanger model by grey-box approach.

2.1.1 State of the art

In the literature, heat exchangers have already been the subject of identification studies. In [Jacobsen and Skogestad, 1994] and [Pfortner et al., 2022], the authors propose an approach which consists in estimating the model from the input-output data. The resulting models, known as the "black-box models", are easy to handle in the context of control design and applications ([Babuška, 1998]). However, the estimated parameters may not have any physical interpretation. In [Jonsson et al., 1992], [Jonsson and Palsson, 1994], [Mutambara and Al-Haik, 1999] and [Jonsson et al., 2007], the heat exchanger parameters are estimated recursively (by Kalman filtering and recursive least squares) from a lumped parameter model, obtained by splitting the heat exchanger into supposedly homogeneous volumes. The algorithms used are interesting because they allow the operating point variations to be monitored over time. And moreover, the (on-line) estimated parameters have a physical meaning; this can be useful especially for the validation step where it helps to assess the success of the estimation process. However, the model that serves as a calibration does not take into account the distributed nature (typically temperature gradients) of heat exchangers. [Farah et al., 2016] proposed an approach for estimating the parameters of a hyperbolic model of a co-current heat exchanger, that consists in reducing the PDEs of the model using reinitialized partial moments. This approach is efficient since the introduction of moments transforms the parameter estimation problem into a linear regression problem that can be solved by a simple least squares algorithm. Yet it requires a significant number of thermocouples to be placed along the heat exchanger, so as to characterize temperature gradients. In [Hamze et al., 2018] the authors propose a parametric estimation approach of the hyperbolic model of a counter-current heat exchanger from a delay model obtained by replacing the spatial differential operators by their average. They propose to use a gradient-descent optimization method to estimate the parameters. The results obtained are quite close to those obtained from a set of ODEs. In the same PDE context, [Burns et al., 2018] proposes an approach that consists in transforming the parabolic PDEs of a counterflow heat exchanger into a set of ordinary differential equations (ODEs) obtained by finite element, on which the Levenberg-Marquart algorithm (see [Marquardt, 1963]) is used to estimate the parameters. The Levenberg-Marquardt algorithm has good convergence properties (especially for ill-conditioned problems). Given that it is a local algorithm, it may converge only if we have a reasonable first estimate. So the search for this first estimate is a big challenge for its use. However, the context in which it is used by [Burns et al., 2018] does not pose any initialization problem because the algorithm is applied to a simulator whose true values of the parameters are known a priori. This does not correspond to the real situation for which, in general, there is no a priori knowledge of the order of magnitude of the parameters.

2.1.2 Contributions and organization of the chapter

In this work the Levenberg-Marquardt algorithm is used to estimate the parameters of the hyperbolic model of a counterflow heat exchanger modeled in the previous chapter, from laboratory data. In our approach, the first estimate is obtained by replacing the spatial derivative operators by global differences, which leads to a system of two sets of linear ODEs with respect to the parameters on which the least squares method is used.
A necessary condition for carrying out an identification procedure is the identifiability property of the system studied. The question is whether it is possible to obtain an estimate of the parameters of the system from the information contained in the experimental data. In the works cited above on heat exchangers, none mentions this problem. And yet, it is crucial for interpreting and determining confidence in model parameter values [Walter and Pronzato, 1997], [Vanrolleghem et al., 1995]. In our study, structural identifiability analysis, in which one assumes data to be noiseless and arbitrarily fine-grained, is made assuming that we have a number of measurements that can characterize the temperature gradient of each fluid. While the practical identifiability is checked by the analysis of the output sensitivity functions.

Besides the conclusion, the chapter is divided into two sections organized as follows.

- Section 2.2: This section is for general system identification information. We begin by justifying the identifiability study by explaining why in general the parameter estimation problem constitutes an ill-posed problem in the sense of Hadamard. In Section 2.2.1 we introduce the concept of structural and practical identifiability, as well as some tools to verify them. Section 2.2.2 deals with parametric estimation methods.
- Section 2.3: This section concerns the application of the theories of Section 2.2 to the case of the counterflow heat exchanger model. In Section 2.3.1, we transform the heat exchanger PDE model into a set of ODEs by the method of lines because it makes it possible to take advantage of the integration algorithms of the ODEs. In Section 2.3.2, we establish the structural identifiability of the PDE model, and next, we study the practical identifiability of the system by considering the steady-state model. Section 2.3.4 deals with the estimation and the validation of the heat exchanger parameters from data collected on the test bench described in Section 2.3.3.

2.2 Theoretical overview of systems identification

Let Υ be a dynamical system with continuous states represented by the following state form:

$$\Upsilon(t,\theta,\xi(0)) : \begin{cases} \dfrac{d\xi}{dt}(t,\theta,u(t)) = f\left(\xi(t,\theta,u(t)),\theta,u(t)\right) \\ y(t,\theta) = h\left(\xi(t,\theta,u(t)),\theta\right) \\ \xi(0) = \xi_0(\theta), \end{cases} \tag{2.2.1}$$

where $t \in [0,+\infty)$, $\theta \in$**P** unknown parameter vector with **P**$\subset \mathbb{R}^p$ the set of admissible parameters, $\xi : [0,+\infty) \times \mathbf{P} \subset \mathbb{R}^n$ the state variable, $u \in \mathbb{R}^{n_u}$ the control (or input) variable, $y \in \mathbb{R}^{n_y}$ the output variable and $\xi_0(\theta)$ the initial condition. f and h correspond to the state function and the output function, respectively.
Generally, the determination of $y(t,\theta)$ which constitutes the resolution of the direct problem, supposes to know the input vector, the vector of parameters as well as the initial condition. In practice, the parameters and initial conditions (especially for infinite-dimensional systems) are unknown or poorly understood. Thus we are led to solve a priori an inverse problem, which is to determine the causes from the effects. This assumes that some state variables are accessible to measurement. The inverse problem that interests us is that of identifying the vector of parameters θ. In most applications, the physical quantity θ is not directly accessible for measurement. It is always possible to determine θ by observing other quantities, typically the output variable y, which are linked to it by physical laws, $y = h(\theta)$. In that case, the concept of inverse problem corresponds to the idea of inverting these physical laws to indirectly access the variable of interest θ, knowing y. Parametric identification consists of choosing a model structure, estimating (minimizing a quadratic cost) θ and characterizing the uncertainty on the estimate from the measurements on ξ_0, u, and y.
In general, solving an inverse problem such as parametric identification is very delicate because these problems are generally ill-posed in the sense of Hadamard [Hadamard, 1923].

Definition 2.2.1 (Well-posedness)**.** *The evolution problem is well-posed in the sense of Hadamard is a problem whose:*

- *the solution exists (existence),*
- *the solution is unique (uniqueness),*
- *the solution depends continuously on the parameters, the sources (or input variables) and the initial condition (stability).*

Indeed, for a given physical system, the experimental data that we have are generally noisy and there is no guarantee that such data come from this system.

Even if the solution exists, it is possible that different parameter configurations have provided the same observations. In addition, the stability of the solution is not always ensured because of the available data which are different from the real data. This problem is usually called "identifiability" or "estimability" [Walter and Pronzato, 1997], [Ljung, 1999, Chapter 4]. The study of the identifiability of the parameters of the model before their estimation proves to be an essential task.

2.2.1 Identifiability

Identifiability is the study of the feasibility of estimating model parameters. The main question of the identifiability analysis can be summarized as follows [Dochain, 2008, Chapter 3]: *let us assume that a certain number of variables of the state of the model are available to the required extent; on the basis of the structure of the model (structural identifiability) or on the basis of the type and quality of the data available (practical identifiability), can we expect to be in a position to attribute a unique value to each parameter of the model by means of parameter estimation?*
In the rest of this subsection we discuss the notion of structural identifiability and practical identifiability.

2.2.1.1 Structural identifiability

Structural identifiability is an important property when evaluating system parameters, as it guarantees uniqueness of parameters. Intuitively the parameters of a system are structurally identifiable, if it is theoretically possible to determine them in a unique way from the observations of the inputs and outputs supposed to be perfectly known, by placing oneself in an ideal non-noisy framework, where the sought model corresponds to that of the studied system. Before giving the formal definition of structural identifiability, let us first recall that of the notion of experience and structural model associated with the system $\Upsilon(t, \theta, \xi_0(\theta), u)$.

Definition 2.2.2 (Experiment)**.** *An experiment on* $[0, +\infty)$ *for the system* Υ *is the pair* (ξ_0, u) *such that for any* $\theta \in \boldsymbol{P}$, $\Upsilon(t, \theta, \xi_0(\theta), u)$ *has a unique solution* $\xi(t, \theta)$ *defined on* $[0, +\infty)$.

Definition 2.2.3 (Structural model)**.** *Let us consider an experiment* (ξ_0, u) *on* $[0, +\infty)$ *for the system* $\Upsilon(t, \theta, \xi_0(\theta), u)$. *We call structural model associated with experiment* (ξ_0, u), *the application*

$$\mathcal{M}(t, \theta, \xi_0, u) : \theta \in \boldsymbol{P} \longmapsto \Upsilon(t, \theta, \xi_0(\theta), u).$$

Now let us introduce the concept of parametric identifiability of the model $\mathcal{M}(t, \theta, \xi_0, u)$.

Definition 2.2.4 (Identifiability). *Let* (ξ_0, u) *be an experiment on* $[0, +\infty)$ *of the physical system* $\Upsilon(t, \theta, \xi_0(\theta), u)$.

- *Model* $\mathcal{M}(t, \theta, \xi_0, u)$ *is said to be identifiable in* $\theta \in \boldsymbol{P}$ *if and only if:*

$$\left\{\forall \hat{\theta} \in \boldsymbol{P},\, y(\cdot, \theta) = y(\cdot, \hat{\theta})\right\} \Longrightarrow \theta = \hat{\theta}, \quad \forall t \in [0, +\infty).$$

- *Model* $\mathcal{M}(t, \theta, \xi_0, u)$ *is said to be identifiable on* $\boldsymbol{P}$ *if it is identifiable in all parameter* $\theta \in \boldsymbol{P}$.

In the literature there are several methods or tests that allow to verify the structural identifiability. And the methods are differentiated according to whether the structure of the model is linear or nonlinear, finite-dimensional or infinite-dimensional. For finite dimensional linear or nonlinear systems, the structural identifiability is well understood, there are many tests available for this, and the methods used are those based on the Laplace transform, generating series and the Taylor series expansion (see, for example [Bellman and Åström, 1970], [THOWSEN, 1978], [Pohjanpalo, 1978], [Cobelli et al., 1979], [Godfrey and DiStefano, 1985], [Walter and Pronzato, 1996], [Orlov et al., 2002]). Only one, based on the Taylor series expansion of the observations, is directly applicable to nonlinear models or time-varying systems. This approach assumes that the output variable $y(t, \theta)$ and its successive derivatives are known, and the identifiability study consists of the evaluation at the origin of $y(t, \theta)$ and its derivatives, and solving the resulting system of equations for a limited number of derivatives of $y(t, \theta)$[i]. A sufficient condition of structural identifiability is that there is a unique solution for this equation system.
As far as infinite dimensional systems are concerned, the problem is generally complex because of the distributed nature of system state variables which call upon advanced mathematical theories to solve this kind of inverse problem [Nakagiri, 1983]. In [Nakagiri, 1993] there is a review of Japanese results from 1980 to 1992, concerning problems of identifiability of linear PDE models using spectral theory. For parabolic models, one finds there studies based on the spectral approach of the Sturm-Liouville operator. In general, when it comes to linear systems (parabolic or hyperbolic), Carleman's inequalities (estimates) are widely used (see e.g. [Klibanov, 1992], [Klibanov, 2000]). Carleman estimates are a powerful tool which was originally proposed by [Carleman, 1939] for proofs of uniqueness results for ill-posed Cauchy problems. Today, the applications of these inequalities are diverse and numerous [Perasso, 2009]. In addition to uniqueness results, they are also used in control theory or for the study of inverse problems for PDEs (see for example [Tucsnak and Weiss, 2009, Chapter 9 and Appendix IV] and [Mauffrey, 2012]).

[i]In practice it is recommended to use a number of derivatives of $y(t, \theta)$ at least equal to the number of unknown parameters [Dochain, 2008, Chapter 3].

2.2.1.2 Practical identifiability

Practical identifiability is complementary to structural identifiability, in the sense that we are interested now in the impact of the available experimental data concerned with the identifiability of the model studied's model parameters. Suppose the observed (vector) output satisfies

$$y(t_k) = y_m(t_k, \theta^*) + \zeta(k), \qquad k = 1, 2, 3, \cdots, N, \tag{2.2.2}$$

where $y_m(t_k, \theta^*)$ is the output of a deterministic model (the vector of corresponding quantities computed by the model $\mathcal{M}(t, \theta, \xi_0, u)$), θ^* is the true value of the parameters and $\zeta(k)$ belongs to a sequence of independent random variables with probability densities $\pi_{\zeta_k}(\zeta(k))$, given by

$$\pi_{\zeta_k}(\zeta_1, \zeta_2, \cdots, \zeta_N) = \prod_{j=1}^{N} \pi_{\zeta_k}(\zeta(k)). \tag{2.2.3}$$

Let us consider the output error:

$$\varepsilon(t_k, \theta) = y(t_k) - y_m(t_k, \theta). \tag{2.2.4}$$

For the true value of the parameters, the error is $\varepsilon(t_k, \theta^*) = \zeta(k)$. Thus, for N observations, the likelihood is written

$$\pi_y(y|\theta) = \prod_{k=1}^{N} \pi_{\zeta_k}(\varepsilon(t_k, \theta)) = \prod_{k=1}^{N} \pi_{\zeta_k}(y(t_k) - y_m(t_k, \theta)). \tag{2.2.5}$$

Then the log-likelihood is expressed as the sum of the terms each of which is associated with the output error:

$$\ln \pi_y(y|\theta) = \sum_{k=1}^{N} \ln \pi_{\zeta_k}(y(t_k) - y_m(t_k, \theta)). \tag{2.2.6}$$

Suppose $\zeta(k)$ is a vector of N random variables with zero mean and variance σ_k^2. First let us consider the case where σ_k^2 is an unknown vector. In that case one could think of creating an extended parametric vector $\theta_e = \left(\theta^\top, \sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_N\right)^\top$, and estimating it in the sense of maximum likelihood. Taking into account the stochastic character of θ_e, this problem is ill-posed in the sense of Definition 2.2.1, and θ_e is not identifiable in the sense of Definition 2.2.4. Indeed, the number of parameters to be estimated is $N + \dim(\theta)$, whereas only N data are available.
Now consider that σ_k is known and constant, and the noise $\zeta(k)$ follows a Gauss's law. In that case, the associated probability density is:

$$\pi_{\zeta_k}(\zeta(k)) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left[-\frac{1}{2}\left(\frac{\zeta(k)}{\sigma_k}\right)^2\right]. \tag{2.2.7}$$

Thus, the likelihood $\pi_y(y|\theta)$ of the N observations of the output, given by the expression (2.2.5) becomes:

$$\pi_y(y|\theta) = \prod_{k=1}^{N} \pi_{\zeta_k} \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left[-\frac{1}{2}\left(\frac{y(t_k) - y_m(t_k,\theta)}{\sigma_k}\right)^2\right] \tag{2.2.8}$$

this means that the measurements $y(t_k)$ are independent and distributed according to the normal distribution $\mathcal{N}\left(y_m(t_k,\theta), \sigma_k^2\right)$. It follows that the log-likelihood is written:

$$\ln \pi_y(y|\theta) = \sum_{k=1}^{N}(\bullet) - \frac{1}{2}\sum_{k=1}^{N}\left(\frac{y(t_k) - y_m(t_k,\theta)}{\sigma_k}\right)^2, \tag{2.2.9}$$

with $(\bullet)$ an independent term of θ. The estimate of θ in the sense of the maximum likelihood $\hat{\theta}_{\mathrm{ml}}$ is the argument of the maximum of the criterion:

$$J(\theta) = \frac{1}{2}\sum_{k=1}^{N}\left(\frac{y(t_k) - y_m(t_k,\theta)}{\sigma_k}\right)^2, \tag{2.2.10}$$

which can be written as a weighted least squares estimator as follows:

$$J(\theta) = \sum_{k=1}^{N}\left(y(t_k) - y_m(t_k,\theta)\right)^\top Q\left(y(t_k) - y_m(t_k,\theta)\right), \tag{2.2.11}$$

where $Q = \sigma^{-2} I_N$, with I_N the identity matrix of order N.
The covariance of any unbiased estimator satisfies the Cramer-Rao inequality [Ljung, 1999][ii]

$$\mathrm{COV}(\hat{\theta}) = \mathrm{E}\left(\left(\hat{\theta} - \theta^*\right)\cdot\left(\hat{\theta} - \theta^*\right)^\top\right) \geq F^{-1}(\theta^*), \tag{2.2.12}$$

with E the mathematical expectation and with F the Fisher information matrix. It expresses the information contained in the experimental data and whose expression is:

$$F = \sum_{k=1}^{N}\left[\frac{\partial y}{\partial \theta}(t_k)\right]^\top Q\left[\frac{\partial y}{\partial \theta}(t_k)\right]. \tag{2.2.13}$$

This matrix is the inverse of the covariance matrix of the estimation errors of the best unbiased linear estimator, typically that of maximum likelihood [Godfrey and DiStefano, 1985]. The terms $\frac{\partial y}{\partial \theta}$ are the sensitivity functions of the output variables. They quantify the dependence of the model predictions with respect to the parameters. In general, the sensitivity functions play a key role in the

[ii] $\geq$ denotes the inequality in the sense of positive semi-definite matrices. It means that $\mathrm{COV}(\hat{\theta}) \geq F^{-1}(\theta^*)$ is a positive semi-definite symmetric matrix.

evaluation of practical identifiability of parameters. An important property is that when the sensitivity functions are proportional, the Fisher information matrix(2.2.12) is singular. The model is thus not identifiable [Petersen, Britta, 2000]. Therefore, one way to study the practical identifiability of a model is to plot sensitivity functions.
Otherwise, an important result of the practical identifiability is the possibility of calculating the parameter variance, which enables us to calculate a confidence interval for them. Indeed, in general the Cramer-Rao bound makes it possible to set a upper bound to the accuracy that can be achieved in the estimation of one or more parameters. One application in pratical identifiability is to ensure that the asymptotic variance on each parameter does not exceed some reasonable limit, usually fixed empirically. An estimate of this bound is given by [Dochain, 2008, Chapter 3]:

$$\mathrm{COV}(\hat{\theta}) = \underbrace{\frac{J(\hat{\theta})}{N - n_p}}_{\sigma^2} \left(\left[\frac{\partial y}{\partial \theta}(t_k) \right]^\top \left[\frac{\partial y}{\partial \theta}(t_k) \right] \right)^{-1}. \qquad (2.2.14)$$

Once the covariance matrix $\mathrm{COV}(\hat{\theta})$ is known, then the confidence intervals can be calculated as follows:

$$\hat{\theta} \pm t_{\alpha;N-n_p} \mathrm{COV}(\hat{\theta})^{1/2}, \qquad (2.2.15)$$

with α a parameter (usually taken equal to 95%) that expresses the confidence level, n_p the number of parameter, and t is obtained from Student's distributions. Recall that the Student distribution is a probability distribution that estimates the value of the mean of a small sample taken from a population that follows a normal distribution and whose standard deviation is unknown. It is in this sense that it is involved in the construction of the confidence intervals of the estimated parameters.

2.2.2 Parametric estimation methods

After having fixed the structure of the model, the second stage consists in estimating the parameters of this model based on the optimization of a chosen criterion.
In this section, two families of identification algorithms are presented: the equation error algorithm [Ljung, 1999], and the output error algorithm [Walter and Pronzato, 1997]. These two algorithms are the most used identification (or estimation) methods. Equation error methods are dedicated to linear models with respect to parameters, that enables the use of simple linear optimization techniques. Their bias is however dependent on the nature of the data noise. Output error methods can be applied to different types of models (linear and nonlinear) and are unbiased. However, it is difficult to ensure their convergence to the global optimum.
In order to present these methods, we assume that the output of the system

(2.2.1) is described as follows:

$$y(k) = h\left(u(k-1), y(k-1,\theta)\right) + \zeta(k), \tag{2.2.16}$$

with $\zeta(k)$ the noise.

2.2.2.1 The equation error algorithm

Equation-error algorithms rely on the minimization of a quadratic criterion to estimate the unknown parameters of the model that describes the system. The parameter vector θ, is described in such a way as to minimize the sum of the squares of the norms of the equation error vectors.

$$J(\theta) = \frac{1}{N+1}\sum_{k=0}^{N} \varepsilon^{\top}(k,\theta)\varepsilon(k,\theta), \tag{2.2.17}$$

with N the number of measures, and $\varepsilon(k,\theta)$ the equation error vectors defined as:

$$\varepsilon(k,\theta) = y(k) - h\left(u(k-1), y(k-1,\theta)\right). \tag{2.2.18}$$

In general, parametric estimation methods are categorized into linear methods and nonlinear methods and follow the linearity and nonlinearity of the models with respect to the vector of parameters, respectively. In the sequel, the system is considered to be linear with respect to the parameters. Thus, the output of the system is given by

$$y(k) = \varphi^{\top}(k)\theta + \zeta(k), \tag{2.2.19}$$

where $\varphi(k)$ is the regressor, and contains the system's input-output measurements.

The problem is to find an estimate $\hat{\theta}$ of θ from measurements $y_m(k)$ and $\varphi(k)$. A simple way to estimate the parameter vector is represented by a measurement number equal to the number of parameters. In this situation $\varphi(k)$ is a square matrix. Also if $\varphi(k)$ is non-singular, equation (2.2.19) can be easily solved to get θ. In practice, measurement noise leads to the use of a larger number of measurements than the size of the parameter vector, so as to obtain a relevant estimate of θ. In that case the solution of the optimization problem is given by the following lemma.

Lemma 2.2.1. *Let us consider the cost function $J(\theta)$ given by equation (2.2.17). If the matrix $\varphi(k)\varphi^{\top}(k)$ is non-negative definite, then the minimization problem of $J(\theta)$ has a unique solution:*

$$\hat{\theta} = \left[\frac{1}{N+1}\sum_{k=0}^{N}\varphi(k)\varphi^{\top}(k)\right]^{-1}\left[\frac{1}{N+1}\sum_{k=0}^{N}\varphi(k)y(k)\right]. \tag{2.2.20}$$

The outputs measured are often noisy. Therefore, an asymptotic estimation bias is added to $\hat{\theta}$. Thus, it is interesting to introduce an instrumental variable to improve the quality of estimation [Ljung, 1999]. Despite the bias problem, equation error methods are widely used in initializing other identification algorithms such as output error algorithms.

2.2.2.2 Output error algorithm

The output error method is probably the most widely used method in parametric estimation of a system. In this method, the vector of estimated parameters is obtained by minimizing a quadratic criterion of the form (2.2.17), except that the error is defined as follows:

$$\varepsilon(k,\theta) = y(k) - h\left(u(k-1), \hat{y}(k-1,\theta)\right), \tag{2.2.21}$$

with $\hat{y}(k-1)$ the simulated output.
Several optimization algorithms can be used. One of the most widely used nonlinear optimization techniques is Levenberg-Marquardt's method [Levenberg, 1944, Marquardt, 1963]. This method has become a standard technique for nonlinear least squares problems, adopted in a wide range of disciplines. It is an iterative method which allows to locate the minimum of a function, with several variables, which is expressed as the sum of the squares of the nonlinear functions with real values. Levenberg-Marquardt's method makes it possible to combine the convergence properties guaranteed by the gradient descent and Gauss-Newton's algorithm [Ake, 1996, Chapter 9].
In the gradient descent method, the parameter vector θ is iteratively estimated as follows:

$$\hat{\theta}(k+1) = \hat{\theta}(k) - \eta J'(\hat{\theta}), \tag{2.2.22}$$

with

$$J'(\hat{\theta}) = \frac{\partial J}{\partial \hat{\theta}} = -2\frac{1}{N+1}\sum_{k=0}^{N} \frac{\partial \varepsilon(k,\theta)}{\partial \theta}\varepsilon(k,\hat{\theta})$$

the gradient, and η is the convergence step in the k iteration.
In the Gauss-Newton's method, θ is iteratively estimated as follows:

$$\hat{\theta}(k+1) = \hat{\theta}(k) - \left(J''^{-1}(\hat{\theta}) \cdot J'(\hat{\theta})\right), \tag{2.2.23}$$

with

$$J''(\hat{\theta}) = \frac{\partial^2 J}{\partial \hat{\theta} \partial \hat{\theta}} = 2\frac{1}{N+1}\sum_{k=0}^{N} \frac{\partial \varepsilon^\top(k,\theta)}{\partial \theta}\frac{\partial \varepsilon(k,\theta)}{\partial \theta}$$

the Hessian of $J(\hat{\theta})$.
Levenberg's method [Levenberg, 1944], proposes a concatenation of these two

equations into a single:

$$\hat{\theta}(k+1) = \hat{\theta}(k) - \left(\left(J''(\hat{\theta}) + \lambda(k) I \right)^{-1} J'(\hat{\theta}) \right), \tag{2.2.24}$$

with $\lambda(k)$ a factor adjusted with each new iteration and that allows gradual switching between the two methods. The most significant change was made by Marquardt [Marquardt, 1963], replacing the identity matrix I in equation (2.2.24) with $\mathrm{diag}\left(J''(\hat{\theta})\right)$:

$$\hat{\theta}(k+1) = \hat{\theta}(k) - \left(\left(J''(\hat{\theta}) + \lambda(k) \cdot \mathrm{diag}\left(J''(\hat{\theta})\right) \right)^{-1} J'(\hat{\theta}) \right). \tag{2.2.25}$$

The latter takes the name of *Levenberg-Marquardt's method*.
This method works as follows. When we are too far from the minimum (i.e. the error is very large) we increase the value of λ, which allows the Levenberg method to switch to the gradient method, and when we are close to the minimum (i.e. the error is very small) we decrease the factor λ, which neglects its term in front of the Hessian term, in this case, Gauss's method takes the process and accelerates the optimization operation towards the minimum. It should be mentioned that this type of method does not guarantee global convergence [Dochain, 2008, Chapter 3]. The optimal minimum point found, θ^*, strongly depends on the starting point of the algorithm, θ_0. Despite this drawback, the Levenberg-Marquardt method guarantees a robust convergence, and does not require any particular assumption on the specifications of the measurement noise.

Having at our disposal a brief presentation of the tools necessary to implement a parametric identification process, the next section presents the application of these techniques to a heat exchanger.

2.3 Identification of heat exchanger model

In this part, we deal with the standard model in process engineering introduced in Chapter 1:

$$\frac{\partial x}{\partial t} = \underbrace{\begin{pmatrix} -v_1 & 0 \\ 0 & v_2 \end{pmatrix}}_{\Lambda} \frac{\partial x}{\partial z} + \underbrace{\begin{pmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{pmatrix}}_{M} x(z,t), \tag{2.3.1}$$

associated to the boundary conditions:

$$x_1(0,t) = x_{1,in}(t), \qquad x_2(1,t) = x_{2,in}(t), \tag{2.3.2}$$

for all $z \in [0, 1]$. In the case of this model, the parameters to identify are $\theta = \{v_1, v_2, \alpha_1, \alpha_2\}$. The set of admissible values for θ is $\mathbb{R}^4_+$.
In the following, we will first approximate the model (2.3.1)-(2.3.2) into a lumped parameter model, before considering a study of identifiability and parametric estimation.

2.3.1 Numerical approximation of the model

In this part, we shall reduce the equations (2.3.1)-(2.3.2) of the heat exchanger to a system of ordinary differential equations. There are several possibilities for this. In our case we opt for the solution by the method of lines (MOL) [Wouwer et al., 2001], [Maidi et al., 2008]. This method is simple to implement and avoids the complexity of the computations encountered by adopting other approaches such as weighted residual methods and variational methods [Burns and Cliff, 2014b], [Burns and Cliff, 2014a], [Burns et al., 2013]. In particular, it makes it possible to take advantage of integration algorithms dedicated to ODEs. Typically discretization is performed in space by using a finite difference scheme and the system of ordinary differential equations is integrated over time using available algorithms.
Within the framework of this work, taking into account the direction of circulation of the internal fluid which enters the heat exchanger at the temperature $x_{1,in}$ at the point $z = 0$, and external fluid which enters at the temperature $x_{2,in}$ at point $z = 1$, we approximate the spatial derivative operators by forward and backward discretization scheme, respectively:

$$\frac{\partial x_1}{\partial z} = \frac{x_{1,j} - x_{1,j-1}}{\Delta z}, \qquad \frac{\partial x_2}{\partial z} = \frac{x_{2,j+1} - x_{2,j}}{\Delta z}.$$

By taking n finite elements (with $n = \frac{L=1}{\Delta z}$, and Δz the discretization step) the balance equations (2.3.1)-(2.3.2) become:

$$\begin{cases} \dfrac{dx_{1,j}}{dt} = -v_1 \dfrac{x_{1,j} - x_{1,j-1}}{\Delta z} + \alpha_1(x_{2,j} - x_{1,j}) \\ \qquad\qquad \text{with } j = 1, \cdots, n, \\ \dfrac{dx_{2,j}}{dt} = v_2 \dfrac{x_{2,j+1} - x_{2,j}}{\Delta z} + \alpha_2(x_{1,j} - x_{2,j}) \\ \qquad\qquad \text{with } j = 0, \cdots, n-1. \end{cases} \tag{2.3.3}$$

Considering $x(t) = [x_{1,1}(t), \cdots, x_{1,n}(t), x_{2,1}(t), \cdots, x_{2,n}(t)]^\top$ we can rewrite (2.3.3) in the matrix form:

$$\frac{dx(t)}{dt} = A_n x(t) + B_1 x_{1,in}(t) + B_2 x_{2,in}(t). \tag{2.3.4}$$

The matrices A_n, B_1 and B_2 are expressed as follows:

$$A_n = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$

where $K_{11} = K_{11}^1 + K_{11}^2$, with $K_{11}^1 = -\left(\frac{v_1}{\Delta z} + \alpha_1\right) \cdot I$, and $K_{11}^2 = \frac{v_1}{\Delta z}\mathcal{O}_1$, $K_{12} = \alpha_1 \mathcal{O}_2$, $K_{21} = \alpha_2 \mathcal{O}_1$, $K_{22} = K_{22}^1 + K_{22}^2$, with $K_{22}^1 = -\left(\frac{v_2}{\Delta z} + \alpha_2\right) \cdot I$ and $K_{22}^2 = \frac{v_2}{\Delta z}\mathcal{O}_2$. The matrices $\mathcal{O}_1$ and $\mathcal{O}_2$ are given by:

$$\mathcal{O}_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$\mathcal{O}_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and $B_1 = \begin{pmatrix} \frac{v_1}{\Delta z} & \cdots & \alpha_2 & \cdots & 0 \end{pmatrix}^\top$, $B_2 = \begin{pmatrix} 0 & \cdots & \alpha_1 & \cdots & \frac{v_2}{\Delta z} \end{pmatrix}^\top$.
Thus, we obtain a system that can be solved easily by using for example the Matlab subroutine *ode*45.

2.3.2 Identifiability

We study in this section the structural and practical identifiability of the heat exchanger model (2.3.1)-(2.3.2). The approach using Carleman's inequalities for the structural identifiability is very computationally intensive. Given the linearity of the model, we propose an approach based on integration by parts and the boundedness of the integral operators.
The following proposition will allow us to conclude on the structural identifiability of the heat exchanger model.

Proposition 2.3.1. *Consider the counterflow heat exchanger model (2.3.1)-(2.3.2), with* $\boldsymbol{P} = \left\{\theta = (v_1, v_2, \alpha_1, \alpha_2) \in \mathbb{R}_+^4\right\}$ *the set of admissible parameters. Then, the model (2.3.1)-(2.3.2) is identifiable on* $\boldsymbol{P}$.

Proof. First note that the model of the heat exchanger includes first order differential operators. To make them disappear we apply the integral calculus $\int_0^T \int_0^1 tz \bullet dxdt$ to (2.3.1). We obtain:

$$\begin{aligned}\underbrace{\int_0^T \int_0^1 tz \frac{\partial x_1}{\partial t} dzdt}_{\phi_{11}} = -v_1 \underbrace{\int_0^T \int_0^1 tz \frac{\partial x_1}{\partial z} dzdt}_{\phi_{12}} \\ + \alpha_1 \underbrace{\int_0^T \int_0^1 tz \left(x_2(z,t) - x_1(z,t)\right) dzdt}_{\phi_{13}}\end{aligned} \tag{2.3.5}$$

and

$$\begin{aligned}\underbrace{\int_0^T \int_0^1 tz \frac{\partial x_2}{\partial t} dzdt}_{\phi_{21}} = v_2 \underbrace{\int_0^T \int_0^1 tz \frac{\partial x_2}{\partial z} dzdt}_{\phi_{22}} \\ + \alpha_2 \underbrace{\int_0^T \int_0^1 tz \left(x_1(z,t) - x_2(z,t)\right) dzdt}_{\phi_{23}}.\end{aligned} \tag{2.3.6}$$

Integrating by part on $[0,T] \times [0,1]$ the expressions containing partial derivative operators we obtain:

$$\begin{aligned}\phi_{11} &= \left[\int_0^1 tzx_1(z,t)dz\right]_0^T - \int_0^T \int_0^1 zx_1(z,t)dzdt \\ &= T\int_0^1 zx_1(z,T)dz - \int_0^T \int_0^1 zx_1(z,t)dzdt,\end{aligned} \tag{2.3.7}$$

$$\begin{aligned}\phi_{12} &= \int_0^T tx_1(1,t)dt - \int_0^T \int_0^1 tx_1(z,t)dzdt \\ \phi_{21} &= T\int_0^1 zx_2(z,T)dz - \int_0^T \int_0^1 zx_2(z,t)dzdt \\ \phi_{22} &= \int_0^T tx_2(1,t)dz - \int_0^T \int_0^1 tx_2(z,t)dzdt,\end{aligned} \tag{2.3.8}$$

where ϕ_{ij} the bounded functions on $L^2(0,1)$, with $L^2(0,1) = L^2((0,1);\mathbb{R})$ that denotes the Hilbert space of measurable square-integrable function with values in $\mathbb{R}$. With these expressions, the equations of system (2.3.5)-(2.3.6) are written as linear regressions:

$$\begin{cases}\phi_{11} = -v_1\phi_{12} + \alpha_1\phi_{13}\\ \phi_{21} = v_2\phi_{22} + \alpha_2\phi_{23}.\end{cases} \tag{2.3.9}$$

If we have non-noisy measurements that can characterize the temperature gradients $x_1(z,t)$ and $x_2(z,t)$, the parameters of the heat exchanger can be estimated uniquely using least squares:

$$\begin{cases}\begin{pmatrix}v_1\\ \alpha_1\end{pmatrix} = \left[\begin{pmatrix}-\phi_{12}\\ \phi_{13}\end{pmatrix}\begin{pmatrix}-\phi_{12} & \phi_{13}\end{pmatrix}\right]^{-1}\begin{pmatrix}-\phi_{12}\\ \phi_{13}\end{pmatrix}\phi_{11},\\ \begin{pmatrix}v_2\\ \alpha_2\end{pmatrix} = \left[\begin{pmatrix}\phi_{22}\\ \phi_{23}\end{pmatrix}\begin{pmatrix}\phi_{22} & \phi_{23}\end{pmatrix}\right]^{-1}\begin{pmatrix}\phi_{22}\\ \phi_{23}\end{pmatrix}\phi_{21}.\end{cases}$$

Then, we conclude that the parameters v_1, v_2, α_1, and α_2 of the heat exchanger model are identifiable. □

Now, let us study the practical identifiability of the heat exchanger model (2.3.1)-(2.3.2), i.e. if the model parameters can in fact be recovered uniquely from the given data. The approach that will be used is based on the sensitivity functions of the outputs of the heat exchanger. The analysis will be carried out by considering the stationary model corresponding to (2.3.1)-(2.3.2). The analytical approach will be compared to the numerical approach derived from the approximate model (2.3.4). Indeed, with regard to hyperbolic systems, it happens that the analytical results do not corroborate with the numerical results [Trefethen and Embree, 2005]. In this case, the interest of this comparison is to ensure that the identifiability of the system is not lost by switching to the discrete model.
The stationary problem of (2.3.1)-(2.3.2) is defined by

$$\begin{cases}\Lambda\dfrac{d\overline{x}}{dz} + M\overline{x}(z) = 0\\ x_1(0) = x_{1,in},\ x_2(1) = x_{2,in}\end{cases} \tag{2.3.10}$$

whose solutions $\overline{x}(z)$ represent the profile of temperatures at equilibrium, that are given for all $\overline{x}(z)$ and $z \in [0,1]$ by $\overline{x}(z) = \mathrm{e}^{\Lambda^{-1}\tilde{M}z}\,x(0)$, with $\tilde{M} = -M$.
We apply the Laplace transform to the spatial variable z, we have:

$$\begin{aligned}\mathrm{e}^{\Lambda^{-1}\tilde{M}z} &= \mathscr{L}_\xi^{-1}\left\{\begin{pmatrix}\xi+\beta_1 & -\beta_1\\ \beta_2 & \xi-\beta_2\end{pmatrix}^{-1}\right\}\\ &= \mathscr{L}_\xi^{-1}\left\{\frac{1}{\xi\left(\xi-(\beta_2-\beta_1)\right)}\begin{pmatrix}\xi-\beta_2 & \beta_1\\ -\beta_2 & \xi+\beta_1\end{pmatrix}\right\}\end{aligned}$$

where ξ is the Laplace variable. By applying the inverse transform to this last expression we obtain:

$$\begin{cases}\overline{x}_1(z) = \xi_{11}(z)x_1(0) + \xi_{12}(z)x_2(0)\\ \overline{x}_2(z) = \xi_{21}(z)x_1(0) + \xi_{22}(z)x_2(0)\end{cases} \tag{2.3.11}$$

with

$$\begin{aligned}
\xi_{11}(z) &= \frac{1}{\beta_2-\beta_1}\left(\beta_2 - \beta_1 e^{(\beta_2-\beta_1)z}\right),\\
\xi_{12}(z) &= \frac{\beta_1}{\beta_2-\beta_1}\left(e^{(\beta_2-\beta_1)z} - 1\right),\\
\xi_{21}(z) &= \frac{\beta_2}{\beta_2-\beta_1}\left(1 - e^{(\beta_2-\beta_1)z}\right),\\
\xi_{22}(z) &= \frac{1}{\beta_2-\beta_1}\left(\beta_2 e^{(\beta_2-\beta_1)z} - \beta_1\right),
\end{aligned}$$

Taking into account the configuration of the system, the boundary condition is expressed by

$$\begin{pmatrix} x_1(0)\\ x_2(0)\end{pmatrix} = \begin{pmatrix} 1 & 0\\ (0 \quad 1)\cdot \exp(\Lambda^{-1}\tilde{M})\end{pmatrix}^{-1} \begin{pmatrix} x_1(0)\\ x_2(1)\end{pmatrix}, \tag{2.3.12}$$

where $(0 \quad 1)\exp(\Lambda^{-1}\tilde{M}) = (\xi_{21}(1)\, \xi_{22}(1))$. By replacing (2.3.12) in (2.3.11) we get

$$\begin{aligned}
\overline{x}_1(z) &= \left[\xi_{11}(z) - \xi_{12}(z)\frac{\xi_{21}(1)}{\xi_{22}(1)}\right] x_1(0) + \frac{\xi_{12}(z)}{\xi_{22}(1)} x_2(1)\\
&= \frac{1}{\beta_1 - \beta_2\,\mathrm{e}^{(\beta_2-\beta_1)}}\left(\beta_1\,\mathrm{e}^{(\beta_2-\beta_1)z} - \beta_2\,\mathrm{e}^{(\beta_2-\beta_1)}\right) x_1(0)\\
&\quad + \frac{\beta_1}{\beta_1 - \beta_2\,\mathrm{e}^{(\beta_2-\beta_1)}}\left(1 - \mathrm{e}^{(\beta_2-\beta_1)z}\right) x_2(1),\\
\overline{x}_2(z) &= \left[\xi_{21}(z) - \xi_{22}(z)\frac{\xi_{21}(1)}{\xi_{22}(1)}\right] x_1(0) + \frac{\xi_{22}(z)}{\xi_{22}(1)} x_2(1)\\
&= \frac{\beta_2}{\beta_1 - \beta_2\,\mathrm{e}^{(\beta_2-\beta_1)}}\left(\mathrm{e}^{(\beta_2-\beta_1)z} - \mathrm{e}^{(\beta_2-\beta_1)}\right) x_1(0)\\
&\quad + \frac{1}{\beta_1 - \beta_2\,\mathrm{e}^{(\beta_2-\beta_1)}}\left(\beta_1 - \beta_2\,\mathrm{e}^{(\beta_2-\beta_1)z}\right) x_2(1),
\end{aligned} \tag{2.3.13}$$

with $\beta_1 = \frac{\alpha_1}{v_1}$, and $\beta_2 = \frac{\alpha_2}{v_2}$.
Evaluating these expressions at the locations $z = 1$ and $z = 0$ for $\overline{x}_1(z)$ and

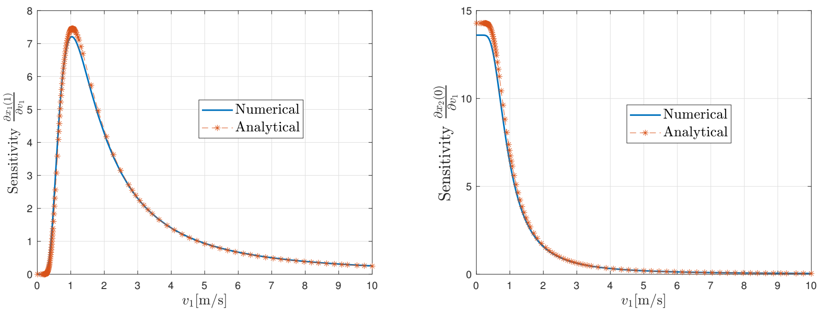
$\overline{x}_2(z)$, respectively, yields the input-output relationship:

$$\begin{cases} x_{1,out} = \dfrac{(\beta_1-\beta_2)\,\mathrm{e}^{(\beta_2-\beta_1)})}{\beta_1-\beta_2\,\mathrm{e}^{(\beta_2-\beta_1)}}x_{1,in} + \dfrac{\beta_1\left(1-\mathrm{e}^{(\beta_2-\beta_1)}\right)}{\beta_1-\beta_2\,\mathrm{e}^{(\beta_2-\beta_1)}}x_{2,in}, \\ x_{2,out} = \dfrac{\beta_2\left(1-\mathrm{e}^{(\beta_2-\beta_1)}\right)}{\beta_1-\beta_2\,\mathrm{e}^{(\beta_2-\beta_1)}}x_{1,in} + \dfrac{\beta_1-\beta_2}{\beta_1-\beta_2\,\mathrm{e}^{(\beta_2-\beta_1)}}x_{2,in}. \end{cases} \tag{2.3.14}$$

To compare them to the outputs of the approximated model we replace the stationary solution of the equation (2.3.4) in the output function $y = Cx$. So the output of the steady-state model (2.3.4) is given by:

$$y = -CA_n^{-1}\left(B_1x_{1,in} + B_2x_{2,in}\right), \tag{2.3.15}$$

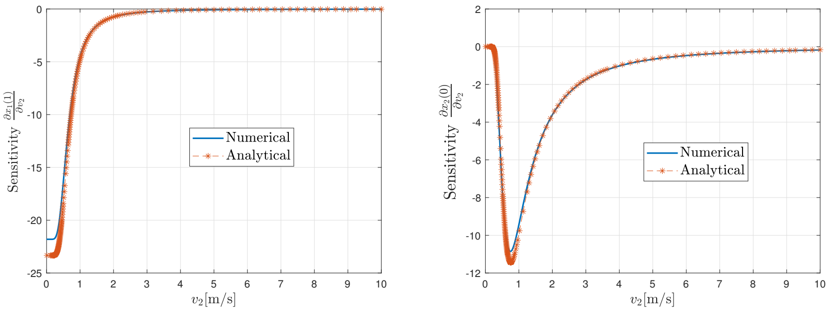
with $C = \begin{pmatrix} 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{2\times n}$ the output matrix.



(a) Sensitivity of output variable $\dfrac{\partial x_1(1)}{\partial v_1}$. **(b)** Sensitivity of output variable $\dfrac{\partial x_2(0)}{\partial v_1}$.

Figure 2.1: Sensitivity functions of output variables with respect to parameter v_1 obtained by an analytical approach and by numerical calculation ($n = 50$), for numerical values: $v_2 = 1.1m/s, \alpha_1 = 3.5s^{-1}$ and $\alpha_2 = 3s^{-1}$.

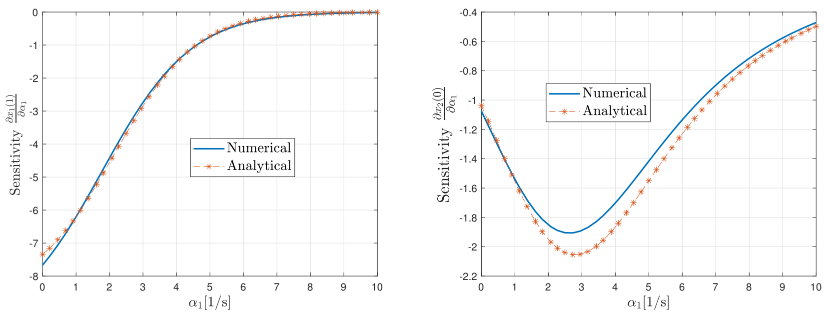
The output sensitivity functions can be obtained using the Symbolic Math Toolbox of Matlab© dedicated to symbolic computation. We can observe by analyzing the above relationships that the sensitivity functions depend on the value of the model parameters.
Figures 2.1a to 2.4b illustrate the output sensitivity functions obtained by the two approaches. One can observe that these functions differ from each other and are independent. This is indicating that the parameters of the heat exchanger model are not correlated, and that they can be identified in practice (a unique estimate would be possible). As mentioned in [Vanrolleghem et al., 1995] and [Petersen, Britta, 2000, Chapter 4], the sensitivity functions emphasize the experimental conditions under which the dependency of the outputs on the parameters is the largest, and thereby under which conditions



(a) Sensitivity of output variable $\frac{\partial x_1(1)}{\partial v_2}$. **(b)** Sensitivity of output variable $\frac{\partial x_2(0)}{\partial v_2}$.

