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Abstract

This paper presents a strategic growth model that analyzes the impact of endogenous preferences on equilibrium dynamics by employing the tools provided by lattice theory and supermodular games. Supermodular game structure of the model let us provide monotonicity results on the greatest and the least equilibrium without making any assumptions regarding the curvature of the production function. We also sharpen these results by showing the differentiability of the value function and the uniqueness of the best response correspondence almost everywhere. We show that, unlike globally monotone capital sequences obtained under corresponding optimal growth models, a non-monotonic capital sequence can be obtained. We conclude that the rich can help the poor avoid poverty trap whereas even under convex technology, the poor may theoretically push the rich to her lower steady state.

Keywords: Lattice programming, Endogenous time preference
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1 Introduction

The classical optimal growth models focus on the convex structures of the technology and preferences that guarantee the monotonical convergence of the sequence of optimal stocks towards a unique steady state. Such a structure imply that convergence is assured under perfect competition, constant or diminishing returns with no external effects and the same constant discount rate, independent of initial conditions. However, growth is uneven among countries (see Quah, 1996; Barro, 1997; Barro and Sala-i-Martin, 1991), regional disparities are persistent and income inequality is severe among the individuals.

To account for non-convergent growth paths among countries and regions, a variety of one-sector optimal growth models that incorporate some degree of market imperfections based on technological external effects and increasing returns have been presented. Within a model of capital accumulation with convex-concave production function, Dechert and Nishimura (1983), Mitra and Ray (1984) have characterized optimal paths and prove the existence of threshold

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effect that generates development or poverty traps (see Azariadis and Stachurski, 2005, for a recent survey). In these models, an economy with low initial capital stock converge to a steady state with low per capita income, while an economy with high initial capital stock converge to a steady state with high per capita income.

On the preference side, the endogeneity of time preference is put forward to show the economies' dependence on initial endowments. While Mantel (1998) consider discount factor as a function of consumption, Stern (2006), along the lines of Becker and Mulligan (1997), let individuals spend resources to increase the appreciation of the future. Le Van et. al. (2009) adapt the classic optimal growth model to include an endogenous rate of time preference depending on the stock of capital. Under these various forms of endogenous discounting, multiplicity of steady states and conditionally sustained growth are shown.

Although initial conditions are important for growth, poverty should not be a curse. Indeed, overcoming regional disparities in per-capita income is a matter of major concern for the governments. Cohesion policies targeting regional disparities aim to connect economically the rich regions with the other regions, so that the laggard regions may benefit from and contribute to the overall economic performance. For example, when the Portugal and Spain joined the EU in 1986, their GDP per capita was - on a purchasing power basis - at 53 % and 70 % of EU average, respectively. In 2008, their GDP per capita - on a purchasing power basis - attained 70% and 90% of the EU-15, (see Eurostat figures, 2009). Provided their history and political transition period, it is obvious that both countries would have grown out of the EU frame. However, EU membership has fostered their growth through various channels including transfers. During the 1994-1999, the structural and cohesion funds amounted 3.3 % of Portuguese GDP and 1.5 % of Spanish GDP. The percentage of public investment financed by EU funds reached average values of 42 % in Portugal and 15 % in Spain (Royo and Manuel, 2003).

In order to overcome regional disparities substantial fiscal transfers flow from relatively rich regions to relatively poor ones. It has been estimated that around 1.3 trillion euros have been transferred to the East after the German reunification (Barrell and Velde, 2000). Due to this effort living standards in the East have risen with GDP per capita attaining 70% of western Germans, however there still exists a gap between East and West. Taking into account that 4% of German GDP is annually consumed by the costs of reunification, the lack of economic blossoming in the East could prevent Germany, and West Germany in particular, from attaining a higher growth trajectory path. Financial transfers to the Mezzogiorno, southern part of Italy, are another example where the burden is put on the large agent in size. There are funds flowing to the South in the form of national health service, unemployment insurance and public expenditures from the central government (Faini et al., 1993). Moreover, Alesina et. al. (1999) shows the use of public employment as a subsidy from the North to the less wealthy South.

On the other hand, richer region would be small in size and may contribute relatively too much to the common goal, deterring its own growth. Catalonia suffers a slow but persistent drop in GDP, attaining its historical minimum within Spain in 2001, 18.6 % of national GDP (Pons-i-Novell and Tremosa-i-Balcells, 2005). Although one can find many reasons that could explain this slowdown, one can not deny the role of the fiscal deficit amounted to 8.9 % of

Catalan GDP (Alcaide and Alcaide, 2002) in the slowdown process. Indeed, according to Ros et. al. (2003), Catalan GDP would have been a 31.3 % bigger in 2000 if the Catalan fiscal deficit had been reinvested in public capital in Catalonia.