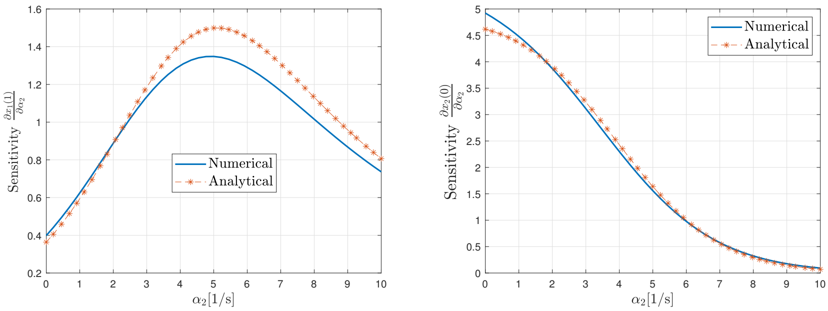
Figure 2.2: Sensitivity functions of output variables with respect to parameter v_2 obtained by an analytical approach and by numerical calculation ($n = 50$), for numerical values: $v_1 = 1m/s, \alpha_1 = 3.5s^{-1}$ and $\alpha_2 = 3s^{-1}$.

most information can be obtained on the parameters. For example, figures 2.1a, 2.2b, 2.3b and 2.4a show peaks indicating that at some point the sensitivity with respect to the parameters is important. Thus, a possible approach for improving the information contained in the data is to choose the sampling instants corresponding to the zones where the parameters have a prominent influence, i.e. when the sensitivity is high. This could be very helpful to avoid large estimation errors. On the other hand, the figures 2.1b, 2.2a, 2.3a and 2.4b do not exhibit any peak. In this case the information contained in the data is distributed homogeneously; and so, the information provided by the data set is better exploited.
Furthermore, note the existence of significant differences between the numeri-



(a) Sensitivity of output variable $\frac{\partial x_1(1)}{\partial \alpha_1}$. **(b)** Sensitivity of output variable $\frac{\partial x_2(0)}{\partial \alpha_1}$.

Figure 2.3: Sensitivity functions of output variables with respect to parameter α_1 obtained by an analytical approach and by numerical calculation ($n = 50$), for numerical values: $v_1 = 1m/s, v_2 = 1.1m/s$ and $\alpha_2 = 3s^{-1}$.



(a) Sensitivity of output variable $\frac{\partial x_1(1)}{\partial \alpha_2}$. **(b)** Sensitivity of output variable $\frac{\partial x_2(0)}{\partial \alpha_2}$.

Figure 2.4: Sensitivity functions of output variables with respect to parameter α_2 obtained by an analytical approach and by numerical calculation ($n = 50$), for numerical values: $v_1 = 1m/s, v_2 = 1.1m/s$ and $\alpha_1 = 3.5s^{-1}$.

cal approach and the analytical approach. The main differences relate to the sensitivities with respect α_1 and α_2. That is mainly due to the fact that these are phenomenological parameters that concentrate a high degree of uncertainty. But the results should be improved by increasing the number of discretization points.

2.3.3 Experimental setup

The heat exchanger that will be used in the case study is a small water to water tubular heat exchanger from Armfield, type HT31, with dimensions $160 \times 630 \times 393$ mm, and consists of two concentric (coaxial) tubes carrying the hot and cold fluids. It is a part of a heat exchanger setup located at the Department of Electromechanics, Polytechnics faculty, University of Lubumbashi in DR Congo. A partial view of the system is given in figure 2.5. This test bench makes it possible to study the characteristic properties of tubular, cross-flow or parallel heat exchangers. For its operation, this system contains two circuits.

- **Hot water circuit**: This circuit has heating elements and water contained in a glass tank. The heating elements are composed of two resistors of 1KW each connected in series on a voltage of 230V, which are controlled by a thermostat with a temperature threshold of 75°C. On start-up, the heated water is sucked in and delivered to the heat exchanger through a volumetric pump driven by a DC motor. After heat exchange, it is evacuated to the tank for a new cycle. The hot water flow $\dot{m}_1$ at the inlet of heat exchanger is controlled by a solenoid valve.
- **Cold water circuit**: This circuit is composed of a refrigeration unit ensuring the cooling of the water to a set temperature. After cooling, the

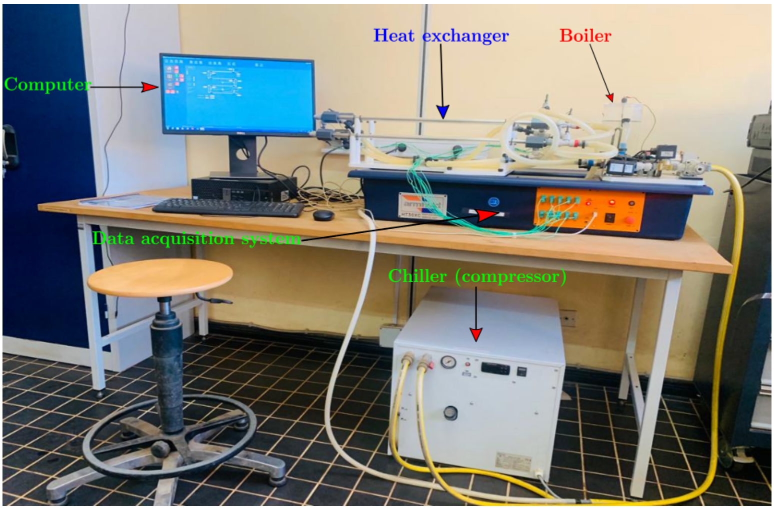


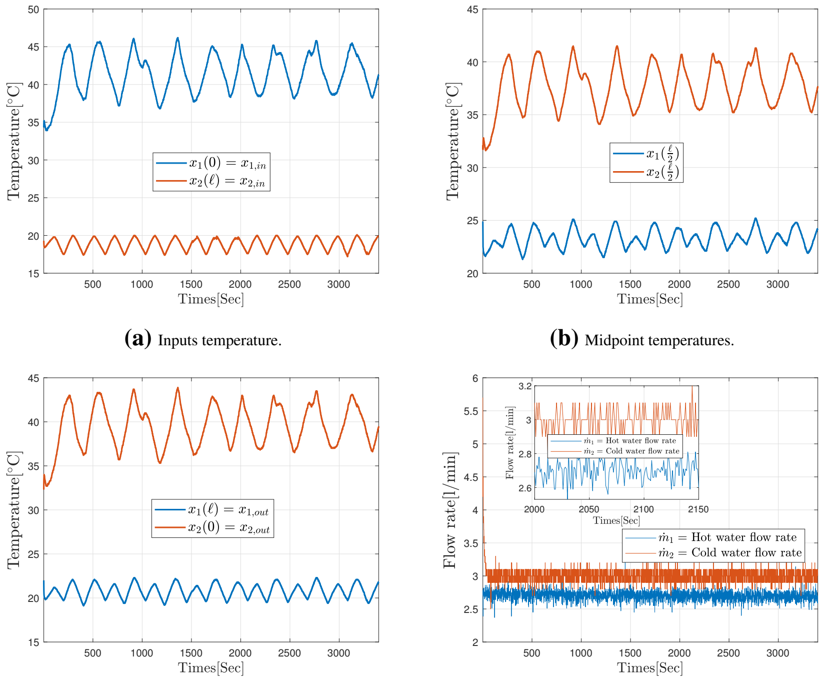
Figure 2.5: Partial view of the heat exchanger test bench in Unilu-lab.

water is routed to the inlet of the heat exchanger via a solenoid valve. The cold water flow $\dot{m}_2$ at the inlet of the exchanger depends essentially on the position of the solenoid valve.

For control or monitoring purposes, the heat exchanger is equipped with sensors supplying signals converted into voltage standardized in different ranges, and which are transmitted to the data acquisition device. The main variables measured as well as their ranges of variations are:

- The temperatures of the hot and cold fluid at three points along the exchanger: $z = 0, z = \frac{L}{2}$ and $z = L$. The signals provided by the thermocouples are transmitted to the data acquisition system in the voltage range $[-5 \quad 5V]$, which correspond to the temperature range $[0 \quad 133°C]$.
- Two flow meters are included (for both fluids). Operating range 0.2 to 9 L/min, resolution 0.1 L/min.

In this miniature version the tubes are separated into sections to reduce the overall length and to enable the temperature at points along both fluid streams to be measured. The inner tube is used for the hot fluid and the outer annulus for cold fluid. This minimises heat loss from the exchanger without the need for additional insulation. The inner tube is constructed from stainless steel and the outer annulus from clear acrylic, providing to minimising thermal losses. The heat transfer area is 0.02m^2.
To identify the parameters of the system, we launched an acquisition campaign of 3400 samples of the variables of the heat exchanger. On these data, two thirds will be used for the calibration stage and simple-validation, and one third



(a) Inputs temperature. **(b)** Midpoint temperatures.

(c) Output temperatures. **(d)** Input mass flow rates.

Figure 2.6: The experiment used in the parameter estimation.

for the (cross)-validation of the model. During the test, the heat exchanger was excited over a certain operating range by modifying successively the compression ratio of the cold water circuit and the heating power of the hot water circuit, so as to vary the inlet temperatures, and changing both of the mass flows rates. The temperature measurements are in degrees Celsius and the accuracy is in the second decimal, while the flow measurements are in $\mathrm{l/min}$ and the accuracy is in the first decimal. The sampling interval was one second and constant throughout the experiment. Figures 2.6a, 2.6b,2.6c, and 2.6d show the inlet temperatures, the midpoint temperatures ($z = \frac{L}{2}$), the outlet temperatures, and mass flow rates for each fluid, respectively. Note that the outlet temperature of the cold fluid is higher than that of the hot fluid; this constitutes one of the main advantages of the counterflow configuration.

The conceptual problem statement of the model parameter estimation and model structure is given below.

2.3.4 Parameter estimations and model validation

Once the identifiability of the parameters has been established, comes the stage of their estimation. Given the nonlinearity of the model output with respect to the parameters $\theta = [v_1, v_2, \alpha_1, \alpha_2]^\top$ (according to the expressions (2.3.14)), the algorithm that will be used is that of Levenberg-Marquardt. However, this being a local algorithm, it only converges if we have a reasonable first estimate of our parameters. To obtain a first approximation of the parameters to be identified, first we approximate the model (2.3.1)-(2.3.2) by global differences (see e.g. [Dochain, 1994], [Renou, 2000, Chapter 3]). This model reduction technique consists in expressing the energy balances of the PDEs model at the outlet of the heat exchanger. This leads to the following equations:

$$\begin{cases} \left.\dfrac{\partial x_1}{\partial t}\right|_{z=1} = -v_1 \left.\dfrac{\partial x_1}{\partial z}\right|_{z=1} + \alpha_1 \left(x_2(z,t) - x_1(z,t)\right)|_{z=1} \\ \left.\dfrac{\partial x_2}{\partial t}\right|_{z=0} = v_2 \left.\dfrac{\partial x_2}{\partial z}\right|_{z=0} + \alpha_2 \left(x_1(z,t) - x_2(z,t)\right)|_{z=0} . \end{cases} \tag{2.3.16}$$

At the heat exchanger outlet, the partial spatial derivatives can be approximated by the following overall differences:

$$\begin{cases} \left.\dfrac{\partial x_1}{\partial z}\right|_{z=1} \approx x_1(1,t) - x_1(0,t) = y_1(t) - x_{1,in}(t) \\ \left.\dfrac{\partial x_2}{\partial z}\right|_{z=0} \approx x_2(1,t) - x_2(0,t) = x_{2,in}(t) - y_2(t). \end{cases} \tag{2.3.17}$$

These global differences are then introduced into the energy balances. This allows to obtain the following linear reduced model:

$$\begin{cases} \dfrac{dy_1(t)}{dt} = -v_1 \left(y_1(t) - x_{1,in}(t)\right) + \alpha_1 \left(x_{2,in}(t) - y_1(t)\right) \\ \dfrac{dy_2(t)}{dt} = v_2 \left(x_{2,in}(t) - y_2(t)\right) + \alpha_2 \left(x_{1,in}(t) - y_2(t)\right) . \end{cases} \tag{2.3.18}$$

We note that under some conditions this model is equivalent to the lumped-parameter one given by [Zitte, 2021, Chapter 1], where the heat transfer is assumed to be uniform along the heat exchanger. Moreover since the reduced model (2.3.18) is linear in parameters θ, these can be estimated by least squares as follows:

$$\begin{cases} \begin{pmatrix} \hat{v}_1 \\ \hat{\alpha}_1 \end{pmatrix} = \left[\sum_{k=1}^{N} T\phi_k\phi_k^\top\right]^{-1} \sum_{k=1}^{N} T\phi_k y_m^1(k+1) \\ \begin{pmatrix} \hat{v}_2 \\ \hat{\alpha}_2 \end{pmatrix} = \left[\sum_{k=1}^{N} T\varphi_k\varphi_k^\top\right]^{-1} \sum_{k=1}^{N} T\varphi_k y_m^2(k+1), \end{cases} \tag{2.3.19}$$

with T the sampling period, and

$$\phi_k = \begin{pmatrix} y_m^1(k) - y_m^1(k-1) - x_{1,in}(k) \\ x_{2,in}(k) - y_m^1(k) + y_m^1(k-1) \end{pmatrix}$$
$$\varphi_k = \begin{pmatrix} x_{2,in}(k) - y_m^2(k) + y_m^2(k-1) \\ x_{1,in}(k) - y_m^2(k) - y_m^2(k-1) \end{pmatrix}.$$

$y_m^1(t)$ and $y_m^2(t)$ are the components of the vector of the measured outputs:

$$y_m(t) = \begin{pmatrix} x_1(1,t) = x_{1,out}(t) = y_m^1(t) \\ x_2(0,t) = x_{2,out}(t) = y_m^2(t) \end{pmatrix}. \tag{2.3.20}$$

So the first estimate of the parameters is equal to

$$\theta_0 = [0.0914 \quad 0.0949 \quad 0.9029 \quad 0.9103]^\top.$$

Now, the parameter estimation can be performed by Levenberg-Marquardt method using the first estimate of the vector θ_0 as initialization vector. To implement this algorithm on the heat exchanger model, the system (2.3.4) was solved using the following initial condition [Besson et al., 2006]:

$$\begin{cases} x_1(z) = x_{1,in}\left[\left(\mathrm{e}^1\,\dfrac{x_{2,in}}{x_{1,in}} - 1\right)z^2 + 1\right]\mathrm{e}^{-z^2}, \\ \\ x_2(z) = x_{2,in}\exp\left[(1-z)^2\ln\left(\dfrac{x_{1,in}}{x_{2,in}}\right)\right]. \end{cases} \tag{2.3.21}$$

To assess the quality of the parameter estimate, we use the mean quadratic error [Babuška, 1998, Page: 236]:

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(y_m(t_i) - y(t_i,\tilde{\theta})\right)^2}, \tag{2.3.22}$$

and the performance index VAF (Variance Accounted For) described by the expression

$$\mathrm{VAF} = 100 \times \left[1 - \frac{\mathrm{var}\left(y_m - y(\tilde{\theta})\right)}{\mathrm{var}(y_m)}\right]. \tag{2.3.23}$$

The ideal values for these criteria are respectively 0% and 100%. Table 2.1 presents the two performance criteria used in this study for different numbers of discretization points. The analysis of the results of this table shows that the physical parameters of the heat exchanger are well estimated by using a reduced number of discretization points. More specifically, we have a better result by considering 15 discretization points (or by placing 2×15 sensors

Table 2.1: Performance criteria

n	VAF$[\%]$	RMSE$[\%]$
5	$\begin{cases} \mathrm{Vaf}(x_1(1,t)) = 97.3450 \\ \mathrm{Vaf}(x_2(0,t)) = 80.8427 \end{cases}$	$\begin{cases} \mathrm{Rmse}(x_1(1,t)) = 1.8682 \\ \mathrm{Rmse}(x_2(0,t)) = 1.4162 \end{cases}$
10	$\begin{cases} \mathrm{Vaf}(x_1(1,t)) = 97.3018 \\ \mathrm{Vaf}(x_2(0,t)) = 80.2698 \end{cases}$	$\begin{cases} \mathrm{Rmse}(x_1(1,t)) = 0.5897 \\ \mathrm{Rmse}(x_2(0,t)) = 0.5424 \end{cases}$
15	$\begin{cases} \mathrm{Vaf}(x_1(1,t)) = 97.8603 \\ \mathrm{Vaf}(x_2(0,t)) = 92.4458 \end{cases}$	$\begin{cases} \mathrm{Rmse}(x_1(1,t)) = 0.5521 \\ \mathrm{Rmse}(x_2(0,t)) = 0.2871 \end{cases}$
21	$\begin{cases} \mathrm{Vaf}(x_1(1,t)) = 97.5344 \\ \mathrm{Vaf}(x_2(0,t)) = 88.3034 \end{cases}$	$\begin{cases} \mathrm{Rmse}(x_1(1,t)) = 1.5906 \\ \mathrm{Rmse}(x_2(0,t)) = 1.1140 \end{cases}$
30	$\begin{cases} \mathrm{Vaf}(x_1(1,t)) = 96.9238 \\ \mathrm{Vaf}(x_2(0,t)) = 86.1865 \end{cases}$	$\begin{cases} \mathrm{Rmse}(x_1(1,t)) = 2.1741 \\ \mathrm{Rmse}(x_2(0,t)) = 1.3367 \end{cases}$
40	$\begin{cases} \mathrm{Vaf}(x_1(1,t)) = 87.8702 \\ \mathrm{Vaf}(x_2(0,t)) = 81.1439 \end{cases}$	$\begin{cases} \mathrm{Rmse}(x_1(1,t)) = 5.3234 \\ \mathrm{Rmse}(x_2(0,t)) = 1.7226 \end{cases}$

along the heat exchanger).

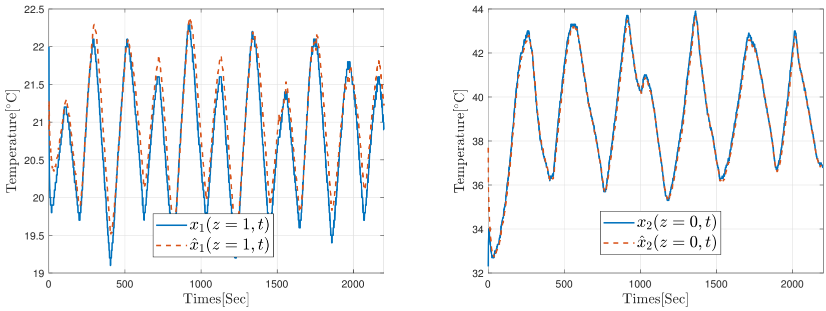
The results of the simulation with the estimated parameters by Levenberg-Marquardt'algorithm for 15 discretization points are illustrated in Figures 2.7a and 2.7b, on the data set used for calibration. It can be observed that the dynamics of the outputs of the heat exchanger are well reproduced. The distributed temperatures corresponding to this experience are represented in Figures 2.8a and 2.8b.

The last step in the identification process is cross validation, whose the aim

Table 2.2: Estimated values and performance coefficients.

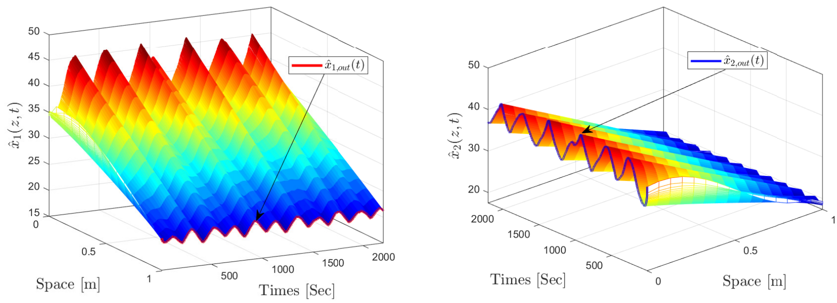
Parameters	Unity	Parameter value $\pm$ confidence intervals
v_1	m/s	$0.0938 \pm 4 \cdot 10^{-4}$
v_2	m/s	$0.0946 \pm 5 \cdot 10^{-4}$
α_1	s^{-1}	$0.9011 \pm 3 \cdot 10^{-4}$
α_2	s^{-1}	$0.9091 \pm 3 \cdot 10^{-4}$

is to demonstrating the predictive capability of the estimated model on the (one-third) dataset not having been used for the estimation of the parameters. This can be seen in Figures 2.9a and 2.9b. The corresponding distributed temperatures are given in Figures 2.10a and 2.10b. All that remains is



(a) Measured and predicted hold fluid temperature. **(b)** Measured and predicted cold fluid temperature.

Figure 2.7: Data fitting in a simple validation.



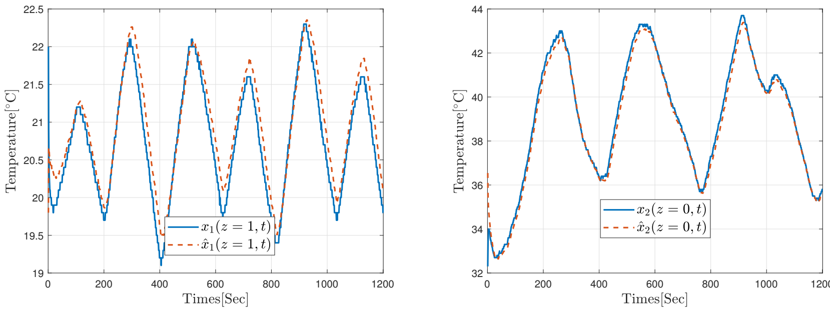
(a) Distributed predicted hold fluid temperature. **(b)** Distributed predicted cold fluid temperature.

Figure 2.8: Distributed temperatures for the simple validation.

to assess the confidence that can be had in numerical values of estimated parameters. This can be obtained by using Fisher's information matrix. The Fisher information matrix corresponding with this experiment is given by:

$$F = \begin{pmatrix} 4,0604 \cdot 10^7 & 1,8244 \cdot 10^7 & 1,8145 \cdot 10^7 & 1,9548 \cdot 10^7 \\ * & 3,4635 \cdot 10^7 & 1,7566 \cdot 10^7 & 1,8610 \cdot 10^7 \\ * & * & 3,4635 \cdot 10^7 & 1,7538 \cdot 10^7 \\ * & * & * & 3,5538 \cdot 10^7 \end{pmatrix}. \quad (2.3.24)$$

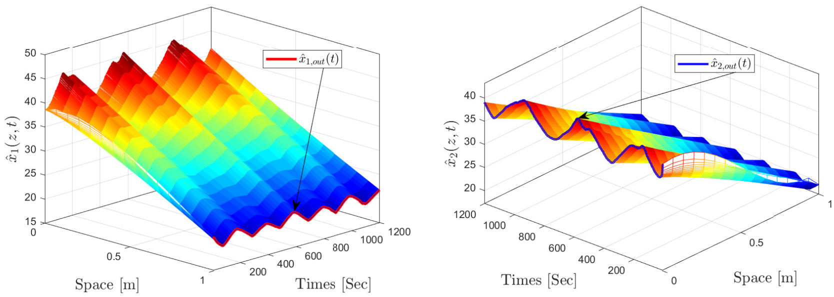
Note that the coefficients of this matrix are very large. By the fact that the inverse of this matrix corresponds to the covariance matrix of $\hat{\theta}$, we can therefore expect more confidence in the estimated values of the parameters. This is confirmed by Table 2.2 where we note very small confidence intervals, which reflect a good confidence of the estimated parameters. Moreover, one can notice that the numerical values of the parameters obtained are very close, i.e: $v_1 \approx v_2$, and $\alpha_1 \approx \alpha_2$. This is due to the fact that the physico-chemical properties of the hot and cold fluids considered in this study are identical.



(a) Measured and predicted hold fluid temperature.

(b) Measured and predicted cold fluid temperature.

Figure 2.9: Data fitting in cross validation.



(a) Distributed predicted hold fluid temperature.

(b) Distributed predicted cold fluid temperature.

Figure 2.10: Distributed temperatures for cross validation.

2.4 Conclusion

The aim of this chapter was to estimate the parameters of a counterflow heat exchanger, in the case where the dynamics is described by a system of two hyperbolic partial differential equations. This study was carried out using data collected on a laboratory test bench. Before estimating the parameters involved in the heat exchanger model, a study of the uniqueness of the parameters, typically of structural and practical identifiability, was carried out. Next, one of the nonlinear optimization methods, precisely that of Levenberg-Marquardt, was used on a set of ordinary differential equations obtained by the method of lines to estimate the parameters of the model. To circumvent the initialization problem of the latter, the least squares method was used on the global difference model. Finally, the results of the proposed identification approach were validated using laboratory data, and checking some statistical properties.

Part II

Analysis

3

Dynamical analysis in Hilbert space

Contents

3.1 Introduction

We consider in this chapter the standard model in process engineering of the counterflow heat exchanger

$$\partial_t x = \Lambda \partial_z x + Mx(z,t), \tag{3.1.1}$$

where $\Lambda = \begin{bmatrix} -v_1 & 0 \\ 0 & v_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $M = \begin{bmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, and we associated to this equation the following boundary conditions:

$$x_1(0,t) = x_{1,in}(t), \quad x_2(1,t) = x_{2,in}(t), \tag{3.1.2}$$

where $x_{1,in}$ and $x_{2,in}$ are the inlet temperatures of the heat exchanger. Here, we consider $x_{2,in} = u \in \mathbb{R}$ as a control variable and $x_{1,in}$ as a constant disturbance. The objective pursued in this chapter is to analyze some fundamental properties related to the control theory of the model (3.1.1)-(3.1.2) that can allow to set up (as we will do in the last two chapters of this work) adequate control laws. The state variables being of infinite dimension, the analysis will be made in an

adequate Hilbert space.
If the contributions centered on models with lumped parameters encountered a vast research activity, it is not the case for distributed parameter models that yet provide the advantage to emphasize the presence of temperature gradients within the system. In the context of PDEs model, it is worth noting that the scientific literature on the dynamical properties on counterflow heat exchangers is less prolific than that on parallel-flow [Sano, 2007a, Sano, 2007b, Sano and Nakagiri, 2009], [Lu and Chen, 2010], [Sano, 2013]. The main reason is the singularity of the matrix M, which coupled with the countercurrent configuration means that the resulting system cannot be put in lower triangular form; in other words, Eq. (3.1.1)-(3.1.2) generate a non-normal semigroup, i.e. the semigroup $(\mathbb{T}(t))_{t\geq 0}$ does not commute with its adjoint ($\mathbb{T}\mathbb{T}^* \neq \mathbb{T}^*\mathbb{T}$) [Curtain and Zwart, 2020, Definition A.3.67]. In this case the spectral theorem that guarantees the decomposition (diagonalization) of the operators no longer applies. This unfortunately prevents to fully take advantage of the theory dedicated to this type of case [Curtain and Zwart, 1995], [Pazy, 1983]. In addition, the fact that the semigroup is non-normal may have an important impact on the study of the stability. It is known in this case that the knowledge of the spectrum alone is not sufficient to understand its action [Trefethen, 1997]. The observation is reflected by essential differences between the analytical results and those obtained by numerical simulation. Indeed, equations of the form $d\xi/dt = A\xi(t)$, where A is a matrix or a linear operator, is one of the most important of these problems. These equations lead back to the study of the semigroups $\exp(At)$. The image and the spectrum of A allow to determine the growth bound of $\|e^{At}\|$. If A is normal then it is well known that $\|(\lambda I - A)^{-1}\| = 1/\mathrm{dist}(\lambda, \sigma(A))$ and therefore the norm of the resolvent would be completely determined by the spectrum. However, if A is non-normal, then the norm of the resolvent $\|(\lambda I - A)^{-1}\|$ can deviate from $1/\mathrm{dist}(\lambda, \sigma(A))$ drastically and hence the study of the pseudo-spectrum will be unavoidable.
In the available literature on the control of heat exchangers, several papers concentrate on the stability analysis of networks of heat exchangers controlled by proportional control [Xu and Sallet, 2002], [Tchousso et al., 2009]. With the same type of control action, [Grabowski, 2007] studies the stability by a frequency domain approach, while preceding the study with an admissibility analysis of the observation operator (it is assumed that there is only one measure available). It should also be specified that this study is done in the case where the equations of the system define a bilinear system.

3.1.1 Contributions and organization of the chapter

The first contribution of this chapter is the generation of an exponentially stable positive semigroup. Regarding positivity, the goal is to show that for any positive initial state, the state trajectories are entirely in the positive cone of the state space; this ensures that the model has a consistent physical interpretation. In particular, given that the semigroup generator is non-normal, the Lyapunov stability is completed by a pseudo-spectral analysis.

Next we show that the system is well posed in the sense of [Tucsnak and Weiss, 2014], when it is controlled and observed through unbounded operators.
Finally we show the controllability and the exact observability in finite-time using the multiplier method proposed by [Coron, 2007], which allows to establish the observability inequalities that are essential for the proof of controllability and observability of distributed parameter systems.
The chapter is organized as follows: In Section 3.2 we define the abstract Cauchy problem and we prove the existence and uniqueness of a stable positive solution; and next we study the well-posedness of the controlled and observed system. In section 3.3, we study the controllability and observability, in the case where measurement and control are performed out at the boundary.

3.1.2 Notations

Throughout this chapter and even in the sequel to this work, I_n denotes, the identity matrix of order n, $\mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) and $\mathbb{R}^n$ (or $\mathbb{C}^n$) denote the set real (or complex) of $n-$order matrices, respectively. $L^2(0,1) = L^2\left((0,1);\mathbb{C}\right)$ denotes the Hilbert space of measurable square-integrable function with values in $\mathbb{C}$. $H^1(0,1) = H^1\left((0,1);\mathbb{C}\right)$ is the Sobolev space of absolutely continuous $\mathbb{C}-$valued functions whose derivatives are in $L^2(0,1)$. $Z = H^1(0,1) \oplus H^1(0,1)$ and $X = L^2(0,1) \oplus L^2(0,1)$ are Hilbert spaces such that the injection $Z \subset X$ is continuous. We denote by $\rho(A)$ the resolvent set of A a linear operator, and, for any $\lambda \in \rho(A)$, the domain $\mathcal{D}(A) \subset X$ equipped with the norm $\|g\|_1 = \|(\lambda I - A)g\|_X$ will be denoted by X_1. The completion of X equipped with the norm $\|g\|_{-1} = \|(\lambda I - A)^{-1}g\|_X$ will be denoted by X_{-1}. The spaces U and Y are Hilbert spaces of control and observation values, respectively.
Reminders about the semigroup theory and and well-posedness problem are given in Appendix A.

3.2 Well-posedness

Let $L \in \mathcal{L}(Z,X)$ and $G \in \mathcal{L}(Z,\mathbb{R})$ be the operators of the abstract boundary control system in the sense of [Tucsnak and Weiss, 2009, Chapter 8], defined from (3.1.1)-(3.1.2) as follows:

$$\begin{cases} \dot{\xi}(t) = L\xi(t), \quad t \in [0,+\infty) \\ \xi(0) = \xi_0, \\ G\xi(t) = u(t), \end{cases} \tag{3.2.1}$$

with $L\xi = \Lambda\dfrac{d\xi}{dz} + M\xi$ and $\xi(t) = (\xi_1(t)\ \xi_2(t))^\top$ the vector of state variables.
The system (3.2.1) can be put into the standard form of a state space system,

namely: $\dot{\xi}(t) = A\xi(t)$, where the linear (unbounded) state operator A is defined by:

$$A = \begin{pmatrix} -v_1\dfrac{d}{dz} - \alpha_1 \cdot I & \alpha_1 \cdot I \\ \alpha_2 \cdot I & v_2\dfrac{d}{dz} - \alpha_2 \cdot I \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.2.2}$$
$$:= L_{|X_1} \in \mathcal{L}(X_1, X),$$

on its the domain $\mathcal{D}(A) = \mathcal{D}(A_{11}) \oplus \mathcal{D}(A_{22}) = \Big\{\xi = (\xi_1\ \xi_2)^\top \in X :$
ξ absolutely continuous (a.c.), $\dfrac{d\xi}{dz} \in X$ and $\xi_1(0) = 0 = \xi_2(1)\}$.
In this section, we show that the model introduced in the previous subsection is well-posed. By well-posedness we mean the existence and the uniqueness of the solution. We first show that the dynamic linear operator A, is the infinitesimal generator of an exponentially stable positive C_0-semigroup. Positivity means that for any positive initial condition and any positive input, the corresponding state trajectory and output are positive. Thus, this ensures that the model has a consistent physical interpretation as we are dealing with always positive temperatures as part of the heat exchangers. In particular, the stability of this semigroup is supplemented by a spectral analysis of its generator. Then, we show that the Cauchy problem (3.2.1) can be written as a well-posed boundary control system.

3.2.1 Uncontrolled system

Before stating the general theorem, let us start by looking for a condition on the parameters of the model (3.1.1)-(3.1.2) guaranteeing the existence of the state trajectories of the model. For that, let us consider the stationary solution $\overline{x}(z) = \exp(-\Lambda^{-1}Mz)x(0)$ of the system (3.1.1)-(3.1.2). Taking into account the configuration of the system, the boundary condition is expressed by

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (0 \quad 1) \cdot \exp(-\Lambda^{-1}M) \end{pmatrix}^{-1} \begin{pmatrix} x_{1,in} \\ x_{2,in} \end{pmatrix}. \tag{3.2.3}$$

According to [Prieur and Winkin, 2018, Lemma1] it is easy to verify that the determinant of the matrix (3.2.3) given by $\det := \beta_1 - \beta_2 e^{\beta_2 - \beta_1}$ is non-zero if and only if:

$$\beta_1 \neq \beta_2. \tag{3.2.4}$$

We will give in the next chapter a thermodynamic interpretation of this condition. Then by [Prieur and Winkin, 2018, Theorem 1] we can state and prove the following theorem.

Theorem 3.2.1. *If $\beta_1 \neq \beta_2$, the linear operator A defined by (3.2.2) generates a positive exponentially stable C_0-semigroup $\mathbb{T}(t)$ of bounded linear operators on X, and there exists $\mu > 0$ such that, for all $t \in [0, +\infty)$, $\|\mathbb{T}(t)\|_X \leq \exp(-\mu t)$.*

The proof of this theorem will be done in four steps:

1. The semigroup generated by the operator A is positive (see Lemma 3.2.1),
2. The operator A has a compact resolvent (see Lemma 3.2.2),
3. The semigroup generated by the operator A is exponentially stable (see Lemma 3.2.4),
4. The operator A^* generates a positive exponentially stable C_0-semigroup (see Proposition 3.2.1).

Lemma 3.2.1. *The linear operator A generates a positive C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$ on X.*

Proof. We establish the proof in four steps:

a) The operator $\tilde{A} = A - \tilde{M} - \omega I$, where $\tilde{M} = \begin{pmatrix} 0 & \alpha_1 I \\ \alpha_2 I & 0 \end{pmatrix}$ and $\omega = -\min(\alpha_1, \alpha_2)$, is dissipative. For all $\xi \in \mathcal{D}(\tilde{A}) = \mathcal{D}(A)$,

$$\begin{aligned} \mathfrak{Re}\langle (\tilde{A}\xi(z))^\top, \xi(z)\rangle_X \quad &\leq \frac{1}{2}\left[-v_1\xi_1^2(z) + v_2\xi_2^2(z)\right]_0^1 \\ &\quad -\alpha_1 \left|\xi_1\right|^2_{L^2(0,1)} - \alpha_2 \left|\xi_2\right|^2_{L^2(0,1)} \\ &\leq -\omega \left\|\xi\right\|_X^2 . \end{aligned}$$

b) $\mathrm{Ran}(I - \tilde{A}) = X$.
The inclusion $\mathrm{Ran}(I - \tilde{A}) \subset X$ is directly deduced from the definition of the domain of $\tilde{A}$. The inclusion $\mathrm{Ran}(I - \tilde{A}) \supset X$ means that $(I - \tilde{A})$ is surjective, i.e., for all $g \in X$ there exists $\xi \in \mathcal{D}(\tilde{A})$ such that $(I - \tilde{A})\xi(z) = g(z)$. For that, let us first show that $(I - \tilde{A}_{11})$ is surjective. In other words, we have to solve the differential equation

$$v_1 \frac{d\xi_1}{dz} + (1 + \alpha_1 + \omega)\xi_1(z) = g_1(z).$$

The solution is given by

$$\xi_1(z) = e^{-\frac{1+\alpha_1+\omega}{v_1}z}\xi_1(0) + \frac{1}{v_1}\int_0^z e^{-\frac{1+\alpha_1+\omega}{v_1}(z-\eta)} g_1(\eta)d\eta.$$

If the latter is in the domain of $\tilde{A}_{11}$, then $\xi_1(0) = 0$, which implies that:

$$\xi_1(z) = \frac{1}{v_1}\int_0^z e^{-\frac{1+\omega+\alpha_1}{v_1}(z-\eta)} g_1(\eta)d\eta,$$

is surjective. Similarly, we prove the surjectivity of $(I - \tilde{A}_{22})$ by verifying that for $\xi_2 \in \mathcal{D}(\tilde{A}_{22})$, the solution of

$$-v_2 \frac{d\xi_2}{dz} + (1 + \alpha_2 + \omega)\xi_2(z) = g_2(z),$$

is given by

$$\xi_2(z) = \frac{1}{v_2} \int_z^1 e^{\frac{1+\omega+\alpha_2}{v_2}(z-\eta)} g_2(\eta) d\eta.$$

c) The operator A is the (infinitesimal) generator of a C_0-semigroup on X. Indeed, by using Lumer-Phillips theorem, it follows, from assertion a) and b) that the operator $\tilde{A} + \omega I$ generates an exponentially stable C_0-semigroup $(\tilde{\mathbb{T}}(t))_{t \geq 0}$ such that $\|\tilde{\mathbb{T}}(t)\| \leq e^{\omega t}$. Moreover, $\|\tilde{M}\xi\|^2 = \|\alpha_1 \xi_2\|^2_{L^2(0,1)} + \|\alpha_2 \xi_1\|^2_{L^2(0,1)} = \alpha_1^2 \|\xi_2\|_{L^2(0,1)} + \alpha_2^2 \|\xi_1\|^2_{L^2(0,1)} \leq \max(\alpha_1^2, \alpha_2^2)\|\xi\|^2_X$. Thus, $\tilde{M}$ is (linear and) bounded with $\|\tilde{M}\| \leq \max(\alpha_1, \alpha_2)$. In addition there exists $\xi \in X$ such that $\|\xi\| = 1$ and $\|\tilde{M}\xi\|^2 = \max(\alpha_1^2, \alpha_2^2)$. Hence, $\|\tilde{M}\| = \sup_{\substack{\xi \in X, \\ \xi \neq 0}} \frac{\|\tilde{M}\xi\|}{\|\xi\|} = \max(\alpha_1, \alpha_2)$. Consequently, by a standard result from [Curtain and Zwart, 2020, Theorem 5.3.1] it follows that, the operator A generates a C_0-semigroup $(\mathbb{T}(t))_{t \geq 0}$ such that $\|\mathbb{T}(t)\| \leq e^{(\omega+\nu)t}$, where $\nu = \|\tilde{M}\| = \max(\alpha_1, \alpha_2)$ and $\omega = -\min(\alpha_1, \alpha_2)$. In particular, $(\mathbb{T}(t))_{t \geq 0}$ is a C_0-semigroup of contraction whenever $\alpha_1 = \alpha_2$.
d) $\mathbb{T}(t)$ is a positive.
According to [Engel and Nagel, 2000, Chapter 4, Theorem 1.8] $\mathbb{T}(t)$ is positive if and only if $(sI - A)^{-1}$ is also positive for s sufficiently large. This can be shown by checking the positivity of the diagonal operators A_{11} and A_{22}, and then using [Engel and Nagel, 2000, Corollary 1.11, page 356]. In that result, it is stated that, if an operator generates a positive semigroup and another is linear bounded positive, then the sum of the two operators generates a positive C_0-semigroup.
Observe that, for any $y \in X$ such that $y_1 \geq 0$ and $y_2 \geq 0$ (i.e on $[0, 1]$),

$$\left((sI - A_{11})^{-1} y_1\right)(z) = \frac{1}{v_1} \int_0^z e^{-\frac{1+\alpha_1}{v_1}(z-\eta)} y_1(\eta) d\eta \geq 0 \tag{3.2.5}$$

and

$$\left((sI - A_{22})^{-1} y_2\right)(z) = \frac{1}{v_2} \int_z^1 e^{\frac{1+\alpha_2}{v_2}(z-\eta)} y_2(\eta) d\eta \geq 0 \tag{3.2.6}$$

Hence the C_0-semigroup generated by $\tilde{A} = \text{diag}(A_{11}, A_{22})$ is positive. By [Engel and Nagel, 2000, Corollary 1.11, page 356] it follows that the generator $A = \tilde{A} + \tilde{M}$ generates a positive C_0-semigroup, since the (multiplication) operator $\tilde{M}$ is positive and bounded. □

Lemma 3.2.2. *The operator A has a compact resolvent.*

Proof. It suffices to observe that the resolvent operator associated with A is an integral operator of Hilbert-Schmidt type. This operator is given for any $s \in \rho(A)$ and $y = (y_1, y_2)^\top \in X$, by

$$\begin{aligned}\xi(z) &= \begin{pmatrix}\xi_1(z)\\ \xi_2(z)\end{pmatrix} = (sI - A)^{-1}y(z)\\ &= e^{\tilde{A}(s)z}\begin{pmatrix}0\\ \xi_2(0)\end{pmatrix} + \int_0^z e^{\tilde{A}(s)(z-\eta)}y(\eta)d\eta,\end{aligned}$$

with

$$\xi_2(0) = \left[e^{-\tilde{A}(s)}\right]_2 \cdot \left\{\begin{pmatrix}\xi_1(1)\\ 0\end{pmatrix} - \int_0^1 e^{\tilde{A}(s)(1-\eta)}y(\eta)d\eta\right\}$$

where $\tilde{A}(s) = \begin{pmatrix} -\frac{s+\alpha_1}{v_1} & -\frac{\alpha_1}{v_1} \\ \frac{\alpha_2}{v_2} & \frac{s+\alpha_2}{v_2}\end{pmatrix}$, and $y(z) = \left[\frac{y_1(z)}{v_1} \quad -\frac{y_2(z)}{v_2}\right]^\top$. $[M]_2\cdot$ denotes the second row of a 2×2 matrix M. The components of the transition matrix $e^{\tilde{A}(s)z}$ are:

$$\begin{aligned}
\phi_{11}(z,s) &= \frac{\mathrm{e}^{\alpha(s)z}}{2\beta(s)}\left(2\beta(s)\cosh\left(\beta(s)z\right) - \gamma(s)\sinh\left(\beta(s)z\right)\right)\\
\phi_{12}(z,s) &= \frac{\beta_1}{\beta(s)}\,\mathrm{e}^{\alpha(s)z}\sinh\left(\beta(s)z\right)\\
\phi_{21}(z,s) &= -\frac{\beta_2}{\beta(s)}\,\mathrm{e}^{\alpha(s)z}\sinh\left(\beta(s)z\right)\\
\phi_{22}(z,s) &= \frac{\mathrm{e}^{\alpha(s)z}}{2\beta(s)}\left(2\beta(s)\cosh\left(\beta(s)z\right) + \gamma(s)\sinh\left(\beta(s)z\right)\right).
\end{aligned}$$

with: $\beta_1 = \frac{\alpha_1}{v_1}$, $\beta_2 = \frac{\alpha_2}{v_2}$, $\eta_{11} = \frac{s+\alpha_1}{v_1}$, $\eta_{22} = \frac{s+\alpha_2}{v_2}$, $\alpha(s) = \frac{1}{2}(\eta_{22} - \eta_{11})$, $\beta(s) = \frac{\sqrt{\Delta(s)}}{2}$, $\Delta(s) = \frac{1}{2}(\eta_{11}+\eta_{22})^2 - 4\beta_1\beta_2$, $\gamma = \eta_{11}(s) + \eta_{22}(s)$. □

Moreover, it follows from Lemma 3.2.1, Lemma 3.2.2 and C_0-semigroup theory [Curtain and Zwart, 1995], that the following corollary, holds.

Corollary 3.2.1. *For $\xi(0) = \xi_0 \in \mathcal{D}(A)$, the abstract system (L, G) given by (3.2.1) admits a unique solution $\xi \in C^0\left([0,+\infty), \mathcal{D}(A)\right) \cap C^1\left([0,+\infty), X\right)$. Furthermore,*

$$\xi(t) = \mathbb{T}(t)\xi_0, \qquad \forall\, t \in [0,+\infty).$$

In the sequel, it will be assumed that the following two conditions hold and it will be shown that they ensure the stability of the C_0-semigroup $\mathbb{T}(t)$ introduced in Lemma 3.2.1 above.