Reducing regional disparities is a more serious concern for developing countries, where these disparities are two to six times more than in the developed countries (Shankar and Shah, 2003, 2008). In such countries inequalities beyond a threshold may pose a threat to the economic and political stability of the country. The poorest regions may consider such inequalities as manifestation of regional injustice and call for radicalism and drastic redistribution policies. On the other hand, the richest regions may view a union with the poorest regions as a threat to their prosperity in the long run (Shankar and Shah, 2009). For example, in India, a threat to its territorial integrity comes from the rich south who no longer wants to subsidize poor regions of the North.

While the convergence literature focuses on sources and remedies of differences in factor productivity, the interaction among heterogeneous agents and the consequences of such an interaction has been left unexplored. The examples above, particularly the examples on developed countries suggest that the outcomes of the cohesion policies may not be attributable solely to the factor productivity differences. It is the purpose of this paper to analyze the impact of endogenous preferences on equilibrium dynamics in a strategic growth model. We make our analysis while abstracting from any source of factor productivity differences for both simplicity and being able to distinguish the consequences of strategic interaction and endogeneity of time preference from the factors dominating convergence literature.

In line with the empirical studies concluding that the rich are more patient than the poor (see Lawrence, 1991; Samwick, 1998) and in parallel to the idea that the stock of wealth is a key to reach better health services and better insurance markets, we consider that the discount factor is increasing in the stock of wealth. In such a model, consumption profiles of the agents depend on each other and agents set their consumption strategies considering the strategies of the others. We introduce heterogeneity by letting two agents differ in their initial endowment, their share of aggregate income, and therefore in their subjective discount rates. We analyze whether an agent with a low initial capital stock can avoid poverty trap that she would face while acting alone. Moreover, we analyze the implications of an interaction among agents on the relatively richer one, particularly focusing on if the agent with a low initial capital stock can pull the rich to her lower steady state that she would never face while acting by herself.

In a standard optimal growth model with geometric discounting and the usual convexity assumptions on preferences and technology, the optimal path is easily found by differentiating the value function. However, in our model, the objective function includes multiplication of a discount function. This generally destroys the usual concavity argument which is used in the proof of the differentiability of value function and the uniqueness of the optimal paths (see Benveniste and Scheinkman, 1979; Araujo, 1991). Since we can not use classical convex analysis under the potential lack of concavity and the differentiability of the value functions, we employ the theory of monotone comparative statics and supermodular games based on order and monotonicity properties on lattices.

The framework of analysis to study supermodular games has been developed by Topkis (1968, 1978, 1979), Vives (1990), Milgrom and Roberts (1990) and provided comprehensively by Topkis(1998). By allowing very general strategy spaces, supermodular games provide a rich structure to be utilized in the analysis of many diverse models including dynamic games.

We establish the supermodular game structure of our model by showing that actions of the agents are strategic complements so that the best response correspondence of a player to the actions of rivals is increasing in their level. Supermodular game structure of the model let us provide existence and monotonicity results on the greatest and the least equilibrium. We sharpen these results by showing the differentiability of the value function and the uniqueness of the best response correspondence almost everywhere, following from Amir (1996) and Le Van and Dana (2003). Our approach also allows us to provide our results without imposing any assumption on the form of technology.

We show that agents may differ in their capital accumulation decisions. While one of the agents may accumulate, the other agent may deaccumulate the capital stock at the first period. After the first period, they both either accumulate or deaccumulate through the entire duration of the game. This allows us to obtain a non-monotonic capital sequence in contrast with globally monotone capital sequences obtained under the corresponding optimal growth model (Le Van et. al., 2009). While abstracting from the curvature of the technology, we provide that the interaction between the rich and the poor may help the poor avoid poverty traps that she would face while acting alone whereas the poor can theoretically push the rich to her lower steady state even under convex technology.

The article is organized as follows. The next section introduces the model. Tools needed while utilizing the supermodularity of the game; equilibrium dynamics and the steady state analysis have been discussed in Section 3. Finally, the Section 4 concludes.

2 Model

We consider an intertemporal one sector model of a private ownership economy a la Arrow-Debreu with a single good x_t, and two infinitely lived agents (households), $i = 1, 2$.

The single commodity is used as capital, along with labor, to produce output. Labor is presumed to be supplied in fixed amounts, and capital and consumption are interpreted in per-capita terms. The production function net of depreciation is given by $f(x_t)$. We assume that each agent receives a constant share, θ^i of the output (see Debreu, 1959). The amount of current resources not consumed by an agent is saved as capital until the next period. Given the optimal decisions of the rival, each agent chooses a path of consumption $\boldsymbol{c}^i = \left\{c_t^i\right\}_{t\geq 0}$ so as to maximize a discounted sum of instantaneous utilities. The real valued function u gives the instantaneous utility from consumption.

In accordance with these, the problem of each agent ($i = 1, 2$) can be formalized as follows:

$$\max_{\left\{c_t^i, x_{t+1}^i\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(x_s^i) \right) u^i(c_t^i), \tag{1}$$

subject to

$$\forall t,\ c_t^i + x_{t+1}^i \le \theta^i f(x_t^i + x_t^j),\ c_t^i \ge 0,\ x_t^i \ge 0, \tag{2}$$
$$x_0^i \ge 0,\ \boldsymbol{x}^j = \left\{x_t^j\right\} \ge 0, \text{given},$$

where $j \neq i \in \{1, 2\}$ and the real valued function $\beta(x_s^i)$ is the level of discount on future utility. Agents may only differ in their initial endowment, their share of output, and therefore in their subjective discount rates.

We make the following assumptions regarding the properties of the discount, utility and the production functions.

Assumption 1 *$u : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, concave, strictly increasing, $u'(0) = +\infty$ and $u(0) = 0$.*

Assumption 2 *$\beta : \mathbb{R}_+ \to \mathbb{R}_{++}$ is continuous, concave, strictly increasing and $\beta'(0) = +\infty$.*

Assumption 3 *$f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing and there exists an $\bar{x}$ such that $\beta\left(\bar{x}\right) < 1$ and $f(x) < x$ whenever $x > \bar{x}$.*

On the contrary to the standard optimal growth models, we assume that the rate of time preference is endogenous depending on the path of investment. Our assumption that β is strictly increasing in own capital implies a wealth effect in discounting. The assumption that engaging in some activity or sacrificing may alter the discount factor each period has been widely considered in a single decision maker problem (see Hamada and Takeda, 2008, for a recent survey). It has been shown that low levels of initial capital leads to a poverty trap so that paths of deaccumulation would be optimal. However, the impact of recursive preferences in the case of strategic growth games have been left almost unexplored. It is the purpose of this paper to fill this gap in the literature and analyze the consequences of recursive preferences on equilibrium dynamics in this discrete time dynamic game.

2.1 Non-cooperative difference game and equilibrium strategies

We adopt the noncooperative Nash equilibrium concept, in which case the strategy choice is simultaneous and each agent is faced with a single criterion optimization problem with the strategies of the rival taken to be fixed at their equilibrium values. This amounts to characterizing the optimal decisions of each player conditional on the decisions of the rival.

Noting that the constraints will be binding at the optimum as utility and the discount function is strictly increasing, we introduce the function U^i defined on the set of agent i's feasible sequences as

$$U^i\left(\boldsymbol{x}^i, \boldsymbol{x}^j\right) = \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(x_s^i)\right) u^i(\theta^i f(x_t^i + x_t^j) - x_{t+1}^i).$$

The following proposition ensures that, if an agent has an initial positive capital stock, independent of the rivals' choices, he will have a positive consumption and investment path through his trajectory.

Proposition 1 *(i) Given x_0^i and $\boldsymbol{x}^j$, there exists an optimal path $\boldsymbol{x}^i$. The associated optimal consumption path, $\mathbf{c}^i$ is given by*

$$c_t^i = \theta^i f(x_t^i + x_t^j) - x_{t+1}^i, \forall t.$$

(ii) If $x_0^i > 0$, every solution $(\boldsymbol{x}^i, \boldsymbol{c}^i)$ to the problem of agent i satisfies

$$c_t^i > 0,\ x_t^i > 0, \forall t.$$

Proof. *See Appendix.* ■

2.2 Bellman Equation and Best Response Correspondance

As the recursive structure of the standard optimal growth models is preserved by our model, in order to express the optimal decisions of each player conditional on the decisions of the rival, we will utilize the value function and the best response correspondance. Each agent i defines the value function as the solution to his dynamic optimization problem given the initial state and the decisions of his rival j, with $j \neq i$,

$$V^i(x_0^i \mid \boldsymbol{x}^j) = \max_{\{x_{t+1}^i\}_{t=0}^{\infty}} \left\{ \begin{array}{r} \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(x_s^i) \right) u(\theta^i f(x_t^i + x_t^j) - x_{t+1}^i) \\ \mid\ x_t^i \geq 0, \forall t \end{array} \right\} \quad (3)$$