Assumption 3.2.1. *Let*

$$P = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}, \tag{3.2.7}$$

be a diagonal positive definite matrix. There exists a positive constant $\mu > 0$ such that

$$PM + M^\top P \leq -2\mu P. \tag{3.2.8}$$

Assumption 3.2.2. *There are two positive constants $\lambda_{min}(P) = min\{p_1, p_2\}$, $\lambda_{max}(P) = max\{p_1, p_2\}$ a minimum and maximum eigenvalues of P respectively, such that for all $\xi \in X$ the following inequality is satisfied:*

$$\lambda_{min}(P)\, \|\xi\|_X^2 \leq \xi^\top P \xi \leq \lambda_{max}(P)\, \|\xi\|_X^2 \,. \tag{3.2.9}$$

Remark 3.2.1. *Condition (3.2.8) is equivalent to the existence of a Lyapunov functional for the open-loop system (3.1.1)-(3.1.2). It is found in [Bastin and Coron, 2016, Chapter 5] within the framework of hyperbolic systems of conservation laws. The inequality (3.2.9) is a derivability condition of the Lyapunov functional. The computation of the positive definite matrix P can be done using the YALMIP toolbox [Lofberg, 2004].*

Lemma 3.2.3. *The C_0-semigroup $\mathbb{T}(t)$ generated by A is exponentially stable.*

Proof. For all $\xi \in X$, by a straightforward calculation we have:

$$\begin{aligned}
\langle A\xi, P\xi\rangle + \langle P\xi, A\xi\rangle &= \int_0^1 (A\xi(z))^\top P\xi(z)dz + \int_0^1 (P\xi(z))^\top A\xi(z)dz \\
&= \int_0^1 \xi^\top(z) \left(\Lambda P \frac{d}{dz} + M^\top P\right) \xi(z)dz \\
&\quad + \int_0^1 \xi^\top(z) \left(\Lambda P \frac{d}{dz} + PM\right) \xi(z)dz \\
&= \xi^\top(1)\Lambda P\xi(1) - \xi^\top(0)\Lambda P\xi(0) \\
&\quad + \int_0^1 \underbrace{\xi^\top(z) \left(PM + M^\top P\right) \xi(z)}_{\overset{\text{Eq.(3.2.8)}}{\leq} -2\mu\xi^\top P\xi} dz \\
&\leq -\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)} \|\xi\|_X^2.
\end{aligned}$$

Thus, by [Curtain and Zwart, 1995, Theorem 5.1.3] we conclude the proof. □

Lemma 3.2.4. *Let $A \in \mathcal{D}(A)$. For all $\zeta \in X$, adjoint operator of A is defined by*

$$A^* \zeta = -\Lambda \frac{d\zeta}{dz} + M^\top \zeta, \tag{3.2.10}$$

with his domain $\mathcal{D}(A^*) = \Big\{ \zeta = (\zeta_1, \zeta_2)^\top \in X : \zeta \text{ a.c.}, \ \frac{d\zeta}{dz} \in X$ *and* $\zeta_1(1) = 0 = \zeta_2(0) \}$.

Proof. Indeed, it suffices to observe that the usual pairing identity $\langle A^*\zeta, \xi \rangle = \langle \zeta, A\xi \rangle$ holds $\forall \zeta \in \mathcal{D}(A^*)$ and $\forall \xi \in \mathcal{D}(A)$. To check this, let us consider the operator A^* given by (3.2.10) on its domain $\mathcal{D}(A^*)$ and observe that, for all $\xi \in \mathcal{D}(A)$ and for all $\zeta \in \mathcal{D}(A^*)$ we have that,

$$\begin{aligned}
\langle A^*\zeta, \xi \rangle &= \int_0^1 (A^*\zeta(z))^\top \xi(z) dz = \int_0^1 \left(-\Lambda \frac{d\zeta}{dz} + M^\top \zeta(z) \right)^\top \xi(z) dz \\
&= - \int_0^1 \left(\Lambda \frac{d\zeta}{dz} \right)^\top \xi(z) dz + \int_0^1 \left(M^\top \zeta(z) \right)^\top \xi(z) dz \\
&= - \int_0^1 \frac{d\zeta^\top}{dz} \Lambda \xi(z) dz + \int_0^1 \zeta^\top(z) M \xi(z) dz.
\end{aligned} \tag{3.2.11}$$

The last equality comes from the fact that Λ is a real-valued diagonal matrix, and therefore $\Lambda^\top = \Lambda$. Let us integrate by part the first term on $[0, 1]$. Thus, we get:

$$\begin{aligned}
\langle A^*\zeta, \xi \rangle &= \Big[-\zeta^\top(z) \Lambda \xi(z) \Big]_0^1 + \int_0^1 \zeta^\top(z) \Lambda \xi(z) dz + \int_0^1 \zeta^\top(z) M \xi(z) dz \\
&= -\zeta^\top(1) \Lambda \xi(1) + \zeta^\top(0) \Lambda \xi(0) + \int_0^1 \zeta^\top(z) \left(\Lambda \frac{d}{dz} + M \right) \xi(z) dz.
\end{aligned}$$

Now, using the definition of $\mathcal{D}(A^*)$, we get, for all $\zeta \in \mathcal{D}(A^*)$,

$$\begin{aligned}
-\zeta^\top(1) \Lambda \xi(1) + \zeta^\top(0) \Lambda \xi(0) = v_1 \underbrace{\zeta_1(1)}_{=0} \xi_1(1) - v_2 \zeta_2(1) \xi_2(1) \\
- v_1 \zeta_1(0) \xi_1(0) + v_2 \underbrace{\zeta_2(0)}_{=0} \xi_2(0).
\end{aligned}$$

Combining the above expression and the definitions of A and its domain $\mathcal{D}(A)$

yields the following identity, for all $\xi \in \mathcal{D}(A)$ and for all $\zeta \in \mathcal{D}(A^*)$:

$$\begin{aligned}
\langle A^*\zeta, \xi\rangle &= -v_1\zeta_1(0)\underbrace{\xi_1(0)}_{=0} - v_2\zeta_2(1)\underbrace{\xi_2(1)}_{=0} + \int_0^1 \zeta^\top(z)\left(\Lambda\frac{d}{dz} + M\right)\xi(z)dz \\
&= \int_0^1 \zeta^\top(z)\left(\Lambda\frac{d}{dz} + M\right)\xi(z)dz \\
&= \langle \zeta, A\xi\rangle.
\end{aligned}$$

Thus we have that the operator A^* defined in (3.2.10) and its domain $\mathcal{D}(A^*)$ characterize the adjoint operator of A. □

The following proposition shows that A^* generates a positive exponentially stable C_0-semigroup.

Proposition 3.2.1. *For all $\zeta \in \mathcal{D}(A^*)$ the dynamic operator $A^* \in \mathcal{D}(A^*)$ generates an exponentially stable positive C_0-semigroup $\mathbb{T}^*(t)$ on X.*

Proof. First, the generation of the C_0-semigroup $\mathbb{T}^*(t)$ is deduced from a classical result on the duality between $\mathbb{T}(t)$ and $\mathbb{T}^*(t)$; see [Curtain and Zwart, 1995, Theorem 2.2.6]. In addition let us show that $\mathbb{T}^*(t)$ is positive. For that it suffices to consider $g = (g_1, g_2)^\top \in X$, with $g_1 \geq 0$ and $g_2 \geq 0$. We can verify that

$$\left((sI - A_{11}^*)^{-1}g_1\right)(z) = \frac{1}{v_1}\int_z^1 e^{\frac{s+\alpha_1}{v_1}(z-\eta)}g_1(\eta)d\eta \geq 0,$$

and

$$\left((sI - A_{22}^*)^{-1}g_2\right)(z) = \frac{1}{v_2}\int_0^z e^{-\frac{s+\alpha_2}{v_2}(z-\eta)}g_2(\eta)d\eta \geq 0.$$

So $\mathrm{diag}(A_{11}^*, A_{22}^*)$ generates a positive semigroup. Given that $\alpha_1 I$ and $\alpha_2 I$ are bounded and positive, then by [Engel and Nagel, 2000, Corollary 1.11, page 356] $\mathbb{T}^*(t)$ is positive.
Finally, $\mathbb{T}^*(t)$ is exponentially stable since for all $t \geq 0$, $\|\mathbb{T}^*(t)\| = \|\mathbb{T}(t)\|$ and the C_0-semigroup $\mathbb{T}(t)$ is exponentially stable. □

3.2.2 Spectral analysis

The dynamic operator A defined by (3.2.2) has a set of eigenvalues that we want to compute and analyze.
By definition, the eigenvalues of A are the numbers $\lambda \in \mathbb{C}$ such as the boundary value problem:

$$\begin{cases} A\begin{pmatrix}\xi_1(z)\\ \xi_2(z)\end{pmatrix} = \lambda\begin{pmatrix}\xi_1(z)\\ \xi_2(z)\end{pmatrix} \\ \xi_1(0) = 0,\ \xi_2(1) = 0, \end{cases} \tag{3.2.12}$$

has a non-trivial solution $(\xi_1(z), \xi_2(z))^\top$. The problem (3.2.12) is a linear boundary value problem. The general solution of this differential problem is of the form: $\xi(z) = \exp\left(\Lambda^{-1}(\lambda I - M)z\right) = \begin{pmatrix} \phi_{11}(z,\lambda) & \phi_{12}(z,\lambda) \\ \phi_{21}(z,\lambda) & \phi_{22}(z,\lambda) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, where $C_1, C_2 \in \mathbb{C}$ are arbitrary constants and where:

$$\begin{aligned}
\phi_{11}(z,\lambda) &= \frac{p_1(\lambda) - \frac{\lambda+\alpha_2}{v_2}}{p_1(\lambda) - p_2(\lambda)} e^{p_1(\lambda)z} - \frac{p_2(\lambda) - \frac{\lambda+\alpha_2}{v_2}}{p_1(\lambda) - p_2(\lambda)} e^{p_2(\lambda)z}, \\
\phi_{12}(z,\lambda) &= \beta_1 \frac{e^{p_1(\lambda)z} - e^{p_2(\lambda)z}}{p_1(\lambda) - p_2(\lambda)}, \quad \phi_{21}(z,\lambda) = \beta_2 \frac{e^{p_2(\lambda)z} - e^{p_1(\lambda)z}}{p_1(\lambda) - p_2(\lambda)}, \\
\phi_{22}(z,\lambda) &= \frac{p_1(\lambda) - \frac{\lambda+\alpha_1}{v_1}}{p_1(\lambda) - p_2(\lambda)} e^{p_1(\lambda)z} - \frac{p_2(\lambda) + \frac{\lambda+\alpha_1}{v_1}}{p_1(\lambda) - p_2(\lambda)} e^{p_2(\lambda)z},
\end{aligned}$$

with

$$\begin{aligned}
p_1(\lambda) &= \frac{1}{2}\left(\frac{\lambda+\alpha_2}{v_2} - \frac{\lambda+\alpha_1}{v_1} + \sqrt{\Delta(\lambda)}\right), \\
p_2(\lambda) &= \frac{1}{2}\left(\frac{\lambda+\alpha_2}{v_2} - \frac{\lambda+\alpha_1}{v_1} - \sqrt{\Delta(\lambda)}\right), \\
\Delta(\lambda) &= \left(\frac{\lambda+\alpha_2}{v_2} + \frac{\lambda+\alpha_1}{v_1}\right)^2 - 4\frac{\alpha_1\alpha_2}{v_1 v_2}.
\end{aligned}$$

By imposing on this general solution the homogeneous boundary conditions $\xi_1(0) = 0$ and $\xi_2(1) = 0$, we obtain the following linear system:

$$\begin{pmatrix} 1 & 0 \\ (0 \quad 1)\cdot \exp\left(\Lambda^{-1}(\lambda I - M)\right) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.2.13}$$

Thus the complex number λ is an eigenvalue of A if and only if it is a solution of the characteristic equation:

$$\varrho(\lambda) := \det \begin{bmatrix} 1 & 0 \\ (0 \quad 1)\cdot \exp\left(\Lambda^{-1}(\lambda I - M)\right) \end{bmatrix} = 0,$$

i.e,

$$\varrho(\lambda) :\stackrel{\Delta}{=} k_1(\lambda)e^{p_1(\lambda)} + k_2(\lambda)e^{p_2(\lambda)} = 0, \tag{3.2.14}$$

where the functions k_i, $i = 1, 2$, are given by

$$k_1(\lambda) = \frac{p_1(\lambda) + \frac{\lambda+\alpha_1}{v_1}}{p_1(\lambda) - p_2(\lambda)} \text{ and } k_2(\lambda) = \frac{p_2(\lambda) + \frac{\lambda+\alpha_1}{v_1}}{p_2(\lambda) - p_1(\lambda)}.$$

Observe that for $p_1(\lambda) = p_2(\lambda)(\Delta(\lambda) = 0)$, the eigenvalues are

$$\lambda_n = -\frac{v_1\alpha_2 + v_2\alpha_1}{v_1 + v_2} \pm 2\frac{v_1 v_2}{v_1 + v_2}\sqrt{\beta_1\beta_2 - n^2\pi^2}, \quad \text{with } n \in \mathbb{N}.$$

In this case, the operator A having a compact resolvent (see Lemma 3.2.2), it satisfies the spectral growth property [Curtain and Zwart, 1995, Theorem 5.1.6], i.e. its growth bound ω_0 coincides with its spectral bound:

$$\omega_0 = \sup_{n\in\mathbb{N}_0}\mathfrak{Re}(\lambda_n) = -\frac{\left(\sqrt{v_2\alpha_1} - \sqrt{v_1\alpha_2}\right)^2}{v_1 + v_2}.$$

A more general result (in the case where $p_1(\lambda) \neq p_2(\lambda)$) concerning the stability of the spectrum and the boundedness of the norm of the resolvent operator is established in [Xu and Sallet., 1993]. Moreover, A being an non-normal operator, the knowledge of the spectrum alone is not enough to explain its behavior. Small disturbances such as approximation errors can strongly influence the behavior of the less stable modes. The pseudo-spectrum approach proces to be an important tool to study this type of case [Trefethen, 1997], [Trefethen and Embree, 2005].

If A is a closed linear operator that is densely defined on X, the ε-pseudospectrum family of A is defined by

$$\sigma_\varepsilon(A) := \left\{\lambda \in \mathbb{C} : \left\|(\lambda I - A)^{-1}\right\| \geq \frac{1}{\varepsilon}\right\},\ \varepsilon > 0.$$

The study of the ε-pseudospectrum refers to the study of the level curves of the norm of the resolvent operator, which makes it possible to emphasize its spectral instability with respect to disturbances.

Figures 3.1, 3.2 and 3.3 show the spectrum and the pseudospectrum (level curves) of A. We note that all eigenvalues are located in the left complex half-plane. Moreover when $\varepsilon \to 0$ the norm of the resolvent remains bounded; therefore, small disturbances are not enough to induce system instability. Thus we conclude that the linear operator A is less sensitive to small disturbances, and that the discretization scheme used guarantees the stability of the spectrum and the boundedness of the resolvent with respect to the imaginary axis.

3.2.3 Well-posedness of boundary control system

In this subsection we give some technical results to underline the existence and the uniqueness of the solution of (3.1.1)-(3.1.2), when the system is controlled and observed by unbounded operators. The results that we are going to show are essentially inspired by the article [Prieur and Winkin, 2018].

Proposition 3.2.2. *If $\beta_1 \neq \beta_2$, then the abstract Cauchy system (L, G) defined by (3.2.1) is a boundary control system. The control operator is given by $B = \begin{pmatrix}0 & v_2\delta_1\end{pmatrix}^\top \in \mathcal{L}(U, X_{-1})$, with δ_1 the Dirac function at $z = 1$.*

Proof. Note that the model (3.1.1)-(3.1.2) fits into the general framework of linear hyperbolic systems described by [Prieur and Winkin, 2018], for which the well-posedness of the boundary control system is established based on [Tucsnak and Weiss, 2009, Chapter 10]. Let us now show that the abstract

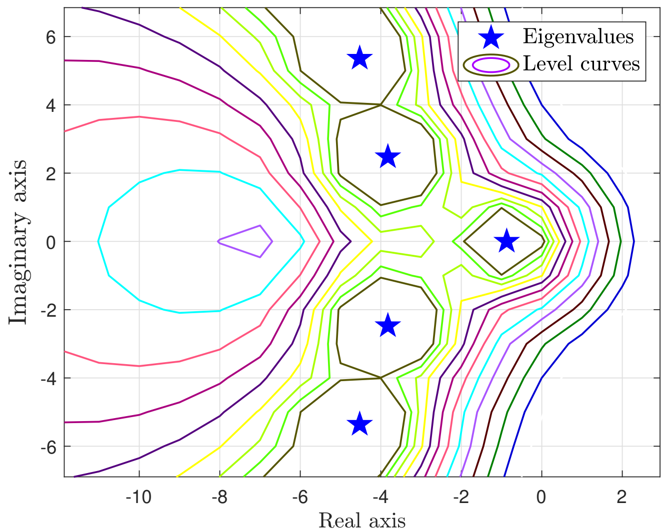


Figure 3.1: Pseudospectra for $\varepsilon^{-1} = 10^{-0.1:0.05:0.5}$.

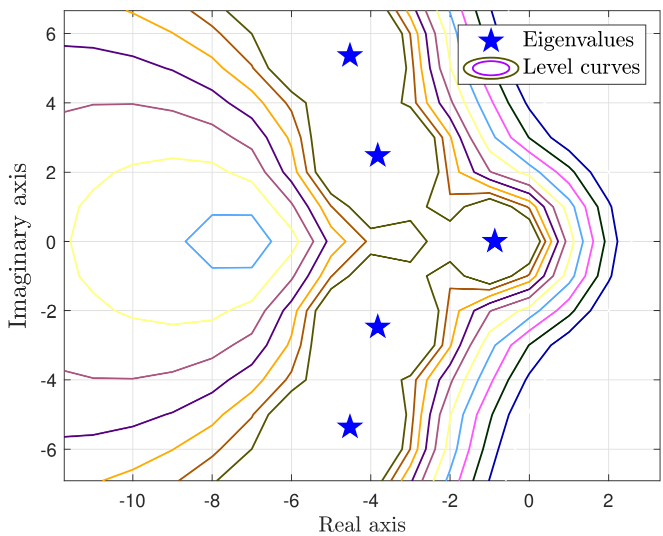


Figure 3.2: Pseudospectra for $\varepsilon^{-1} = 10^{-0.01:.05:0.5}$.

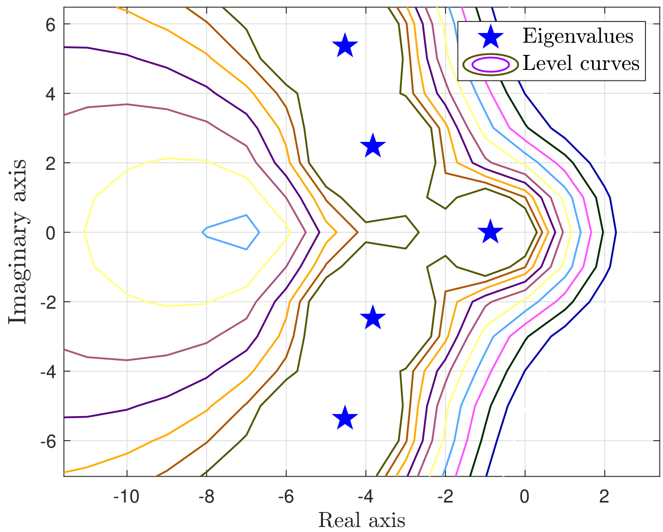


Figure 3.3: Pseudospectra for $\varepsilon^{-1} = 10^{-0.001:.05:0.5}$.

Cauchy system (L, G) is equivalent to $\dot{\xi}(t) = A\xi(t) + Bu(t)$. Then, given that $A = L_{|X_1}$ the left term of the equation (A.2.6) is

$$\langle L\xi(z), \zeta(z)\rangle = \int_0^1 \left(\Lambda \frac{d\xi}{dz} + M\xi(z)\right)^\top \zeta(z) dz. \tag{3.2.15}$$

The first right-hand side term of the equation (A.2.6) can expanded by integration by part and by using $\zeta \in \mathcal{D}(A^*)$ as follows:

$$\begin{aligned}\langle \xi(z), A^*\zeta(z)\rangle &= \int_0^1 \xi^\top(z) \left(-\Lambda \frac{d\zeta}{dz} + M^\top \zeta(z)\right) dz \\ &= -\left[\xi^\top(z)\Lambda\zeta(z)\right]_0^1 + \int_0^1 \left(\Lambda \frac{d\xi}{dz} + M\xi(z)\right)^\top \zeta(z) dz \\ &= -\left(v_1\xi_1(0)\zeta_1(0) + v_2\xi_2(1)\zeta_2(1)\right) \\ &\quad + \int_0^1 \left(\Lambda \frac{d\xi}{dz} + M\xi(z)\right)^\top \zeta(z) dz.\end{aligned} \tag{3.2.16}$$

Inserting the expressions from (3.2.15) and (3.2.16) into (A.2.6) yields

$$\langle G\xi(z), B^*\zeta(z)\rangle = v_1\xi_1(0)\zeta_1(0) + v_2\xi_2(1)\zeta_2(1),$$

and by identification we have

$$B^*\zeta = v_2\zeta_2(1), \quad \text{i.e.,} \quad B = (0, \quad v_2\delta_1)^\top. \tag{3.2.17}$$

□

It follows that the following corollary holds.

Corollary 3.2.2. *For $\xi(0) = \xi_0 \in Z$, the abstract Cauchy system (L, G) associated with the initial condition ξ_0 admits a unique solution $\xi \in C^0([0, +\infty), Z) \cap C^1([0, +\infty), X)$. Furthermore,*

$$\xi(t) = \mathbb{T}(t)\xi_0 + \int_0^t \mathbb{T}(t-\tau)Bu(\tau)d\tau, \quad \forall t \in [0, +\infty).$$

Remark 3.2.2. *The admissibility of the unbounded control operator ensures obtaining solutions $\xi(\cdot)$ of square integrable for any initial condition ξ_0 in the state space X. This property is also useful for studying the controllability of infinite dimensional systems.*

Let us consider the output equation

$$y = C\xi = \xi_1(1), \tag{3.2.18}$$

with $C \in \mathcal{L}(X_1, \mathbb{R})$ the observation operator. Let us show that this operator is an admissible operator in the sense of the following definition [Tucsnak and Weiss, 2009, Definition 4.3.1]. For that, we start by introducing the *state-output* operator $\Psi_\tau \in \mathcal{L}(X_1, L^2([0, \infty), Y))$ defined by:

$$(\Psi_\tau \xi_0)(t) = \begin{cases} C\mathbb{T}_\tau \xi_0, & \forall t \in [0, \tau], \\ 0, & \forall t > \tau \end{cases} \tag{3.2.19}$$

Then the following definition is considered.

Definition 3.2.1 (Admissible observation operator)**.** *C is an admissible observation operator for $\mathbb{T}(t)$, if for some $\tau > 0$ the state-output operator Ψ_τ has a continuous extension to X; i.e., for $\tau > 0$ there exists a constant $k > 0$ such that*

$$\int_0^\tau \|C\mathbb{T}(t)\xi_0\|_Y^2 \, dt \le k \|\xi_0\|_X^2, \quad \forall \xi_0 \in \mathcal{D}(A).$$

Proposition 3.2.3. *The output operator of the heat exchanger defined by (3.2.18) is an admissible observation operator.*

Proof. It suffices to prove the result when $\tilde{M} = 0$, and to extend it in the case where $\tilde{M} \neq 0$, taking advantage of the fact that $\tilde{M}$ is bounded and positive.

Let us consider

$$\begin{aligned}\frac{\partial \xi_1}{\partial t} &= -v_1 \frac{\partial \xi_1}{\partial z} - \alpha_1 \xi_1(z,t) \\ \frac{\partial \xi_2}{\partial t} &= v_2 \frac{\partial \xi_2}{\partial z} - \alpha_2 \xi_2(z,t).\end{aligned} \tag{3.2.20}$$

By multiplying the first equation of (3.2.20) by $\xi_1(z,t)$ and the second by $\xi_2(z,t)$, and by integrating the expressions obtained on $[0,T] \times [0,1]$ we have:

$$\begin{aligned}v_1 \int_0^T \xi_1^2(1,t)dt &= \int_0^1 \xi_1^2(z,0)dz - \int_0^1 \xi_1^2(z,T)dz \\ &\quad - 2\alpha_1 \int_0^T \int_0^1 \xi_1^2(z,t)dzdt \\ &\leq \|\xi_{10}(z)\|_{L^2(0,1)}, \\ v_2 \int_0^T \xi_1^2(0,t)dt &= \int_0^1 \xi_2^2(z,0)dz - \int_0^1 \xi_2^2(z,T)dz \\ &\quad - 2\alpha_2 \int_0^T \int_0^1 \xi_2^2(z,t)dzdt \\ &\leq \|\xi_{20}(z)\|_{L^2(0,1)}.\end{aligned}$$

Considering the C_0-semigroup $\tilde{\mathbb{T}}(t)$ and the output equation (3.2.18), we get:

$$\int_0^T \left\| C\tilde{\mathbb{T}}(t)\xi_0 \right\|^2 dt \leq k \|\xi_{10}\|^2_{L^2(0,1)}, \quad \forall\, k > 0.$$

With the Definition 3.2.1 we have the admissibility of C for the semigroup $\tilde{\mathbb{T}}(t)$ generated by $\tilde{A}$. $\tilde{M}$ being positive and bounded operator, by additive disturbance with $\tilde{A}$ will not make C inadmissible[a]. So for all $t \in [0,T]$, C is admissible for $\mathbb{T}(t)$. □

[a] The same argument is used by [Prieur and Winkin, 2018] in the context of the of the admissibility of control operator for the linear hyperbolic systems.

Remark 3.2.3. *The admissibility of the unbounded observation operator ensures obtaining an output function $y(\cdot)$ L^2-integrable for any initial condition in the state space X. This property is useful for studying the observability of infinite dimensional systems.*

In order to guarantee the regularity of the control-observation system, we shall use a Yosida approximation of the output operator C [Weiss, 1994, Dehaye and Winkin, 2016]:

$$C_\Lambda \xi = \lim_{\lambda \to +\infty} \lambda C (\lambda I - A)^{-1} \xi = \xi_1(1). \tag{3.2.21}$$

The advantage of using C_Λ is due to the fact that unlike C it is bounded. Indeed, as the observation is made on ξ_1, we can consider:

$$\lambda \left(\lambda I - A_{11}\right)^{-1} \xi_1(z) = \frac{\lambda}{v_1} \int_0^1 e^{-\frac{\lambda+\alpha_1}{v_1}(1-z)} \xi_1(z) dz.$$

By integrating by part on $[0,1]$ we have:

$$\begin{aligned}
\lambda \left(\lambda I - A_{11}\right)^{-1} \xi_1(z) &= \frac{\lambda}{v_1} \left[\frac{v_1}{\lambda+\alpha_1} e^{-\frac{\lambda+\alpha_1}{v_1}(1-z)} \xi_1(z) \right]_0^1 \\
&\quad - \frac{\lambda}{v_1} \int_0^1 \frac{v_1}{\lambda+\alpha_1} e^{-\frac{\lambda+\alpha_1}{v_1}(1-z)} \frac{d\xi_1}{dz} dz \\
&= \frac{\lambda}{\lambda+v_1} \left[\xi_1(1) - e^{-\frac{\lambda+\alpha_1}{v_1}} \xi_1(0) \right. \\
&\quad \left. - \int_0^1 e^{-\frac{\lambda+\alpha_1}{v_1}(1-z)} \frac{d\xi_1}{dz} dz \right].
\end{aligned}$$

The last term of the previous equation can be bounded by Cauchy-Schwartz inequality as follows:

$$\begin{aligned}
\left| \int_0^1 e^{-\frac{\lambda+\alpha_1}{v_1}(1-z)} \frac{d\xi_1}{dz} dz \right| &\leq \left(\int_0^1 e^{-\frac{\lambda+\alpha_1}{v_1}(1-z)} dz \right)^{1/2} \left\| \frac{d\xi_1}{dz} \right\|_{L^2(0,1)} \\
&= \sqrt{\frac{v_1}{\lambda+\alpha_1} \left(1 - e^{-\frac{\lambda+\alpha_1}{v_1}} \right)} \left\| \frac{d\xi_1}{dz} \right\|_{L^2(0,1)}.
\end{aligned}$$

At the same time, one can verify that:

$$\lim_{\lambda \to +\infty} \lambda \left(\lambda I - A_{11}\right)^{-1} \xi_1(z) = \xi_1(1).$$

Theorem 3.2.2. *The system (3.1.1)-(3.1.2) associated with (3.2.18) constitutes a regular system in the sense of [Tucsnak and Weiss, 2014], and is written in the state space X as follows:*

$$\left\{ \begin{array}{ll} \dfrac{d\xi}{dt} = A\xi(t) + Bu(t), & \forall\, t \in [0, +\infty) \\ y(t) = C_\Lambda \xi(t). & \end{array} \right. \tag{3.2.22}$$

According to the theory presented in [Weiss, 1994, Tucsnak and Weiss, 2014] and the Theorem 3.2.2, we obtain the following results on the representation of the input-output transfer function. This proposition turns out to be very useful both from a theoretical and a computational point of view (see Chapter 5).

Proposition 3.2.4. *The transfer function of heat exchanger model is given by*

$$g_{12}(s) = C_\Lambda(sI - A)^{-1}B = \frac{2\beta_1 \sinh(\beta(s))}{2\beta(s)\cosh(\beta(s)) + \gamma(s)\sinh(\beta(s))}. \tag{3.2.23}$$

Furthermore, $g_{12}(s)$ is analytic and bounded in the entire half-plane $\Re e(s) \geq \omega_0 > \omega(A)$, where $\omega(A)$ denotes the decay rate of the C_0-semigroup generated by A.

The calculation of $g_{12}(s)$ is performed in Appendix B.

3.3 Controllability and observability

Systems analysis consists of a set of concepts allowing a classification of systems with a view to a better understanding of their operation. In the case of distributed parameter systems, many concepts have been introduced and explored, allowing better control strategies for these systems. Among these concepts, those of controllability and observability have generated a large literature, mainly in the case of linear systems [El Jai and Pritchard, 1988], [Amouroux et al., 1994], [Winkin et al., 2000], [Zerrik et al., 2004], [El Jai and Hamzaoui, 2009].
Controllability and observability issues related to the heat exchanger model considered in this work will therefore be addressed in this section.

3.3.1 Controllability

Recall that a system is said to be controllable if, for any desired state ξ_d in a state space X, there exists an input u that allows the system to be transferred from its initial state ξ_0 at time $t = 0$ to the desired state ξ_d at final time $t = T$. Several works relating to this notion have been developed. There are two important points regarding controllability of distributed parameter systems. The first concerns a characterization of exact controllability by an inequality constructed from the adjoint problem. The second, a sufficient condition for exact controllability; it concerns the compactness of the input-state operator [Curtain and Zwart, 1995, Chapter 4], [Tucsnak and Weiss, 2009, Chapter 11]. An approximate concept, weak or approximate controllability, introduced for systems that are not exactly controllable, is better suited to the case of distributed parameter systems. Approximate controllability consists in choosing a control input that transfers the system to a state close to the desired state. This can be expressed mathematically by the fact that the image of the control operator is dense everywhere in the state space. Although this is the most adequate way for PDE models, we give in what follows a (theoretical) result which characterizes the exact controllability of the model (3.1.1)-(3.1.2). Before that, let us specify that, generally to study the controllability of a system

we go through the dual problem. The reason is that using the direct problem amounts to looking for solutions in a big space which is a completion of the space X, and inconvenient to manipulate.
For all $\zeta = (\zeta_1\ \zeta_2)^\top \in \mathcal{D}(A^*)$ we define the adjoint problem of (3.1.1)-(3.1.2) by:

$$\begin{cases} \dot{\zeta}(t) = A^*\zeta(t), & t \in [0, T] \\ \zeta(0) = \zeta_0. \end{cases} \tag{3.3.1}$$

The proof of the following proposition is based on the multiplier method [Coron, 2007, Chapter 2].

Proposition 3.3.1. *The heat exchanger system, i.e.* $\Sigma(A, B, -)$ *is exactly controllable.*

Proof. As the control variable $u(t)$ acts directly on the state $\xi_2(z,t)$, then on $\xi_1(z,t)$ via the coupling matrix M, the observability inequality can be constructed from the transport equation of the state $\zeta_2(z,t)$. For that, let's consider

$$\begin{cases} \partial_t \zeta_2(z,t) = -v_2 \partial_z \zeta_2(z,t) \\ \quad \zeta_2(0,t) = 0, \quad t \in [0, T] \\ \quad \zeta_2(z,0) = \zeta_{20}(z); \quad z \in [0, L], \end{cases} \tag{3.3.2}$$

and let $e(t) = \int_0^L \zeta_2^2(z,t)dz$ be partial energy. We multiply the first equation of (3.3.2) by $\zeta_2(z,t)$, and integrate on $[0, L]$, we get

$$\begin{aligned} \int_0^L \zeta_2(z,t)\frac{\partial \zeta_2}{\partial t} dz &= -v_2 \int_0^L \zeta_2(z,t) \frac{\partial \zeta_2}{\partial z} dz \\ \frac{d}{dt}\int_0^L \zeta_2^2(z,t)dz &= -v_2 \left[\zeta_2^2(z,t)\right]_0^L \\ &= -v_2 \zeta_2^2(L,t) \\ e(t) &= e(0) - v_2 \int_0^t \zeta_2^2(L,\tau)d\tau, \quad \text{with } \tau \in [0,t]. \end{aligned} \tag{3.3.3}$$

Now let us multiply the first equation of (3.3.2) by $(z-L)\zeta_2(z,t)$ and integrate on $[0, L]$, we obtain

$$\begin{aligned} v_2 \int_0^L (z-L)\zeta_2(z,t)\frac{\partial \zeta_2}{\partial z} dz &= -\int_0^L (z-L)\zeta_2(z,t)\frac{\partial \zeta_2}{\partial t} dz \\ v_2 \left[(z-L)\zeta_2^2(z,t)\right]_0^L - v_2 \int_0^L \zeta_2^2(z,t)dz &= -\frac{d}{dt}\int_0^L (z-L)\zeta_2^2(z,t)dz \\ e(t) &= \frac{1}{v_2}\frac{d}{dt}\int_0^L (z-L)\zeta_2^2(z,t)dz \\ \int_0^T e(t)dt \le \frac{L}{v_2}\int_0^L \zeta_2(z,0)dz &= \frac{L}{v_2} e(0). \end{aligned} \tag{3.3.4}$$

From (3.3.3) and (3.3.4), we get observability inequality:

$$\int_0^T \|B^*\mathbb{T}^*(t)\zeta\|^2 \, dt \geq C_T \, \|\zeta_2(\,\cdot\,,t)\|_{L^2(0,L)}, \tag{3.3.5}$$

with C_T given by:

$$C_T = \frac{1}{v_2}\sqrt{\frac{v_2 T - L}{T}}. \tag{3.3.6}$$

Since B and B^* are admissible, then the asumptions of [Tucsnak and Weiss, 2009, Theorem 7.4.1] are satisfied, i.e (A^*, B^*) is exactly observable. Thus by duality the pair (A, B) is exactly controllable for any time sufficiently large such that $T > \frac{L}{v_2}$. □

Remark 3.3.1. *As previously said, the operator B is not full rank. This situation may suggest that the number of boundary controls (which typically corresponds to one) is not sufficient for the exact controllability of the system. But let us note that by the structure of the coupling matrix M we know how to reach any target, provided that the time $T > 0$ is sufficiently large.*

3.3.2 Observability

In system theory, knowing the state of the system is very important for the implementation of controllers. Unlike lumped-parameter systems, systems modelled by PDEs do not offer the possibility of measuring the complete state of the system. The problem that we address in this section is that of the reconstruction (or observability) of the state of the system, when the measurement is made through localized sensor in $z = L$ for $\xi_1(z, t)$.
Let us start by giving the following observability inequality.

Proposition 3.3.2. *The heat exchanger system, i.e. $\Sigma(A, -, C)$ is exactly observable.*

Proof. By using the direct problem $\Sigma(A, -, C)$ and by a similar development to that of Proposition 3.3.1 we obtain:

$$\int_0^T \|C\xi(t)\|_Y^2 \, dt \geq C_T \, \|\xi_{10}(\,\cdot\,)\|^2_{L^2(0,L)}, \tag{3.3.7}$$

where C_T is a positive constant. With the admissibility of $C \in \mathcal{L}(X_1, Y)$ and as $\mathbb{T}(t)$ is invertible, then by [Tucsnak and Weiss, 2009, Remark 6.1.2], (A, C) is exactly observable on X at time $T > 0$. □

3.4 Conclusion

In this chapter we have studied the useful properties of control theory for a counterflow heat exchanger. This system is known to be poorly conditioned. The goal was there to try to understand the impact on certain dynamic properties in particular: positivity, stability, controllability and observability, so as to identify possible problems which could be compensated by control laws. For that, firstly we showed that the uncontrolled system generates an exponentially stable positive semigroup. Particular attention has been paid to spectral analysis; we were able to show that despite the poor conditioning of the system, stability in simulation was always guaranteed. Next, we have shown that the system is well posed when it is controlled and observed through unbounded operators. With the admissibility of control and observation operators, by the multiplier method we have proved exact controllability and observability.

4

Thermodynamics analysis

Contents

4.1 Introduction

4.1.1 State of the art

HEAT exchangers are one of the most important equipment in industry, where irreversible thermodynamic processes and transformations occur. The analysis of irreversibilities that lead to the degradation of energy, has been made by several authors. For example, [Bejan, 1977] analyzes a counterflow heat exchanger for gas-to-gas application. He introduces a dimensionless number, called "number of entropy production", to quantify the irreversibilities. In [Bejan, 1996, Chapter 5], [Mohamed, 2005], [Guo et al., 2010] authors studied chronologically in various situations the entropy production in balanced and imbalanced counterflow heat exchangers, and whether viscous dissipation phenomena are considered or not. They show that in the absence of viscous dissipation phenomena, balanced exchangers, i.e. whose fluids have the same thermal heat capacity flow rates, correspond to the equipartition of the transferred flow, but not of the thermodynamic driving force or of the entropy production. Recall that according to Ilya Prigogine, a system or process that achieves equipartition over time is a so-called steady-state process, and therefore appears to minimize the overall entropy production [Prigogine, 1947]. Systems where entropy production is thus equidistributed are not feasible in practice. Their interest is that they serve as reference cases, that we seek to approach.

Regarding the problem of heat dissipation in balanced or symmetrical counterflow heat exchangers, some authors propose "entransy" as an alternative quantity to entropy production. The concept of entransy is recent. It was introduced by [Guo et al., 2003, Guo et al., 2007] by analogy between electrical conduction and thermal conduction. In this analogy the heat flux and the temperature correspond to the electric current and to the electric tension, respectively. Thus they deduce a thermal quantity that they call "entransy" equivalent to the electrical potential energy applications. Nowadays, this concept is widely used in energy. The main interest of using entransy as an irreversibility criterion comes from some properties of variational nature (see in [Guo et al., 2007], [Mingtian, 2011a], the entransy minimum and maximum dissipation principle). In the context of heat exchangers, it was used by [Mingtian, 2011b] and [Shaojun et al., 2016] for a study of minimum energy dissipation. The concept of entransy and entransy dissipation is still in development and is not unanimous among scientists. Although its use in the context of optimization works nevertheless seems promising (see e.g. [Kostic, 2017]), the concept of entransy remains very controversial [Grazzini et al., 2013, Bejan, 2014, Herwig, 2014, Sekulic et al., 2015]. For [Bejan, 2014] for example, the entransy is not a physical quantity and the entransy dissipation would only be a multiple of the entropy production. Thus, to study the energy dissipation and the stability of systems in the thermodynamic sense, it is wise to use thermodynamic functions accepted by a wide panel of researchers.

Regarding optimization problem of balanced or symmetrical counterflow heat exchangers, [Nummedal and Kjelstrup, 2001] and [Johannessen et al., 2002] have studied the produced entropy minimization problem using the Euler-Lagrange method which is a classical approach in variational calculus. From these studies it appears that the solution to the minimization problem is that at imposed heat transfer flux, the thermodynamic driving force must be constant and equal to its average. Other works concerning the minimization of the irreversibility of counterflow heat exchangers focus on their architecture (the ratio of the two-channel spacings, the total heat transfer area between the two streams); see [Ordóñez and Bejan, 2000].

4.1.2 Contributions and organization of the chapter

In view of the work cited above, an important factor in the quantification of irreversibilities in heat exchangers is the "imbalance factor". It is the ratio between the thermal capacity flow (also called calorific flow rate) of the hot and cold fluids. This factor is well understood when the dynamics of the heat exchanger are described by a lumped parameter model, but remains poorly

known in an infinite dimensional context. Indeed, the thermal capacity flow of fluids do not appear (explicitly) in distributed parameter models. This limits the application of the established results to the type of models used in this work.
In this chapter we perform a thermodynamic analysis of an infinite dimensional heat exchanger model. First (see Section 4.2), we show that the thermodynamic analysis of the heat exchanger from the well-posedness condition (see equation (3.2.4)) is equivalent to that made from the thermal capacity flows rates of the fluids by [Bejan, 1977, Balaji et al., 2021, Ogunedo, 2020]. In other words, the ratio $\frac{(\beta_j=\alpha_j/v_j)_{\min}}{(\beta_j=\alpha_j/v_j)_{\max}}$ defines the imbalance factor of a counterflow heat exchanger described by PDEs. Moroever, unlike the works that only use the number of entropy production as a criterion for measuring irreversibility, we also use the entropy production. This gives a practical meaning to the study of existence and uniqueness of solution carried out in Chapter 3. Next in the Section 4.3, we investigate and establish the passivity of the heat exchanger by using entropy as the storage function and entropy production as the dissipation function. Finally in the Section 4.4, we study the dynamic stability of the heat exchanger in a thermodynamic sense, i.e by studying the extremum of a state function. Recall that the existing results in the literature on this subject (see [Besson et al., 2006] and Chapter 3) use Lyapunov functions constructed from the square of the intensive quantities, typically temperatures. Such functions are not energy functions in the physical or thermodynamic sense. In this chapter we use as Lyapunov function the thermodynamic availability, which is a function that depends on the entropy as well as the irreversible entropy production of the system [Alonso and Ydstie, 2001]. The major interest of resorting to this kind of functions or even to certain thermodynamic potentials (associated with the second law of thermodynamic) lies in the fact that they make it possible to find the stability conditions of the non-equilibrium systems under the effect of fluctuations of the thermodynamic quantities: temperature, pressures and concentrations, due in particular to the random movements of the molecules. This is the junction point between *thermodynamic stability* and *dynamic stability* (otherwise called stability of small oscillations), where the use of Lyapunov functions find their importance.

4.2 Steady state analysis

Regardless of the nature of the fluid, according to the relationships between the thermal capacity flows rate of the fluids, it is possible to describe three cases of heat transfer (see [Balaji et al., 2021], [Ogunedo, 2020]): the first two $q_{c_1} \neq q_{c_2}$ correspond to the case where heating and cooling vary differently along the heat exchanger, and the third $q_{c_1} = q_{c_2}$ to the case where they vary in the same way. So it is clear that in these three cases the distribution of entropy production $\sigma(z)$ along the exchanger is not the same. Unfortunately the thermal capacity flow rates of the fluids do not appear explicitly in the model (1.3.9)-(1.3.10). In what follows, considering the stationary problem corresponding to the problem (1.3.9)-(1.3.10), we are going to show that it is possible to obtain the same conclusions by exploiting the condition of existence and uniqueness of solutions obtained in the previous chapter.

4.2.1 Equilibrium profiles analysis

The stationary problem of (1.3.9)-(1.3.10) is defined by

$$\begin{cases} \Lambda \dfrac{d\overline{x}}{dz} + M\overline{x}(z) = 0 \\ x_1(0) = x_{1,in},\ x_2(L) = x_{2,in}. \end{cases} \tag{4.2.1}$$

For $z \in [0, L]$ we have the following expressions for the components $\overline{x}_1(z)$ and $\overline{x}_2(z)$:

$$\begin{aligned} \overline{x}_1(z) = & \frac{1}{\beta_1 - \beta_2\, \mathrm{e}^{(\beta_2-\beta_1)L}} \left(\beta_1\, \mathrm{e}^{(\beta_2-\beta_1)z} - \beta_2\, \mathrm{e}^{(\beta_2-\beta_1)L}\right) \\ & x_1(0) + \frac{\beta_1}{\beta_1 - \beta_2\, \mathrm{e}^{(\beta_2-\beta_1)L}} \left(1 - \mathrm{e}^{(\beta_2-\beta_1)z}\right) x_2(L), \\ \overline{x}_2(z) = & \frac{\beta_2}{\beta_1 - \beta_2\, \mathrm{e}^{(\beta_2-\beta_1)L}} \left(\mathrm{e}^{(\beta_2-\beta_1)z} - \mathrm{e}^{(\beta_2-\beta_1)L}\right) x_1(0) \\ & + \frac{1}{\beta_1 - \beta_2\, \mathrm{e}^{(\beta_2-\beta_1)L}} \left(\beta_1 - \beta_2\, \mathrm{e}^{(\beta_2-\beta_1)z}\right) x_2(L), \end{aligned} \tag{4.2.2}$$

with $\beta_1 = \frac{\alpha_1}{v_1}$, and $\beta_2 = \frac{\alpha_2}{v_2}$.
We saw in the previous chapter by applying the results of the article by [Prieur and Winkin, 2018] that the standard process engineering model (1.3.9)-(1.3.10) is well-posed if and only if:

$$\beta_1 \neq \beta_2. \tag{4.2.3}$$

We are going to show that this condition on the physical parameters of the system plays an important role in understanding the thermodynamics of heat exchangers, and that is equivalent to $\dot{q}_{c_1} = \dot{m}_1 c_{p_1} \neq \dot{q}_{c_2} = \dot{m}_2 c_{p_2}$ the inequality found in several papers: [Bejan and Kraus, 2003, Chapter 11], [Forsberg, 2021], [Wilmot and Christians, 1983], [Knissel and Peußner, 2018]. Three cases can be studied. If $\beta_1 > \beta_2$; this means that the amount of heat that can be transmitted is limited by fluid 1. It is this fluid that undergoes the greatest temperature variation along the heat exchanger. Its outlet temperature $x_{1,out}$ is close to the inlet temperature $x_{2,in}$, as shown in figure 4.1. If $\beta_2 > \beta_1$ we have the reverse scenario, see figure 4.2. From figures 4.1 and 4.2 we note also that the outlet of the cold fluid is greater than that of the hot fluid; this is one of the main advantages of the counterflow configuration. In contrast, the thermal pinch $\Delta \mathrm{T}_{\min}$ is located at the outlet of the hot or cold fluid, depending

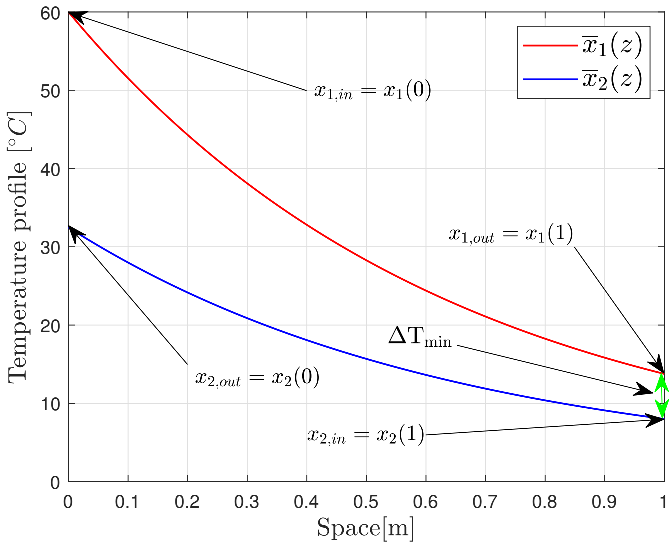


Figure 4.1: Equilibrium profiles for: $x_{1,in} = 60°\mathrm{C}$, and $x_{2,in} = 10°\mathrm{C}$, in the case $\beta_1 > \beta_2$.

on whether $\beta_1 > \beta_2$ or $\beta_2 > \beta_1$, respectively.
Note that in general for $\beta_1 \neq \beta_2$, the temperature difference along the heat exchanger is obtained from the expressions of the trajectories (4.2.2), and is equal to:

$$\Delta \mathrm{T}(z) = \frac{(\beta_1 - \beta_2)\, \mathrm{e}^{(\beta_2-\beta_1)z}}{\beta_1 - \beta_2\, \mathrm{e}^{(\beta_2-\beta_1)L}} \left(x_1(0) - x_2(L)\right). \tag{4.2.4}$$

If, moreover, $L \to \infty$, we have: $\left.(\Delta \mathrm{T}(z))\right|_{z=L} (= x_1(0) - x_2(L)) \to 0$. In this case, for $\beta_1 > \beta_2$ are given by:

$$\begin{cases} x_1(L) = x_2(L) \\ x_2(0) = \dfrac{\beta_2}{\beta_1} x_1(0) + \dfrac{\beta_1 - \beta_2}{\beta_1} x_2(L), \end{cases} \tag{4.2.5}$$

and when $\beta_2 > \beta_1$:

$$\begin{cases} x_1(L) = \dfrac{\beta_2 - \beta_1}{\beta_2} x_1(0) + \dfrac{\beta_1}{\beta_2} x_2(L) \\ x_2(0) = x_1(0). \end{cases} \tag{4.2.6}$$

It is noted that the hot (cold) fluid outlet is at the same temperature as the cold (hot) fluid inlet, by the fact that the temperature difference tends to infinity at

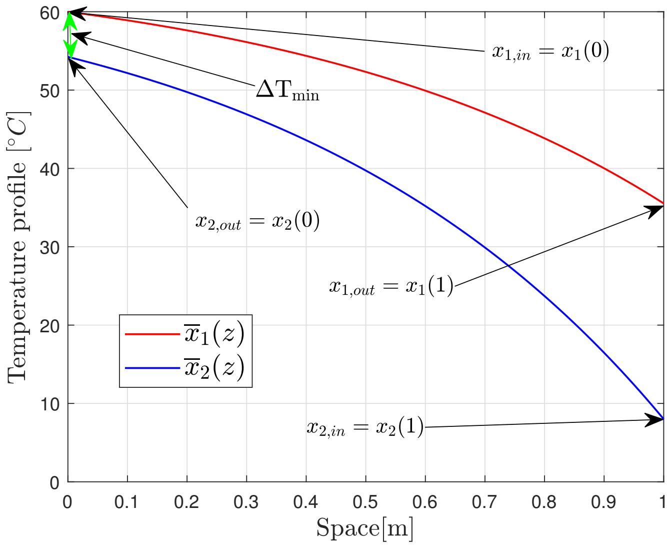
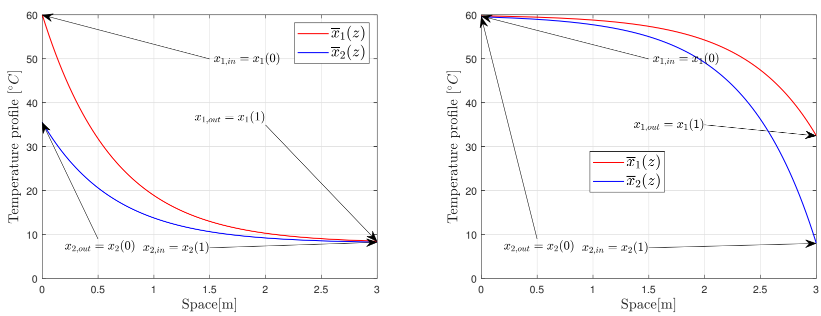


Figure 4.2: Equilibrium profiles for: $x_{1,in} = 60$°C, and $x_{2,in} = 10$°C, in the case $\beta_2 > \beta_1$.

the same time as $L \to \infty$; see Figures 4.3a and 4.3b for $L = 3m$. This shows that *the lower the thermal pinch, the higher the transferred thermal heat*. It can be deduced that the thermal pinch is an efficiency factor for an imbalanced exchanger.



(a) Equilibrium profiles: $\beta_1 > \beta_2$, for $L = 3m$.

(b) Equilibrium profiles: $\beta_2 > \beta_1$, for $L = 3m$.

Figure 4.3: Equilibrium profiles for inlet temperatures $x_{1,in} = 60°C$, $x_{2,in} = 8°C$, and parameter values: $v_1 = 0.9749m/s, v_2 = 1.6513m/s, \alpha_1 = 3.2389s^{-1}$ and $\alpha_2 = 2.903s^{-1}$.

Now let us see what happens to the condition (4.2.3) when it is not verified.

For $\beta_1 = \beta_2 = \beta$ the problem defined by (4.2.1) becomes singular. In that case, the temperature profiles are calculated using a series expansion (of first order) of $e^{\Lambda^{-1}\tilde{M}z} = I + \Lambda^{-1}\tilde{M}z$. Then, we obtain:

$$\begin{cases} \overline{x}_1(z) = \left(1 - \beta z + \dfrac{\beta^2 L}{1+\beta L} z\right) x_1(0) + \dfrac{\beta z}{1+\beta L} x_2(L), \\ \overline{x}_2(z) = \beta \left(\dfrac{L(1+\beta z)}{1+\beta L} - z\right) x_1(0) + \dfrac{1+\beta z}{1+\beta L} x_2(L). \end{cases} \tag{4.2.7}$$

Figure 4.4a shows that the temperature difference remains uniform along the heat exchanger by the fact that, we have symmetrical flows. In other words, the counterflow heat exchanger works particularly well when $\beta_1 = \beta_2$ because warming varies in the same way as cooling (absence of pinch thermal; indeed $\Delta\mathrm{T}(z) = \frac{1}{1+\beta L}(x_1(0) - x_2(L)$ is constant). However, even if the length is infinite the heat exchanger is still irreversible; in that case the trajectories of $x_1(z)$ and $x_2(z)$ are collinear ($x_1(L) = x_2(L)$, $x_2(0) = x_1(0)$).

Remark 4.2.1.

- *The (unrealistic) infinite length counterflow heat exchanger where $\beta_1 = \beta_2$ is reversible: the outlet of the cold fluid is at the same temperature as the inlet of the hot fluid simply because the temperature difference $\Delta\mathrm{T}$ which controls the thermal exchanged flux tends towards zero at the same time that L tends towards infinity. This is the condition of reversible exchange in thermodynamics.*

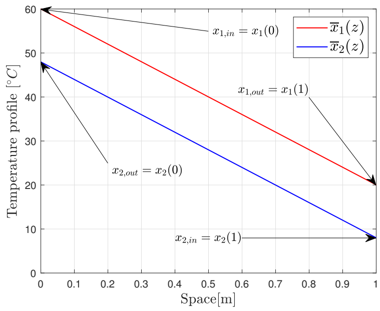
- *The counterflow heat exchanger even of infinite length is irreversible if $\beta_1 \neq \beta_2$; the ratio $(\beta_j)_{\min}/(\beta_j)_{\max}$ characterizes the irreversibility. Thus, we have just given meaning to the condition of existence and uniqueness of solution (4.2.3) of the standard model in process engineering (1.3.9)-(1.3.10).*

Remark 4.2.2. *Some authors talk about balanced counterflow heat exchanger to refer to case $\beta_1 = \beta_2$, and imbalanced to refer to case $\beta_1 \neq \beta_2$ [Bejan, 1996, Chapter 5], [Mohamed, 2005], [Guo et al., 2010]. Although they don't say it, this can be explained by the fact that the temperature difference along a counterflow heat exchanger is isothermal ($\Delta\mathrm{T}(z)$ is constant) in the first case, while in the second case it is nonisothermal.*

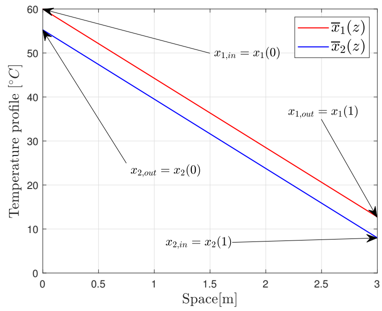
4.2.2 Effectiveness of heat exchanger

To analyze the operation of a heat exchanger, it is often convenient to use the concepts of effectiveness and numbers of heat transfer units, that we introduce below.

We define the *effectiveness ε of an heat exchanger* as the ratio of the thermal power actually exchanged, to that which would be exchanged ideally if we used the total difference between the inlet temperatures of the hot fluids and



(a) Equilibrium profiles: $\beta_1 = \beta_2$, for $L = 1m$.



(b) Equilibrium profiles: $\beta_1 = \beta_2$, for $L = 3m$.

Figure 4.4: Equilibrium profiles for inlet temperatures $x_{1,in} = 60°C$, $x_{2,in} = 8°C$, and parameter values: $v_1 = 0.9749m/s, v_2 = 1.6513m/s, \alpha_1 = 3.2389s^{-1}$ and $\alpha_2 = 2.903s^{-1}$.

cold [Bejan and Kraus, 2003, Chapter 11]. Recall that the thermal power transferable is given by:

$$\begin{aligned} q_{th} &= U_g A_s \int_0^L \frac{(\beta_1 - \beta_2)\,\mathrm{e}^{(\beta_2-\beta_1)z}}{\beta_1 - \beta_2\,\mathrm{e}^{(\beta_2-\beta_1)L}} dz\,(x_1(0) - x_2(L)) \\ &= U_g A_s \frac{\left(e^{(\beta_2-\beta_1)L} - 1\right)}{\beta_2 e^{(\beta_2-\beta_1)L} - \beta_1}\,(x_1(0) - x_2(L))\,. \end{aligned} \tag{4.2.8}$$

The ideally transferable thermal power is obtained in a heat exchanger of infinite length, and it is obtained from the equation (4.2.8):

$$\begin{aligned} q_{th(max)} &= \frac{U_g A_s}{(\beta_j)_{\max}}\,(x_1(0) - x_2(L)) \\ &= \left(\dot{q}_{c_j}\right)_{\min}\,(x_1(0) - x_2(L))\,. \end{aligned} \tag{4.2.9}$$

It is calculated based on $(\beta_j)_{\max}$ or $(\dot{q}_{c_j})_{\min}$, because the product $(\dot{q}_c)\cdot(x_1(0) - x_2(L))$ having to be the same for both fluids, the smallest (largest) temperature variation must necessarily relate to the fluid which has the largest β_j or smaller thermal capacity flow rate. So we have:

$$\varepsilon :\overset{\Delta}{=} \frac{(\dot{q}_{c_j}\Delta\mathrm{T})_{\text{fluid}}}{(\dot{q}_{c_j})_{\min}\,(x_1(0) - x_2(L))} = \frac{\Delta\mathrm{T}_{\text{max fluid}}}{x_1(0) - x_2(L)}, \tag{4.2.10}$$

where $\Delta\mathrm{T}_{\text{max fluid}}$ denotes the inlet-outlet temperature difference of the fluid that has the largest β_j or smallest thermal capacity flow rate. The expression

(4.2.10) is also written:

$$\varepsilon :\stackrel{\Delta}{=} \frac{x_1(0) - x_1(L)}{x_1(0) - x_2(L)}, \text{ for } \beta_1 > \beta_2 \quad \varepsilon :\stackrel{\Delta}{=} \frac{x_2(0) - x_2(L)}{x_1(0) - x_2(L)}, \text{ for } \beta_2 > \beta_1. \tag{4.2.11}$$

Note that the thermal power exchanged is given by

$$q_{ex} = \dot{q}_{c_1}(x_1(0) - x_1(1)) = \dot{q}_{c_2}(x_2(0) - x_2(L)). \tag{4.2.12}$$

Finally, the following dimensionless ratio is called the *number of transfer units*:

$$N_{tu} :\stackrel{\Delta}{=} \frac{A_t U_g}{(\dot{q}_{c_j})_{\max}} = (\beta_j)_{\min} \cdot L. \tag{4.2.13}$$

This parameter provides an indication of the size of the heat exchanger.
The effectiveness of the heat exchanger ε is written as a function of the number of transfer units N_{tu} as follows [Bejan, 1987]:

$$\varepsilon :\stackrel{\Delta}{=} \frac{1 - \exp\left[-(1-\varpi)N_{tu}\right]}{1 - \varpi\exp\left[-(1-\varpi)N_{tu}\right]}, \qquad \text{for } \beta_1 \neq \beta_2, \tag{4.2.14}$$

with ϖ denoting the "imbalance rate" of the heat exchanger:

$$\varpi \stackrel{\Delta}{=} \frac{(\dot{q}_{c_j})_{\min}}{(\dot{q}_{c_j})_{\max}} = \frac{(\beta_j)_{\min}}{(\beta_j)_{\max}}, \tag{4.2.15}$$

and where the smaller of the two thermal capacity flow rates has been placed in the numerator. We obviously have:

$$0 \leq \varpi \leq 1. \tag{4.2.16}$$

When $\beta_1 = \beta_2$, the efficiency as a function of N_{tu} is given by:

$$\varepsilon :\stackrel{\Delta}{=} \frac{N_{tu}}{1 + N_{tu}}, \qquad \varepsilon \in [0, 1[. \tag{4.2.17}$$

The relationship (4.2.14) is shown in Figure 4.5. In particular, we see that the gain in effectiveness of an heat exchanger increases as the number of transfer units is increased (for example, by increasing the length or the exchange area).

Discourse on the Method
The NTU method provides an elegant and quick answer to most of the problems that arise in engineering studies relating to heat exchangers. These are subdivided into two main classes:

1. ***Design issue*** *in which the inlet temperatures and an outlet temperature are imposed, the velocities (mass flow rates) being known.* ***The issue is:*** *select the most appropriate heat exchanger model and find its size,*

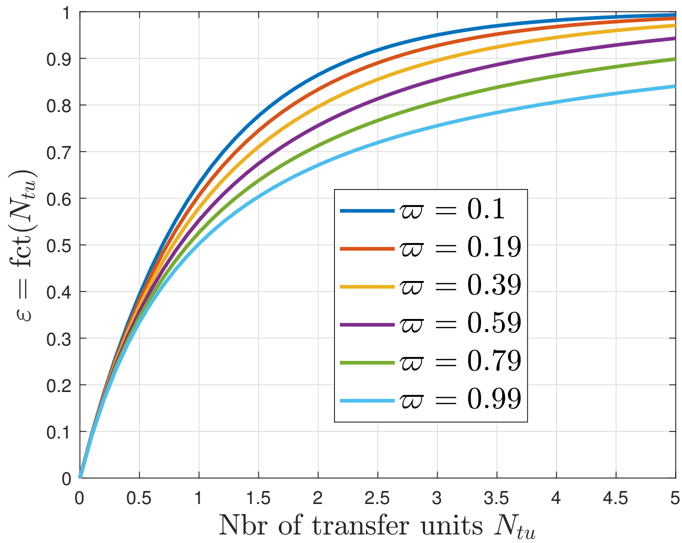


Figure 4.5: Effectiveness $\varepsilon = f(N_{tu})$ of counterflow heat exchanger.

i.e. the length L or the area A_t necessary to obtain the desired outlet temperature. The method to use is to calculate ϖ and ε, then $N_{tu}(\varepsilon)$, from which we get L or A_t. It is in this framework that optimization research most often fits [Nummedal and Kjelstrup, 2001], [Ordóñez and Bejan, 2000], [Johannessen et al., 2002].

2. ***Performance issues** where the data are exchanger model and size, mass flow rates and inlet temperatures. It is then a question of determining the thermal power exchanged q_{ex} and the outlet temperatures. The expressions (4.2.15) and (4.2.13) are used to calculate ϖ and N_{tu}, and we deduce $\varepsilon(N_{tu})$ from expression (4.2.14); the two unknown outlet temperatures are given by (4.2.11), and q_{ex} is obtained from the expression (4.2.12).*

4.2.3 Thermal irreversibility

We have just seen that depending on the conditions on the system parameters, three different behaviors can be identified. We shall now see what this implies about the entropy generated.
At steady state, equation (1.3.4) is written

$$\sigma(z) = A_t U_g \frac{(x_1(z) - x_2(z))^2}{x_1 x_2}. \tag{4.2.18}$$

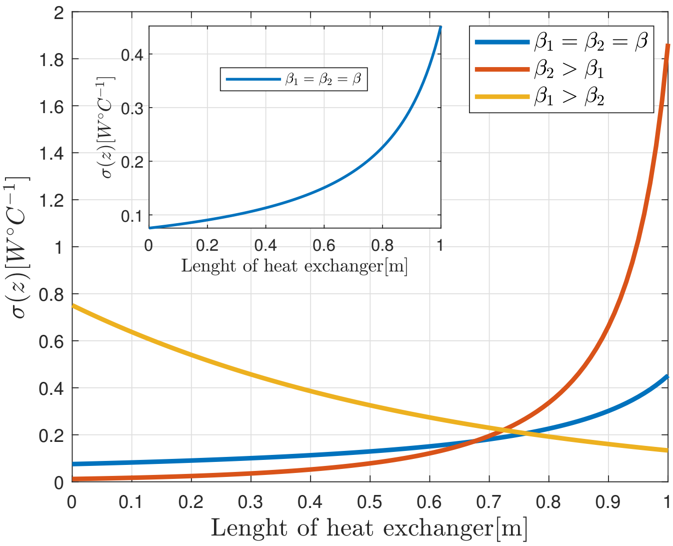


Figure 4.6: Density of entropy generation $\sigma(z)$ of balanced and imbalanced counter-flow heat exchanger for $L = 1m$.

Thus the local entropy production is easily obtained by successively substituting in (4.2.18) the expressions of the trajectories (4.2.2). Figure (4.6) represents the local entropy production profiles $\sigma(z)$ in the cases $\beta_1 \neq \beta_2$ and $\beta_1 = \beta_2$. We observe that this function is positive and convex. It is mainly for this reason that the entropy production can be used in some cases as a Lyapunov function (see e.g. [Favache and Dochain, 2009], [García-Sandoval et al., 2016]). Furthermore, from the Figure 4.6 an important conclusion is that, although the heat flux transferred $q(z)$ is uniform in the case of a balanced heat exchanger ($\beta_1 = \beta_2$), the entropy production along the heat exchanger axis is not. This is due in particular to the fact that the thermodynamic driving force $\left(\frac{1}{x_2(z)} - \frac{1}{x_1(z)}\right)$ is non-uniform. We note also that the balanced counterflow heat exchanger corresponds to a minimum entropy production regime.

In Figure 4.7 we also see that the irreversibility decreases when $L \mapsto +\infty$. Thus, related to the analysis of temperature profiles we conclude that, to harness the full potential of heat recovery or heat transfer it is essential to carry out an analysis upstream in order to find the right balance between "length (or heat transfer surface area)" and "thermal pinch", and ultimately to reduce investment costs.

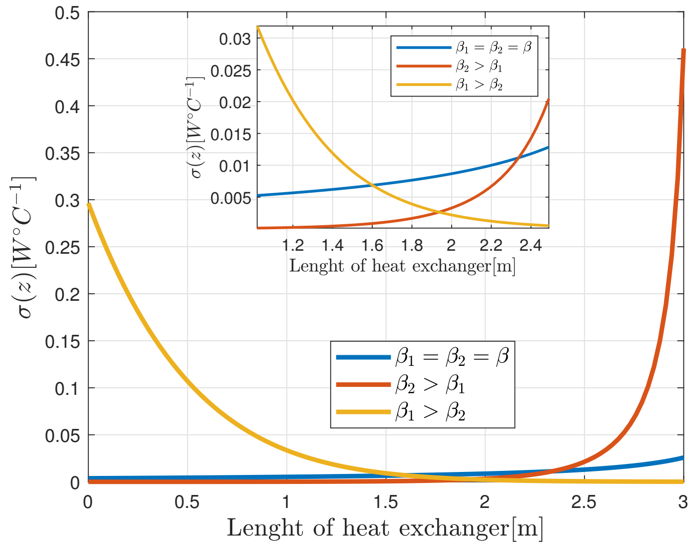


Figure 4.7: Local entropy production $\sigma(z)$ of balanced and imbalanced counterflow heat exchanger for $L = 3m$.

Remark 4.2.3. *From Remark 4.2.1 and from the second law of thermodynamics it follows that:*

- *A balanced heat exchanger ($\beta_1 = \beta_2$) of infinite length is in an equilibrium state characterized by maximum entropy and zero entropy production ($x_1(z) = x_2(z)$ implies that $\sigma(z) = A_t U_g \frac{(x_1(z)-x_2(z))^2}{x_1 x_2} = 0$).*
- *An imbalanced heat exchanger ($\beta_1 \neq \beta_2$) is in an non-equilibrium state. But it is locally in an equilibrium state, precisely at the thermal pinch point when the length is infinite.*

Similarly, the global entropy production Σ_s can be expressed as a function of the temperatures x_1 and x_2. Indeed, knowing that $ds_j = \rho_j V_j c_{p_j} \frac{1}{x_j} dx_j$, equation (1.3.4) can also be written in steady state:

$$\begin{aligned} \sigma(z) &= v_1 \frac{ds_1}{dz} - v_2 \frac{ds_2}{dz} \\ &= A_t U_g \left(\frac{1}{\beta_1} \frac{dx_1}{dz} - \frac{1}{\beta_2} \frac{dx_2}{dz} \right). \end{aligned} \tag{4.2.19}$$

Thus, from (4.2.19) the global entropy production becomes:

$$\begin{aligned}\Sigma_s &= \int_0^L \sigma(z)dz = A_s U_g \left(\frac{1}{\beta_1} \ln x_1(z)|_0^L - \frac{1}{\beta_2} \ln x_2(z)|_0^L \right) \\ &= A_s U_g \left[\frac{1}{\beta_1} \ln \left(\frac{x_1(L)}{x_1(0)} \right) + \frac{1}{\beta_2} \ln \left(\frac{x_2(0)}{x_2(L)} \right) \right].\end{aligned} \tag{4.2.20}$$

This expression is positive when β_1 and β_2 are positive. Using the effectiveness given by (4.2.10), we introduce a dimensionless number called "entropy generation number" introduced by [Bejan, 1977], and comparing the entropy production of a "real" situation to that of an "ideal" situation, which is written in our case:

$$\begin{aligned}N_s &= \frac{\Sigma_s}{(\dot{q}_{c_j})_{\max}} = (\beta_j)_{\min} \left[\frac{1}{\beta_1} \ln \left(\frac{x_1(L)}{x_1(0)} \right) + \frac{1}{\beta_2} \ln \left(\frac{x_2(0)}{x_2(L)} \right) \right] \\ &= (\beta_j)_{\min} \left\{ \frac{1}{\beta_1} \ln \left[1 - \varpi\varepsilon \left(1 - \frac{x_2(L)}{x_1(0)} \right) \right] + \frac{1}{\beta_2} \ln \left[1 + \varepsilon \left(\frac{x_1(0)}{x_2(L)} - 1 \right) \right] \right\} \\ &= (\beta_j)_{\min} \left\{ \frac{1}{\beta_1} \ln \left[1 - \varpi\varepsilon \left(1 - \frac{1}{\alpha} \right) \right] + \frac{1}{\beta_2} \ln \left[1 + \varepsilon(\alpha - 1) \right] \right\} \\ &= (\beta_j)_{\min} \ln \left[\left(1 - \varpi\varepsilon \left(1 - \frac{1}{\alpha} \right) \right)^{1/\beta_1} (1 + \varepsilon(\alpha - 1))^{1/\beta_2} \right],\end{aligned} \tag{4.2.21}$$

with $\alpha = \frac{x_1(0)}{x_2(L)}$ the ratio of inlet temperatures. For balanced counterflow heat exchanger ($\beta_1 = \beta_2 = \beta$), (4.2.21) becomes:

$$N_s = \ln \left[1 + \varepsilon \frac{(1+\varepsilon)(\alpha - 1)^2}{\alpha} \right]. \tag{4.2.22}$$

The evolution of the entropy generation number N_s as a function of the effectiveness and temperature ratio of the heat exchanger is presented in Figure 4.8 and 4.9, for the imbalanced and balanced exchanger, respectively. In the first case, the entropy generation number increases with the increase of effectiveness ε and temperature ratio α. In particular, we observe that N_s vanishes when $\varepsilon = 0$. In the case of the balanced counterflow heat exchanger (Figure 4.9), we observe that entropy generation number N_s has a parabolic behavior, and symmetrical with respect to the axis passing through its maximum value that corresponds to $\frac{dN_s}{d\varepsilon} = 0 \Leftrightarrow (1 - 2\varepsilon)(\alpha - 1)^2 = 0 \Longrightarrow \varepsilon = 0.5$. Furtheremore, we note that N_s is minimal both when the effectiveness ε is maximal and (unexpectedly) minimal. Indeed, The behavior of N_s in the interval $0.5 \leq \varepsilon \leq 1$ is consistent. However, when $\varepsilon \to 0$ the behavior of N_s is surprising and nor intuitively obvious, because we expect N_s to increase monotonically as the exchanger length decreases. This situation was called by [Bejan, 1996, Chapter 5] **entropy generation paradox** in a balanced heat exchanger. It was verified

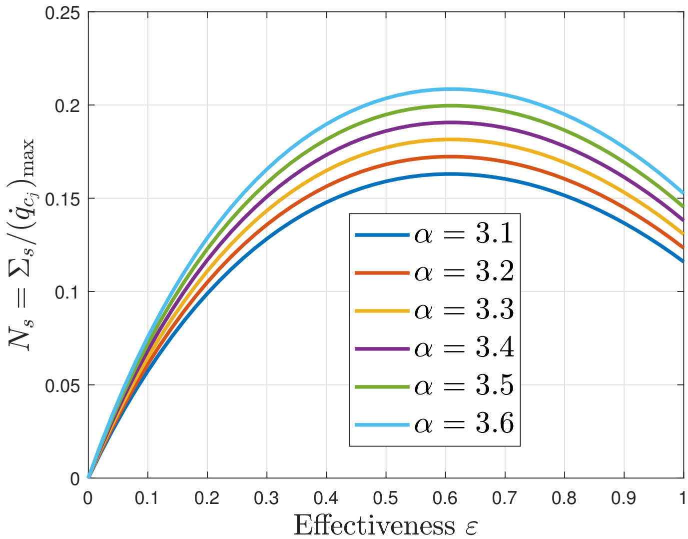


Figure 4.8: Entropy generation number N_s as a function of the effectiveness ε of imbalanced counterflow heat exchanger ($\beta_2 \neq \beta_1$).

in several studies [Mohamed, 2005], [Guo et al., 2010].
In general, whether in the case of a balanced or imbalanced heat exchanger, from expression (4.2.10), zero effectiveness implies that

$$\begin{cases} x_1(L) = x_1(0) \\ x_2(0) = x_2(L). \end{cases} \tag{4.2.23}$$

In other words, there is no heat transfer. This confirms Bejan's arguments (see [Bejan, 1996, page 124-127]). He explains that the lower-left corner of Figures 4.8 and 4.9 is technically correct, because heat exchangers, in general, contribute to the overall irreversibility of the processes that incorporate them. So an absent heat exchanger ($\varepsilon = 0$) can only contribute zero irreversibility ($N_s = 0$) as a heat exchanger. Nevertheless, it should be specified that the reason mentioned by him which explains the absent heat exchanger and which leads to zero effectiveness is that $A_s = 0$. Although A_s does not appear explicitly in the expressions of the trajectories of the system, the conclusion remains the same in the context of our study by considering that $L = 0$.

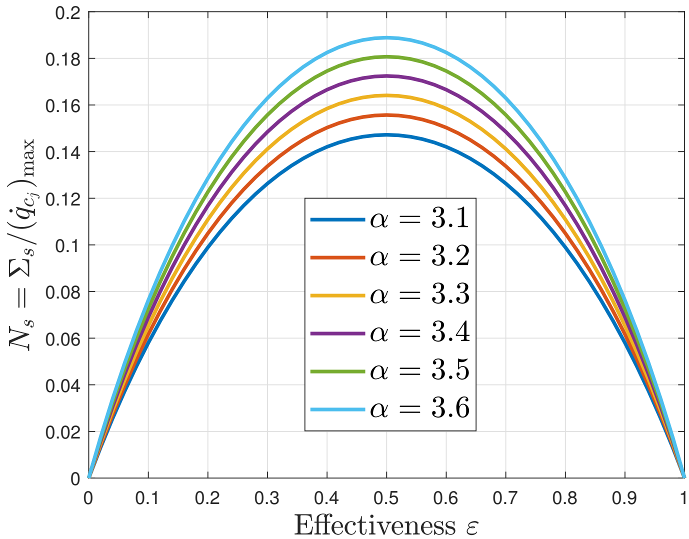


Figure 4.9: Entropy generation number N_s as a function of the effectiveness ε of balanced counterflow heat exchanger ($\beta_2 = \beta_1 = \beta$).

4.3 Passivity analysis

Passive systems are a class of dynamical systems in which the energy exchanged with the environment plays a central role. Most industrial processes (chemical reactor, bioreactor, heat exchanger, etc.) belong to this class of systems. In passive systems the rate at which the energy flows into the system is not less than the increase in storage. In other words, a passive system cannot store more energy than what is supplied from the outside, with the difference being the dissipated energy [Ortega et al., 1998].
Let us start by recalling the definition of the passivity ([der Schaft, 1996]).

Definition 4.3.1. *Let us consider a dynamical system $\Gamma(t, \xi(t))$ with input variable $u(t) \in \mathbb{R}^\ell$, output variable $y(t) \in \mathbb{R}^p$ and the state variable $\xi \in L^2(0,1)$. We consider as a supply rate function $w(t) \in \mathbb{R}^\ell \times \mathbb{R}^p \mapsto \mathbb{R}$. We say that $\Gamma(t, \xi(t))$ is passive with respect to the supply rate function $w(t)$ if there exists a non-negative function $W(\xi(t))$ called storage function, such that for all $u(t) \in \mathbb{R}^\ell$, $y(t) \in \mathbb{R}^p$ and $\xi_0 \in L^2(0,1)$:*

$$W(\xi(T)) - W(\xi_0) \leq \int_0^T u^\top(t) y(t) dt. \tag{4.3.1}$$

Let us recall the energy model of the heat exchanger:

$$\begin{cases} \dfrac{\partial s_1}{\partial t} = -v_1 \dfrac{\partial s_1}{\partial z} - \dfrac{q(z,t)}{x_1(z,t)} \\ \dfrac{\partial s_2}{\partial t} = v_2 \dfrac{\partial s_2}{\partial z} + \dfrac{q(z,t)}{x_2(z,t)}. \end{cases} \tag{4.3.2}$$

Proposition 4.3.1. *If $\beta_1 \neq \beta_2$, the thermodynamic model (4.3.2) of counterflow heat exchanger is passive with the entropy functional as the storage function and entropy production functional as the dissipation function.*

Proof. Let us consider the following auxiliary variables:

$$W(t) := \overline{S} - S(t),$$

and

$$w(z,t) := \overline{s} - s(z,t),$$

where $\overline{S}$ and $\overline{s}$ are arbitrarily large constant values of entropy such that for $t \in [0, +\infty)$, $W(t)$ and $w(z,t)$ are nonnegative. Then

$$\begin{aligned} \frac{dW(t)}{dt} &= -\int_0^1 \left(\frac{\partial s_1}{\partial t} + \frac{\partial s_2}{\partial t} \right) dz \\ &= -\int_0^1 \left(-v_1 \frac{\partial s_1}{\partial z} + v_2 \frac{\partial s_2}{\partial z} + \left(\frac{1}{x_2} - \frac{1}{x_1} \right) q(z,t) \right) dz \\ &= -\int_0^1 \left(-v_1 \frac{\partial s_1}{\partial z} + v_2 \frac{\partial s_2}{\partial z} + \sigma(z,t) \right) dz \\ &= [v_1 s_1(z,t) - v_2 s_2(z,t)]_0^1 - \int_0^1 \sigma(z,t) dz. \end{aligned} \tag{4.3.3}$$

Given that the dissipation function $\sigma(z,t) \geq 0$, then according to definition 4.3.1 the following inequality of the passivity holds:

$$\begin{aligned} \frac{dW(t)}{dt} &\leq [v_1 s_1(z,t) - v_2 s_2(z,t)]_0^1 \\ &= w\left(s_{in}(t), s_{out}(t)\right), \end{aligned}$$

and for $t \in [0, T]$ its integral form is:

$$W(T) - W(0) \leq \int_0^T w\left(s_{in}(t), s_{out}(t)\right) dt.$$

Note that $w\left(s_{in}(t), s_{out}(t)\right)$ can be seen as a bounded supply rate function. □

In this proof, the interest of the variable $W(t)$ was to formalize the passivity in the sense of Definition 4.3.1. Note also that for an isolated system (with $v_1 = v_2$), $s_{in}(t) = s_{out}(t)$, and that implies that the supply rate function $w(t)$ is zero, and $\frac{dW(t)}{dt} = -\int_0^1 \sigma(z,t)dz = -\Sigma_s(t)$.

4.4 Thermodynamic stability analysis

The second law of thermodynamics establishes the irreversibility of physical phenomena. According to this law, entropy can be seen as an extensive variable, or as a thermodynamic property. For an isolated system it can be shown that the thermodynamic equilibrium corresponds to the maximum of the entropy ($\Sigma_s \geq 0$) seen as a thermodynamic property, i.e. to the concavity of entropy with respect to its extensive variables (U, V, N). From the concavity of the entropy we derive a general result of thermodynamic stability of an equilibrium point [Callen, 2006, Chapter 8].

Definition 4.4.1 (Thermodynamic stability)**.** *An equilibrium point is stable if the entropy always remains under the tangent plane of the entropy curve taken at the considered equilibrium point.*

In this section we investigate the thermodynamic stability of an imbalanced counterflow heat exchanger using the thermodynamic availability, which is a function associated with entropy and which we present in the following.

4.4.1 Distributed thermodynamic availability function

We present in this subsection the thermodynamic availability function and its properties. This function is related to the entropy and irreversible entropy production. It was introduced by [Alonso and Ydstie, 2001] for the stabilization of finite dimensional thermodynamic systems. The results obtained have been extended by [Ruszkowski et al., 2005] for the study of the passivity of distributed parameter systems, precisely those of the diffusion-reaction type. The convexity of this function is based on the concavity of the entropy. Note that it is also possible to define an availability function in energy vision from the internal energy [Willems, 1972]. In this case the convexity of the availability function will be based on the convexity of internal energy.
Our development starts with a definition and a reminder of the properties of this function in entropic vision.
Consider $\mathbf{h}(z,t) \in Z$ as the state vector (extensive properties vector) of the entropy function (as state function of $\mathbf{h}(z,t)$) $s(z,t) \in Z$.

Definition 4.4.2 (Local availability). *Let $\tilde{w}(z,t) = w(\boldsymbol{h}(z,t)) - w(\boldsymbol{h}_d(z))$ be the difference between the profile of the intensive variables of the system $w(\boldsymbol{h}(z,t))$ and the desired time invariant intensive variables $w(\boldsymbol{h}_d(z))$, where the desired steady state $\boldsymbol{h}_d(z)$ is defined in a Sobolev space Z.*
We call local thermodynamic availability, the function $a(\boldsymbol{h}(z,t))$ defined with respect to $\boldsymbol{h}_d(z)$ as follows:

$$a(\boldsymbol{h}(z,t)) = -\tilde{w}(z,t)^\top \boldsymbol{h}(z,t). \tag{4.4.1}$$

Therefore, the global availability is given by

$$\mathcal{A}(t) = \int_0^1 a(\mathbf{h}(z,t))dz, \qquad t \in [0,+\infty). \tag{4.4.2}$$

We give some properties of local availability that will be used later.

Property 1. *If the entropy is strictly concave, then*

1. $a(\boldsymbol{h}(z))$ *is strictly convex,*
2. $a(\boldsymbol{h}(z)) > 0$, $\boldsymbol{h}(z) \neq \boldsymbol{h}_d(z)$,
3. $a(\boldsymbol{h}(z)) = 0$, $\boldsymbol{h}(z) = \boldsymbol{h}_d(z)$,

This property follows from the concavity of entropy and that entropy must remain under the tangent plane through reference state $\mathbf{h}_d(z)$.

Property 2. *The differential form $a(\boldsymbol{h}(z,t)) = -\tilde{w}(z,t)^\top \boldsymbol{h}(z,t)$ is homogeneous function of degree 1 with respect to $\boldsymbol{h}(z,t)$.*

This property derives from the fact that entropy is a homogeneous extensive function of degree 1 with respect to its extensive variables (U,V,N), i.e., with respect to the quantity of matter of the system considered.

Property 3. *Let the function $f(\boldsymbol{h},\boldsymbol{h}_d) = -\tilde{w}\tilde{\boldsymbol{h}}$ with $\tilde{w} = w(\boldsymbol{h}) - w(\boldsymbol{h}_d)$ and $\tilde{\boldsymbol{h}} = \boldsymbol{h} - \boldsymbol{h}_d$. If s is strictly concave, then:*

1. $0 \leq a(\boldsymbol{h}) \leq f(\boldsymbol{h},\boldsymbol{h}_d)$,
2. $\tilde{w} = -\mathcal{P}\tilde{\boldsymbol{h}}$, *where $\mathcal{P} \in \mathbb{R}^{n\times n}$ is a positive definite matrix.*

Property 4. *The local availability $a(\boldsymbol{h})$ verified:*

$$0 \leq \gamma_0 \tilde{\boldsymbol{h}}^\top \tilde{\boldsymbol{h}} \leq a(\boldsymbol{h}) \leq \gamma_1 \tilde{\boldsymbol{h}}^\top \tilde{\boldsymbol{h}}, \tag{4.4.3}$$

for two positive constants γ_0 and γ_1. The equality holds for $\boldsymbol{h}(z) = \boldsymbol{h}_d(z)$ only.

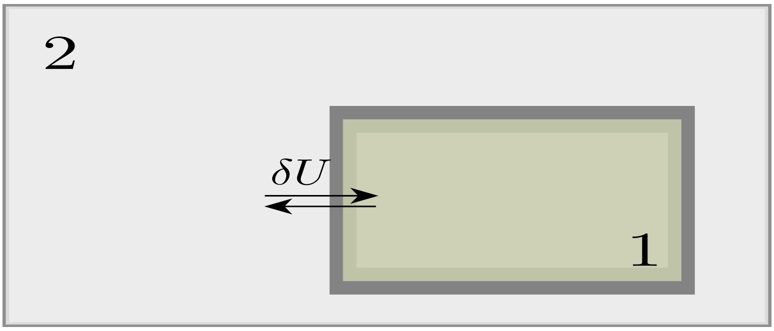


Figure 4.10: Thermal fluctuations around equilibrium.

The properties we have just stated are established in [Alonso and Ydstie, 2001]. These properties show that thermodynamic availability is a non-negative function. If the entropy is strictly concave, the availability vanishes at the reference point $\mathbf{h}_d(z)$. However, in a general context, entropy is not strictly concave. The solution proposed by [Jillson and Erik Ydstie, 2007] to make the entropy strictly concave was to constrain an extensive variable to remain constant, i.e. to impose at least one of the following constraints in order to fix the homogeneity ratio:

$$\begin{cases} \text{the internal energy } U = \text{Constant}, \\ \text{the total number of moles } N = \text{Constant}, \\ \text{the volume } V = \text{Constant}, \\ \text{the total mass } m = \text{Constant}. \end{cases} \tag{4.4.4}$$

In what follows we give the stability condition with respect to thermal fluctuations, from which we will deduce the extensive variable to be fixed which makes it possible to make the entropy strictly concave.

4.4.1.1 Thermal Stability condition

We know that in an isolated system, entropy increases to its maximum value. A fluctuation can therefore only decrease entropy; in response to this fluctuation, irreversible processes associated with entropy production bring the system back to equilibrium. Conversely, if the fluctuations are amplified, this means that the system is not in the state of equilibrium corresponding to a maximum of entropy: the equilibrium state is stable with respect to disturbances. In this subsection we give a thermal stability condition with respect to thermal fluctuations of the thermodynamic systems.
Let us consider a situation where a temperature fluctuation occurs in a small region of the system depicted in the Figure 4.10. This fluctuation comes from a flow of energy δU from one region to another, and leads to a variation of the

temperature δT in the small region (the indices correspond to the two regions). The total entropy of the system is:

$$S = S_1 + S_2, \tag{4.4.5}$$

with S_1 a function of $U_1, V_1, \cdots$, and S_2 a function of $U_2, V_2, \cdots$ The variation of the entropy δS due to the flow of energy corresponds to the Taylor expansion of S from its equilibrium value:

$$\delta S = S - S_{\text{eq}} = \frac{\partial S_1}{\partial U_1}\delta U_1 + \frac{\partial S_2}{\partial U_2}\delta U_2 + \frac{\partial^2 S_1}{\partial U_1^2}\frac{(\delta U_1)^2}{2} + + \frac{\partial^2 S_2}{\partial U_2^2}\frac{(\delta U_2)^2}{2} + \cdots \tag{4.4.6}$$

All derivatives are evaluated at steady state. Since the total energy of the system remains constant, $\delta U_1 = -\delta U_2 = \delta U$. Moreover, since $(\partial S/\partial U)_{V,N} = 1/T$, the equation (4.4.6) can be written:

$$\delta S = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\delta U + \left(\frac{\partial}{\partial U_1}\frac{1}{T_1} + \frac{\partial}{\partial U_2}\frac{1}{T_2}\right)\frac{(\delta U)^2}{2} + \cdots \tag{4.4.7}$$

We can now identify the first two terms of the entropy production of (4.4.7), δS and $\delta^2 S$, and express them in terms of the disturbance δU:

$$\delta S = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\delta U, \tag{4.4.8}$$

$$\frac{1}{2}\delta^2 S = \left(\frac{\partial}{\partial U_1}\frac{1}{T_1} + \frac{\partial}{\partial U_2}\frac{1}{T_2}\right)\frac{(\delta U)^2}{2}. \tag{4.4.9}$$

At equilibrium, all thermodynamic forces being zero, the temperature of the system is uniform. So we have $T_1 = T_2$, and the expression (4.4.8) is zero ($\delta S = 0$). Let us then consider the second term. We observe that

$$\frac{\partial}{\partial U}\frac{1}{T} = -\frac{1}{T^2}\frac{\partial T}{\partial U} = -\frac{1}{T^2}\frac{1}{C_V N}, \tag{4.4.10}$$

where C_V is the heat capacity at constant volume ($C_V = Nc_V$, where c_V is the specific heat and N the number of moles). Similarly, if the variation of the temperature of the small region is δT, we have $\delta U_1 = C_{V_1}(\delta T)$, where C_{V_1} is the heat capacity of this region. All the derivatives being taken at equilibrium, we thus obtain for the expression (4.4.9)

$$\frac{1}{2}\delta^2 S = -\frac{C_{V_1}(\delta T^2)}{2T^2}\left(1 + \frac{C_{V_1}}{C_{V_2}}\right). \tag{4.4.11}$$

This is by definition a local fluctuation, and system 1 is small compared to system 2, $C_{V_1} << C_{V_2}$, the second therm in the bracket can be ignored. So we have the equilibrium state stability condition:

$$\frac{1}{2}\delta^2 S = -\frac{C_{V_1}(\delta T^2)}{2T^2} < 0. \tag{4.4.12}$$

This condition requires that the constant volume heat capacity (and therefore the specific heat) be positive. It is an important physical condition, which expresses that the equilibrium state is stable with respect to thermal fluctuations. This consolidates the results obtained by [Glansdorff and Prigogine, 1971, Chapter 4].
Related to the condition (4.4.4) we deduce that for thermal stability problems, the entropy is strictly concave if and only if the total volume is constant.