V^i is well defined, non-negative, continuous and strictly increasing. The satisfaction of Bellman's equation is also straightforward (see Stokey and Lucas, 1989 and Le Van and Dana, 2003):

$$V^i\left(x_t^i \mid \boldsymbol{x}^j\right) = \max_{\{x_{t+1}^i\}_{t=0}^{\infty}} \left\{ u(\theta^i f(x_t^i + x_t^j) - x_{t+1}^i) + \beta(x_{t+1}^i) V^i\left(x_{t+1}^i \mid \boldsymbol{x}^j\right) \right\} \quad (4)$$

The best response correspondance of agent i, $\mu^i : \mathbb{R}_+ \to \mathbb{R}_+$, is defined as follows:

$$\mu^i\left(x_t^i \mid \boldsymbol{x}^j\right) = \arg\max \left\{ \begin{array}{r} u(\theta^i f(x_t^i + x_t^j) - y^i) + \beta(y^i) V^i\left(y^i \mid \boldsymbol{x}^j\right) \\ \mid y^i \in \left[0, \theta^i f(x_t^i + x_t^j)\right] \end{array} \right\} \quad (5)$$

The non-emptiness and the closedness of the best response correspondence and its equivalance with the optimal path follow easily from the continuity of the value function by a standard application of the theorem of the maximum (see Stern, 2006 and Le Van and Dana, 2003). Accordingly, if the game admits an equilibrium then the optimal choices of agent i will be prescribed by the solution $V^i\left(x_0^i \mid \boldsymbol{x}^j\right)$ of the functional equation (4).

In accordance with these, a sequence $\mathbf{x}^i$ is the maximizing control strategy of agent i if it satisfies

$$\forall t,\ V^i\left(x_t^i \mid \boldsymbol{x}^j\right) = u(c_t^i) + \beta(x_{t+1}^i) V^i\left(x_{t+1}^i \mid \boldsymbol{x}^j\right),\ \textit{where } x_{t+1}^i \in \mu_t^i\left(x^i \mid \boldsymbol{x}^j\right).$$

In a standard optimal growth model with geometric discounting and the usual concavity assumptions on preferences and technology, the optimal policy correspondence, μ is single valued and the properties of the optimal path is easily found by using the first order conditions together with envelope theorem by differentiating the value function. However, in our model, the objective function includes multiplication of a discount function. This generally destroys the usual concavity argument which is used in the proof of the differentiability of value function and the uniqueness of the optimal paths (see Benveniste and Scheinkman, 1979; Araujo, 1991).

3 Dynamic Properties of Strategic Equilibria

In order to prove existence of equilibrium and characterize the qualitative properties of the agents' optimal choices, we will employ lattice programming and Topkis' theorems on supermodular games (see Topkis, 1998) as we need to surmount the potential lack of concavity and the differentiability of the value functions. The following section outlines the properties of the supermodular games we will use in our analysis.

3.1 Supermodular Games and Topkis' Theorem

Definition 1 *A* ***non-cooperative game*** $(N, S, \{f_i : i \in N\})$ *is a supermodular game if the set S of feasible joint strategies is a sublattice of $\mathbb{R}^m$ (or of $\prod_{i \in N} \mathbb{R}^{m_i}$), the payoff function $f_i(y_i, x_{-i})$ is supermodular in y_i on S_i for each x_{-i} in S_{-i} and each player i, and $f_i(y_i, x_{-i})$ has increasing differences in (y_i, x_{-i}) on $S_i \times S_{-i}$ for each i.*

Under some regularity conditions, the following therom implies that the set of equilibrium points for a supermodular game is a nonempty complete lattice.

Theorem 1 *(Topkis, 1998) Consider a supermodular game $(N, S, \{f_i : i \in N\})$ for which the set S of feasible joint strategies is nonempty and compact and the payoff function $f_i(y_i, x_{-i})$ is upper semicontinuous in y_i on $S_i(x_{-i})$ for each x_{-i} in S_{-i} and i, then the set of equilibrium points is a nonempty complete lattice and a greatest and a least equilibrium point exist.*

We refer to two theorems while obtaining the results on equilibrium points. The first one gives us sufficient conditions on a parameterized collection of supermodular games such that the greatest and the least equilibrium points for the game corresponding to each particular parameter increase as the parameter increases. The second one gives us sufficient conditions for an optimization problem to obtain a stronger result ensuring that the best response correspondance of an agent in its initial capital is increasing. This fact will be crucial in proving the uniqueness of the equilibrium point where the greatest and the least equilibrium points coincide.