4.4.1.2 Dynamics of thermodynamic availability

The time derivative of the local availability can be obtained from the infinite dimensional Gibbs equation [Zhou et al., 2021]. Considering $\tilde{w}(z,t) = w(\mathbf{h}(z,t)) - w(\mathbf{h}_d(z))$, we can write:

$$\frac{Da}{Dt} = \frac{D\left(-\tilde{w}^\top \mathbf{h}\right)}{Dt} = -\frac{D\tilde{w}^\top}{Dt}\mathbf{h} - \tilde{w}^\top \frac{D\mathbf{h}}{Dt}. \tag{4.4.13}$$

By replacing the operator $\frac{D}{Dt}$ on the left and right of equation (4.4.13) by its expression $\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}$ we obtain:

$$\frac{\partial a}{\partial t} = -\frac{\partial \tilde{w}^\top}{\partial t}\mathbf{h} - \tilde{w}^\top \frac{\partial \mathbf{h}}{\partial t}. \tag{4.4.14}$$

According to the Gibbs-Duhem relationship $\frac{\partial \tilde{w}^\top}{\partial t}\mathbf{h} = -v\frac{\partial \tilde{w}^\top}{\partial z}\mathbf{h}$, at constant pressure and density, $v\frac{\partial \tilde{w}^\top}{\partial z}\mathbf{h} = 0$ (see [Zhou et al., 2021]). Considering this fact, equation (4.4.14) becomes:

$$\begin{aligned}\frac{\partial a}{\partial t} &= v\frac{\partial \tilde{w}^\top}{\partial z}\mathbf{h} - \tilde{w}^\top \frac{\partial \mathbf{h}}{\partial t} \\ &= -\tilde{w}^\top \frac{\partial \mathbf{h}}{\partial t}.\end{aligned} \tag{4.4.15}$$

From (4.4.2) and (4.4.15) the time derivative of global availability function $\mathcal{A}(t)$ has the following expression:

$$\frac{d\mathcal{A}(t)}{dt} = \int_0^1 \frac{\partial a}{\partial t} dz = -\int_0^1 \tilde{w}^\top \frac{\partial \mathbf{h}}{\partial t} dz. \tag{4.4.16}$$

4.4.2 Stability of heat exchanger

In this subsection we are interested in the stability in the thermodynamic sense of the imbalanced counterflow heat exchanger, using thermodynamic availability function previously presented.
Let us consider the entropy representation of heat exchanger:

$$\begin{cases} \dfrac{\partial h_1}{\partial t} = -v_1 \dfrac{\partial h_1}{\partial z} - q(z,t) \\ \dfrac{\partial h_2}{\partial t} = v_2 \dfrac{\partial h_2}{\partial z} + q(z,t), \end{cases}$$

that can be written in compact form as follows:

$$\frac{\partial h}{\partial t} = \Lambda \frac{\partial h}{\partial z} + gq(z,t), \qquad \text{with } g = (-1 \quad 1)^\top . \tag{4.4.17}$$

We saw previously that to examine the stability of a system by exploiting the thermodynamic availability, one of the extensive variables of the entropy function must be fixed. According to the modelling assumptions of the heat exchanger considered in this study, V and N are constant (according to the modelling assumptions). Thus, we establish the following result.

Proposition 4.4.1. *If $\beta_1 \neq \beta_2$, the dynamic system (4.4.17) of imbalanced heat exchanger is globally asymptotically stable at h_d.*

Proof. Let us consider $\tilde{w}^\top(z,t) = \left(\frac{1}{\tilde{x}_1} \; \frac{1}{\tilde{x}_2}\right)$, with $\tilde{x}_1(z,t) = x_1(z,t) - x_{1d}(z)$, $\tilde{x}_2(z,t) = x_2(z,t) - x_{2d}(z)$, and $\tilde{h} = (\tilde{h}_1 \; \tilde{h}_2)^\top$ with $\tilde{h}_1 = h_1 - h_{1d}$ and $\tilde{h}_2 = h_2 - h_{2d}$. Substituting the dynamical system (4.4.17) into the expression of the global availability derivative (4.4.16) we get the following:

$$\begin{aligned} \frac{d\mathcal{A}}{dt} &= \int_0^1 \frac{\partial a}{\partial t} dz = -\int_0^1 \tilde{w}^\top(z,t) \frac{\partial \tilde{h}}{\partial t} dz \\ &= -\int_0^1 \tilde{w}^\top(z,t) \left(\Lambda \frac{\partial \tilde{h}}{\partial z} + gq(z,t) \right) dz. \end{aligned}$$

By integrating by part in the domain $[0,1]$ the first term on the right which represents the convection flux, we obtain:

$$\begin{aligned} \frac{d\mathcal{A}}{dt} = &- \left[\tilde{w}^\top(z,t) \Lambda \tilde{h}(z,t) \right]_0^1 + \int_0^1 \Big(\tilde{h}^\top(z,t) \Lambda \frac{\partial \tilde{w}}{\partial z} \\ &+ \tilde{w}^\top(z,t) gq(z,t) \Big) dz. \end{aligned}$$

Note that $\displaystyle\int_0^1 \tilde{w}^\top gq dz = \int_0^1 \left(\frac{1}{\tilde{x}_2} - \frac{1}{\tilde{x}_1} \right) q dz = \tilde{\Sigma}_s(t) \geq 0$ corresponds to

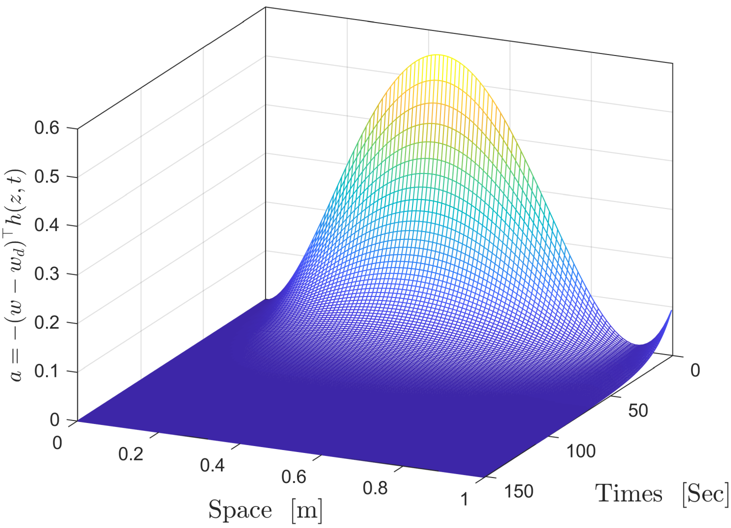


Figure 4.11: Local availability of imbalanced counterflow heat exchanger ($\beta_1 \neq \beta_2$).

the irreversible entropy production at the reference point (h_{1d}, h_{2d}). Thus,

$$\frac{d\mathcal{A}}{dt} \leq -\left[\tilde{w}^\top(z,t)\Lambda\tilde{h}(z,t)\right]_0^1 + \int_0^1 \left(\tilde{h}^\top(z,t)\Lambda\frac{\partial \tilde{w}}{\partial z}\right) dz.$$

Property 3 tells us that $\tilde{w}(z,t) = -\mathcal{P}\tilde{h}(z,t)$, with $\mathcal{P} \in \mathbb{R}^{2\times 2}$ a symmetric positive definite matrix. So we can write:

$$\begin{aligned}\frac{d\mathcal{A}}{dt} &= \left[\tilde{h}^\top(z,t)\Lambda\mathcal{P}\tilde{h}(z,t)\right]_0^1 - \int_0^1 \tilde{h}^\top(z,t)\Lambda\mathcal{P}\frac{\partial \tilde{h}}{\partial z}dz \\ &\leq -\int_0^1 \tilde{h}^\top(z,t)\Lambda\mathcal{P}\frac{\partial \tilde{h}}{\partial z}dz \leq 0.\end{aligned}$$

Finally, if $h \mapsto h_d$, $\tilde{h}(\cdot,t) \mapsto 0$, then $\lim\limits_{t\mapsto+\infty} \mathcal{A}(t) = 0$, which implies the global asymptotically stability of heat exchanger system. □

In order to assess the stability of the heat exchanger, simulation tests were carried out. The simulation parameters are listed in Table 4.1, and the initial

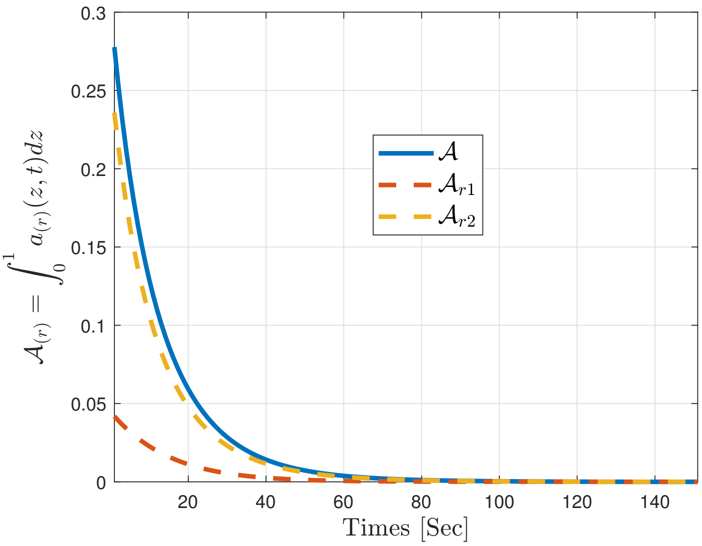


Figure 4.12: Global availability $\mathcal{A}$ and global reduced availability $\mathcal{A}_{r1}$ and $\mathcal{A}_{r2}$ of imbalanced counterflow heat exchanger ($\beta_1 \neq \beta_2$).

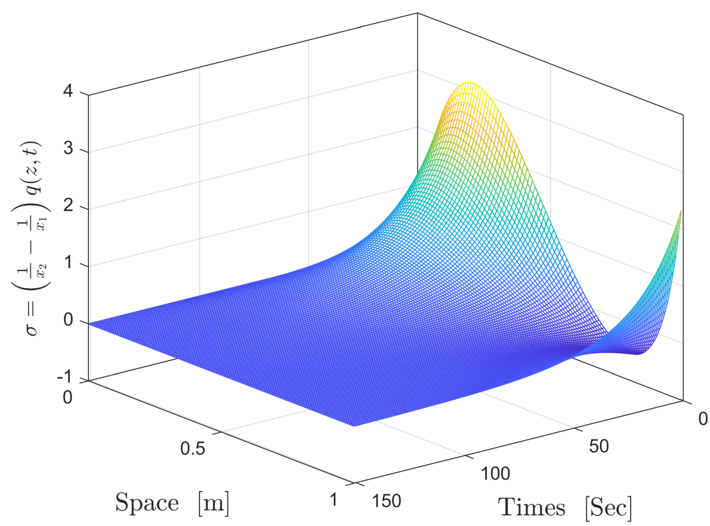


Figure 4.13: Local irreversible entropy production of imbalanced counterflow heat exchanger ($\beta_1 \neq \beta_2$).

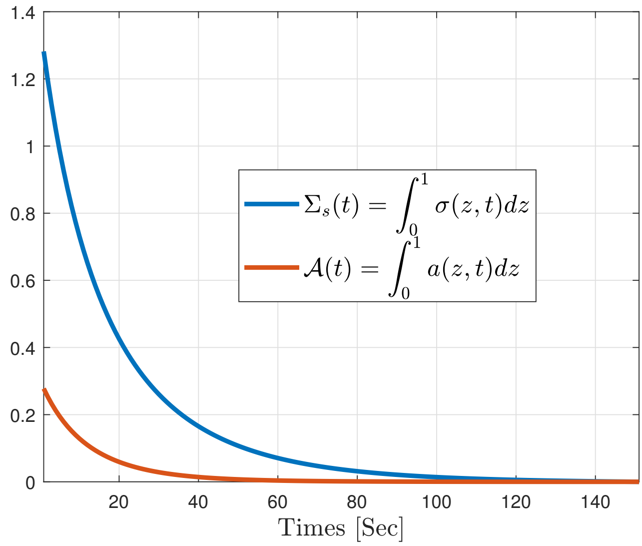


Figure 4.14: Global irreversible entropy production $\Sigma_s(t)$ and global availability $\mathcal{A}(t)$ of imbalanced counterflow heat exchanger ($\beta_1 \neq \beta_2$).

condition inspired by [Besson et al., 2006] was considered:

$$\begin{cases} x_1(z) = x_{1,in}\left[\left(\mathrm{e}^1\,\dfrac{x_{2,in}}{x_{1,in}} - 1\right)z^2 + 1\right]\mathrm{e}^{-z^2}, \\ x_2(z) = x_{2,in}\exp\left[(1-z)^2\ln\left(\frac{x_{1,in}}{x_{2,in}}\right)\right]. \end{cases} \tag{4.4.18}$$

Figure 4.11 shows the time and spatial evolutions of the local availability

Table 4.1: Heat exchanger parameters & simulation conditions

Description	Symbol	Value
Boundary condition for internal fluid	$x_1(0,t)$	$60°\mathrm{C}$
Boundary condition for extternal fluid	$x_2(1,t)$	$8°\mathrm{C}$
Heat transfer coefficient of internal fluid	α_1	$0.9011 s^{-1}$
Heat transfer coefficient of external fluid	α_2	$0.9091 s^{-1}$
Velocity of internal fluid	v_1	$0.0938 m/s$
Velocity of internal fluid	v_2	$0.0946 m/s$

function $a(z,t)$. It converges globally and asymptotically to zero along the counterflow heat exchanger. Figure 4.12 shows that the global availability function $\mathcal{A}(t)$ converges asymptotically to zero. Note that this function can be broken down into two contributions:

$$\begin{aligned}\mathcal{A}(t) &= -\int_0^1 \begin{pmatrix}\frac{1}{\tilde{x}_1} & \frac{1}{\tilde{x}_2}\end{pmatrix}\begin{pmatrix}\tilde{h}_1 \\ \tilde{h}_2\end{pmatrix} dz \\ &= -\int_0^1 \frac{1}{\tilde{x}_1}\tilde{h}_1(z,t)dz - \int_0^1 \frac{1}{\tilde{x}_2}\tilde{h}_2(z,t)dz,\end{aligned} \tag{4.4.19}$$

the first of which noted $\mathcal{A}_{r1}(t)$, corresponds to the thermal contribution of the internal fluid, and the second noted $\mathcal{A}_{r2}(t)$, to the thermal contribution of the external fluid. These global reduced availability functions are depicted in Figure 4.12. We note that they converge towards zero with a smaller amplitude than $\mathcal{A}(t)$. In particular, the amplitude of $\mathcal{A}_{r2}$ is greater than that of $\mathcal{A}_{r2}$. This is due to the fact that according to the simulation conditions listed in table 4.1, the following condition on the thermodynamic driving forces of the heat exchanger is verified: $(1/x_2) > (1/x_1)$. In other words, the whole of the entropy produced $\Sigma_s(t)$ is transferred to the external fluid.
Figure 4.13 illustrates the distributed time response of local irreversible entropy production $\sigma(z,t)$. We also note that it converge toward zero, when $t \mapsto +\infty$. The global irreversible entropy production $\Sigma_s(t)$ is compared to the global availability $\mathcal{A}(t)$ of the heat exchanger in Figure 4.14. We note that $\mathcal{A}(t)$ converge faster towards zero than $\Sigma_s(t)$ with also a smaller amplitude.

4.5 Conclusion and suggestions for future research

This chapter was dedicated to the thermodynamic analysis of the distributed parameter model of a counterflow heat exchanger. This device is essential for energy applications. First, we performed an analysis of the equilibrium profiles from a certain condition on the system parameters, that guarantees the existence and uniqueness of the solution of the PDEs model. This analysis highlighted the importance of thermal pinch as a factor in energy efficiency for irreversible heat exchanger. Next, we successively studied passivity and asymptotic stability, by using the entropy as a storage function entropy production as a dissipation function for to proof the passivity, and the thermodynamic availability function for the stability.

Note however that the entropic analysis as done in Section 4.2 only takes into account the aspects related to the second law of thermodynamics. In this sense, for a more complete analysis, it would be interesting to use an exergy approach. The interest of this method lies mainly in the fact that its implementation gives both qualitative and quantitative information on energy transfers. Thus, beyond the concepts related to the second law of thermodynamics, it also covers the concepts of the first law that the entropic analysis does not take into account.

Moreover, although passivity is established using aspects related to the second law of thermodynamics, its use in the context of the boundary control deserves further investigation. The main problem is that the boundary thermodynamic driving forces do not appear explicitly in the integral entropy balance.

Part III

Control

5

Boundary control

Contents

5.1 Introduction

In this chapter, we are interested in the control of heat exchangers. In most industrial applications, most often the thermal flow (which crosses the exchange surface which separates the fluids) is used to cool or heat a fluid which can neither be cooled nor heated, or difficult to cool or heat directly. In terms of investment and quality, the advantages of heat exchangers are well established [Azad and Amidpour, 2011]. In general, they are sized to operate in steady state (open-loop). However, performance deteriorates when disturbances due to the input quantities (temperatures, flow rates) and the fouling appear; hence the need to replace them each time, which has a negative economic impact. Therefore, the control of the latter is essential as an alternative and adequate solution.

The problem of controlling heat exchangers has been addressed by many researchers using different methods: either by synthesizing controllers from the infinite dimensional system, or by relying on spatial reduction methods. To control the temperature of the internal fluid (typically the hot fluid), two characteristic variables can be manipulated: the external fluid temperature (cold fluid)([Maidi et al., 2009], [Burns and Zietsman, 2016]) or its flow velocity ([Grabowski, 2007], [Aulisa et al., 2016], [Sandoval, 2021]). In the literature, the second strategy is not highly considered for two reasons: using the velocity as a control variable amounts to modifying the flow regime which can change from laminar to turbulent, with impact on the control law which in this case will require an adaptation of the parameters of the system, which is not always easy to get [Maidi et al., 2009, Maidi et al., 2010], [Aulisa

et al., 2016]. In addition, using the velocity amounts to considering the inlet temperature of the external fluid as constant, which imposes limitations for the control variable, and therefore for the setpoints to be reached. On the other hand, the manipulation of the external temperature makes it possible to work with constant hydrodynamic parameters whose variations constitute disturbances which can be rejected by the controller.
There are several control strategies for these devices. A majority is based on the notion of relative degree, which consists in differentiating the output variable until the control variable explicitly appears [Maidi et al., 2009]. These techniques provide interesting simulation results; yet their implementation requires at least two sensors (due to the discretization of the transport operator), depending on the assumptions made on the measured output. In general the proposed solutions consist in coupling the control laws to a state observer [Zobiri et al., 2017]. In this chapter we propose a boundary control scheme. It appears that the control action is precisely one of the boundary condition of the PDE system. The control action that is derived is indeed a Proportional-Integral (PI)-controller (so as to cancel the closed loop static error). The boundary control strategy has the advantage of simplicity of implementation since it only requires the measurement of the state variables at the boundaries.
Furthermore, unlike the velocity or flow manipulation control approach which is done by valves, temperature manipulation is done by a dynamic actuator which can be a compressor when the control objective is to cool, or a boiler when the control objective is to heat. See [Burns and Cliff, 2014a, Burns and Zietsman, 2016] for more information.

5.2 PI-Boundary output feedback

Extent of finite dimensional system control theory, the proportional integral (PI)-controller is mainly used to control an output subject to disturbances. The output is typically a state of the PDE at the boundary. The control law includes a term proportional to the error between the output to be controlled and the desired output, as well as an integral term of this error.
To give a clearer idea, consider the model of the heat exchanger described in chapter 1

$$\begin{cases} \dfrac{\partial \xi}{\partial t} = \Lambda \dfrac{\partial \xi}{\partial z} + M\xi(z,t) \\ \xi_1(0,t) = \xi_{1,in}(t), \ \xi_2(1,t) = u(t), \end{cases} \tag{5.2.1}$$

evolving for $0 \leq z \leq 1$ and $t \geq 0$, and where $\xi(z,t) = (\xi_1(z,t)\ \xi_2(z,t))^\top \in \mathbb{R}^2$ represent the vector of the distributed temperature. $\xi_{1,in}(t) \in \mathbb{R}$ is the inlet temperature of the internal fluid which is considered here as a constant disturbance. The outlet temperature $\xi_1(1,t)$ of the internal fluid is to be stabilized at reference value by a control applied at $\xi_2(1,t) = u(t)$. We associate to the boundary condition $\xi_2(1,t)$ a PI-controller whose expression is given in [Bastin and Coron, 2016], as follows:

$$\begin{cases} \xi_2(1,t) = K_p\xi_1(1,t) + K_i\chi(t) \\ \partial_t\chi(t) = \xi_1(1,t) - y_d(t), \end{cases} \tag{5.2.2}$$

with $K_p \in \mathbb{R}$ the proportional gain and K_i the integral gain, $y_d(t) \in \mathbb{R}$ the reference, and where $\chi(t) \in \mathbb{R}$ is a new state variable related to the integral action.

In general, when a dynamic system is controlled by a PI or other type controller, we have to guarantee that the controlled system is stable. A controller can be used to stabilize an unstable system, just as a stable system can be destabilized by a controller, if it is not suitable. It is therefore necessary to be careful, even when the eigenvalues of the open-loop system are stable. Our first concern is to analyse the exponential stability of the system (5.2.1)-(5.2.2) according to the following definition.

Let $\mathbb{X} = L^2(0,1) \times L^2(0,1) \times \mathbb{R}$ the Hilbert space (of the closed-loop system (5.2.1)-(5.2.2)) equipped with the scalar product

$$\langle(\xi,\chi),(\phi,\varphi)\rangle_{\mathbb{X}} = \int_0^1 \xi_1(z)\phi_1(z)dz + \int_0^1 \xi_2(z)\phi_2(z)dz + \chi\varphi,$$

endowed with the norm $\|f\|^2_{\mathbb{X}} = \|f_1\|^2_{L^2(0,1)} + \|f_2\|^2_{L^2(0,1)} + |f_3|^2_{\mathbb{R}}$ defined for any function $f \in \mathbb{X}$.

Definition 5.2.1. *The closed-loop system (5.2.1)-(5.2.2) is exponentially stable for the $\mathbb{X}-$norm if there exist two constants $C > 0$ and $\nu > 0$ such that, for all $\xi_0(z) \in L^2\left((0,1);\mathbb{R}^2\right)$ and $\chi_0 \in \mathbb{R}$, the solution of the Cauchy problem (5.2.1)-(5.2.2) satisfies*

$$\|\xi(\cdot,t),\chi(t)\|_{\mathbb{X}} \leq Ce^{-\nu t}\,\|\xi_0(z),\chi_0\|_{\mathbb{X}}.$$

The objective is to calculate the values of the feedback gains K_p and K_i that ensure stability in the sense of the Definition 5.2.1, and regulation. In the literature, there are two types of approach:

- The frequency approach, based on the study of the characteristic equation associated with the dynamics of the output (see e.g. [Litrico and Fromion, 2009a], and the books [Litrico and Fromion, 2009c], [Bastin and Coron, 2016, Chapter 2 & 3]). This approach is generally dedicated to PDEs with delay.
- The time-based approach, in which we find approaches based on the use of Lyapunov functions (see e.g. [de Halleux et al., 2003], [Santos et al., 2008], and the reference book [Bastin and Coron, 2016]), and others on semigroup theory (see e.g [Pohjolainen, 1982]).

In the following we shall consider the temporal approaches because unlike the frequency approaches, they make it possible to find sufficient stability conditions.

5.2.1 A Lyapunov approach

One way to prove the stability of a dynamical system is to use Lyapunov's theory. This method is among other things used for nonlinear dynamical systems. It can be shown that if a problem is well-posed, Lyapunov's theorem in finite dimension adapts in the following way [Coron et al., 2007]. *The equilibrium state $\xi(z,t) = 0$ is exponentially stable in $L^2(0,1)$ if there exists a continuously differentiable function $V(\xi(z,t))$ such that*:

- $V(0) = 0$,
- $V(\xi(z,t)) > 0, \ \forall \xi(z,t) \neq 0 \in L^2(0,1)$,
- $\dot{V}(0) = 0$,
- $\dot{V}(\xi(z,t)) < 0, \ \forall \xi(z,t) \neq 0 \in L^2(0,1)$.

The function $V(\xi(z,t))$ is called a Lyapunov's function. It represents the overall energy of the system, and can be likened to a standard norm. Note that if we have furthermore that

$$\dot{V}(\xi(z,t)) < -\mu V(\xi(z,t)), \qquad \text{with } \mu > 0,$$

then the equilibrium is also exponentially stable. A difficulty of this theorem is that it does not provide the function $V(\xi(z,t))$. In general, we choose a function of the form $V(\xi(z,t)) = \xi(z,t)^\top P(z)\xi(z,t)$, with $P(z)$ a strictly positive definite matrix. Another weakness of this approach is that unlike the frequency approach, it only provides sufficient stability conditions. Indeed, if there exists a Lyapunov function $V_1(\xi(z,t))$ under some conditions C_1, nothing says that there does not exist another one $V_2(\xi(z,t))$ under the conditions $C_2 \neq C_1$. The conditions C_1 are therefore sufficient, but not necessary.
Despite this, the Lyapunov approach is widely used, and gives conclusive practical results. In [Santos et al., 2008] for example, a linear system of two conservation laws is boundary controlled by PI-regulators. A Lyapunov function, including the gains of proportional and integral actions is synthesized. These gains are then calculated to make the Lyapunov function strict. The control design method is illustrated with an hydraulic application, namely the level and flow regulation in a reach of the Sambre river and in the micro-channel of Valence, respectively through simulations and experimentations.
In the following, this approach will be applied to the model (5.2.1) of the counterflow heat exchanger.

5.2.1.1 PI-controller

Let us go back to the problem presented at the beginning of the Section 5.2. Consider the model given by the system of equations (5.2.1) and the PI-controller applied to the boundary condition $\xi_2(1,t)$ and whose expression is given by the system of equation (5.2.2).
Then the main result of our chapter is given in the following theorem.

Theorem 5.2.1. *If $\beta_1 \neq \beta_2$, and for any initial condition $(\xi_{10}(z), \xi_{20}(z), \chi_0)^\top \in \mathbb{X}$ satisfying the first derivative as well as the compatibility conditions of (5.2.2), there exist positive constants μ, k, K_p and K_i such that:*

a) there exists an exponentially stable equilibrium state denoted $(\overline{\xi}, \overline{\chi})$ and satisfying closed-loop dynamics (5.2.1)-(5.2.2). More precisely, we have:

$$\left\|(\xi, \chi) - (\overline{\xi}, \overline{\chi})\right\|_{\mathbb{X}} \leq ke^{-\mu t}\left\|(\xi_0, \chi_0) - (\overline{\xi}, \overline{\chi})\right\|_{\mathbb{X}}.$$

b) if $(\xi_{10}(z), \xi_{20}(z), \chi_0)^\top$ satisfies the C^0 and C^1- compatibility conditions and it is in $\mathbb{X}_1$ with $\mathbb{X}_1 = H^1(0,1) \times H^1(0,1) \times \mathbb{R}$, then for a given desired output $y_d \in \mathbb{R}$, the closed-loop (5.2.1)-(5.2.2) ensures that $\lim_{t\to+\infty} \|y(t) - y_d(t)\|_{\mathbb{R}} = 0$.

The proof of Theorem 5.2.1 is established using three lemmas:

- Lemma 5.2.1 shows the existence and uniqueness of the equilibrium for which the closed-loop system is well-posed.
- Lemma 5.2.2 shows the exponential stability of the closed-loop system, in the sense of the $\mathbb{X}-$norm.
- Lemma 5.2.4 shows that if $(\xi_{10}(z), \xi_{20}(z), \chi_0)^\top$ satisfies the compatibility conditions of [Bastin and Coron, 2016, Theorem A .1], then the exponential stability of the closed-loop system implies regulation.

Lemma 5.2.1. *If $\beta_1 \neq \beta_2$, then there exists an equilibrium state $(\overline{\xi}(z), \overline{\chi})$ for the dynamical system satisfying equations (5.2.1)-(5.2.2). Furthermore, the dynamical system built around this equilibrium is well-posed in $\mathbb{X}$.*

Proof. First, the equilibrium state must verify

$$\begin{cases} \Lambda \dfrac{d\overline{\xi}}{dz} + M\overline{\xi}(z) = 0, \\ \xi_1(0) = \xi_{1,in}, \quad \xi_2(1) = u, \end{cases} \tag{5.2.3}$$

for all $z \in [0,1]$. According to the Theorem 3.2.1 the problem (5.2.3) have a unique solution iff $\beta_1 \neq \beta_2$, and this solution is given by $\overline{\xi}(z) = \Phi(z)\xi(0)$, with $\Phi(z) = \exp(-\Lambda^{-1}Mz) : [0,1] \mapsto \mathbb{R}^{2\times 2}$, and where $(\xi_1(0) \quad \xi_2(0))^\top = \begin{pmatrix} 1 & 0 \\ (0 \quad 1)\exp(-\Lambda^{-1}M) \end{pmatrix}^{-1} \begin{pmatrix} \xi_1(0) \\ \xi_2(1) \end{pmatrix}$. Moreover, for $\partial_t \chi(t) = 0$, we have $\overline{\chi} = \frac{1}{K_i}(\xi_2(1) - K_p\xi_1(1))$ which can be expressed in terms of inlet temperatures as follows:

$$\overline{\chi} = \frac{1}{K_i}\left\{\frac{K_p}{\beta_1 - \beta_2 e^{(\beta_2-\beta_1)}}\left(\beta_2 e^{(\beta_2-\beta_1)} - \beta_1 e^{(\beta_2-\beta_1)}\right)\xi_{1,in}\right.$$
$$\left. + \left(1 + \frac{\beta_1}{\beta_1 - \beta_2 e^{(\beta_2-\beta_1)}}\left(1 - e^{(\beta_2-\beta_1)}\right)\right)\xi_{2,in}\right\}.$$

We can then consider the dynamics of the system around this equilibrium by considering $\xi'(z,t) = \xi(z,t) - \overline{\xi}(z)$ and $\chi'(t) = \chi(t) - \overline{\chi}$. These new variables are solution of the system

$$\begin{cases} \partial_t \xi'(z,t) = \Lambda \partial_z \xi'(z,t) + M\xi'(z,t) \\ \quad \partial_t \chi'(t) = \xi_1'(1,t), \quad z \in [0,1], \text{ and } t \in [0,+\infty), \\ \quad \xi_1'(0,t) = \xi_{1,in}(t) \\ \quad \xi_2'(1,t) = u(t) = K_p\xi'(1,t) + K_i\chi'(t). \end{cases} \tag{5.2.4}$$

By [Bastin and Coron, 2016, Theorem A.6] we have that system (5.2.4) is well-posed in $\mathbb{X}$, i.e. $\xi(t) \in C^0([0,+\infty), X) \cap C^1([0,+\infty); X)$ and $\chi \in C^1([0,+\infty), \mathbb{R})$. □

Let us now show that the obtained equilibrium $(\overline{\xi}, \overline{\chi})$ is exponentially stable. For that, let $V(\xi_1, \xi_2, \chi) : \mathbb{X} \to \mathbb{R}^3$ be an Lyapunov functional candidate (inspired by [Santos et al., 2008]) defined as follows:

$$\begin{aligned} V(t) &= \int_0^1 \begin{pmatrix} \xi_1' e^{-\frac{\mu z}{2}} \\ \xi_2' e^{\frac{\mu z}{2}} \\ \chi'(t) \end{pmatrix}^\top \begin{pmatrix} p_1 & 0 & 0 \\ * & p_2 & 0 \\ * & * & p_3 \end{pmatrix} \begin{pmatrix} \xi_1' e^{-\frac{\mu z}{2}} \\ \xi_2' e^{\frac{\mu z}{2}} \\ \chi'(t) \end{pmatrix} dz \\ &= \int_0^1 \left(p_1 \xi_1'^2(z,t) e^{-\mu z} + p_2 \xi_2'^2(z,t) e^{\mu z}\right) dz + p_3 \chi'^2(t) \\ &= V_1(t) + V_2(t) + V_3(t), \qquad p_1, p_2 \text{ and } p_3 > 0. \end{aligned} \tag{5.2.5}$$

Lemma 5.2.2. *If $\beta_1 \neq \beta_2$, there exists $K_p > 0$ and $K_i > 0$ such that for all $(\xi_0', \chi_0') \in \mathbb{X}$, the dynamical system (5.2.4) built around the equilibrium $(\overline{\xi}, \overline{\chi})$ is exponentially stable in $\mathbb{X}$[a].*

[a] To lighten the writing in the proof, we will use $\xi(z,t)$ instead of $\xi'(z,t)$ and $\chi(t)$ instead of $\chi'(t)$. In addition, we consider that $y(t) = 0$.

Proof. By deriving $V(t)$ along the trajectories of controlled system, and by integrating by part the distributed state variables on $[0,1]$ with the boundary conditions of (5.2.4) we obtain (we assume that the disturbance $\xi_1(0,t)=0$):

$$
\begin{aligned}
\dot V_1(t) &= 2p_1\int_0^1 \xi_1(z,t)\left(-v_1\partial_z\xi_1(z,t)+\alpha_1(\xi_2(z,t)-\xi_1(z,t))\right)e^{-\mu z}dz\\
&= -v_1p_1\left[\xi_1^2(z,t)e^{-\mu z}\right]_0^1-\mu v_1p_1\int_0^1\xi_1(z,t)^2e^{-\mu z}dz\\
&\quad +2\alpha_1p_1\int_0^1\xi_1(z,t)\left(\xi_2(z,t)-\xi_1(z,t)\right)e^{-\mu z}dz\\
&= -p_1v_1\xi_1^2(1,t)e^{-\mu}-p_1(2\alpha_1+v_1\mu)\int_0^1\xi_1^2(z,t)e^{-\mu z}dz\\
&\quad +2\alpha_1p_1\int_0^1\xi_1(z,t)\xi_2(z,t)e^{-\mu z}dz,\\
\dot V_2(t) &= 2p_2\int_0^1\xi_2(z,t)\left(v_2\partial_z\xi_2(z,t)+\alpha_2(\xi_1(z,t)-\xi_2(z,t))\right)e^{\mu z}dz\\
&= v_2p_2\left[\xi_2^2(z,t)e^{\mu z}\right]_0^1-\mu v_2p_2\int_0^1\xi_2^2(z,t)e^{\mu z}dz\\
&\quad +2\alpha_2p_2\int_0^1\xi_2(z,t)\left(\xi_1(z,t)-\xi_2(z,t)\right)e^{\mu z}dz\\
&= -p_2v_2\left[\xi_2^2(0,t)-\left(K_p^2\xi_1^2(1,t)+2K_pK_i\xi_1(1,t)\chi(t)+K_i^2\chi^2(t)\right)e^{\mu}\right]\\
&\quad -p_2(2\alpha_2+v_2\mu)\int_0^1\xi_2^2(z,t)e^{\mu z}dz+2p_2\alpha_2\int_0^1\xi_1(z,t)\xi_2(z,t)e^{\mu z}dz\\
\dot V_3(t) &= 2p_3\chi(t)\xi_1(1,t).
\end{aligned}
$$

By regrouping all these terms, we rewrite $\dot V(t)$ as the sum of an integral and a term containing only the boundary variables:

$$
\begin{aligned}
\dot V(t) = &-\int_0^1\begin{pmatrix}\xi_1(z,t)\\ \xi_2(z,t)\end{pmatrix}^\top Q_1\begin{pmatrix}\xi_1(z,t)\\ \xi_2(z,t)\end{pmatrix}dz\\
&-\begin{pmatrix}\xi_1(1,t)\\ \chi(t)\\ \xi_2(0,t)\end{pmatrix}^\top Q_2\begin{pmatrix}\xi_1(1,t)\\ \chi(t)\\ \xi_2(0,t)\end{pmatrix},
\end{aligned}
\tag{5.2.6}
$$

where

$$
Q_1=\begin{pmatrix}p_1(2\alpha_1+v_1\mu)e^{-\mu z} & -(\alpha_1p_1e^{-\mu z}+\alpha_2p_2e^{\mu z})\\ * & p_2(2\alpha_2+v_2\mu)e^{\mu z}\end{pmatrix},
$$

$$
Q_2=\begin{pmatrix}a_{11} & a_{12} & 0\\ * & a_{22} & 0\\ * & * & p_2v_2\end{pmatrix},\quad \text{with}\ \begin{cases}a_{11}=p_1v_1e^{-\mu}-p_2v_2K_p^2e^{\mu}\\ a_{12}=-(p_2v_2K_iK_pe^{\mu}+p_3)\\ a_{22}=-p_2v_2K_i^2e^{\mu}.\end{cases}
$$

$\dot{V}(t)$ is negative if the matrices Q_1 and Q_2 are positive definite, which is the case if their traces and their determinants are positive.
Let us start with matrix Q_1. Its trace will be positive if $v_{1\to 2}, \alpha_{1\to 2}, p_{1\to 2}$ and μ are positive. Its determinant is given by

$$\begin{aligned}\Delta_{Q_1}(\mu) &= p_1p_2(2\alpha_1 + v_1\mu)(2\alpha_2 + \mu v_2) - \left(\alpha_1 p_1 e^{-\mu z} + \alpha_2 p_2 e^{\mu z}\right)^2 \\ &= \mu p_1 p_2 \left[2(\alpha_1 v_2 + \alpha_2 v_1) + v_1 v_2 \mu\right] - \left(\alpha_1 p_1 e^{-\mu z} - \alpha_2 p_2 e^{\mu z}\right)^2 .\end{aligned}$$

If we choose p_1 and p_2 such as $\alpha_1 p_1 = \alpha_2 p_2$, we can show that there is a μ small enough for $\Delta_{Q_1}(\mu)$ to be positive $\forall z \in [0,1]$. Indeed under this condition, $\Delta_{Q_1}(0) = 0$, and $\dfrac{d\Delta_{Q_1}}{d\mu}(0) = 2p_1^2(\alpha_1 v_2 + \alpha_2 v_1) > 0$. So we have an increasing function at the origin and thus positive for sufficiently small μ.
By applying Sylvester's criterion on Q_2 we obtain the following conditions:

$i)$ $0 < K_p < e^{-\mu}\sqrt{\frac{v_1 p_1}{v_2 p_2}}$,

$ii)$ $-v_2 p_2 e^{\mu} a_{11} K_i^2 - (v_2 p_2 K_p e^{\mu} K_i + p_3)^2 > 0$

The last condition $ii)$ can be seen as a second order equation in K_i. Its discriminant is given by

$$\Delta_{K_i}(K_p) = 4 v_2 p_2 p_3^2 \left(v_2 p_2 e^{\mu} K_p^2 - v_1 p_1 e^{-\mu}\right) > 0. \tag{5.2.7}$$

Under this condition $ii)$ admits two positive roots K_{i-}, K_{i+}, for the $K_p \in \left]0, e^{-\mu}\sqrt{\dfrac{v_1 p_1}{v_2 p_2}}\right[$ according to $i)$. Consequently, $i)$ and $ii)$ imply that for any initial condition in $\mathbb{X}$, there are two constants $C, \nu > 0$ such that $\forall\, t \in [0, +\infty)$

$$\|\xi_1(\cdot, t), \xi_2(\cdot, t), \chi(t)\|_{\mathbb{X}} \leq C e^{-vt} \|\xi_{10}(\cdot), \xi_{20}(\cdot), \chi_0\|_{\mathbb{X}} . \tag{5.2.8}$$

□

Thus, with Lemmas 5.2.1 and 5.2.2 we have proved point *a)* of the main Theorem 5.2.1. It only remains to prove the regulation of the controlled system.
Before stating the last proposition of the proof of the main theorem, let us consider the following lemma on Sobolev's embedding theorem which allows to deduce the regulation from the exponential stability [Trinh et al., 2018], [Terrand-Jeanne et al., 2020].

Lemma 5.2.3. *Let $f : [0,1] \to \mathbb{R}$ be a smooth function in $H^1(0,1)$. Then, there exists a positive real number $k > 0$ such that*

$$\|f\|_{L^\infty(0,1)} \leq k \|f\|_{H^1(0,1)}.$$

It follows the following result:

Lemma 5.2.4. *Let $y_d(t) \in \mathbb{R}$ be the desired output. The closed-loop system (5.2.4) with the following control law*

$$u(t) = K_p \xi_1'(1,t) + K_i \int_0^t \xi_1'(1,\tau)d\tau, \tag{5.2.9}$$

where $\xi_1'(1,t) = y(t) - y_d(t)$, is globally asymptotically stable in $y_d(t)$.

Proof. Consider the smooth initial condition (ξ_0', χ_0') in $\mathbb{X}_1$ $(with\ \mathbb{X}_1 = H^1(0,1) \times H^1(0,1) \times \mathbb{R})$ and satisfying the C_0 and C_1 compatibility conditions of (5.2.4). So, by [Bastin and Coron, 2016, Theorem A.1], (5.2.4) is well-posed in $\mathbb{X}_1$, with

$$\begin{aligned}&\xi(t) \in C^0\left([0,+\infty); H^1((0,1);\mathbb{R}^2) \cap C^1\left([0,+\infty); L^2((0,1);\mathbb{R}^2)\right)\right),\\ &\chi \in C^1([0,+\infty),\mathbb{R}).\end{aligned}$$

By an approach similar to Lemma 5.2.2 we can show that the dynamical system (5.2.4) is exponentially stable in $\mathbb{X}_1$. That implies that

$$\lim_{t\to+\infty} \|\xi_1'(z,t)\|_{H^1(0,1)} = 0, \qquad \lim_{t\to+\infty} \|\xi_2'(z,t)\|_{H^1(0,1)} = 0.$$

Now employing the Sobolev embedding one can find that for $\forall z \in [0,1]$

$$\lim_{t\to+\infty} \xi_1'(z,t) = 0, \qquad \lim_{t\to+\infty} \xi_2'(z,t) = 0,$$

which implies that along the closed loop dynamics we have

$$y(t) - y_d(t) = \lim_{t\to+\infty} \xi_1'(1,t) = 0.$$

□

5.2.1.2 Numerical simulations

To assess the performance of the controller, simulation tests were carried out. These relate to the problems of tracking and rejection of the disturbance.
For the set point tracking test, set point steps of $5°C$ ($y_d(t) = 25°C$) and $-2°C$ ($y_d(t) = 18°C$) were imposed at times $t = 320s$ and $t = 650s$. For the parameters p_1, p_2 and μ of the considered Lyapunov functional, we solved the following LMI: $PM + M^\top P \leq -2\mu P$, using the YALMIP toolbox proposed by [Lofberg, 2004], and we found: $P = \mathrm{diag}(3.7646,\ 4.9098)$ and $\mu = 0.0012$. Moreover we took $p_3 = 1$. From condition i), the numerical value of proportional gain is $K_p = e^{-\mu}\sqrt{\frac{v_1 p_1}{v_2 p_2}} = 0.8787$. The numerical value of K_i is obtained by considering the maximum of ii): $K_{i-} = 0.189,\ K_{i+} = 0.2214$. The Figure 5.1 gives the variation of the outlet temperature $\xi_1(1,t)$, and that of the jacket $\xi_2(1,t)$. We see that the PI-controller ensures good tracking of the imposed set point, with physically admissible variations of the manipulated

quantity $\xi_2(1,t)$. The three-dimensional plot of the spatio-temporal evolution of the temperatures $\xi_1(z,t)$ and $\xi_2(z,t)$ of the heat exchanger (Figures 5.2a and 5.2b) during reference tracking confirms the correct behavior and performance of the PI-controller.
During the control law synthesis step, we assumed that the inlet temperature $\xi_{1,in}(t)$ of the internal fluid was a constant disturbance. Now let us evaluate the performance of the controller based on the variation of $\xi_{1,in}(t)$. To do this, a variation of -40% in the inlet temperature $\xi_{1,in}(t)$, was introduced at time $t = 500s$. The evolutions of the output, of the reference (constant) and of the control input are given by Figures 5.3a and 5.3b, and spatio-temporal evolutions in the Figure 5.3c and 5.3d. The obtained results show that the PI-regulator offers good performance since the effect of the disturbance is better damped and quickly rejected. This can be justified using the interaction analysis between the disturbance $\xi_{1,int}(t)$ and the controlled output $\xi_{1,out}(t)$. The approach used in this work consists in representing the Nyquist array of the transfer function $g_{12}(s)$ between $\xi_{1,out}(s)$ and $\xi_{2,in}(s)$, superposed by the Gershgorin circles of the transfer function $g_{11}(s)$ between $\xi_{1,out}(s)$ and $\xi_{1,in}(s)$; s is the Laplace variable[i]. The coordinates of the center of a circle are the real and imaginary parts of the transfer function $g_{12}(s)$ and the radius R of the circle is the sum of the modules of $g_{11}(s)$. The radius $R(\omega)$ is given by the following formula:

$$R(\omega) = |g_{11}(j\omega)|, \qquad \text{with } \omega \in [0, +\infty).$$

In the context of boundary control of heat exchanger, the variable pairing is based on the principle of main non-diagonal dominance, i.e. the best pairing is selected such that the Gershgorin's bands do not enclose the origins of the corresponding main non-diagonal Nyquist graphs. Figure 5.4 depict map for the Nyquist array of $g_{12}(s)$ with Gershgorin's bands respectively over the frequency range $\omega \in [10^{-2}, 10^{2}]$. Examination of this figure shows that the input variable $\xi_{1,in}$ weakly affects the controlled variable $\xi_{1,out}$ because the Gershgorin's bands do not enclose the origin of the complex plane.
Furthermore, although the disturbance is introduced at time $t = 500s$, its influence on the controlled output appears from time $t = 570s$. This is justified by the fact that the transfer between the disturbance $\xi_{1,in}(t)$ and controlled output $\xi_{1,out}(t)$ is characterized by a pure delay; see the transfer function $G_{11}(z,s)$ and $g_{11}(s)$ in Appendix B.1.

The heat exchanger is often connected to other thermal equipment which affects its operation [Burns and Cliff, 2014a]. For example, the velocity of the fluid to be cooled is linked to the equipment located upstream of the heat exchanger. Therefore, the controller must react appropriately to variations in the velocities of each fluid. To evaluate the performance of the PI-controller in this situations, velocity variations of the order of $\pm 35\%$ were operated at time $t = 250s$. According to the results obtained (Figures 5.5a to 5.5d), it is remarkable that despite the variations introduced, the tracking of the set point and the rejection of disturbance are perfectly ensured, and the variation of the

[i]The calculation of transfer functions is performed in the Appendix B.

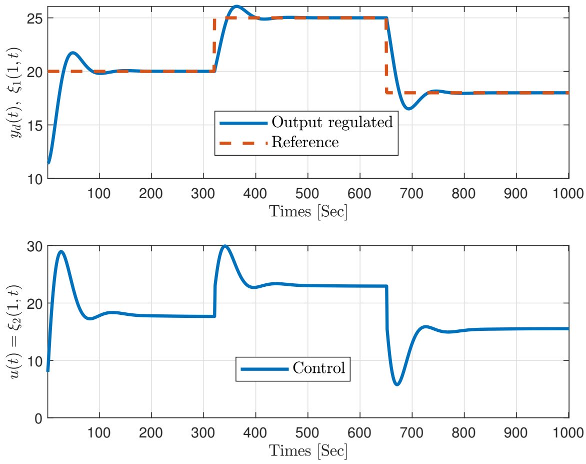
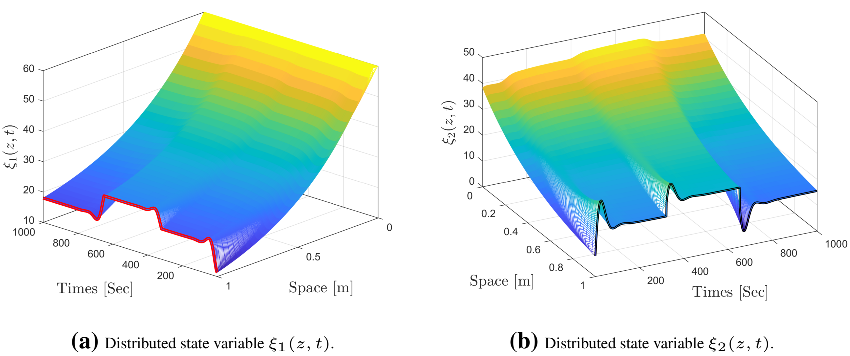


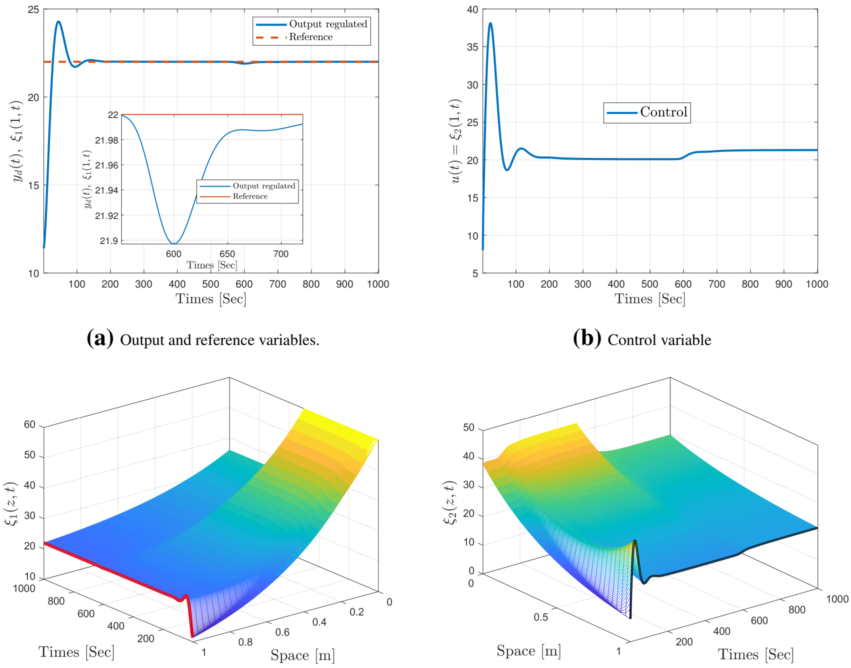
Figure 5.1: Reference, Output and control variables.

control variable $u(t)$ remains physically acceptable, and compensates for the disturbing effect induced by the variations of the speeds v_1 and v_2.
To finish this simulation step, we perform a robustness test of the controller with respect to the variations of the heat exchange coefficients α_1 and α_2. Indeed the variations of these parameters are generally induced by fluctuations of the input quantities of the heat exchanger. For this test, all the parametric changes occur at time $t = 300s$, after having imposed a variation of each parameter of $\mp 30\%$. Figures 5.6a-5.6d show the results obtained. It is remarkable that the control law succeeds in rejecting variations in the heat exchange coefficients. Nevertheless, as for the velocities, the introduced variations generate transiently a modification of the profiles of the temperatures.



(a) Distributed state variable $\xi_1(z,t)$. **(b)** Distributed state variable $\xi_2(z,t)$.

Figure 5.2: Distributed state variables for the reference tracking test.



(a) Output and reference variables.

(b) Control variable

(c) Distributed state variable $\xi_1(z,t)$.

(d) Distributed state variable $\xi_2(z,t)$.

Figure 5.3: Disturbance rejection $\xi_{1,in}$.

5.2.2 State-space approach

We propose to implement on the output feedback a proportional-integral controller defined by:

$$\begin{cases} u(t) = K_p(C_\Lambda \xi(t) - y_d(t)) + k_i K_I \chi(t) \\ \dfrac{d\chi}{dt} = C_\Lambda \xi(t) - y_d(t), \end{cases} \tag{5.2.10}$$

where $K_p \in \mathbb{R}$ is the proportional gain of the regulator, $k_i \in \mathbb{R}$ the turning parameter and $K_I \in \mathbb{R}$ the constant (vector) matrix. $y_d(t) \in \mathbb{R}$ is the reference and $\chi(t) \in \mathbb{R}$ the new state variable which corresponds to the integral of $C_\Lambda \xi(t) - y_d(t)$. The standard diagram of such a control approach is given in Figure 5.7. The closed-loop system is defined by

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} \xi \\ \chi \end{pmatrix} = \begin{pmatrix} A + BK_pC_\Lambda & Bk_iK_I \\ C_\Lambda & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \chi \end{pmatrix} - \begin{pmatrix} BK_p \\ 1 \end{pmatrix} y_d(t) \\ y(t) = C_\Lambda \xi(t). \end{cases} \tag{5.2.11}$$

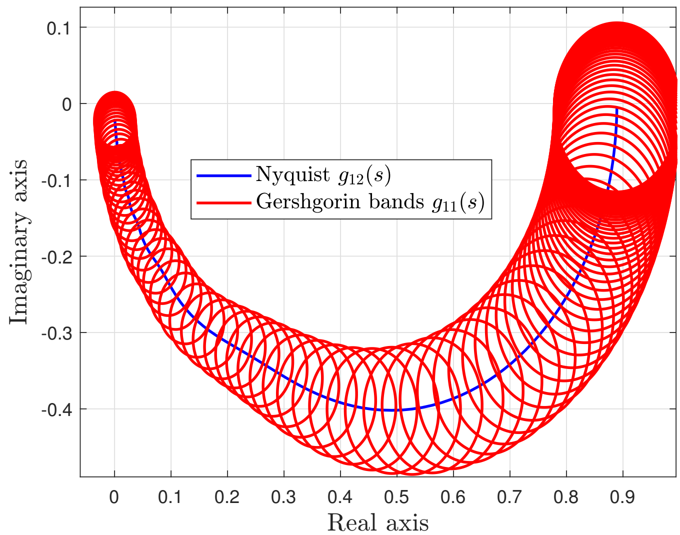
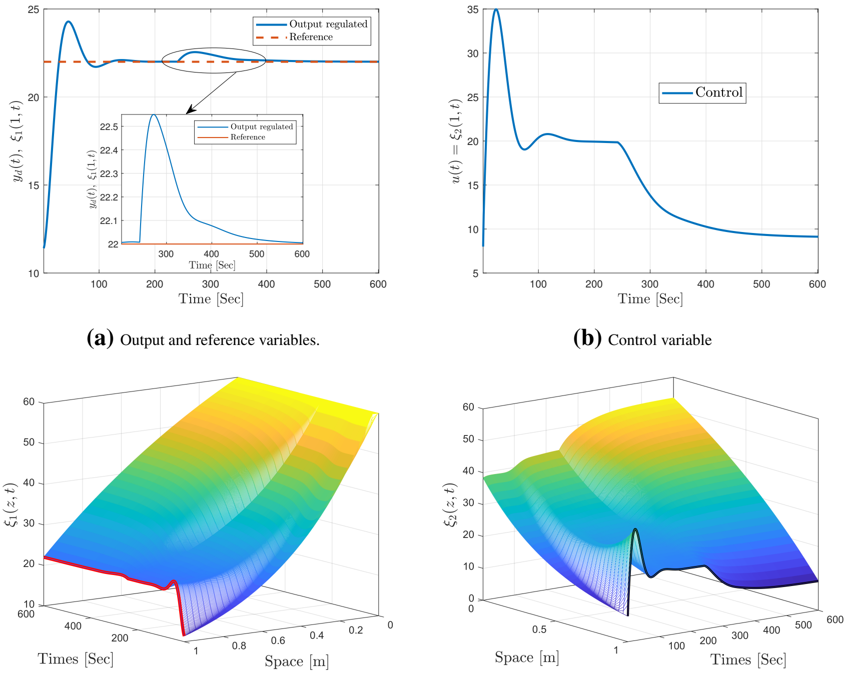


Figure 5.4: Gershgorin bands of open-loop system

The difference with the control approach developed in subsection 5.2.1 is that here we look for conditions on the controller parameters by exploiting the heat exchanger state model.
The aim is to choose K_p, k_i, and K_I such that:

1. The equilibrium state of the system (5.2.11) is exponentially stable. In other words, there are two positive scalars μ and ν such that $\forall\ t \in [0, +\infty)$ and $\xi_0 \in \mathcal{D}(A)$
$$\|\xi(t)\|_X \leq \nu e^{-\mu t}\|\xi_0\|.$$
2. The output of the system converges to $y_d(t)$, i.e.
$$\lim_{t\to+\infty} \|C_\Lambda \xi(t) - y_d(t)\| = 0, \quad t \in [0, \infty).$$
3. Stability and regulation must be guaranteed for all $\xi_0 \in \mathcal{D}(A)$, and this despite the constant disturbances acting on the system.

The mathematical formulation of the PI regulation problem was established by [Pohjolainen, 1982] for systems evolving in a Banach space. One of the assumptions of [Pohjolainen, 1982] is the analyticity of the semigroup. When the state space is a Hilbert space, this constraint was lifted by [Xu and Jerbi, 1995], by an application of Huang's theorem [Huang, 1985]. In [Xu and Jerbi, 1995], the semigroup analyticity assumption is replaced by exponential stability. Nevertheless the authors consider bounded input and output operators.



(a) Output and reference variables.

(b) Control variable

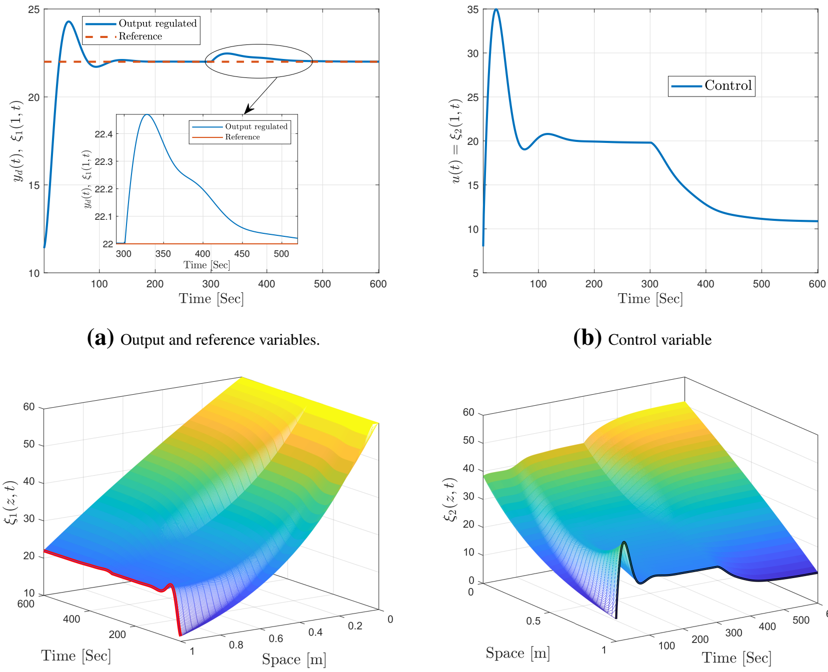
(c) Distributed state variable $\xi_1(z, t)$.

(d) Distributed state variable $\xi_2(z, t)$.

Figure 5.5: Case of parameter change v_1 and v_2 of the two fluids with constant set point.

This does not correspond to the case of the hyperbolic model that we will describe in the following. For systems described with unbounded input and output operators, [Xu and Sallet, 2014] show that the results of [Xu and Jerbi, 1995] can be applied provided that these operators are admissible for the semigroup and by using a Yosida approximation of output operator (see [Weiss, 1994], [Dehaye and Winkin, 2016]), so as to guarantee the regularity of the control-observation system. However, the expression for the integral gain of the controller given in [Xu and Jerbi, 1995] is a function of the norm of the resolvent of an unbounded operator, which is not easy to compute. In the following section we use a recent result of [Terrand-Jeanne et al., 2020] which results from a combination of Lyapunov's approach and operator perturbation theory.

The open-loop system being exponentially stable, we can treat separately the choice of the integral part of the controller and that of the proportional part [Terrand-Jeanne et al., 2020].



(a) Output and reference variables.

(b) Control variable

(c) Distributed state variable $\xi_1(z,t)$.

(d) Distributed state variable $\xi_2(z,t)$.

Figure 5.6: Case of parameter change α_1 and α_2 of the two fluids with constant set point.

5.2.2.1 Integral part of the PI-controller

Consider the control law:

$$\begin{cases} u(t) = k_i K_I \chi(t) \\ \dfrac{d\chi}{dt} = C_\Lambda \xi(t) - y_d(t). \end{cases} \tag{5.2.12}$$

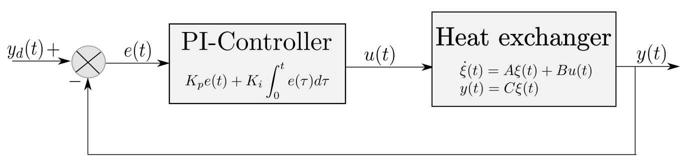


Figure 5.7: PI-regulation diagram by output feedback.

The closed-loop system (5.2.11) becomes:

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix}\xi\\ \chi\end{pmatrix} = \underbrace{\begin{pmatrix}A & Bk_iK_I\\ C_\Lambda & 0\end{pmatrix}}_{A_e}\begin{pmatrix}\xi\\ \chi\end{pmatrix} - \begin{pmatrix}0\\ 1\end{pmatrix} y_d(t) \\ y(t) = C_\Lambda \xi(t), \end{cases} \tag{5.2.13}$$

with $A_e \in \mathcal{D}(A_e) = \{\psi \in \mathbb{X} \text{ and } \frac{d\xi}{dz} \in X \text{ a.c.}, \xi_1(0) = 0; \xi_2(1) = 0\}$. The model being well posed, it remains to determine the expression of the integral gain allowing to guarantee the stability of the closed-loop system (5.2.13).
Note that the operator A_e defined in (5.2.13) can be written as follows: $A_e = A_1 + k_i A_2$, where $A_1 = \begin{pmatrix}A & 0\\ C_\Lambda & 0\end{pmatrix}$ generates a C_0-semigroup and represents the open-loop system ($k_i = 0$) of (5.2.13), and $A_2 = \begin{pmatrix}0 & BK_I\\ 0 & 0\end{pmatrix}$ bounded in $\mathbb{X}$. It comes from the form of A_1 that its spectrum is written as: $\sigma(A_1) = \sigma(A) \cup \{0\}$. Since A generates an exponentially stable semigroup (see Theorem 3.2.1), then a vertical line can separate $\sigma(A)$ and the isolated point $\{0\}$. [Pohjolainen, 1982] shows that this separation remains valid for a certain value of k_i lower than a limit value that we note k_i^*. To find the expression of k_i^* we shall use the solution based on the theory of the perturbation of operators. In this context we consider that the operator of the closed-loop system A_e is a disturbance of the operator of the open-loop system.
Under the assumption that A generates an analytical semigroup (meromorphic), B and C_Λ are admissible operators and $C_\Lambda A^{-1} B \neq 0$, [Xu and Sallet, 2014] showed that the integral gain which stabilizes the system is equal to: $k_i K_I = k_i \left(C_\Lambda A^{-1} B\right)^{-1}$, with k_i such that $0 < k_i < k_i^*$, and

$$k_i^* = \min_{\lambda \in \rho(A_1)} \left(\max \|A_2\| \|\mathrm{R}(\lambda, A_1)\| + 1\right)^{-1}.$$

To avoid having to calculate the norms of operator A_2 and resolvent $\mathrm{R}(\lambda, A_1)$, which are not easy to calculate, we use the recent result based on Lyapunov's approach [Terrand-Jeanne et al., 2020, Theorem 2].

Lemma 5.2.5. *If B and C_Λ are admissible operators for $\mathbb{T}(t)$, with $\mathbb{T}(t)$ analytical C_0-semigroup, then there exist $\mu > 0$ and a positive self adjoint operator $\mathcal{P}(z) \in \mathcal{L}(X)$ such that*

$$k_i^* = \frac{\nu}{2\left\|C_\Lambda A^{-1}\right\| \left\|\mathcal{P}B\left(C_\Lambda A^{-1}B\right)^{-1}\right\|}. \tag{5.2.14}$$

The self-adjoint positive operator $\mathcal{P}$ considered here is the Lyapunov operator; the following theorem gives its expression.

Theorem 5.2.2. *For all $\xi \in \mathcal{D}(A)$ the operator $A \in \mathcal{D}(A)$ generates an exponentially stable C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$ on $L^2(0,1) \times L^2(0,1)$. Moreover, the Lyapunov operator is given by*

$$\mathcal{P}(z) = \begin{pmatrix} \frac{1}{2\alpha_1}\left(1 - e^{-2\beta_1(1-z)}\right) & 0 \\ 0 & \frac{1}{2\alpha_2}\left(1 - e^{-2\beta_2 z}\right) \end{pmatrix}. \tag{5.2.15}$$

Proof. By [Curtain and Zwart, 1995, Lemma 4.1.24] we know that if $\mathcal{P}\mathcal{D}(A) \subset \mathcal{D}(A^*)$ the positive self-adjoint operator $\mathcal{P} \in \mathcal{L}(X)$ is a solution of

$$A^*\mathcal{P}\xi + \mathcal{P}A\xi = -\xi, \quad \text{for } \xi \in \mathcal{D}(A). \tag{5.2.16}$$

Let $\mathcal{P} = \text{diag}(p_1, p_2)$. By developing equation (5.2.16) we obtain:

$$\begin{cases} \left(A_{11}^* p_1 + p_1 A_{11} + 1\right)\xi_1 = 0 \\ \left(A_{12}^* p_2 + p_1 A_{12}\right)\xi_2 = 0 \\ \left(A_{21}^* p_1 + p_2 A_{21}\right)\xi_1 = 0 \\ \left(A_{22}^* p_2 + p_2 A_{22} + 1\right)\xi_2 = 0, \end{cases} \tag{5.2.17}$$

where $A_{11}^* = v_1\frac{d}{dz} - \alpha_1$ defined on $\mathcal{D}(A_{11}^*) = \{\xi_1 \in L^2(0,1) : \xi_1 \text{ a.c.}, \frac{d\xi_1}{dz} \in L^2(0,1) \text{ and } \xi_1(1) = 0\}$, and $A_{22}^* = -v_2\frac{d}{dz} - \alpha_2 \in \mathcal{D}(A_{22}^*) = \{\xi_2 \in L^2(0,1) : \xi_2 \text{ a.c.}, \frac{d\xi_2}{dz} \in L^2(0,1) \text{ and } \xi_2(0) = 0\}$. Since $\frac{d\,\cdot}{dz}p_1(z)\cdot I = \frac{dp_1}{dz}\cdot I + p_1(z)\cdot\frac{d\,\cdot}{dz}$ and $\frac{d\,\cdot}{dz}p_2(z)\cdot I = \frac{dp_2}{dz}\cdot I + p_2(z)\cdot\frac{d\,\cdot}{dz}$, one obtains that p_1 and p_2 are solution of the following differential system:

$$\begin{cases} v_1\dfrac{dp_1}{dz} - 2\alpha_1 p_1(z) + 1 = 0, & p_1(1) = 0 \\ v_2\dfrac{dp_2}{dz} + 2\alpha_2 p_2(z) - 1 = 0, & p_2(0) = 0. \end{cases} \tag{5.2.18}$$

We can see that for $z \in [0,1]$, $p_1(z) \geq 0$ and $p_2(z) \geq 0$. □

The $\left(C_\Lambda A^{-1}B\right)^{-1}$ can be deduced from the expression of the transfer function given by:

$$g_{12}(s) = \frac{2\beta_1 \sinh\left(\beta(s)\right)}{2\beta(s)\cosh\left(\beta(s)\right) + \gamma(s)\sinh\left(\beta(s)\right)}. \tag{5.2.19}$$

Thus,

$$K_I = \frac{(\beta_2 - 2\beta_1)e^{\beta_2-\beta_1} - \beta_1}{\beta_1\left(e^{\beta_2-\beta_1} - 1\right)}. \tag{5.2.20}$$

$C_\Lambda A^{-1}$ is mapping $u' \to y$ defined by the following ordinary differential equation:

$$\begin{cases} \Lambda \dfrac{d\xi}{dz} + M\xi = 0, \\ (\xi_1(0) \quad \xi_2(1))^\top = u', \\ y = \xi_1(1). \end{cases}$$

So,

$$C_\Lambda A^{-1} y = \begin{pmatrix} \dfrac{\beta_1\beta_2(e^{(\beta_2-\beta_1)}-1)}{\beta_2-\beta_1}h - \dfrac{\beta_2-\beta_1 e^{(\beta_2-\beta_1)}}{\beta_2-\beta_1} \\ \beta_1 h \end{pmatrix}^\top y, \tag{5.2.21}$$

with $h = \dfrac{1-e^{(\beta_2-\beta_1)}}{(\beta_2-2\beta_1)e^{(\beta_2-\beta_1)}-\beta_1}$. Since A_e is exponentially stable, then the closed-loop system has robustness against external disturbances and modelling errors. In particular when the disturbance is assumed to be constant, we have $\lim\limits_{t\to+\infty} \|y(t)-y_d\| = 0$ (see [Pohjolainen, 1982] for the proof).

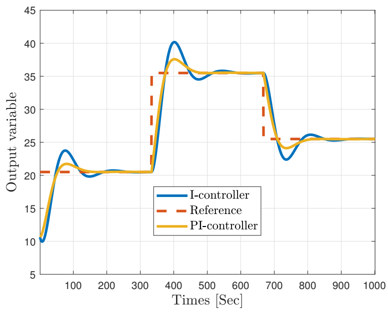
5.2.2.2 Proportional part of the PI-controller

It is more difficult to choose well the proportional gain K_p in our case. Without assuming that the operator A is the generator of an exponentially stable semigroup we cannot prove the same spectrum separation result as in [Pohjolainen, 1982, Theorem 3.6] for the perturbed operator $A + BK_pC_\Lambda$. In the case where the operator is regular spectral and the operator C_Λ satisfies some condition associated with the spectrum $\sigma(A)$ (see [Xu and Sallet, 1992]). Theorem 3.6 of [Pohjolainen, 1982] still gives a good choice of the proportional gain. In general we only have the following result [Xu and Jerbi, 1995, Theorem 2].

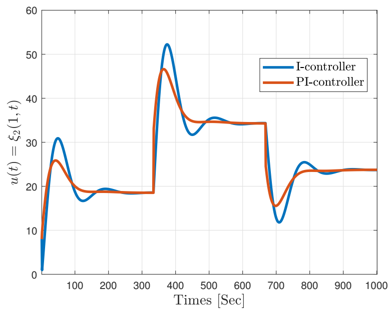
Theorem 5.2.3. *Assume that the operator A is the generator of an exponentially stable C_0-semigroup on X and C_Λ is $A-$admissible. There exists a constant number $K_p^* > 0$ such that for each $K_p \in [-K_p^*, K_p^*]$, the perturbed operator $A + BK_pC_\Lambda$ still generates an exponentially stable C_0-semigroup.*

5.2.2.3 Numerical simulation

The numerical value of $k_iK_I = 0.9k_i^*K_I = 0.1788$ is obtained from expressions (5.2.14), (5.2.15), (5.2.20) and (5.2.21), for a value of $k_i^* = 0.0886$. We note that we have the same order of magnitude as the integral gain obtained using Lyapunov's approach. Furthermore, the numerical value of the proportional gain considered is that obtained using Lyapunov's approach. According to [Pohjolainen, 1982, Terrand-Jeanne et al., 2020], for a linear (infinite-dimensional) system when the open-loop system is exponentially stable, a simple integral controller is enough to stabilize the system and to achieve regulation or set point tracking objectives. Figures 5.8a and 5.8b, compare the efficiency of an I-controller and PI-controller in a setpoint tracking framework. Even if in both cases the objective is achieved, as far as the I-controller is concerned, strong oscillations of the controlled and manipulated variable can be observed during the transient periods corresponding to the set point changes. In addition, in this case we note that the response of the system is similar to that of a non-minimum phase system. Figure 5.9a-5.9b and 5.10a-5.10b show the distribution temperatures $\xi_1(z,t)$ and $\xi_2(z,t)$ in both cases.

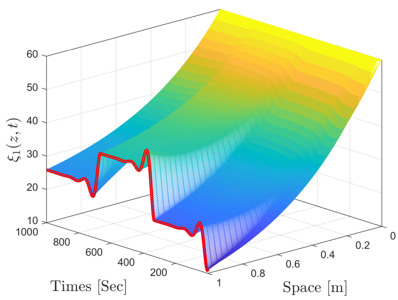


(a) Output variable $y(t)$ and reference $y_d(t)$.

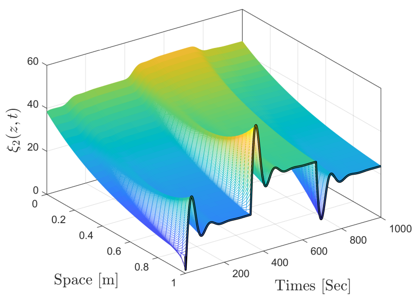


(b) Control variable $u(t)$.

Figure 5.8: Performance of I-controller and PI-controller.



(a) Distributed state $\xi_1(z,t)$.

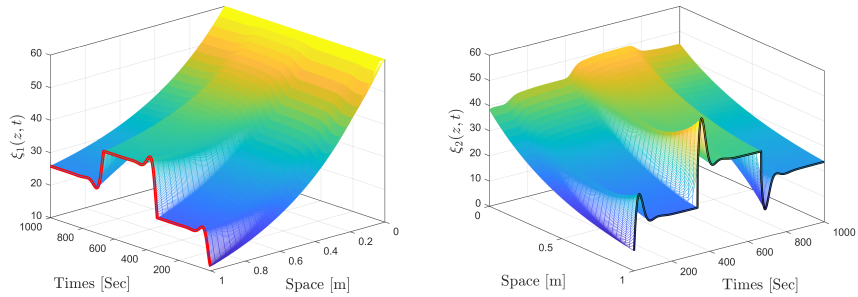


(b) Distributed state $\xi_2(z,t)$.

Figure 5.9: I-controller, distributed state variable.

5.3 Conclusion

As we have seen during this chapter, the boundary output feedback control strategy offers an interesting alternative for the control of systems of conservation laws, when observation is done at the boundary of the system. This



(a) Distributed state $\xi_1(z,t)$. **(b)** Distributed state $\xi_2(z,t)$.

Figure 5.10: PI-controller, distributed state variable.

method fits perfectly into the framework of heat exchangers. The study was carried out using an infinite-dimensional model of counterflow heat exchangers to which we associated a PI-controller to one of the boundary conditions. We have given conditions on the parameters of the regulator which make it possible to guarantee the stability of the closed-loop system and a zero tracking error, by successively using a Lyapunov approach and a state-space approach. The proposed approaches were illustrated in simulation. The results confirmed the efficiency of the boundary output feedback control strategy. Indeed, in addition to the rapid rejection of the disturbing effect due to the variation of the inlet temperature of the internal fluid of the heat exchanger, the dynamic behavior of the regulated output is better damped and the constraints of the regulation variable are less energy. Moreover, it has been shown that the PI-controller has interesting properties to cope with variations in the transport velocities of each fluid.
In the next chapter we propose a control law which allows to take into account the complete state of the system.

6

LQ-OPTIMAL CONTROL

Contents

6.1 Introduction

In this chapter we are interested in the LQ-optimal control of the counterflow heat exchanger. The choice of this approach is based on the fact that it offers better performance than many others (see for example [Hasan et al., 2016] where a comparison is made with the backstepping approach).
Linear quadratic (LQ)-optimal control is an old field in control theory. However its application to infinite dimensional systems constitutes a very active research area. To solve an optimal control problem two methods exist in the literature: the spectral factorization method (see [Callier and Winkin, 1990], [Callier and Winkin, 1992], [GRABOWSKI, 1994], [Callier and Dumortier, 1996], [Callier and Winkin, 2002]), and that based on the resolution of an operator Riccati equation (ORE) [Curtain and Zwart, 1995, Chapter 5], [Morris, 2020, Chapter 4]. The first is based on the description of the system in the frequency domain and uses several computation steps, including the resolution of a Diophantine equation, which is not easy to solve when it comes to infinite dimensional systems for which the transfer functions are irrational. On the other hand it is always possible to solve an ORE, especially when the functions are developable in Fourier series in a well-defined Hilbert space. Generally a control

law developed from this approach requires complete knowledge of the state of the system for its implementation, what is not possible, generally speaking, for distributed parameter systems. One of the solutions consists in coupling the control law to a state observer whose role is to provide the estimated state. In the context of hyperbolic systems this strategy was developed recently by [Aksikas, 2020], [Aksikas, 2021]. The approach consists in solving the LQ-optimal problem related to the direct problem and to its dual, in order to determine the optimal feedback operator as well as the optimal output injection operator.
In this chapter this control approach is applied to the heat exchanger model considered in this work. Regarding optimal control, the existing results were developed from a model approximated by ordinary differential equations [Burns and Cliff, 2014b], [Burns and Cliff, 2014a]. The originality of our approach lies in the resolution of LQ-optimal problem from the PDEs model of the heat exchanger. Furthermore, this control law is coupled with an integral action, in order to guarantee the rejection of the constant disturbances and cancellation of the tracking error.
The chapter is organized as follows: In Section 6.2 we recall the theoretical foundations of the LQ-optimal control of infinite-dimensional systems. Section 6.3 is intended to apply this theory to the heat exchanger model considered in this work.

6.2 Optimal Linear-Quadratic controller design

6.2.1 Finite-time LQ-optimal control

Consider a standard controlled system

$$\begin{cases} \dfrac{d\xi}{dt} = A\xi(t) + Bu(t), \qquad \xi(0) = \xi_0, \\ y(t) = C\xi(t), \end{cases} \tag{6.2.1}$$

where A with domain $\mathcal{D}(A)$ generates a strongly continuous semigroup $\mathbb{T}(t)$ on a Hilbert space X and $B \in \mathcal{L}(U, X)$ where the control space U is another Hilbert space, $C \in \mathcal{L}(X, Y)$ (with Hilbert space Y) weights the state over the time interval $[0, T]$. For all $t \in [0, T]$ the state trajectory of the state linear system are given by

$$\begin{cases} \xi(t) = \mathbb{T}(t)\xi_0 + \displaystyle\int_0^t \mathbb{T}(T-\tau)Bu(\tau)d\tau \\ y(t) = C\xi(t). \end{cases} \tag{6.2.2}$$

Consider the cost functional

$$J(u,\xi_0;T) = \langle \xi(T), \Pi\xi(T)\rangle + \int_0^T \langle C\xi(\tau), C\xi(\tau)\rangle + \langle u(\tau), Ru(\tau)\rangle d\tau, \tag{6.2.3}$$

where $\xi(t)$ is given by (6.2.2) and $R \in \mathcal{L}(U)$ is a self-adjoint positive definite operator weighting the control, $\Pi \in \mathcal{L}(X)$ is a self-adjoint and nonnegative operator weighting the final state. The operator R, C, and Π are design parameters and are defined based on the control system objectives. Our control problem is, given a $\xi_0 \in X$, to find an optimal control $u \in L^2((0,T);U)$ that minimizes the cost functional $J(u,\xi_0;T)$ over all trajectories of system (6.2.1). Since for a given $\xi_0 \in X$ the trajectories of (6.2.1) are completely determined by the inputn it follows that this is equivalent to minimizing the cost functional (6.2.3) over all $u \in L^2((0,T);U)$ subject to the constraint 6.2.2).

Theorem 6.2.1. *Consider the differential Riccati equation*

$$\frac{d}{dt}\langle \Pi(t)\xi_1, \Pi(t)\xi_2\rangle + \langle A\xi_1, \Pi(t)\xi_2\rangle + \langle \Pi(t)\xi_1, A\xi_2\rangle + \langle C\xi_1, C\xi_2\rangle \\ - \langle B^*\Pi(t)\xi_1, R^{-1}B^*\Pi(t)\xi_2\rangle = 0, \quad \xi_1,\xi_2 \in \mathcal{D}(A). \tag{6.2.4}$$

The differential equation (6.2.4) has a unique semi-definite solution $\Pi(t) \in \mathcal{L}(X,X)$, $\Pi(t)(\mathcal{D}(A)) \subset \mathcal{D}(A^*)$, unique in the class of continuous, self-adjoint operators on X such that for all $t \in]0,T[$, $\langle \Pi(t)\xi_1, \xi_2\rangle$ is differentiable for $\xi_1, \xi_2 \in \mathcal{D}(A)$, and

$$\min_{u\in L^2((0,T);U)} J(u,\xi_0,T) = \langle \Pi(0)\xi_0, \xi_0\rangle,$$

and, defining

$$K(t) = R^{-1}B^*\Pi(t), \tag{6.2.5}$$

the optimal control is $u(t) = -K(t)\xi(t)$, where $\xi(t)$ is given by (6.2.2).

6.2.2 Infinite-time LQ-optimal control

The optimal control $u(t) = -R^{-1}B\Pi(t)\xi(t)$ minimizes the cost functional (6.2.3) on the finite time interval $[0,T]$, but we are also interested in the long-term behaviour as $T \to \infty$, in particular, the stability properties of the closed-loop system. To this end, in this section, we shall consider the linear quadratic control problem on the infinite-time interval, $[0,\infty)$, for the state linear system (6.2.1).

The infinite-time LQ-controller design objective is to find a control $u(t)$ so that the cost functional

$$J(u,\xi_0) = \int_0^\infty \langle C\xi(\tau), C\xi(\tau)\rangle + \langle u(\tau), Ru(\tau)\rangle d\tau, \tag{6.2.6}$$

where $R \in \mathcal{L}(U)$ is a self-adjoint operator control, and $C \in \mathcal{L}(X, Y)$ (with Hilbert space Y) is minimized, subject to $\xi(t)$ given by (6.2.2).

Definition 6.2.1. *The dynamic system (6.2.1) with cost functional (6.2.6) is optimizable if for every $\xi_0 \in X$ there exists $u \in L^2((0,\infty); U)$ such that the cost is finite.*

We remark that if (6.2.1) is exponentially stabilizable, then it satisfies the finite-cost condition.

Theorem 6.2.2. *If (6.2.1) with cost functional (6.2.6) is optimizable and (A, C) is detectable, then for every $\xi_0 \in X$ the cost functional (6.2.6) has a minimum. Furthermore, there exists a self-adjoint nonnegative operator $\Pi \in \mathcal{L}(X)$ such that*

$$\min_{u \in L^2((0,\infty);U)} J(u, \xi_0) = \langle \xi_0, \Pi\xi_0 \rangle.$$

Also, $\Pi(\mathcal{D}(A)) \subset \mathcal{D}(A^*)$ and Π is the unique nonnegative solution to the operator equation

$$\langle A\xi_1, \Pi\xi_2\rangle + \langle \Pi\xi_1, A\xi_2\rangle + \langle C\xi_1, C\xi_2\rangle - \langle B^*\Pi\xi_1, R^{-1}B^*\Pi\xi_2\rangle = 0, \quad (6.2.7)$$

for all $\xi_1, \xi_2 \in \mathcal{D}(A)$. Defining

$$K = R^{-1}B^*\Pi, \quad (6.2.8)$$

the corresponding optimal control is $u(t) = -K\xi(t)$ and $A - BK$ generates an exponential stable semigroup $\mathbb{T}_{(A-BK)}(t)$.

Theorem 6.2.2 implies that, as for finite-dimensional systems, optimality is equivalent to exponential stabilizability. The Riccati operator equation (6.2.7) is equivalent to

$$\left[A^*\Pi + \Pi A - \Pi B R^{-1} B^* \Pi + C^* C\right] \xi = 0, \qquad \forall\, \xi \in \mathcal{D}(A). \quad (6.2.9)$$

6.3 Optimal control of heat exchanger

In this section we will apply the optimal control approach to the model of the heat exchanger whose equations are:

$$\partial_t \xi(z,t) = \Lambda \partial_z \xi(z,t) + M\xi(z,t), \quad (6.3.1)$$

where $\Lambda = \begin{bmatrix} -v_1 & 0 \\ 0 & v_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $M = \begin{bmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $z \in [0,1]$, $t \geq 0$, $\xi_1(z,t), \xi_2(z,t) \in \mathbb{R}$ denote the distributed state variables of each component typically temperatures, v_i and α_i the positive parameters. We associate to this equation the following boundary conditions:

$$\xi_1(0,t) = \xi_{1,in}(t), \ \xi_2(1,t) = \xi_{2,in}(t), \tag{6.3.2}$$

where $\xi_{1,in}$ and $\xi_{2,in}$ are the inlet temperatures of the heat exchanger, and we consider that $\xi_{1,in}$ is constant, and $\xi_{2,in}(t) = u(t) \in U$ as a control variable at time $t > 0$.
In Chapter 3 we wrote the counterflow heat exchanger model in the abstract form of a boundary control and observation system, with unbounded control and observation operators. The standard theory dedicated to the LQ-optimal control approach that we described in the previous section, cannot be applied directly because of the unboundedness of the control operator. In the solution proposed by [Weiss and Weiss, 1997], the admissible state feedback operator is obtained by successfully solving a operator Riccati equation as well as a spectral factorization problem. In order not to resort to additional computation steps, in particular to obtain the spectral factor of the Popov function which requires the description of the system in the frequency domain, we will reduce ourselves to the framework of infinite-dimensional systems with bounded control and observation operators, using the [Fattorini, 1968] approach for boundary control, as described in [Curtain and Zwart, 1995, Chapter 3.3].

6.3.1 Abstract formulation

The above linear system can be formulated as an abstract boundary control problem on the Hilbert space X,

$$\begin{cases} \dot{\xi}(t) = \mathfrak{A}\xi(t), & \xi(0) = \xi_0, \\ \mathcal{B}\xi(t) = u(t), & \\ y(t) = \mathcal{C}\xi(t), & \end{cases} \tag{6.3.3}$$

where the linear operator $\mathfrak{A}$ is defined by

$$\mathfrak{A} = \begin{pmatrix} -v_1 \dfrac{d}{dz} - \alpha_1 \cdot I & \alpha_1 \cdot I \\ \alpha_2 \cdot I & v_2 \dfrac{d}{dz} - \alpha_2 \cdot I \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

on its domain of definition given by

$$\mathcal{D}(\mathfrak{A}) = \mathcal{D}(A_{11}) \oplus \mathcal{D}(A_{22}) = \Big\{ \xi = (\xi_1 \ \xi_2)^\top \in X : \ \xi \text{ absolutely continuous} \\ \text{(a.c.)}, \ \frac{d\xi}{dz} \in X \text{ and } \xi_1(0) = 0 = \xi_2(1) \Big\},$$

where I is the identity operator on the space $L^2(0,1)$. The input operator $\mathcal{B} : X \to U$ is given by

$$\mathcal{B}\xi = \xi_2(1).$$

For all $\xi = (\xi_1\ \xi_2)^\top \in X$, the output operator $\mathcal{C} : X \to Y$ is given by

$$\mathcal{C}\xi = \langle c, \xi_1 \rangle = \int_0^1 c(z)\xi_1(z)dz, \tag{6.3.4}$$

where $c(z)$ is a space-varying continuous function on the interval $[0, 1]$.

6.3.1.1 Triangularized model

In general, whether bounded or unbounded (observation and control) operators, the design of an optimal feedback operator is obtained by solving an Operator Riccati Equation (ORE) [Weiss and Weiss, 1997], [Curtain and Zwart, 1995, Chapter 5]. Results specific to hyperbolic PDEs of the form (6.3.1)-(6.3.2) exist in the literature. In [Aksikas et al., 2009], [Aksikas, 2020], [Aksikas, 2021] it is shown that when the transport velocities of the fluids are different (as in the case of model (6.3.1)-(6.3.2)), there is a diagonal operator, unique solution of the ORE. It is easy to verify that in the case of the model (6.3.1)-(6.3.2), the use of such operator does not guarantee the uniqueness of solution of the ORE (there are two ordinary differential equations, and two coupled compatibility equations impossible to satisfy). This problem can be circumvented by putting the model (6.3.1)-(6.3.2) in a lower triangular form. Although that is impossible by classical approaches (the matrix M is singular, the system generates a non-normal semigroup [Burns and Zietsman, 2016]), we show the possibility of doing it by solving a certain Riccati differential equation. The followed approach is that developed by [Aksikas et al., 2007], and which was applied to the model of a fixed-bed tubular reactor.
Consider the state transformation $\zeta = S\xi$ with $S \in \mathcal{L}(Z)$ a linear bounded operator given by:

$$S := \begin{pmatrix} I & \vartheta \cdot I \\ 0 & I \end{pmatrix}, \tag{6.3.5}$$

with $\vartheta \in [0, 1]$ a C^1-bounded function which satisfies certain conditions. Applying the state transformation defined by S, we find the operator $\tilde{\mathfrak{A}} := S\mathfrak{A}S^{-1}$ given by:

$$\begin{aligned}\tilde{\mathfrak{A}} &= \begin{pmatrix} I & \vartheta \cdot I \\ 0 & I \end{pmatrix} \begin{pmatrix} -v_1 \dfrac{d}{dz} - \alpha_1 \cdot I & \alpha_1 \cdot I \\ \alpha_2 \cdot I & v_2 \dfrac{d}{dz} - \alpha_2 \cdot I \end{pmatrix} \begin{pmatrix} I & \vartheta \cdot I \\ 0 & I \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -v_1 \dfrac{d}{dz} - (\alpha_1 + \alpha_2\vartheta) \cdot I & (*) \\ \alpha_2 \cdot I & v_2 \dfrac{d}{dz} - \alpha_2(1-\vartheta) \cdot I \end{pmatrix},\end{aligned}$$

on its domain $\mathcal{D}(\tilde{\mathfrak{A}}) = \mathcal{D}(\mathfrak{A})$, with $(*)$ equal to:

$$(*) := \left(v_1 \frac{d\vartheta}{dz} - \alpha_2 \vartheta^2(z) - (\alpha_1 + \alpha_2)\vartheta(z) + \alpha_1 \right) \cdot I = 0. \tag{6.3.6}$$

For the operator $\tilde{\mathfrak{A}}$ to be lower triangular, it suffices that ϑ satisfies the following Riccati differential equation (RDE):

$$v_1 \frac{d\vartheta}{dz} - \alpha_2 \vartheta^2(z) - (\alpha_1 + \alpha_2)\vartheta(z) + \alpha_1 = 0, \quad \vartheta(0) = 0, \tag{6.3.7}$$

to which we associate the following linear Hamiltonian system (see [Abou-Kandil et al., 2003, Chapter 3] for more information):

$$\frac{d}{dz}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix} -\frac{\alpha_1+\alpha_2}{v_1} & -\frac{\alpha_2}{v_1} \\ \beta_1 & 0 \end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix}, \begin{pmatrix}\theta_1(0) = 1 \\ \theta_2(0) = 0\end{pmatrix}. \tag{6.3.8}$$

Lemma 6.3.1. *If $\theta(z) = (\theta_1(z)\ \theta_2(z))^\top$ is solution of system (6.3.8), then $\vartheta(z) = \theta_2(z)(\theta_1(z))^{-1}$ is solution of the RDE (6.3.7) for all $z \in [0,1]$ such that $\theta_1(z) \neq 0$.*

Theorem 6.3.1. *The RDE (6.3.7) admits a unique bounded C^1-solution on $[0,1]$ given by:*

$$\vartheta(z) = \alpha_2 \frac{e^{-\frac{\alpha_2}{v_1}z} - e^{-\beta_1 z}}{\alpha_2 e^{-\frac{\alpha_2}{v_1}z} + \alpha_1 e^{\beta_1 z}}. \tag{6.3.9}$$

Consequently, the operator $S \in \mathcal{L}(Z)$ given by (6.3.5) is a similarity transformation which triangularizes the operator $\mathfrak{A}$.