Theorem 2 *Let $S_i^t(x_{-i})$ denote the set of feasible strategies for player i given strategies x_{-i} for the other players and $S_{-i} = \{x_{-i} : S_i(x_{-i}) \text{ is nonempty}\}$. Suppose that T is a partially ordered set and $(N, S^t, \{f_i^t : i \in N\})$ is a collection of supermodular games parameterized by t in T where in game t, the payoff function for each player i is $f_i^t(x)$ and the set of feasible joint strategies is S^t. The set S^t of feasible joint strategies is nonempty and compact for each t in T and is increasing in t on T. Let S_{-i}^t and $S_i^t(x_{-i})$ denote the dependence of S_{-i} and $S_i(x_{-i})$ on the parameter t. For each player i and each x_{-i} in S_{-i}^t, the payoff function $f_i^t(y_i, x_{-i})$ is upper semicontinuous in y_i on $S_i^t(x_{-i})$ for each t in T and has increasing differences in (y^i, t) on $(\cup_{t \in T} S_i^t) \times T$. Then there exists a greatest equilibrium point and a least equilibrium point for each game t in T, and the greatest (least) equilibrium point for game t is incresing in t on T.*

Theorem 3 *Suppose that X is a lattice, T is a partially ordered set, S^t is a subset of X for each t in T, S^t is increasing in t on T, $f(x, t)$ is supermodular* *in x on X for each t in T, and $f(x,t)$ has strictly increasing differences in (x,t) on $X \times T$. If t' and t'' are in T, $t' < t''$, x' is in $argmax_{x \in S^{t'}} f(x,t)$ and x'' is in $argmax_{x \in S^{t''}} f(x,t)$, then $x' \preceq x''$.*

3.2 Equilibrium Dynamics

In this section, supermodularity of our strategic growth game will be established. Then by using this property, existence and monotonicity proofs for the extreme equilibrium points will be shown. Although these proofs are straightforward applications of the theorems above, since they are important to see how the supermodular game structure can be utilized in the analysis of dynamic games, they are provided in detail at the appendix.

In the strategic growth game we have presented, $N = \{i, j\}$ and $S^{(x_t^i | \boldsymbol{x}^j)} = \left[0, \theta^i f(x_t^i + x_t^j)\right]$ denotes the set of feasible strategies for agent i at each point in time. We define $S \subset \mathbb{R}^2$ as the set of feasible joint strategies, i.e., $S^{(x_t^i, x_t^j)} = S^{(x_t^i, | \boldsymbol{x}^j)} \times S^{(x_t^j, | \boldsymbol{x}^i)}$. Agent i's payoff function is defined on $S^{(x_t^i, x_t^j)}$; for every joint strategy $y_t = (y_t^i, y_t^j) \in S^{(x_t^i, x_t^j)}$ agent i receives an utility measured by her payoff function:

$$P^{(x_t^i | \boldsymbol{x}^j)}(y^i) = u(\theta^i f(x_t^i + x_t^j) - y^i) + \beta\left(y^i\right) V^i\left(y^i \mid \boldsymbol{x}^j\right),$$

where $y^j = x_{t+1}^j \in \boldsymbol{x}^j$.

Proposition 2 *The non-cooperative game, $(N = \{i, j\}, S^{(x^i, x^j)}, \{P^{(x^i | \ \boldsymbol{x}^j)} : i \in N\})$ is a supermodular game.*

Proof. See Appendix. ■

For each feasible joint strategy z in $S^{(x^i, x^j)}$, the best joint response correspondence, $\mu(x \mid z \in \boldsymbol{x})$ is defined as the direct product of the individual agents' best response correspondances:

$$\mu(x \mid z \in \boldsymbol{x}) = \arg \max_{y \in S^{(x^i, x^j)}} P^{(x^i | \ \boldsymbol{x}^j)}(y^i) + P^{(x^j | \ \boldsymbol{x}^i)}(y^j).$$

Recall that, $P^{(x_t^i | \boldsymbol{x}^j)}(y^i) = u(\theta^i f(x_t^i + x_t^j) - y^i) + \beta\left(y^i\right) V^i\left(y^i \mid \boldsymbol{x}^j\right)$, and $P^{(x_t^i | \boldsymbol{x}^j)}(y^j) = u(\theta^i f(x_t^i + x_t^j) - y^j) + \beta\left(y^j\right) V^i\left(y^j \mid \boldsymbol{x}^i\right)$, where $z^j = x_{t+1}^j \in \boldsymbol{x}^j$ and $z^i = x_{t+1}^i \in \boldsymbol{x}^i$.