See [Aksikas et al., 2007] for the proof of the existence and uniqueness of a solution $\vartheta(z)$ of the RDE (6.3.7).
Likewise, we define the input operator $\tilde{\mathcal{B}}$ by

$$\tilde{\mathcal{B}}\zeta(t) = \mathcal{B}S^{-1}\zeta(t) = \mathcal{B}\begin{pmatrix}\zeta_1(t) - \vartheta(\cdot)\zeta_2(t) \\ \zeta_2(t)\end{pmatrix} = \zeta_2(1,t), \tag{6.3.10}$$

and $\forall \zeta = (\zeta_1, \zeta_2)^\top \in X$ the observation operator $\tilde{\mathcal{C}} = \mathcal{C}S^{-1} \in \mathcal{L}(X, Y)$ by

$$\tilde{\mathcal{C}}\zeta(t) = \int_0^1 c(z)\zeta_1(z,t)dz - \int_0^1 c(z)\vartheta(z)\zeta_2(z,t)dz. \tag{6.3.11}$$

Since S is a regular and differentiable operator, then for all $\zeta \in X$ the dynamic abstract system (6.3.3) is equivalent to:

$$\begin{cases} \dot{\zeta}(t) = \tilde{\mathfrak{A}}\zeta(t), & \zeta(0) = \zeta_0, \\ \tilde{\mathcal{B}}\zeta(t) = u(t), & \\ y(t) = \tilde{\mathcal{C}}\zeta(t). & \end{cases} \tag{6.3.12}$$

Both $\Sigma(\mathfrak{A}, \mathcal{B}, \mathcal{C})$ and $\Sigma(\tilde{\mathfrak{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ being equivalent, by the Theorem 3.2.1 it follows that:

Corollary 6.3.1 (Well-posedness and stability). *For $\zeta(0) = \zeta_0 \in \mathcal{D}(\tilde{\mathfrak{A}})$, the abstract system (6.3.12) admits a unique solution $\zeta \in C^0\left([0, +\infty), \mathcal{D}(\tilde{\mathfrak{A}})\right) \cap C^1\left([0, +\infty), X\right)$. The mild solution of (6.3.12) is given by:*

$$\zeta(t) = e^{\tilde{\mathfrak{A}}t}\zeta_0, \qquad \forall\, t \in [0, +\infty).$$

Furthermore, the operator $\tilde{\mathfrak{A}}$ generates an exponentially stable C_0−semigroup denoted by $(\tilde{\mathbb{T}}(t))_{t\geq 0} = e^{\tilde{\mathfrak{A}}t}$ on X, i.e. there $N > 0$ and $\mu > 0$ such that

$$\|\tilde{\mathbb{T}}(t)\| \leq Ne^{-\mu t}, \qquad t \geq 0.$$

6.3.1.2 Boundary control

For reachability analysis, it is necessary to return to a "state representation" form: $\dot{\xi}(t) = \mathfrak{A}\xi(t) + Bu(t)$, with homogeneous boundary conditions. This is possible by considering some transformations which consist in extracting the boundary control part of the dynamic model (6.3.1)-(6.3.2) and rewriting it as a boundary control Fattorini model [Fattorini, 1968]. In that case, the goal is to find a bounded operator $\tilde{B} \in \mathcal{L}(U, X)$ such that

a) for all $u(t) \in \mathbb{R}$, $\tilde{B}u \in \mathcal{D}(\tilde{\mathfrak{A}})$,

b) the operator $\tilde{\mathfrak{A}}\tilde{B} \in \mathcal{L}(U, X)$,

c) $\tilde{\mathcal{B}}\tilde{B}u = u$ for all $u \in U$.

If the operator $\tilde{B}$ is chosen such that for all $u \in U$

$$\tilde{B}u(t) = \begin{pmatrix} b_1(z) \\ b_2(z) \end{pmatrix} u(t), \tag{6.3.13}$$

where $b_i(z)$ $(i = 1, 2)$ are continuous functions such that $b_i(1) = 1$, then it can be easily observed that a) and b) hold. Moreover, c) is also satisfied since

$$\tilde{\mathcal{B}}\tilde{B}u = b_2(z = 1)u = u.$$

Now we are in a position to define the new operator

$$\tilde{A} : \mathcal{D}(\tilde{A}) \to X \quad \text{by} \quad \tilde{A}\zeta = \tilde{\mathfrak{A}}\zeta,$$

on its domain $\mathcal{D}(\tilde{A}) = \mathcal{D}(\tilde{\mathfrak{A}}) \cap \ker(\tilde{\mathcal{B}})$. Let us consider the new state $v(t) = \zeta(t) - \tilde{B}u(t)$ and the new input $\tilde{u}(t) = \dot{u}(t)$. Then, by using the augmented state $\zeta^e(t) = \begin{pmatrix} u(t) & v(t) \end{pmatrix}^\top \in X^e := U \oplus X$, the system can be written as follows

$$\begin{cases} \dot{\zeta}^e(t) = A^e\zeta^e(t) + B^e\tilde{u}(t), & \zeta(0) = \zeta_0 \\ y(t) = C^e\zeta^e(t), \end{cases} \tag{6.3.14}$$

where the operator $A^e \in U \oplus \mathcal{D}(\tilde{A})$, together with the operators $B^e \in \mathcal{L}(U, X^e)$ and $C^e \in \mathcal{L}(X^e, Y)$ by

$$A^e = \begin{pmatrix} 0 & 0 \\ \tilde{\mathfrak{A}}\tilde{B} & \tilde{A} \end{pmatrix}, \quad B^e = \begin{pmatrix} I \\ -\tilde{B} \end{pmatrix}, \quad C^e = \tilde{\mathcal{C}}\begin{pmatrix} \tilde{B} & I \end{pmatrix}. \tag{6.3.15}$$

The operator $\tilde{\mathfrak{A}}\tilde{B}$ is given explicitly by

$$\tilde{\mathfrak{A}}\tilde{B} = \begin{pmatrix} -v_1\dfrac{db_1}{dz} - (\alpha_1 + \alpha_2\vartheta(z))b_1 \\ \alpha_2 v_1 + v_2\dfrac{db_2}{dz} - \alpha_2(1 - \vartheta(z))b_2 \end{pmatrix} \cdot I := \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \cdot I.$$

It should be noted that the spectrum of the operator A^e is such that: $\sigma(A^e) = \sigma(\mathfrak{A}) \cup \{0\}$. Despite this, it is always possible to find an operator from Lyapunov who guarantees the exponential stability of the semigroup generated by A^e.

Theorem 6.3.2 (Stabilizability). *Let us consider the operators A^e and B^e given by (6.3.15). Then $\Sigma(A^e, B^e, -)$ is exponentially stabilizable.*

Proof. Let $K^e = (k_1\ K) \in \mathcal{L}(X^e, \mathbb{R})$ with $k_1 \in \mathbb{R}$ and $K \in \mathcal{L}(X, \mathbb{R})$ a bounded operator such that $(A^e - B^eK^e)$ generates an exponentially stable C_0-semigroup $(\mathbb{T}_{(A^e - B^eK^e)}(t))_{\geq 0}$ on X^e. If we choose $k_1 > 0$ and $K = 0$, the resulting closed-loop operator become:

$$A^e - B^eK^e = \begin{pmatrix} -k_1 & 0 \\ \tilde{\mathfrak{A}}\tilde{B} - \tilde{B}k_1 & \tilde{A} \end{pmatrix} = \begin{pmatrix} A^e_{11} & 0 \\ A^e_{21} & A^e_{22} \end{pmatrix}.$$

This operator is a triangular operator whose diagonal operators A^e_{11} and A^e_{22} generate exponentially C_0-semigroup on $\mathbb{R}$ and X (according to the Corollary 6.3.1), respectively. An thus, by [Curtain and Zwart, 1995, Lemma 3.2.2] and for $t \geq 0$, $\mathbb{T}_{(A^e - B^eK^e)}(t)$ is exponentially stable on X^e. □

6.3.2 LQ-optimal feedback control

Having defined the system in the form of a boundary control observation system, we now turn to the calculation of the optimal state feedback operator. The finite-time optimization approach has as solution the differential equation (6.2.4). For PDEs, this differential equation is an operator on an infinite-dimensional Hilbert space and it cannot be solved exactly. In this chapter we are going to consider the approach by optimization over an infinite time whose cost functional is given by (6.2.6), so as to work on a stationary equation defined for all $\zeta^e \in \mathcal{D}(A^e)$ and $\Pi(\mathcal{D}(A^e)) \subset \mathcal{D}(A^{e*})$ by

$$\left[A^{e*}\Pi + \Pi A^e + C^{e*}C^e - \Pi B^e r^{-1} B^{e*}\Pi\right]\zeta^e = 0, \tag{6.3.16}$$

and where r is a positive function. In this case, the quadratic cost functional (6.2.6) is minimized and the closed-loop system is stabilized by the unique control $\tilde{u}(t)$ given for any $t \geq 0$ by

$$\tilde{u} = -K^e_{\text{opt}}\zeta^e(t) = -\frac{1}{r}B^{e*}\Pi\zeta^e(t), \quad \text{with } \zeta^e(t) = e^{(A^e - B^e K^e)t}\zeta^e_0, \tag{6.3.17}$$

where the optimal feedback operator K^e_{opt} was considered positive. Solving the operator Riccati equation (6.3.16) could be a very challenging problem. For linear hyperbolic systems of the form (6.3.1), it is shown in [Aksikas et al., 2009, Theorem 5] and [Aksikas, 2021] that the operator Riccati equation (6.3.16) admits a diagonal solution $\Pi = \operatorname{diag}(\pi_1, \Pi_r) \in \mathcal{L}(X^e)$, with $\Pi_r = \operatorname{diag}(\pi_2, \pi_3) \in \mathcal{L}(X)$. The main argument is based on the commutativity between the Riccati operator Π_r and the transport velocity coefficient matrix Λ. This is used in this work because, even under the assumption $v_1 = v_2$, the counterflow configuration of the heat exchanger would make commutativity between Λ and Π_r possible only if Π_r is diagonal.
Thus, for all $\xi^e \in \mathcal{D}(A^e)$ the operator Riccati equation (6.3.16) can be decoupled and converted to the following operator equations:

$$(\tilde{\mathcal{C}}\tilde{B})^*(\tilde{\mathcal{C}}\tilde{B}) - \pi_1 r^{-1}\pi_1 = 0, \tag{6.3.18}$$

$$\tilde{A}^*\Pi_r + \Pi_r\tilde{A} + \tilde{\mathcal{C}}^*\tilde{\mathcal{C}} - \Pi_r\tilde{B}r^{-1}\tilde{B}^*\Pi_r = 0, \tag{6.3.19}$$

$$(\tilde{\mathfrak{A}}\tilde{B})^*\Pi_r + (\tilde{\mathcal{C}}\tilde{B})^*\tilde{\mathcal{C}} - \pi_1 r^{-1}\tilde{B}^*\Pi_r = 0, \tag{6.3.20}$$

$$\Pi_r(\tilde{\mathfrak{A}}\tilde{B}) + \tilde{\mathcal{C}}^*(\tilde{\mathcal{C}}\tilde{B}) - \Pi_r r^{-1}\tilde{B}\pi_1 = 0. \tag{6.3.21}$$

Now, we can solve these four equations separately. The equation (6.3.18) can be easily solved through straightforward calculations and gives the expression of π_1:

$$\pi_1 = \sqrt{r\left(\langle c, b_1\rangle - \langle c, \vartheta b_2\rangle\right)^2} = \sqrt{r}\left|\langle c, b_1 - \vartheta b_2\rangle\right|. \tag{6.3.22}$$

To solve the operator Riccati equation (6.3.19) we need the adjoint operator $\tilde{A}$. Its expression is stated in the following lemma.

Lemma 6.3.2. *Let $\tilde{A}$, the operator defined by*

$$\tilde{A}\varphi = \underbrace{\begin{pmatrix} -v_1 & 0 \\ 0 & v_2 \end{pmatrix}}_{\Lambda} \frac{d\varphi}{dz} + \underbrace{\begin{pmatrix} -(\alpha_1 + \alpha_2\vartheta)\cdot I & 0 \\ \alpha_2 \cdot I & -\alpha_2(1-\vartheta)\cdot I \end{pmatrix}}_{M_1} \varphi.$$

For all $\varphi \in X$ the adjoint of $\tilde{A}$ is given by $\tilde{A}^*\varphi = -\Lambda\dfrac{d\varphi}{dz} + M_1^\top\varphi$, and its domain $\mathcal{D}(\tilde{A}^*) = \Big\{\varphi = (\varphi_1, \varphi_2) \in X : \varphi \text{ a.c. }, \dfrac{d\varphi}{dz} \in X, \varphi_1(1) = 0, \varphi_2(0) = 0\Big\}$.

By using the expressions of $\tilde{A}$ and its adjoint $\tilde{A}^*$, and since $\frac{d\Pi_r}{dz}\cdot I = -\frac{d\,\cdot}{dz}\cdot\Pi_r\cdot I + \Pi\cdot\frac{d\,\cdot}{dz}$, it can be easily observed that the following identity holds

$$\tilde{A}^*\Pi_r + \Pi_r\tilde{A} = \Lambda\frac{d\Pi_r}{dz} + M_1^\top\Pi_r + \Pi_r M_1. \tag{6.3.23}$$

Then, for all $\zeta \in \mathcal{D}(\tilde{A})$ and $\Pi_r(\mathcal{D}(\tilde{A})) \subset \mathcal{D}(\tilde{A}^*)$ equation (6.3.19) can be converted into the following matrix differential equation:

$$\begin{cases} \Lambda\dfrac{d\Pi_r}{dz}\zeta = -M_1^\top\Pi_r\zeta - \Pi_r M_1\zeta - \tilde{\mathcal{C}}\langle\tilde{\mathcal{C}},\zeta\rangle + r^{-1}\langle\tilde{B},\Pi_r\zeta\rangle\Pi_r\tilde{B}, \\ \pi_2(1) = 0, \quad \pi_3(0) = 0. \end{cases} \tag{6.3.24}$$

Note that the operator Riccati equation (6.3.24) is an integro-differential equation that is difficult to solve. Indeed, since $U = Y = \mathbb{R}$, then

$$\tilde{B}^*\zeta := \int_0^1 (b_1(z)\zeta_1(z) + b_2(z)\zeta_2(z))\,dz,$$

and equation $\tilde{\mathcal{C}}\zeta$ given by (6.3.11), which represent average value of $b_1\zeta_1 + b_2\zeta_2$ and $c\zeta_1 - c\vartheta\zeta_2$ on $[0, 1]$. To simplify the problem, we have approximated the average values by distributed functions; this makes it possible to see $\tilde{B}\tilde{B}^*$ and $\tilde{\mathcal{C}}^*\tilde{\mathcal{C}}$ as matrices instead of operators. In other words:

$$\tilde{B}\tilde{B}^*\zeta :\approx \begin{pmatrix} b_1^2 & b_1 b_2 \\ b_1 b_2 & b_2^2 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix},$$

and

$$\tilde{\mathcal{C}}^*\tilde{\mathcal{C}}\zeta :\approx \begin{pmatrix} c^2 & -\vartheta c^2 \\ -\vartheta c^2 & \vartheta^2 c^2 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}.$$

This argument is commonly used in optimal control of distributed parameter systems [Aksikas, 2020], [Aksikas, 2021], and will be used in the rest of the chapter for the solving of the optimal observer. Consequently, equation (6.3.24) becomes:

$$\begin{cases} v_1 \dfrac{d\pi_2}{dz} - r^{-1} b_1^2 \pi_2^2(z) - 2(\alpha_1 + \alpha_2 \vartheta)\pi_2(z) + c^2 = 0, \\ v_2 \dfrac{d\pi_3}{dz} + b_2^2 r^{-1} \pi_3^2(z) + 2\alpha_2(1-\vartheta)\pi_3(z) - \vartheta^2 c^2 = 0, \\ \alpha_2 \pi_3 - r^{-1} b_1 b_2 \pi_2(z)\pi_3(z) - c^2 \vartheta = 0, \\ \pi_2(1) = 0, \quad \pi_3(0) = 0. \end{cases} \tag{6.3.25}$$

On the other hand, the equations (6.3.19) and (6.3.20) are equivalent, and represent compatibility equations. They can be written explicitly and converted to the following algebraic equations:

$$\begin{cases} \gamma_1 \pi_2 - r^{-1} b_1 \pi_1 \pi_2 + c^2(b_1 - \vartheta b_2) = 0 \\ \gamma_2 \pi_3 - r^{-1} b_2 \pi_1 \pi_3 - c^2 \vartheta (b_1 - \vartheta b_2) = 0. \end{cases} \tag{6.3.26}$$

Now we are able to express the stabilizing compensator based on the state-feedback boundary control of the dynamical system (6.3.3).

Proposition 6.3.1. *Let us consider the boundary control system (6.3.3). If π_1 and $\Pi_r = \operatorname{diag}(\pi_2, \pi_3)$ are the solutions of (6.3.22) and (6.3.25), then the optimal state-feedback control law that minimizes the cost criterion (6.2.6) along the trajectories of the system (6.3.3) is given by*

$$\begin{aligned} u(t) = e^{-\omega t} u(0) &+ \frac{1}{r} \int_0^t \int_0^1 e^{-\omega(t-\tau)} b_1 \pi_2(z) \xi_1(z,\tau)\, dz\, d\tau \\ &+ \frac{1}{r} \int_0^t \int_0^1 e^{-\omega(t-\tau)} (b_1 \vartheta \pi_2(z) + b_2 \pi_3(z)) \xi_2(z,\tau)\, dz\, d\tau, \end{aligned} \tag{6.3.27}$$

with $\omega = \frac{1}{r}\left(\sqrt{r}\,|\langle c, b_1 - \vartheta b_2\rangle| + \langle b_1, b_1\pi_2(z)\rangle + \langle b_2, b_2 \pi_3(z)\rangle\right)$ *a positive constant.*

Proof. Let us consider the dynamic extended system (6.3.14). By using equation (6.3.17), it follows that

$$\begin{aligned} \dot{u}(t) = \tilde{u}(t) &= -\frac{1}{r} B^{e*} \Pi \zeta^e(t) \\ &= -\frac{1}{r} \pi_1(z) u(t) + \frac{1}{r} \tilde{B}^* \Pi_r(z) v(t), \quad \text{with } \zeta^e(t) = (u(t)\ v(t))^\top. \end{aligned}$$

Considering the fact that $v(t) = \zeta(t) - \tilde{B} u(t)$ and expression of π_1 given by

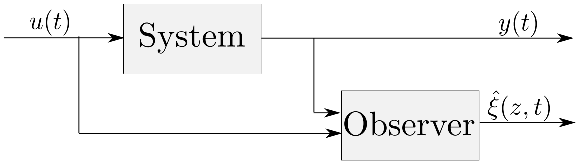


Figure 6.1: Observer principle.

equation (6.3.22), the previously equation become:

$$\begin{aligned}
\dot{u}(t) &= -\frac{1}{r}\left(\sqrt{r}\,|\langle c, b_1\rangle - \langle c, \vartheta b_2\rangle| + \langle b_1, b_1\pi_2(z)\rangle + \langle b_2, b_2\pi_3(z)\rangle\right), \\
&\quad + \frac{1}{r}\langle b_1, \pi_2(z)\zeta_1(z)\rangle + \frac{1}{r}\langle b_2, \pi_3(z)\zeta_2(t)\rangle \\
&= -\omega u(t) + \frac{1}{r}\int_0^1 b_1\pi_2(z)\zeta_1(z,t)dz + \frac{1}{r}\int_0^1 b_2\pi_3(z)\zeta_2(z,t)dz, \\
u(t) &= e^{-\omega t}u_0 + \frac{1}{r}\int_0^t\int_0^1 e^{-\omega(t-\tau)}b_1\pi_2(z)\zeta_1(z,\tau)dzd\tau \\
&\quad + \frac{1}{r}\int_0^t\int_0^1 e^{-\omega(t-\tau)}b_2\pi_3(z)\zeta_2(z,\tau)dzd\tau, \qquad \text{with } u_0 = u(0).
\end{aligned}$$

By the state transformation (6.3.5) it follows that $\zeta_1(z,t) = \xi_1(z,t) + \vartheta\xi_2(z,t)$, $\zeta_2(z,t) = \xi_2(z,t)$, which conclude the proof. □

6.3.3 State observer design

To implement a state feedback control law, it necessary to know the value of the states of a dynamic system in order to control it correctly. However, the dimension of the space of physical observations available to us is always finite and limited in practice, while the state space of the heat exchanger is infinite dimensional. In order to allow the practical application of this control law, we propose to estimate the complete state of the heat exchanger. The control law can therefore be carried out using the estimated state. Thus, the goal of this subsection is to reconstruct the internal state of the heat exchanger using a dynamic algorithm (a dynamic system). This last device is called *observer*.

The detectability of the couple (A^e, C^e) and the linearity of the triplet (A^e, B^e, C^e) allow us to choose a classical Luenberger type observer governed by the following linear differential equation:

$$\begin{cases}
\dfrac{d\tilde{\zeta}^e(t)}{dt} = A^e\tilde{\zeta}^e(t) + B^e\tilde{u}(t) - L^e\left(C^e\tilde{\zeta}^e(t) - y(t)\right), \\
\tilde{\zeta}^e(0) = \tilde{\zeta}^e_0 \in X^e, \quad \forall t \in [0, +\infty),
\end{cases} \tag{6.3.28}$$

where $\tilde{\zeta}^e(t) \in X^e$ is the estimated state, $L^e \in \mathcal{L}(U, X^e)$ is the observer gain, which adjusts the convergence of the state observer. The principle of this observer is presented in Figure 6.1. It consists of a copy of the observing system and a correction term based on the measured output with a positive correction gain.

6.3.3.1 Stability analysis

We are interested in the convergence of the observer whose de dynamics is given by (6.3.28), this is equivalent to the stability of the error system. In fact, by noting by $\varepsilon(t) = \zeta^e(t) - \tilde{\zeta}^e(t)$ the error between the trajectories of the estimated state and the real state, the convergence of the observer is nothing but the stability of the error system whose dynamics is governed by the following equation:

$$\frac{d\varepsilon(t)}{dt} = (A^e - L^e C^e)\,\varepsilon(t). \tag{6.3.29}$$

Note that this dynamic does not depend on the input variable.

Definition 6.3.1. *The system of equations (6.3.28) defines a boundary observer for the system (6.3.1)-(6.3.2) if for all $\tilde{\zeta}_0(z) \in X^e$ and an arbitrary input $\tilde{u}(t) \in U$, the following equation hold:*

$$\lim_{t \to +\infty} \|\zeta^e(\cdot\,, t) - \tilde{\zeta}^e(\cdot\,, t)\|_{X^e} = 0.$$

Then we can state the following result:

Proposition 6.3.2. *Let $\tilde{u}(t) \in U$ be a continuous input. Then system (6.3.28) is an exponential boundary observer for system (6.3.1)-(6.3.2).*

Proof. Since $\Sigma(A^e, -, C^e)$ is observable (detectable), there exists a bounded operator $L^e \in \mathcal{L}(U, X^e)$ such that the linear operator $(A^e - L^e C^e)$ generates an exponentially stable C_0-semigroup $(\mathbb{T}_{(A^e - L^e C^e)}(t))_{t \geq 0}$ on X^e, which satisfies the following relation:

$$\exists\ M_1, \omega_1 > 0 \ \text{ such that } \left\|\mathbb{T}_{(A^e - L^e C^e)}(t)\right\|_{X^e} \leq M_1 e^{-\omega_1 t},$$

which implies:

$$\begin{aligned}
\|\varepsilon(t)\|_{X^e} &\leq \left\|\mathbb{T}_{(A^e - L^e C^e)}(t)\right\|_{X^e} \|\varepsilon_0\| \\
&\leq M_1 e^{-\omega_1 t} \|\varepsilon_0\|_{X^e}.
\end{aligned}$$

Thus, $\varepsilon(t) \longmapsto 0$, when $t \longmapsto +\infty$. $\square$

6.3.3.2 Optimal observer design

The challenge in the design of a state observer is the calculation of the gain which allows the convergence of the estimation error. Regarding the heat exchangers, in [Zobiri et al., 2017] and [Ghousein et al., 2020] the temperatures are estimated by Luenberger observer and backstepping observer, respectively. In the first case, the gain is obtained by a Lyapunov approach, and in the second case by adaptive backstepping approach that can estimate not only the full state of the system, but also unknown in-domain parameters. In this paragraph, the observer gain will be optimally calculated using the duality between observation and LQ-control. This amounts to finding the (optimal) injection operator $L^e_{\text{opt}} \in \mathcal{L}(Y, X^e)$ from the adjoint operator $L^{e*}_{\text{opt}} \in \mathcal{L}(X^e, Y)$ of the LQ-problem associated with the dual system $\Sigma(A^{e*}, C^{e*}, B^{e*})$. In such a context, the Luenberger observer (6.3.28) can be seen as a deterministic Kalman observer.
Let us consider the cost functional

$$\tilde{J}(\tilde{\zeta}^e_0, \tilde{u}^e) = \int_0^\infty \Big(\langle B^{e*}\tilde{\zeta}^e(t), B^{e*}\tilde{\zeta}^e(t)\rangle + \langle \tilde{u}^e(t), \tilde{r}\tilde{u}^e(t)\rangle\Big)\, dt, \tag{6.3.30}$$

associated with the dual problem, where $\tilde{r} > 0$. By [Aksikas, 2021], we know that the unique control law $\tilde{u}^e(t)$ which minimizes cost functional (6.3.30) is given by

$$\tilde{u}^e(t) = -L^{e*}_{\text{opt}}\tilde{\zeta}^e(t) = -\tilde{r}^{-1}C^e\mathcal{Q}\tilde{\zeta}^e(t), \tag{6.3.31}$$

with $\mathcal{Q}(\mathcal{D}(A^{e*})) \subset \mathcal{D}(A^e)$ that satisfies the following ORE:

$$\Big[A^e\mathcal{Q} + \mathcal{Q}A^{e*} + B^eB^{e*} - \mathcal{Q}C^{e*}\tilde{r}^{-1}C^e\mathcal{Q}\Big]\,\tilde{\zeta}^e = 0. \tag{6.3.32}$$

Considering that $\Pi = \operatorname{diag}(q_1, \mathcal{Q}_r) \in \mathcal{L}(X^e)$ with $\mathcal{Q}_r = \operatorname{diag}(q_2, q_3) \in \mathcal{L}(X)$, the operator Riccati equation (6.3.32) can be converted to the following set of equations:

$$I - \tilde{r}^{-1}q_1(\tilde{C}\tilde{B})^*(\tilde{C}\tilde{B})q_1 = 0, \tag{6.3.33}$$
$$\tilde{A}\mathcal{Q}_r + \mathcal{Q}_r\tilde{A}^* + \tilde{B}\tilde{B}^* - \tilde{r}^{-1}\mathcal{Q}_r\tilde{C}^*\tilde{C}\mathcal{Q}_r = 0 \tag{6.3.34}$$
$$q_1(\tilde{\mathfrak{A}}\tilde{B})^* - \tilde{B}^* - \tilde{r}^{-1}q_1(\tilde{C}\tilde{B})^*\tilde{C}\mathcal{Q}_r = 0, \tag{6.3.35}$$
$$(\tilde{\mathfrak{A}}\tilde{B})q_1 - \tilde{B} - \tilde{r}^{-1}\mathcal{Q}_r\tilde{C}^*(\tilde{C}\tilde{B})q_1 = 0. \tag{6.3.36}$$

The explicit expression of q_1 can be deduced from (6.3.33) and is given by

$$q_1 = \sqrt{\tilde{r}}\,(|\langle c, b_1 - b_2\vartheta\rangle|)^{-1}\,. \tag{6.3.37}$$

By following similar process as in the previous section, from (6.3.34) it follows that $q_2(z)$ and $q_3(z)$ are solutions of the following set of differential equations:

$$\begin{cases} v_1 \dfrac{dq_2}{dz} + \tilde{r}^{-1}c^2 q_2^2(z) + 2(\alpha_1 + \alpha_2\vartheta)q_2(z) - b_1^2 = 0 \\ v_2 \dfrac{dq_3}{dz} - \tilde{r}^{-1}c^2\vartheta q_3^2(z) - 2\alpha_2(1-\vartheta)q_3(z) + b_2^2 = 0 \\ \alpha_2 q_3(z) + \tilde{r}^{-1}c^2\vartheta q_2(z)q_3(z) + b_1 b_2 = 0, \\ q_2(0) = 0, \quad q_3(1) = 0. \end{cases} \tag{6.3.38}$$

The compatibility equations (6.3.35) and (6.3.36) can be reduced in the following equations

$$\begin{cases} \gamma_1 q_1 - \tilde{r}^{-1}c^2(b_1 - \vartheta b_2)q_1 q_2 - b_1 = 0 \\ \gamma_2 q_1 + \tilde{r}^{-1}c^2\vartheta(b_1 - \vartheta b_2)q_1 q_3 - b_2 = 0. \end{cases} \tag{6.3.39}$$

Now let us express the optimal observer.

Proposition 6.3.3. *If $\mathcal{Q} = \mathrm{diag}(q_1, \mathcal{Q}_r) \in \mathcal{L}(X^e)$ is the solution of ORE given by (6.3.32), then the optimal state-observer of the dynamic system (6.3.3) is given by*

$$\begin{cases} \dot{\tilde{\xi}}(t) = \mathfrak{A}\tilde{\xi}(t) - \tilde{r}^{-1}\mathcal{Q}_r\mathcal{C}^*(\mathcal{C}\tilde{\xi}(t) - y(t)) \\ \tilde{\xi}(0) = \tilde{\xi}_0. \end{cases} \tag{6.3.40}$$

Proof. First, from (6.3.28) the optimal observer is given by the following dynamic equation:

$$\begin{cases} \dot{\tilde{\zeta}}^e(t) = \left(A^e - L^e_{\mathrm{opt}}C^e\right)\tilde{\zeta}^e(t) + B^e\tilde{u}(t) + L^e_{\mathrm{opt}}y(t), \\ L_{\mathrm{opt}} = \tilde{r}^{-1}\left(q_1(\tilde{\mathcal{C}}\tilde{B})^* \quad \mathcal{Q}_r\tilde{\mathcal{C}}^*\right)^{\top}, \end{cases} \tag{6.3.41}$$

with $L_{\mathrm{opt}} \in \mathcal{L}(X^e)$ the optimal gain of the state observer. If we consider the estimated state as $\tilde{\zeta}^e(t) = (u(t)\ \tilde{v}(t))^{\top} \in X^e$ and the expressions of A^e, B^e and C^e given by (6.3.15), then the equation (6.3.41) becomes:

$$\begin{cases} \dot{u}(t) = -\tilde{r}^{-1}q_1(\tilde{\mathcal{C}}\tilde{B})^*(\tilde{\mathcal{C}}\tilde{B})u(t) - \tilde{r}^{-1}q_1(\tilde{\mathcal{C}}\tilde{B})^*\tilde{\mathcal{C}}\tilde{v}(t) \\ \qquad\quad + \tilde{u}(t) + \tilde{r}^{-1}q_1(\tilde{\mathcal{C}}\tilde{B})^*y(t) \\ \dot{\tilde{v}}(t) = \left(\tilde{\mathfrak{A}}\tilde{B} - \tilde{r}^{-1}\mathcal{Q}_r\tilde{\mathcal{C}}^*(\tilde{\mathcal{C}}\tilde{B})\right)u(t) + \left(\tilde{A} - r^{-1}\mathcal{Q}_r\tilde{\mathcal{C}}^*\tilde{\mathcal{C}}\right)\tilde{v}(t) \\ \qquad\quad - \tilde{B}\tilde{u}(t) + \tilde{r}^{-1}\mathcal{Q}_r\tilde{\mathcal{C}}^*y(t). \end{cases} \tag{6.3.42}$$

Now if we substitute $\tilde{v}(t)$ by $\tilde{\zeta}(t) - \tilde{B}u(t)$ together with straightforward calculations and eliminations, we get that:

$$\begin{cases} \dot{u}(t) = -\tilde{r}^{-1}(\tilde{\mathcal{C}}\tilde{B})^* \left(\tilde{\mathcal{C}}\tilde{\zeta}(t) - y(t)\right) + \tilde{u}(t), \\ \dot{\tilde{\zeta}}(t) = \tilde{\mathfrak{A}}\tilde{\zeta}(t) - \tilde{r}^{-1}\mathcal{Q}_r\tilde{\mathcal{C}}^* \left(\tilde{\mathcal{C}}\tilde{\zeta}(t) - y(t)\right). \end{cases} \tag{6.3.43}$$

By using the state transformation $S \in \mathcal{L}(Z)$ given by equation (6.3.5) with $\tilde{\zeta}(t) = S\tilde{\xi}(t)$, $\tilde{\mathfrak{A}} = S\mathfrak{A}S^{-1}$, $\tilde{\mathcal{C}} = \mathcal{C}S^{-1}$, the above equations lead to $\dot{u}(t) = \tilde{u}(t)$ and the state observer (6.3.40). □

6.3.4 Observer-based feedback control

A question immediately comes to mind as soon as the state observer has been realized: "*is it possible to use the state estimated by the observer (6.3.40) in the control law (6.3.27)*"? This question always has a positive answer within the framework of linear systems thanks to the principle of separation, which specifies that the behavior in steady state in closed-loop of a regulated system with or without observer is the same one (however, the transient behavior is different, as well as the robustness properties) [Gauthier and Kupka, 1992]. The following result allows to conclude on the stability of the controlled dynamical system coupled with the state observer.

Theorem 6.3.3 (Closed-loop stability)**.** *Consider the state linear system $\Sigma(A^e, B^e, C^e)$ given by (6.3.14) and that is exponentially stabilizable and exponentially detectable. If $K^e_{\text{opt}} \in \mathcal{L}(U, X^e)$ and $L^e_{\text{opt}} \in \mathcal{L}(Y, X^e)$ are such that $A^e - B^e K^e_{\text{opt}}$ and $A^e - L^e_{\text{opt}} C^e$ generates an exponentially stable C_0-semigroups, than the controller $u(t) = -K^e_{\text{opt}}\tilde{\zeta}^e(t)$, where $\tilde{\zeta}^e(t)$ is the state of the optimal observer with output injection operator L^e_{opt}, stabilizes the closed-loop system. The stablizing compensator is given by*

$$\begin{cases} \dot{\tilde{\zeta}}^e(t) = \left(A^e - L^e_{\text{opt}}C^e\right)\tilde{\zeta}^e(t) + B^e u(t) + L^e_{\text{opt}} y(t) \\ u(t) = -K^e_{\text{opt}}\tilde{\zeta}^e(t) \\ \tilde{\zeta}^e(0) = \tilde{\zeta}^e_0. \end{cases} \tag{6.3.44}$$

Proof. The closed-loop system is governed by

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} \zeta^e(t) \\ \tilde{\zeta}^e(t) \end{pmatrix} = \begin{pmatrix} A^e & -B^e K^e_{\text{opt}} \\ L^e_{\text{opt}}C^e & A^e - L^e_{\text{opt}}C^e - B^e K^e_{\text{opt}} \end{pmatrix} \begin{pmatrix} \zeta^e(t) \\ \tilde{\zeta}^e(t) \end{pmatrix} \\ \tilde{\zeta}^e(0) = \tilde{\zeta}^e_0, \quad \zeta^e(0) = \zeta^e_0. \end{cases} \tag{6.3.45}$$

We know that $\text{diag}(A^e, A^e)$ is the infinitesimal generator of a C_0-semigroup. Moreover, $\begin{pmatrix} 0 & -B^e K^e_{\text{opt}} \\ L^e_{\text{opt}}C^e & -L^e_{\text{opt}}C^e - B^e K^e_{\text{opt}} \end{pmatrix}$ is linear and bounded oper-

ator. By the bounded perturbation theorem, the closed-loop operator $A_{cl} = \begin{pmatrix} A^e & -B^e K^e_{\text{opt}} \\ L^e_{\text{opt}} C^e & A^e - L^e_{\text{opt}} C^e - B^e K^e_{\text{opt}} \end{pmatrix}$ generates a C_0-semigroup on $X^e \oplus X^e$.
We now show that this operator generates an exponentially stable semigroup. For that, we introduce the linear and bounded invertible operator $\mathcal{T} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \in X^e \oplus X^e$, and where $\mathcal{T}^{-1} = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}$. So, we verify that

$$\mathcal{T}^{-1} A_{cl} \mathcal{T} = \begin{pmatrix} A^e - L^e_{\text{opt}} C^e & 0 \\ L^e_{\text{opt}} C^e & A^e - B^e K^e_{\text{opt}} \end{pmatrix} = \bar{A}_{cl}. \qquad (6.3.46)$$

Given that $A^e - L^e_{\text{opt}} C^e$ and $A^e - B^e K^e_{\text{opt}}$ generates exponentially stable semigroups, then the operator $\bar{A}_{cl}$ generates an exponentially stable C_0-semigroup. As $\mathcal{T}$ is similarity transformation the growth bounds of the semigroups generated by A_{cl} and $\bar{A}_{cl}$ are the same. □

By using the state transformation (6.3.5) and the dynamic compensator (6.3.44), we obtain the dynamic compensator for the model (6.3.3):

$$\begin{cases} \dot{\tilde{\xi}}(t) = \mathfrak{A}\tilde{\xi}(t) - \tilde{r}^{-1}\mathcal{Q}_r \mathcal{C}^*(\mathcal{C}\tilde{\xi}(t) - y(t)), \\ u(t) = e^{-\omega t}u(0) + \dfrac{1}{r}\displaystyle\int_0^t \int_0^1 e^{-\omega(t-\tau)} b_1 \pi_2(z)\tilde{\xi}_1(z,\tau)dzd\tau \\ \qquad + \dfrac{1}{r}\displaystyle\int_0^t \int_0^1 e^{-\omega(t-\tau)} (b_1 \vartheta \pi_2(z) + b_2 \pi_3(z))\tilde{\xi}_2(z,\tau)dzd\tau, \\ \tilde{\xi}(0) = \tilde{\xi}_0, \end{cases} \qquad (6.3.47)$$

with $\omega = \frac{1}{r}\left(\sqrt{r}\,|\langle c, b_1 - \vartheta b_2\rangle| + \langle b_1, b_1\pi_2(z)\rangle + \langle b_2, b_2\pi_3(z)\rangle\right)$ a positive constant.
Note that for a given set point $y_d(t) \in Y$, the control law obtained does not make it possible to cancel the static error of the closed-loop system (6.3.47). Indeed, this is due to the fact that the dynamic model of the heat exchanger does not have open-loop integration. Classical control theory therefore leads us to add an integrator to the control chain by following the diagram in Figure

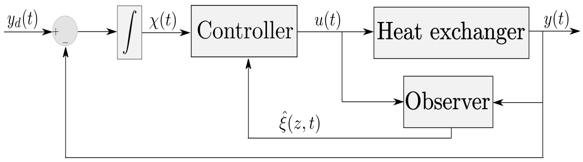


Figure 6.2: Block diagram of observer-based control with integral action.

6.2. Thus, the resulting control law is given by:

$$\begin{cases} \dot{\tilde{\xi}}(t) = \mathfrak{A}\tilde{\xi}(t) - \tilde{r}^{-1}\mathcal{Q}_r\mathcal{C}^*(\mathcal{C}\tilde{\xi}(t) - y(t)), \\ u(t) = e^{-\omega t}u(0) + \dfrac{1}{r}\displaystyle\int_0^t\int_0^1 e^{-\omega(t-\tau)}b_1\pi_2(z)\tilde{\xi}_1(z,\tau)dzd\tau \\ \qquad\qquad + \dfrac{1}{r}\displaystyle\int_0^t\int_0^1 e^{-\omega(t-\tau)}(b_1\vartheta\pi_2(z) + b_2\pi_3(z))\tilde{\xi}_2(z,\tau)dzd\tau + \chi(t), \\ \dfrac{d\chi(t)}{dt} = y_d(t) - y(t), \end{cases} \tag{6.3.48}$$

with $\chi(t) \in \mathbb{R}$ being the tracking error integral.

6.3.5 Numerical simulations

This subsection presents the results of numerical simulation for the linear closed-loop heat exchanger model (6.3.1)-(6.3.2). The model parameter values used here for the numerical simulations are the same ones obtained in Chapter 2.
Before presenting the results of the observer-based optimal control law simulation, let us start by experimenting with the capabilities of the Luenberger Optimal Observer to estimate the state vector (the internal and external temperature distributions of the fluid). For this, the simulations will be carried out using (another) data set taken from the heat exchanger test bench described in Chapter 2. We launched an acquisition campaign of 3600 samples of the inlet and outlet variables. During the test, the sampling interval was one second and constant throughout the experiment. Given the definition of the output function (6.3.4) considered for the control, the challenge is to estimate the complete state from the boundary measurement of the internal temperature of the heat exchanger.
With the choice of weighting function $\tilde{r}(z) = 0.1$, the optimal injection function $q_2(z)$ in Figure 6.3 is obtained. To implement the LQ state-observer, the number of discretization points has been fixed to 30 for the $z-$coordinate points along the heat exchanger, while it has been chosen according to the number of measurements for the $t-$coordinate. The state observer equations

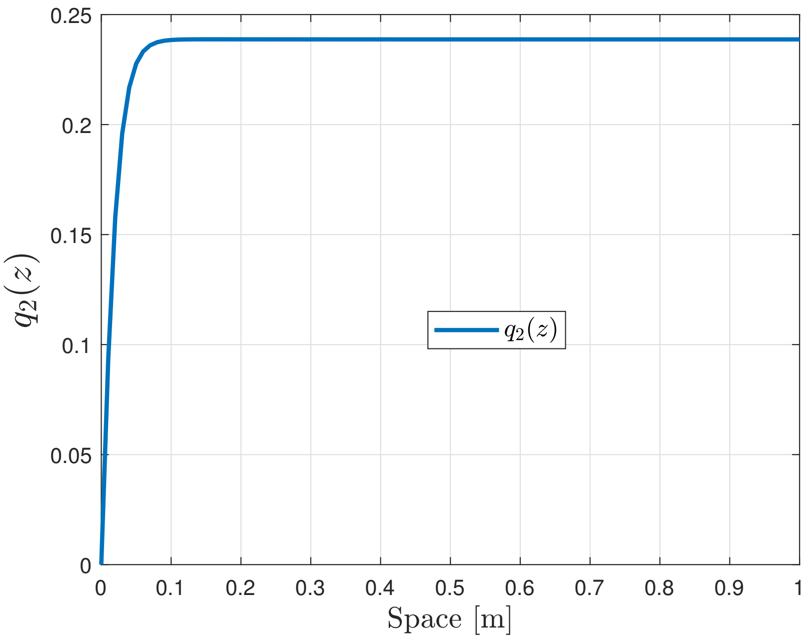
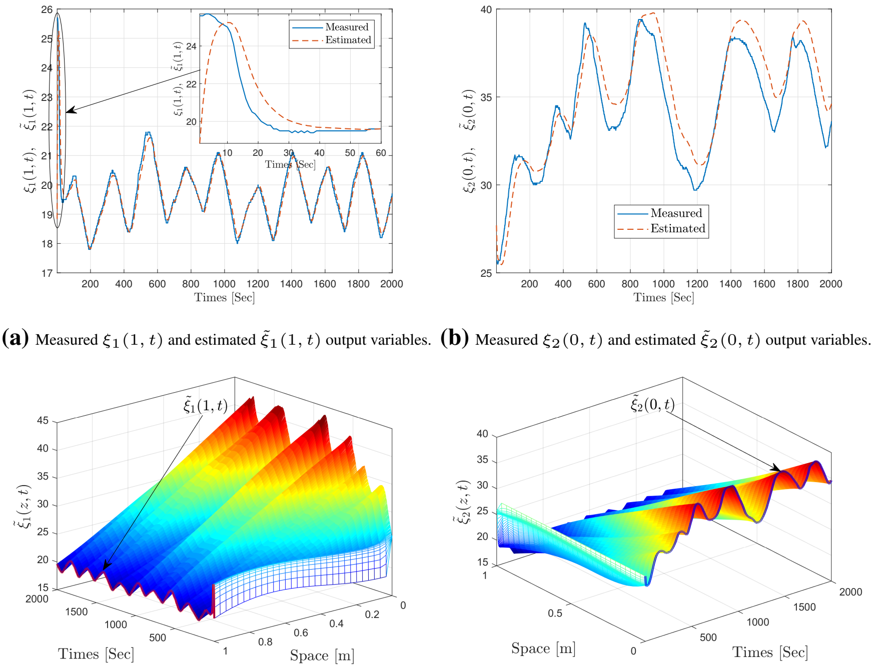


Figure 6.3: LQ-injection function $q_2(z)$ for $\tilde{r}(z) = 0.1$.

have been integrated using the method of lines described in chapter 2. The observer was initialized with outlet temperatures different from those measured. Figure 6.4a compares the measured output $\xi_1^m(t)$ and estimated $\tilde{\xi}_1(1,t)$ by the state-observer (6.3.40) on the first 2000 data collected. We note that the dynamics of the system are well estimated despite the initialization. Figure 6.4b compares the measured output $\xi_2^m(t)$ of the external temperature of the exchanger and that estimated by the observer. Note that the proposed observer has the ability to reconstruct the complete state of the system from a single component measured at the boundary of the heat exchanger, thanks to the coupling matrix M of the model (6.3.1)-(6.3.2). Moreover, the convergence of the observer is good. This can be seen in Figure 6.5 which represents the output error. In particular, for the component $\xi_1^m(t)$ measured and that enters the observer the error tends towards zero, whereas for the component $\xi_2^m(t)$ which does not enter the observer, error tends to a value close to zero. In Figures 6.4c and 6.4d, the surfaces corresponding to the internal and external temperatures are shown, respectively.

Before implementing the control law, we do a last test of the LQ-observer on another dataset. Indeed, as stated in [Dochain, 2008, Chapter 4], the observers must first be validated before they can be used. For this, their predictions must extensively be compared to direct measurements that were not used during the calibration process. Figures 6.6a to 6.6d show the simulation results on the



(a) Measured $\xi_1(1, t)$ and estimated $\tilde{\xi}_1(1, t)$ output variables. **(b)** Measured $\xi_2(0, t)$ and estimated $\tilde{\xi}_2(0, t)$ output variables.

(c) 3D estimated state variable $\tilde{\xi}_1(z, t)$. **(d)** 3D estimated state variable $\tilde{\xi}_2(z, t)$.

Figure 6.4: Calibration of LQ-optimal state-observer.

remaining measured data. It can be seen, as in the first simulations, that the LQ-observer has a good ability to converge and to estimate the set of unmeasured states from the single component $\xi_1^m(t)$ measured at the point $z = 1$. We conclude that the proposed LQ state-observer has been well calibrated and can be used to implement the state feedback control law (6.3.48).