Proposition 3 *Supermodular game $(N, S^{(x^i, x^j)}, \{P^{(x^i | \ \boldsymbol{x}^j)} : i \in N\})$ has the set of equilibrium points which is a nonempty complete lattice and a greatest and a least equilibrium points exist.*

Proof. See Appendix. ■

The existence of equilibria in this class of infinite action dynamic games with imperfect information also follows from Chakrabarti (1999). Given a discrete time dynamic game, it is shown that one can find a finite action game such that every behavior strategy combination of the finite action game can be mimicked by a behavior strategy combination of the original infinite action game in the sense that the resulting payoffs from the behavior strategies cannot differ by more than the small amount ϵ. Moreover, for every strategy combination on a subgame of the original game, it is proven that one can find a behavior strategy of the finite action game which will give payoffs that can not differ by more than ϵ on the subgames.

Accordingly, in our context, as the instantaneous utility functions are continuous and the individual consumption sets are compact sets that depend continuously on the choices made previously, the particular game that we have described above has $\epsilon-perfect$ equilibria (see Chakrabarti, 1999). Such an $\epsilon-perfect$ equilibria can incorporate most of the characteristics of the perfect equilibrium which was used to construct it. Indeed, considering the limits of sequences of outcomes induced by $\epsilon-perfect$ equilibria, with ϵ converging to zero, one can demonstrate that the game admits a subgame perfect equilibrium[1].

Proposition 4 *Let* $X = \{(x^i, x^j) \mid x^i \geq 0,\ x^j \geq 0,\ x^i + x^j \leq f(\max\ (x_0^i + x_0^j, \bar{x}))\}$, *and* $\hat{X} = \left[0,\ f\left(\max(x_0^i + x_0^j,\ \bar{x})\right)\right]$.

i) The greatest (least) equilibrium point for the game $(N,\ S^{(x^i,x^j)}, \{P^{(x^i \mid \boldsymbol{x}^j)} : i \in N\})$ *is incresing in* (x^i, x^j) *on* X.

ii) The greatest (least) equilibrium point for the game $(N, S^x, \{P^{(x \mid \boldsymbol{x}^j)} : i \in N\})$, *where* $x = (x^i + x^j)$ *is increasing in* x *on* $\hat{X}$.

Proof. See Appendix. ■

Next, we provide differentiability of the value function. It will not only help us obtain the Euler conditions easily, but also sharpen the results we get. Since almost everywhere differentiability of the value function is equivalent to almost everywhere uniqueness of the best response correspondance, all the results we get for the extreme equilibrium points hold for the unique equilibrium.

Lemma 1 V^i *is differentiable almost everywhere with*

$$V^{i\prime}(x^i \mid \boldsymbol{x}^j) = u'(\theta f(x^i + x^j) - \mu_i\left(x^i \mid \boldsymbol{x}^j\right))]\theta f'(x^i + x^j)$$

where $\mu_i(x^i \mid \mathbf{x}^j)$ *is the best response correspondance of agent i.*

Proof. See Appendix. ■

Now, by using the differentiability of value function, we could obtain the Euler equation.

Lemma 2 $\forall x \in [0, \max(x_0, \bar{x})]$, $\mu(\cdot)$ *satisfies the Euler equation:*

$$u'[\theta f(x^i + x^j) - \mu_i\left(x^i \mid \boldsymbol{x}^j\right)] = \beta'[\mu_i\left(x^i \mid \boldsymbol{x}^j\right)]V^i[\mu_i\left(x^i \mid \boldsymbol{x}^j\right)] + \\ \beta[\mu_i\left(x^i \mid \boldsymbol{x}^j\right)]V^{i\prime}[\mu_i\left(x^i \mid \boldsymbol{x}^j\right)].$$

Proof. We take the first order derivative of 4 and equate this derivative to zero to obtain the Euler equation. ■

The next result is important to see the possibility of non-monotonic capital sequence in which while one agent accumulates, the other one deaccumulates capital stock at the first period of the game. After the first period, equilibrium dynamics are characterized by the total capital stock.