To illustrate the performance of the integral action LQ-controller, we have chosen the following operating conditions: $\xi_{1,int}(t) = 60°C$, $\xi_2(1, 1) = 8°C$, and $\xi(z, 0) = \tilde{\xi}(z, 0)$. For all $z \in [0, 1]$ the weighting function is chosen as $r(z) = 1.5$. Figure 6.7 shows the resulting LQ-optimal state feedback functions $\pi_1(z), \pi_2(z)$ and $\pi_3(z)$. Note that the function $\pi_1(z)$ is constant and positive as defined by (6.3.22). Observe that the function $\pi_2(z)$ induce a positive spatially varying feedback on the internal temperature $\xi_1(\cdot\,, t)$ (see (6.3.4)). Moroever, observe that the function $\pi_3(z)$ is almost identically zero, i.e., there is a very low gain feedback on the external temperature $\xi_2(\cdot\,, t)$; this result is not surprising in view of the definition of the output function (6.3.4). From the LQ optimal state feedback functions, we get the numerical value of the control law decay constant (6.3.48) as $\omega = 0.8735$. For the set point tracking test, a set point step of $15°C$ ($y_d(t) = 35°C$) and $-10°C$ ($y_d(t) = 25°C$) was imposed at time $t = 168s$ and $t = 335s$. The simulation results are summarized in Figures 6.8a-6.8d. We note good tracking performance (Figure

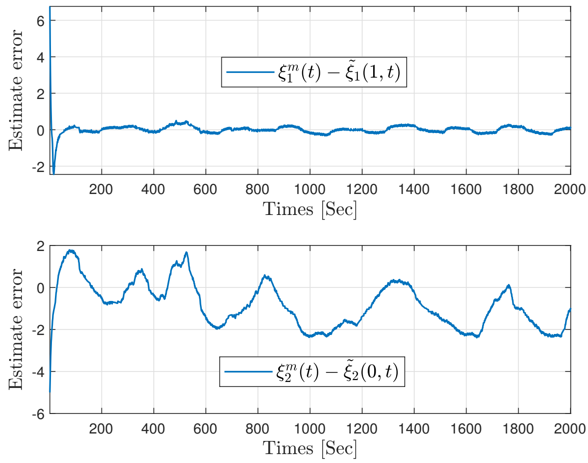
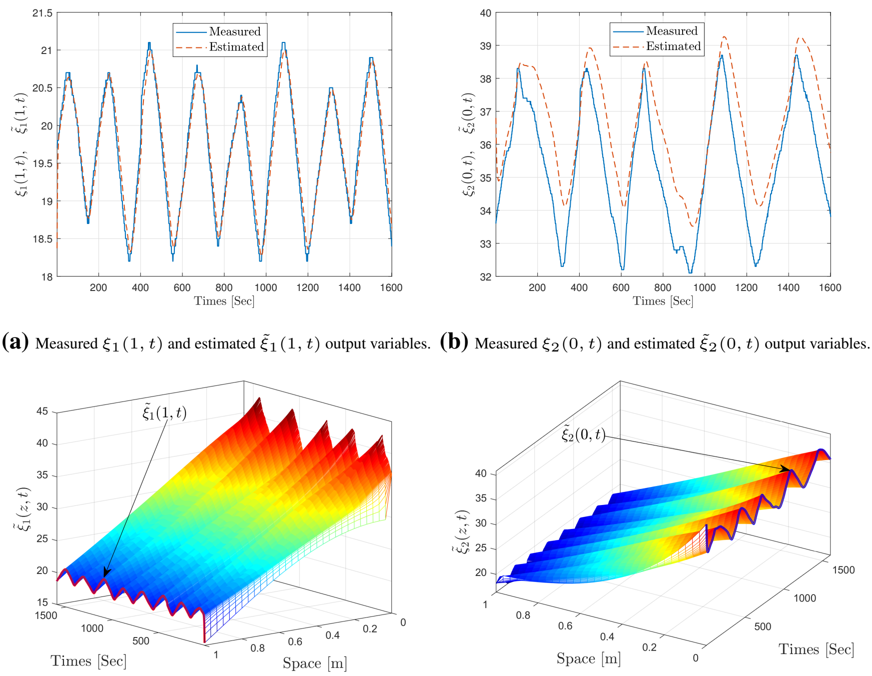


Figure 6.5: Estimate error of each component.

6.8a) and an acceptable variation of the manipulated variable (Figure 6.8b). We end this simulation step by presenting the $\mathbb{X}-$norm of the closed-loop state vector $(\tilde{\xi}_1(\cdot\,,t), \tilde{\xi}_2(\cdot\,,t), \chi(t))$ as a function of time, see Figure 6.9. One observes that the equilibrium corresponding to each reference value is reached exponentially fast on each subinterval.

6.4 Conclusion

In this chapter we have studied the LQ-optimal control problem of a counter-flow heat exchanger. This system, governed by two coupled PDEs, is known to be poorly conditioned. Indeed, the operator which carries the dynamics of the system contains singularities which prevent the system to benefit from the theoretical results which exist in the literature, in particular those concerning the optimal control. By a state transformation based on the resolution of a Riccati differential equation we were able to put the model of the system in the lower triangular form. Next, in order not to resort to additional computation steps, in particular by solving a spectral factorization problem, the model has been rewritten as an abstract boundary control problem with bounded control and observation operators, using the Fattorini approach, from which we developed an LQ-optimal observer-based control law with integral. In particular, the observer was validated using data collected on a laboratory test bench. The simulation results of the control law coupled with the state observer have shown the relevance of the proposed approach.



(a) Measured $\xi_1(1,t)$ and estimated $\tilde{\xi}_1(1,t)$ output variables. **(b)** Measured $\xi_2(0,t)$ and estimated $\tilde{\xi}_2(0,t)$ output variables.

(c) 3D estimated state variable $\tilde{\xi}_1(z,t)$. **(d)** 3D estimated state variable $\tilde{\xi}_2(z,t)$.

Figure 6.6: Validation of LQ-optimal state-observer.

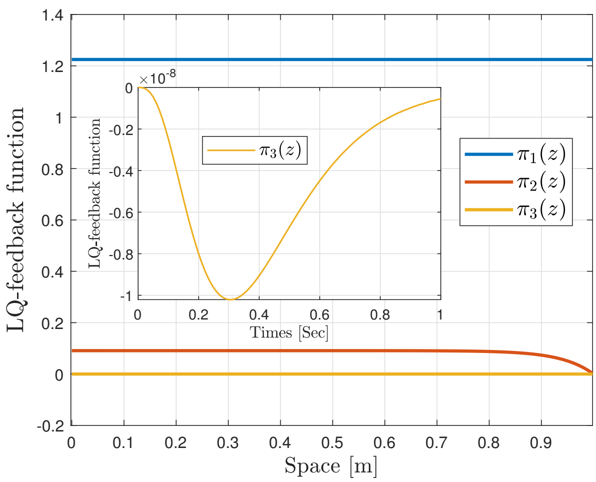
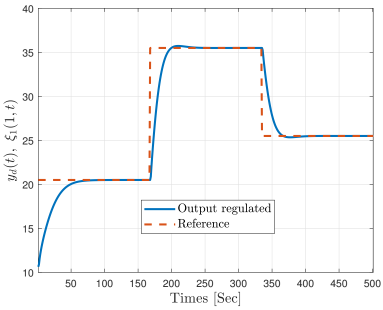
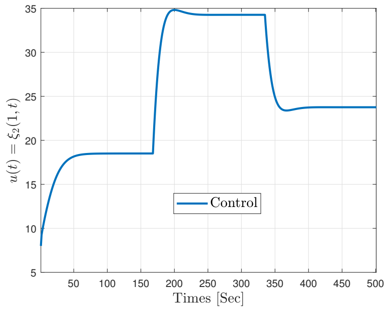


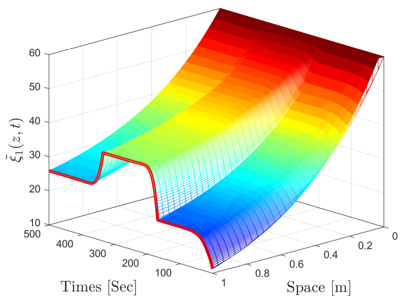
Figure 6.7: LQ-feedback functions $\pi_1(z), \pi_2(z)$ and $\pi_3(z)$ for $r(z) = 1.5$.



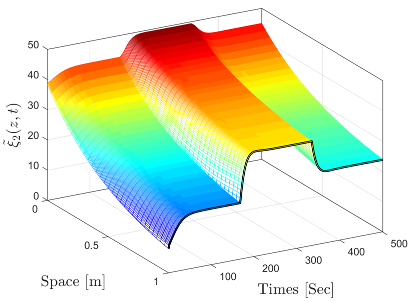
(a) Reference and output variables.



(b) Control variable.



(c) 3D estimated state variable $\tilde{\xi}_1(z, t)$.



(d) 3D estimated state variable $\tilde{\xi}_2(z, t)$.

Figure 6.8: Closed-loop state trajectory.

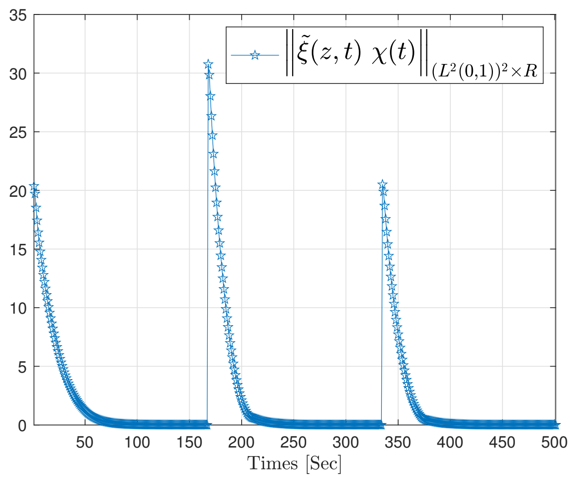


Figure 6.9: $(L^2(0,1), \mathbb{R})^2 \times \mathbb{R}-$norm of the closed-loop state trajectory.

Conclusion and perspectives

In this thesis we were interested in the modelling, analysis and control of heat exchangers. These systems are widely used in energy applications in industry, and belong to the class of infinite dimensional thermodynamic systems. The goal was to develop control approaches that optimize the operation of tubular counterflow heat exchangers. To achieve this, this work was organized into three parts:

1. The first part dealt with the modelling and the identification of the parameters of a heat exchanger.
2. The second part focused on the analysis of the dynamic properties of the counterflow heat exchanger, first looking for the conditions on the parameters that guarantee the existence and uniqueness of the solution.
3. The third part dealt with control issues, in a context of boundary control approach.

The original results presented in this thesis can be roughly summarized by the following points:

1. The infinite-dimensional thermodynamic representation of a counterflow heat exchanger, and the estimation of the model parameters in temperature (standard model in process engineering), while dealing with questions of structural and practical identifiability.
2. Consolidation of existing results relating to the analysis of counterflow heat exchangers in an appropriate Hilbert space by providing a condition on the model parameters (and giving a thermodynamic interpretation to this condition) that guarantees the existence and uniqueness of the solution.
3. The study of passivity and stability in the thermodynamic sense, precisely using the entropy as a storage function, the entropy production as a dissipation function for to proof the passivity, and the thermodynamic availability function for the stability.
4. The use of PI regulation in a boundary control context, while searching the conditions on the controller parameters by Lyapunov method as well as by semigroup techniques, that guarantee the stability of the closed-loop system. The application of LQ-optimal control theory for infinite dimensional systems.

The issues addressed in this thesis can be considered as a first stage in the modelling, analysis and control of heat exchangers in the case where the dynamics are described by partial differential equations. In the future, various issues need to be considered. In this work, parameter estimation was performed assuming that the initial temperature profiles are known. Although the results are promising, this is generally an unrealistic assumption for distributed parameter systems. In particular, we would like to use the adjoint approach based on the calculus of variations and which allows in a single algorithm to estimate the parameters of the model at the same time, as well as the (initial) profiles of the state variables [Nguyen et al., 2016,Nguyen et al., 2018,Mishra et al., 2020].

Regarding the control, this work gives (in addition to the PI control) the tools which make it possible to address the question of optimal control by boundary action, i.e. by manipulation of the temperature. The proposed optimal control approach is based on the using of an diagonal operator as the solution of the operator Riccati equation. Although the simulation results are appreciable, it is difficult to assert that the feedback operator obtained with a Riccati diagonal operator corresponds to the unique feedback operator for the model considered. To validate it, it would be interesting in the future to exploit other optimal control techniques, in particular those by spectral factorization as presented in [Dehaye and Winkin, 2016], to compare the feedback operator obtained with that found in this work.

From the thermodynamic point of view, we established passivity and stability. However, these results were not exploited for the control of the heat exchanger. It would therefore be interesting to address this problem, in particular that of PI-regulation by passivity techniques in the thermodynamic sense, as applied by [Hoang and Dochain, 2013,Zhou et al., 2013,Zárate-Navarro et al., 2019,Zhou et al., 2021] for the stabilization of tubular reactors. For this purpose, it is envisaged to define a set point based on the entropy generation number [Sandoval, 2021].
Finally, the analysis of the equilibrium profiles revealed that the thermal pinch can be seen as a performance factor for the heat exchanger. It is important to remember that inlet temperatures and velocities are the manipulable quantities on which the thermal pinch depends. In this sense, it is therefore possible to consider using mixed control techniques to optimize the effectiveness of the heat exchanger.

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A

Infinite-dimensional systems theory

This part refers to the following books: [Pazy, 1983], [Curtain and Zwart, 1995], [Tucsnak and Weiss, 2009].

A.1 Semigroup theory

A.1.1 Unbounded operators on a Hilbert space

In the following, unless otherwise specified, X represents a Hilbert space.

Definition A.1.1. *A linear application in X is a pair $(A, \mathcal{D}(A))$, where $\mathcal{D}(A)$ is a vector subspace of X, and $A : \mathcal{D}(A) \to X$, is a linear application. We say that A is bounded if $\|A\xi\|$ remains bounded for all ξ in $\{\xi \in \mathcal{D}(A); \|\xi\| \leq 1\}$. Otherwise, we say that A is unbounded.*

Definition A.1.2. *Let be an operator $A : \mathcal{D}(A) \subset X \to X$. Let $A_\lambda = \lambda I - A$ with $\lambda \in \mathbb{C}$. If λ is such that the image of A_λ, denoted by $\mathrm{R}(A_\lambda)$ is dense in X and that A_λ is continuously invertible, then we say that λ is an element of the resolvent set of A, denoted by $\rho(A)$, and we note $\mathrm{R}(\lambda, A) = (\lambda I - A)^{-1}$.*

If $\lambda \notin \rho(A)$, we say that λ belongs to the spectrum of A, denoted $\sigma(A)$, and which breaks down into three subsets:

- $\sigma_p(A)$: contains all values of λ such that A_λ has no inverse. It is the *point spectrum* and its elements are the eigenvalues of A:
$$\sigma_p(A) = \{\lambda \in \mathbb{C} | (\lambda I - A) \text{ is not injective}\}\,.$$
- $\sigma_c(A)$: contains all values of λ for which A_λ has a dense but non-continuous domain inverse. It is the *continuous spectrum*:
$$\begin{aligned}\sigma_c(A) &= \Big\{\lambda \in \mathbb{C} | (\lambda I - A) \text{ is injective}, \overline{\mathrm{Ran}(\lambda I - a)} = X,\\ &\qquad \text{but } (\lambda I - A)^{-1} \text{ is unbounded}\Big\}\\ &= \Big\{\lambda \in \mathbb{C} | (\lambda I - A) \text{ is injective}, \overline{\mathrm{Ran}(\lambda I - a)} = X,\\ &\qquad \text{but } \mathrm{Ran}(\lambda I - A) \neq X\}\,.\end{aligned}$$

- $\sigma_r(A)$: contains all values of λ such that A_λ has an inverse whose domain is not dense in X. It is the *residual spectrum*:

$$\sigma_r(A) = \{\lambda \in \mathbb{C} | (\lambda I - A) \text{ is injective, but } \operatorname{Ran}(\lambda I - A) \text{ is not dense in } X\}.$$

A very important case is when the resolvent $\mathrm{R}(\lambda, A)$ is compact, for some $\lambda \in \rho(A)$, we then have the following theorem.

Theorem A.1.1. *Let A be a closed operator in X such that $\mathrm{R}(\lambda, A)$ is compact, for some $\lambda \in \rho(A)$. Then the spectrum of A is discrete and formed only of finite multiplicity eigenvalues. In addition, $\mathrm{R}(\eta, A)$ is compact for all $\eta \in \rho(A)$.*

A.1.2 Maximal dissipative operators

Definition A.1.3 ([Lumer and Phillips, 1961]). *Let X be a Hilbert space, and A a linear operator in X.*

- *A is said to be dissipative (on X) if*

$$\forall \xi \in \mathcal{D}(A), \quad \mathfrak{Re}\langle A\xi, \xi\rangle \leq 0.$$

- *If moreover $I - A$ is surjective of $\mathcal{D}(A)$ in X, then we say that A is maximally dissipative (on X).*

The following proposition gives a characterization of dissipative operators which is used to prove many results concerning maximal dissipative operators.

Proposition A.1.1. *Let $A : \mathcal{D}(A) \subset X \to X$ be an operator. The following properties are equivalent:*

1. *A is dissipative on X,*
2. *A verifies*
$$\forall \xi \in \mathcal{D}(A), \quad \forall \lambda > 0, \quad \|\xi\|_X \leq \lambda^{-1} \|\lambda \xi - A\xi\|_X,$$
3. *A verifies*
$$\forall \xi \in \mathcal{D}(A), \quad \forall \lambda \in \mathbb{C}_{>0}, \quad \|\xi\|_X \leq (\mathfrak{Re}\lambda)^{-1} \|\lambda \xi - A\xi\|_X,$$
where $\mathbb{C}_{>0} = \{\lambda \in \mathbb{C} / \mathfrak{Re}\lambda > 0\}$.

This characterization of the dissipative operators allows to establish a characterization of the dissipative operators which are maximally dissipative.

Proposition A.1.2. *If A is dissipative then the following assertions are equivalent:*

1. *A is m-dissipative in X.*
2. *$\exists \lambda_0 > 0$ such that $(\lambda_0 I - A)$ is surjective $(\operatorname{Ran}(\lambda_0 I - A) = X)$.*
3. *$\forall \lambda > 0$, $(\lambda I - A)$ is invertible, with bounded inverse in $\mathcal{L}(X)$ and $\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \lambda^{-1}$.*
4. *$\forall \lambda \in \mathbb{C}_{>0}$, $(\lambda I - A)$ is invertible, with bounded inverse in $\mathcal{L}(X)$ and $\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \leq (\mathfrak{Re}\lambda)^{-1}$.*

The Proposition A.1.2 ensures the existence of the adjoint operator of A, when A is m-dissipative, since in this case $\mathcal{D}(A)$ is dense in X. Recall that the adjoint of A is the operator A^* defined as:

$$\mathcal{D}(A^*) = \{\zeta \in X; \exists k < \infty,\ \forall \xi \in \mathcal{D}(A),\ |\langle A\xi, \zeta\rangle| \leq k\|\xi\|_X\}$$

and

$$\langle A^*\zeta, \xi\rangle_X = \langle \zeta, A\xi\rangle_X, \quad \forall \zeta \in \mathcal{D}(A^*),\ \forall \xi \in \mathcal{D}(A).$$

Note that A^* is always closed. Moreover, $\mathcal{D}(A^*)$, endowed with the inner product

$$\langle f, g\rangle_{\mathcal{D}(A^*)} = \langle f, g\rangle_X + \langle A^* f, A^* g\rangle_X,$$

is a Hilbert space.

Proposition A.1.3. *If $A : \mathcal{D}(A) \subset B \to X$ is m-dissipative, then its adjoint $A^* : \mathcal{D}(A^*) \subset X \to X$ is m-dissipative.*

A.1.3 Semigroup of bounded linear operators on X

Definition A.1.4. *A semigroup of bounded operators on X is a family $(\mathbb{T}(t))_{t\geq 0} \subset \mathcal{L}(X)$ of operators verifying the following properties:*

1. $\mathbb{T}(t+s) = \mathbb{T}(t)\mathbb{T}(s), \quad$ *for* $t, s \geq 0$;
2. $\mathbb{T}(0) = I$.

-The semigroup is called strongly continuous at the origin (or of class C^0) if:

$$\lim_{t\to 0} \mathbb{T}(t)\xi = \xi, \qquad \forall \xi \in X. \qquad (\star)$$

-It is said contracting if:

$$\|\mathbb{T}(t)\|_{\mathcal{L}(X)} \leq 1, \qquad \forall t \geq 0.$$

Remark A.1.1. *The condition $(\star)$ implies that, for all $\xi \in X$ the map $\mathbb{T}(\cdot)\xi : [0, +\infty) \to X$ is continuous.*

Definition A.1.5. *We call infinitesimal generator of $\mathbb{T}(t)$, the operator A of domain $\mathcal{D}(A)$ defined by:*

$$\mathcal{D}(A) = \left\{\xi \in X : \lim_{h\to 0^+} \frac{\mathbb{T}(h)\xi - \xi}{h},\ \textit{exists}\right\},$$

and for $\xi \in \mathcal{D}(A)$:

$$A\xi = \lim_{h\to 0^+} \frac{\mathbb{T}(h)\xi - \xi}{h}.$$

We have the following theorem.

Theorem A.1.2. *Let $(\mathbb{T}(t))_{t\geq 0}$ be a semigroup of bounded operators on a Hilbert space X, and let A be its infinitesimal generator. For all $\xi_0 \in \mathcal{D}(A)$, we have:*

1. $\mathbb{T}(t)\xi_0 \in \mathcal{D}(A)$, *for all* $t \geq 0$,
2. $\frac{d}{dt}(\mathbb{T}(t)\xi_0) = A\mathbb{T}(t)\xi_0 = \mathbb{T}(t)A\xi_0, \quad$ *for all* $t > 0$,
3. $\mathbb{T}(t)\xi_0 - \xi_0 = \int_0^t A\mathbb{T}(\tau)\xi_0 d\tau$,
4. *A is closed, $\mathcal{D}(A)$ is everywhere dense in X.*

According to Theorem A.1.2, if A is the generator of a C_0-semigroup on X, then A is closed and densely defined. Conversely, if A is densely closed, there exist necessary and sufficient conditions for it to generate a semigroup on X, they are given by the following two fundamental theorems.

Theorem A.1.3 (Hille-Yosida Theorem). *Let A a closed, densely defined, linear operator on a Hilbert space X. A necessary and sufficient condition for A to generate a C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$ is that there exist two real constants M and ω such that:*

$$\forall \lambda > \omega; \qquad \lambda \in \rho(A),$$

and

$$\|\mathrm{R}(\lambda, A)^r\| \leq \frac{M}{(\lambda-\omega)^r}, \qquad \text{for all } r \geq 1,$$

where $\mathrm{R}(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator. In this case, we have:

$$\|\mathbb{T}(t)\| \leq Me^{t\omega}; \quad \forall t \geq 0.$$

Theorem A.1.4 ([Lumer and Phillips, 1961]). *Let A be a closed m-dissipative operator. Then A generates a strongly continuous semi-group of contraction operators.*

As a corollary to this theorem we have the following result.

Corollary A.1.1. *Let A be a closed dissipative operator. If A and A^* are dissipative, then A generates a C_0-semigroup $\mathbb{T}(t)$.*

Theorem A.1.5. *If $\mathbb{T}(t)$ is a C_0-semigroup with infinitesimal generator A on a Hilbert space X, then $\mathbb{T}^*(t)$ is a C_0-semigroup with infinitesimal generator A^* on X.*

Finally, the following theorem asserts that an infinitesimal generator of a perturbed semigroup by a bounded linear operator, is also a generator of a semigroup.

Theorem A.1.6. *Let X be a Hilbert space and A the infinitesimal generator of a C_0-semigroup $\mathbb{T}(t)$ on X, such that $\|\mathbb{T}(t)\| \leq Me^{\omega t}$. If B is a bounded linear operator on X, then $A+B$ is the infinitesimal generator of an C_0-semigroup $\mathbb{T}_{(A+B)}(t)$ on X, such that $\|\mathbb{T}_{(A+B)}(t)\| \leq Me^{(\omega+M\|B\|)t}$.*

A.2 Boundary control problem

We are now interested in the existence and uniqueness of solutions of systems with inhomogeneous boundary conditions,

$$\begin{cases} \dot{\xi}(t) = L\xi(t), & t \in [0, +\infty), \\ G\xi(t) = u(t), & t \in [0, +\infty), \\ \xi(0) = \xi_0 \in Z, \end{cases} \tag{A.2.1}$$

where $L \in \mathcal{L}(Z, X)$ is a differential operator and $G \in \mathcal{L}(Z, U)$ is a boundary trace operator. In the sequel, we assume that U; Z and X are complex Hibert spaces such that $Z \subset X$, with continuous embedding. We shall call U the input space, Z the solution space and X the state space.

Definition A.2.1. *A boundary control system on $U; Z$ and X is a pair of operators $(L, G) \in \mathcal{L}(Z, U) \times \mathcal{L}(Z, U)$, if there exists a $\lambda \in \mathbb{C}$ such that the following properties hold:*

i) G *is surjective,*

ii) $\operatorname{Ker} G$ *is dense in* X*,*

iii) $\lambda I - L$ *restricted to* $\operatorname{Ker} G$ *is bijective,*

iv) $\operatorname{Ker}(\lambda I - L) \cap \operatorname{Ker} G = \{0\}$.

Our aim is to translate the system (A.2.1) into the familiar form $\dot{\xi}(t) = A\xi(t) + Bu(t)$. The boundary control system will be called well-posed if A generates a semigroup on X and B is admissible for this semigroup, in the sense of the following definition.

Definition A.2.2. *Let $B \in \mathcal{L}(U, X_{-1})$ and $\tau \geq 0$. We define the input-state operator $\Phi_\tau \in \mathcal{L}(L^2([0, \infty); U), X_{-1})$ as follows*

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau - t} Bu(t)dt. \tag{A.2.2}$$

The operator $B \in \mathcal{L}(U, X_{-1})$ is called an admissible control operator for $\mathbb{T}(t)$, if $\operatorname{Im}\Phi_\tau \subset X$.

With the assumptions of the Definition A.2.1, we introduce the Hilbert space X_1 and the operator A by

$$X_1 = \operatorname{Ker} G, \qquad A = L_{|X_1} \in \mathcal{L}(X_1, X). \tag{A.2.3}$$

According to item iii) of Definition A.2.1, $\lambda I - A$ is invertible and $(\lambda I - A)^{-1} \in \mathcal{L}(X, X_1)$, so that the norm on X_1 is equivalent to the norm $\|\xi\|_1 = \|(\lambda I - A)\xi\|_X$. As usual, we define the space X_{-1} as the completion of X with respect to the norm $\|\xi\|_{-1} = \|(\lambda I - A)^{-1}\xi\|_X$. Then A has an extension, also denoted by A, such that $A \in \mathcal{L}(X, X_{-1})$.

Proposition A.2.1. *Let (L, G) be a boundary control system on U; Z and X. Let A and X_{-1} be as introduced earlier. Then there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that*

$$L = A + BG, \tag{A.2.4}$$

where A is regarded as an operator from X to X_{-1}. For every $\lambda \in \rho(A)$ we have that $(\lambda I - A)^{-1}B \in \mathcal{L}(U, Z)$ and

$$G(\lambda I - A)^{-1}B = I, \tag{A.2.5}$$

so that in particular, B is bounded from below.

When L, G, A and B are as in the above proposition, we say that A is the generator of the boundary control system (L, G) and B is the control operator of (L, G).

Remark A.2.1. *It follows from (A.2.5) that B is strictly unbounded with respect to X, meaning that $X \cap BU = \{0\}$.*

Remark A.2.2. *Often we need to express B^* in terms of L and G. Instead of finding the control operator B and then computing its adjoint, it is usually more convenient to use the following formula, which follows from (A.2.4):*

$$\langle L\xi, \zeta\rangle = \langle \xi, A^*\zeta\rangle + \langle G\xi, B^*\zeta\rangle \qquad \forall \xi \in X,\ \zeta \in \mathcal{D}(A^*). \tag{A.2.6}$$

Now that we have made the connection between the problem with inhomogeneous boundary condition (A.2.1) and the Cauchy problem with source term in the case where (L, G) is an inhomogeneous abstract boundary problem, we can apply the results of the previous paragraph to deduce the existence and uniqueness conditions of the solutions of (A.2.1).

Proposition A.2.2. *Let (L, G) be a boundary control system on U, Z and X, with A, B as in Proposition A.2.1. We assume that A is the generator of a strongly continuous semigroup $\mathbb{T}(t)$ on X. Then for every $T > 0$, $\xi(0) \in Z$ and $u \in \mathcal{H}^2((0,T);U)$ which satisfy the compatibility condition $G\xi(0) = u(0)$, the equations (A.2.1) have a unique solution ξ and*

$$\xi \in C([0,T];Z) \cap C^1([0,T];X). \tag{A.2.7}$$

If (L, G) is well-posed, then the same conclusion holds for every $T > 0$, $\xi(0) \in X$ and $u \in \mathcal{H}^1((0,T);U)$ that satisfies $G\xi(0) = u(0)$.

B

Frequency analysis

B.1 Transfer functions

Let us now derive the transfer functions from the heat exchanger model described by the following dynamics equations:

$$\partial_t x = \underbrace{\begin{pmatrix} -v_1 & 0 \\ 0 & v_2 \end{pmatrix}}_{\Lambda} \partial_z x + \underbrace{\begin{pmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{pmatrix}}_{M} x(z,t), \tag{B.1.1}$$

associated with the following boundary conditions:

$$x_1(0,t) = x_{1,in}(t), \quad x_2(1,t) = x_{2,in}(t). \tag{B.1.2}$$

Equation (B.1.1) can be written in the Laplace domain $sx(z,s) - x(z,0) = \Lambda\frac{dx(z,s)}{dz} + Mx(z,s)$, or:

$$\frac{dx(z,s)}{dz} = \Lambda^{-1}(sI - M)x(z,s) - x(z,0). \tag{B.1.3}$$

with $x(z,s)$ the Laplace transform of $x(z,t)$ and s the Laplace variable. To establish the transfer functions of the system, we assume that $x(z,0) = 0$. In that case, the differential equation (B.1.3) admits as solution

$$x(z,s) = \Phi(z,s)x(0,s), \tag{B.1.4}$$

where $\Phi(z,s) = \exp\left(\Lambda^{-1}(sI - M)z\right) \in \mathbb{R}^{2\times 2}$ is the state-transition matrix associated to the differential equation (B.1.3), whose the components are:

$$\begin{aligned}
\phi_{11}(z,s) &= \frac{p_1(s) - \eta_{22}(s)}{p_1(s) - p_2(s)}\,\mathrm{e}^{p_1(s)z} - \frac{p_2(s) - \eta_{22}(s)}{p_1(s) - p_2(s)}\,\mathrm{e}^{p_2(s)z}, \\
\phi_{12}(z,s) &= \beta_1 \frac{\mathrm{e}^{p_1(s)z} - \mathrm{e}^{p_2(s)z}}{p_1(s) - p_2(s)}, \\
\phi_{21}(z,s) &= \beta_2 \frac{\mathrm{e}^{p_2(s)z} - \mathrm{e}^{p_1(s)z}}{p_1(s) - p_2(s)}, \\
\phi_{22}(z,s) &= \frac{p_1(s) + \eta_{11}(s)}{p_1(s) - p_2(s)}\,\mathrm{e}^{p_1(s)z} - \frac{p_2(s) + \eta_{11}(s)}{p_1(s) - p_2(s)}\,\mathrm{e}^{p_2(s)z},
\end{aligned} \tag{B.1.5}$$

with $p_{1,2}(s) = \alpha(s) \pm \beta(s)$, where $\beta(s) = \frac{\sqrt{\Delta(s)}}{2}$, $\Delta(s) = (\eta_{11}(s) + \eta_{22}(s))^2 - 4\beta_1\beta_2$, $\eta_{11}(s) = \frac{s+\alpha_1}{v_1}$, $\eta_{22}(s) = \frac{s+\alpha_2}{v_2}$, $\alpha(s) = \frac{1}{2}(\eta_{22}(s) - \eta_{11}(s))$ and $\gamma(s) = \eta_{11}(s) + \eta_{22}(s)$. The components of the state-transition matrix $\Phi(z,s)$ can be written in the form of hyperbolic functions as follows:

$$
\begin{aligned}
\phi_{11}(z,s) &= \frac{\mathrm{e}^{\alpha(s)z}}{2\beta(s)}\left(2\beta(s)\cosh\left(\beta(s)z\right) - \gamma(s)\sinh\left(\beta(s)z\right)\right)\\
\phi_{12}(z,s) &= \frac{\beta_1}{\beta(s)}\,\mathrm{e}^{\alpha(s)z}\sinh\left(\beta(s)z\right)\\
\phi_{21}(z,s) &= -\frac{\beta_2}{\beta(s)}\,\mathrm{e}^{\alpha(s)z}\sinh\left(\beta(s)z\right)\\
\phi_{22}(z,s) &= \frac{\mathrm{e}^{\alpha(s)z}}{2\beta(s)}\left(2\beta(s)\cosh\left(\beta(s)z\right) + \gamma(s)\sinh\left(\beta(s)z\right)\right).
\end{aligned}
\tag{B.1.6}
$$

Considering the fact that the system has an infinity of states, we can derive the distributed transfer functions, i.e., input-states and lumped transfer functions, i.e., input-output (see [Litrico and Fromion, 2009b], [Litrico and Fromion, 2009a] for applications of open channel flow). To obtain these transfer functions in the (counterflow) configuration of the heat exchanger considered, we need to evaluate equation (B.1.4) at $z = 1$. This lets us have:

$$
x(0,s) = \frac{1}{\phi_{22}(1,s)}x_2(1,s) - \frac{\phi_{21}(1,s)}{\phi_{22}(1,s)}x_1(0,s). \tag{B.1.7}
$$

After elementary manipulations, we obtain the distributed transfer matrix, relating the distributed variables $x_1(z,s)$, $x_2(z,s)$ and the input variables $x_1(0,s)$ and $x_2(1,s)$ at any point z of the heat exchanger:

$$
\begin{pmatrix} x_1(z,s) \\ x_2(z,s) \end{pmatrix} = \begin{pmatrix} G_{11}(z,s) & G_{12}(z,s) \\ G_{21}(z,s) & G_{22}(z,s) \end{pmatrix} \begin{pmatrix} x_1(0,s) \\ x_2(1,s) \end{pmatrix} \tag{B.1.8}
$$

with

$$
\begin{aligned}
G_{11}(z,s) &= \phi_{11}(z,s) - \phi_{12}(z,s)\frac{\phi_{21}(1,s)}{\phi_{22}(1,s)}\\
&= \frac{\gamma(s)\,\mathrm{e}^{\alpha(s)z}\sinh\left(\beta(s)(z-1)\right)}{2\beta(s)\cosh\left(\beta(s)\right) + \gamma(s)\sinh\left(\beta(s)\right)}\\
&\quad + \frac{2\beta(z)\,\mathrm{e}^{\alpha(s)z}\cosh(\beta(s)(z-1))}{2\beta(s)\cosh\left(\beta(s)\right) + \gamma(s)\sinh\left(\beta(s)\right)},\\
G_{12}(z,s) &= \frac{\phi_{12}(z,s)}{\phi_{22}(1,s)} = \frac{2\beta_1\,\mathrm{e}^{\alpha(s)(z-1)}\sinh\left(\beta(s)z\right)}{2\beta(s)\cosh\left(\beta(s)\right) + \gamma(s)\sinh\left(\beta(s)\right)},\\
G_{21}(z,s) &= \phi_{21}(z,s) - \phi_{22}(z,s)\frac{\phi_{21}(1,s)}{\phi_{22}(1,s)},\\
&= \frac{2\beta_2\,\mathrm{e}^{\alpha(s)z}\sinh\left(\beta(s)(z-1)\right)}{2\beta(s)\cosh\left(\beta(s)\right) + \gamma(s)\sinh\left(\beta(s)\right)}\\
G_{22}(z,s) &= \frac{\phi_{22}(z,s)}{\phi_{22}(1,s)} = \mathrm{e}^{\alpha(s)(z-1)}\,\frac{2\beta(s)\cosh\left(\beta(s)z\right) + \gamma(s)\sinh\left(\beta(s)z\right)}{2\beta(s)\cosh\left(\beta(s)\right) + \gamma(s)\sinh\left(\beta(s)\right)}.
\end{aligned}
\tag{B.1.9}
$$

Similarly the input-output transfer functions are given by:

$$
\begin{pmatrix} x_1(1,s) \\ x_2(0,s) \end{pmatrix} = \begin{pmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{pmatrix} \begin{pmatrix} x_1(0,s) \\ x_2(1,s) \end{pmatrix}, \tag{B.1.10}
$$

where

$$
\begin{aligned}
g_{11}(s) &= \phi_{11}(1,s) - \phi_{12}(1,s)\frac{\phi_{21}(1,s)}{\phi_{22}(1,s)} \\
&= \frac{2\beta(s)\,\mathrm{e}^{\alpha(s)}}{2\beta(s)\cosh(\beta(s)) + \gamma(s)\sinh(\beta(s))}, \\
g_{12}(s) &= \frac{\phi_{12}(1,s)}{\phi_{22}(1,s)} = \frac{2\beta_1 \sinh(\beta(s))}{2\beta(s)\cosh(\beta(s)) + \gamma(s)\sinh(\beta(s))}, \\
g_{21}(s) &= -\frac{\phi_{21}(1,s)}{\phi_{22}(1,s)} = \frac{2\beta_2 \sinh(\beta(s))}{2\beta(s)\cosh(\beta(s)) + \gamma(s)\sinh(\beta(s))}, \\
g_{22}(s) &= \frac{1}{\phi_{22}(1,s)} = \frac{2\,\mathrm{e}^{-\alpha(s)}\,\beta(s)}{2\beta(s)\cosh(\beta(s)) + \gamma(s)\sinh(\beta(s))}.
\end{aligned}
\tag{B.1.11}
$$

B.2 Poles and Bode diagrams

The main difficulty with the transfer functions obtained is that they are not rational. This is due to the fact that the model that governs the heat exchanger is characterized by PDEs. Nevertheless, it remains true that the system will be stable if and only if the poles of these transfer functions have negative real parts. The difference with rational transfer functions is that there are infinitely many poles here. We will see that within the framework of the analysis of the uncontrolled system, these poles can be explained analytically in some cases.
Indeed, the characteristic equation is $2\beta(s)\cosh\beta(s) + \gamma(s)\sinh\beta(s) = 0$, which can be rewritten as

$$
\frac{p_1(s) + \frac{s+\alpha_1}{v_1}}{p_1(s) - p_2(s)} e^{p_1(s)} + \frac{p_2(s) + \frac{s+\alpha_1}{v_1}}{p_2(s) - p_1(s)} e^{p_2(s)} = 0 \tag{B.2.1}
$$

Observe that for $\Delta(s) = 0$, the poles of the distributed and input-output transfer functions are

$$
s_n = -\frac{v_1\alpha_2 + v_2\alpha_1}{v_1 + v_2} \pm 2\frac{v_1 v_2}{v_1 + v_2}\sqrt{\beta_1\beta_2 - n^2\pi^2}, \quad \text{with } n \in \mathbb{N}., \tag{B.2.2}
$$

which are located in the left complex half plane for positive values of the model parameters. In the general case ($p_1(s) \neq p_2(s)$) we certainly cannot express in algebraic form the solutions of the equation (B.2.1), but some observations will allow to conclude that the poles are in the left complex half plane.
For $\Delta(s) \neq 0$ we will approximate (B.2.1) by the Taylor expansions:

$$
\begin{aligned}
\sqrt{\Delta(s)} &= s\sqrt{\left(\frac{v_1 + v_2}{v_1 v_2} + \frac{\alpha_1 v_2 + \alpha_2 v_1}{v_1 v_2}\frac{1}{s}\right)^2 - 4\frac{\alpha_1\alpha_2}{v_1 v_2}\frac{1}{s^2}} \\
&= s f(\tfrac{1}{s}).
\end{aligned}
$$

The expansion of $f(\frac{1}{s})$ when $s = \infty$ is given by

$$
\begin{aligned}
f(\tfrac{1}{s}) &= \frac{v_1 + v_2}{v_1 v_2} + \frac{\alpha_1 v_2 + \alpha_2 v_1}{v_1 v_2}\frac{1}{s} + \mathcal{O}(\tfrac{1}{s^2}) \\
&= a + \frac{b}{s} + \mathcal{O}(\tfrac{1}{s^2}).
\end{aligned}
$$

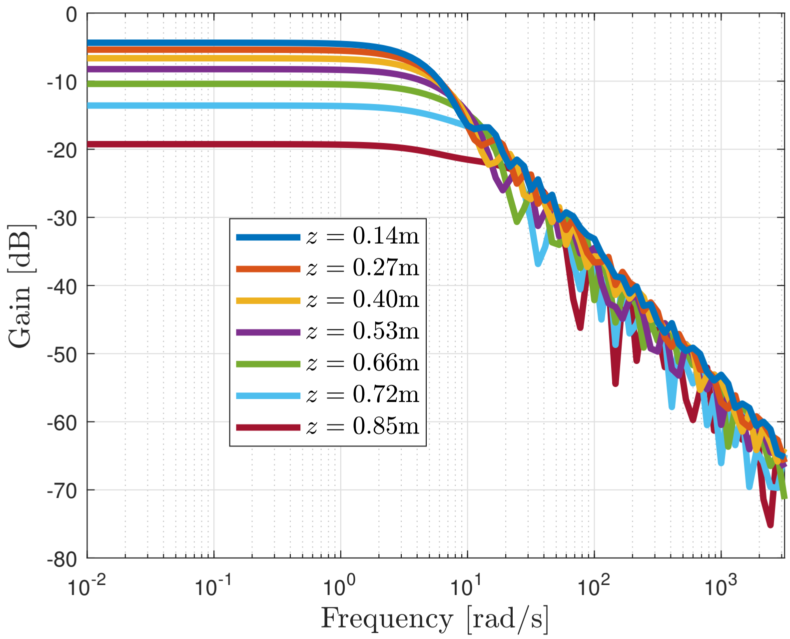

Figure B.1: Bode plot of transfer function $G_{12}(z, s)$

For $|s| \to \infty$ the characteristic equation (B.2.1) becomes:

$$\begin{aligned} e^{2\sqrt{\Delta(s)}} &= \frac{(1 - av_2)s + \alpha_2 - bv_2}{(1 + av_2)s + \alpha_2 - bv_2} \\ &= \frac{1 - av_2}{1 + av_2} + \mathcal{O}(1). \end{aligned}$$

Then the poles of the transfer functions are

$$s_n = -\frac{b}{a} + \frac{1}{a}\ln\sqrt{\frac{1 - av_2}{1 + av_2}} + j\left(n\pi + \frac{\pi}{2}\right), \; n \in \mathbb{Z}. \tag{B.2.3}$$

We can verify that there are located in the left complex half plane. Indeed

$$\frac{1 - av_2}{1 + av_2} \Leftrightarrow -\frac{v_2}{2v_1 + v_2} < 0, \implies \left|\frac{v_2}{2v_1 + v_2}\right| < 1.$$

The poles are therefore all stable in open-loop. Figures B.1 and B.2 give the Bode diagrams of the transfer functions $G_{12}(z, s)$ and $G_{21}(z, s)$ respectively at different points along the heat exchanger, and illustrates the conclusions that can be drawn about the stability of the open-loop system. We note that the frequency behaviour of the (hyperbolic) model considered is comparable to stable first order system. We also note the presence of resonant modes in the higher frequencies.

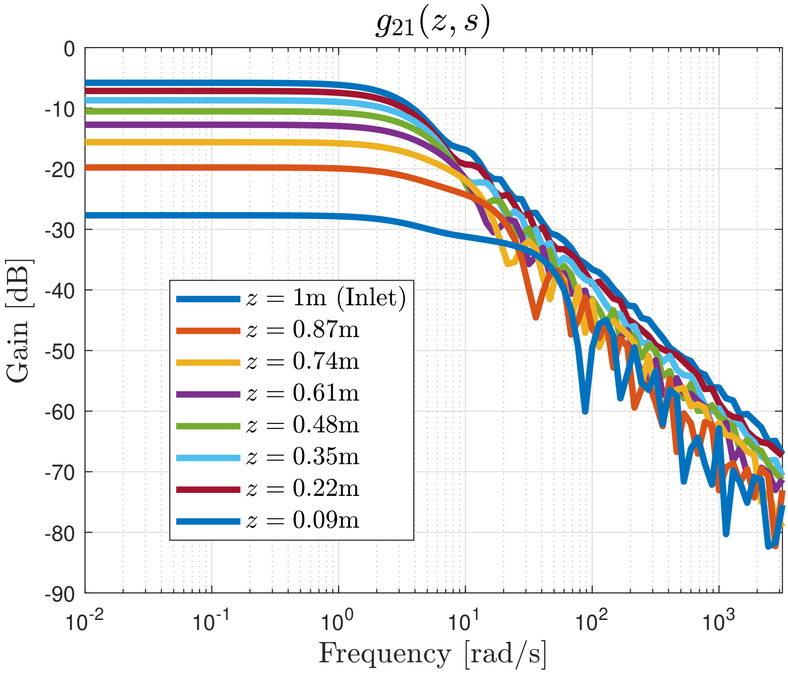

Figure B.2: Bode plot of transfer function $G_{21}(z, s)$

Remark B.2.1. *Note that in general the control operator B of the heat exchanger is never full rank. Indeed, to control the outlet temperatures $x_1(z,t)|_{z=1}$ or $x_2(z,t)|_{z=0}$ we manipulate (except for techniques based on the manipulation of flow rates via transport velocities [Aulisa et al., 2016]) either $x_2(1,t)$ or $x_1(0,t)$, respectively [Maidi et al., 2009], [Burns and Zietsman, 2016]. For this reason the transfer functions to be considered are $G_{12}(z,s)$ and $G_{21}(z,s)$, or $g_{12}(s)$ and $g_{21}(s)$.*