[1]See *Harris (1985), Hellwig and Leininger (1987)* and *Carmona (2003)* for the existence of subgame perfect equilibrium in dynamic games with perfect information. Note that the assumption of perfect information rules out the possibility that agents make simultaneous choices. However, this assumption was used only to show the non-emptiness of a correspondence that specifies a set of outcome paths feasible in any subgame in Harris (1985) and can be dropped by imposing supplementary conditions which guarantee this step.

Proposition 5 *Either both agents accumulate or both agents deaccumulate after the first period of the game. Therefore, we have either* $\left\{(x_t^i,\ x_t^j)\right\}_{t=1}^{\infty} \geq \left\{(x_{t+1}^i, x_{t+1}^j)\right\}_{t=1}^{\infty}$, *or* $\left\{(x_t^i, x_t^j)\right\}_{t=1}^{\infty} \leq \left\{(x_{t+1}^i, x_{t+1}^j)\right\}_{t=1}^{\infty}$.

Proof. *We have four cases to analyze:*

Case1: $x_0^i \geq x_1^i$ *and* $x_0^j \geq x_1^j$. *Since* $x_0^i + x_0^j \geq x_1^i + x_1^j$, *by the monotonicity of the equilibrium in total capital, we have* $x_1^i \geq x_2^i$ *and* $x_1^j \geq x_2^j$. *By the same reason we can conclude that* $x_t^i \geq x_{t+1}^i$ *and* $x_t^j \geq x_{t+1}^j$ *for all* $t \geq 1$.

Case2: $x_0^i \leq x_1^i$ *and* $x_0^j \leq x_1^j$. *Since* $x_0^i + x_0^j \leq x_1^i + x_1^j$, *by the monotonicity of the equilibrium in total capital, we have* $x_1^i \leq x_2^i$ *and* $x_1^j \leq x_2^j$. *Hence we can conclude that* $x_t^i \leq x_{t+1}^i$ *and* $x_t^j \leq x_{t+1}^j$ *for all* $t \geq 1$.

Case3: $x_0^i \geq x_1^i$ *and* $x_0^j \leq x_1^j$ *where* $x_0^i + x_0^j \geq x_1^i + x_1^j$. *Since* $x_0^i + x_0^j \geq x_1^i + x_1^j$, *by the monotonicity of the equilibrium in total capital, we have* $x_1^i \geq x_2^i$ *and* $x_1^j \geq x_2^j$. *After* $t = 1$, *we are in case 1 and conclude that* $x_t^i \geq x_{t+1}^i$ *and* $x_t^j \geq x_{t+1}^j$ *for all* $t \geq 1$.

Case4: $x_0^i \geq x_1^i$ *and* $x_0^j \leq x_1^j$ *where* $x_0^i + x_0^j \leq x_1^i + x_1^j$. *Since* $x_0^i + x_0^j \leq x_1^i + x_1^j$, *by the monotonicity of the equilibrium in total capital, we have* $x_1^i \leq x_2^i$ *and* $x_1^j \leq x_2^j$. *After* $t = 1$, *we are in case 2 and conclude that* $x_t^i \leq x_{t+1}^i$ *and* $x_t^j \leq x_{t+1}^j$ *for all* $t \geq 1$. ■

Remark 1 *None of the results we get in this section, particularly differentiability of value function and monotone dynamics of capital, require the concavity of production function.*

3.3 Steady States

The existence of a steady state follows from Proposition 5 and the fact that $X = \left\{(x^i, x^j) \mid x^i \geq 0,\ x^j \geq 0,\ x^i + x^j \leq f\left(\max(x_0^i + x_0^j, \bar{x})\right)\right\}$ is a compact set. Euler equation provides us the following necessary conditions for the characterization of the steady states:

$$u'[\theta f(x^i + x^j) - x^i] = \beta'(x^i)\frac{u[\theta f(x^i + x^j) - x^i]}{1 - \beta(x^i)} + \theta\beta(x^i)u'(\theta f(x^i + x^j) - x^i)f'(x^i + x^j),$$

and

$$u'[(1-\theta)\, f(x^i + x^j) - x^j] = \beta'(x^j)\frac{u[(1-\theta)\, f(x^i + x^j) - x^j]}{1 - \beta(x^j)} + (1-\theta)\beta(x^i)u'((1-\theta)f(x^i + x^j) - x^j)f'(x^i + x^j).$$

Proposition 6 *Steady states are linearly ordered. That is, for each pair of distinct steady states,* $\hat{x} = (\hat{x}_i, \hat{x}_j)$ *and* $\check{x} = (\check{x}_i, \check{x}_j)$, *either* $\hat{x} > \check{x}$ *or* $\hat{x} < \check{x}$.

Proof. *For a pair of distinct steady states, we have either* $\hat{x}_i + \hat{x}_j = \check{x}_i + \check{x}_j$ *or* $\hat{x}_i + \hat{x}_j \neq \check{x}_i + \check{x}_j$. *Let us assume that they are not linearly ordered. Figure 1 illustrates these cases diagramatically.*

Case 1: $\hat{x}_i + \hat{x}_j = \check{x}_i + \check{x}_j$. *Let's take* $x^n = (\hat{x}_i + \frac{1}{n}, \hat{x}_j)$, *then* $x_i^n + x_j^n > \hat{x}_i + \hat{x}_j$ *and* $x_i^n + x_j^n > \check{x}_i + \check{x}_j$. $\forall$ *n such that* $x_0 = x^n$, *we can conclude* $x_1 \geq \check{x}$ *and*

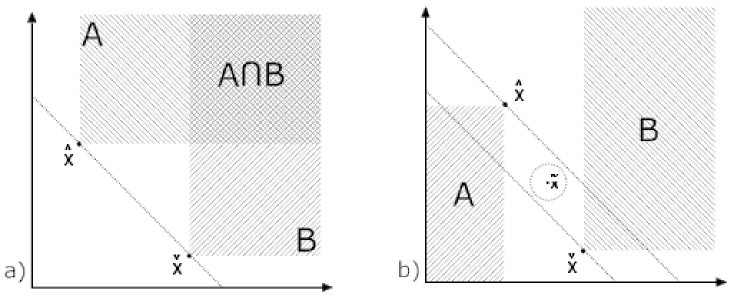

Figure 1: Possible cases if steady states were not linearly ordered

$x_1 \geq \hat{x}$. Hence $x_1 \in A \cap B$ where $A = \{x \mid x \geq \hat{x}\}$ and $B = \{x \mid x \geq \check{x}\}$. By construction, $\lim_{n\to\infty} x^n = \hat{x}$. But, for $x_0 = \hat{x}$, $x_1 = x_0 = \hat{x} \notin A \cap B$ yields a contradiction since best response correspondance is right continuous.

Case 2: $\hat{x}_i + \hat{x}_j \neq \check{x}_i + \check{x}_j$. Without loss of any generality, suppose that $\hat{x}_i + \hat{x}_j > \check{x}_i + \check{x}_j$. Let's $\tilde{x}$ denote $(\frac{(\hat{x}_i+\check{x}_i)}{2}, \frac{(\hat{x}_j+\check{x}_j)}{2})$. There exists ε such that, for $\forall\ x \in B^{\varepsilon}(\tilde{x}_i, \tilde{x}_j)$, we have $\hat{x}_i + \hat{x}_j > \tilde{x}_i + \tilde{x}_j > \check{x}_i + \check{x}_j$. Since $\hat{x}$ is a steady state, for $x_0 = \hat{x}$, we have $x_1 = \hat{x}$. By letting $x_0 = x$, we obtain $x_1 \leq \hat{x}$. In the same way, we can show $x_1 \geq \check{x}$. By almost everywhere uniqueness of the best response correspondances, for some $x_0 = x$, $x_1 \in A \cap B$ where $A = \{x \mid x \leq \hat{x}\}$ and $B = \{x \mid x \geq \tilde{x}\}$. Since $A \cap B = \emptyset$, this contradicts the existence of equilibrium points. ■

Propositions 4 and 5 provide us a way to specify the steady states where capital stock may converge. The next proposition shows that if the initial total capital stock is less than the total capital at the lowest steady state, both agents will accumulate after the first period until capital stock converges to this steady state. Similarly, if the initial total capital stock is more than what we have at the highest steady state, both agents will deaccumulate after the first period until capital stock converges there. For all other cases, capital stock converges either to the highest among stable steady states in which total capital stock is lower than the initial total capital stock or to the lowest among stable steady states in which total capital stock is higher than the initial total capital stock. Moreover, as long as one agent accumulates more than the amount that the other agent deaccumulates, capital stock will converge to the higher of those through a sustained accumulation path.

Proposition 7 *Let $\hat{x}$ denote the lowest among stable steady states where total capital stock is higher than the initial total capital stock and $\check{x}$ denote highest among stable steady states where total capital stock is lower than the initial total capital stock, if they exist. Additionally, let $x_t, \underline{x}$ and $\overline{x}$ denote capital stock at time t, the lowest steady state and the highest steady state, respectively.*

(a) If $x_0^i + x_0^j \leq \underline{x}^i + \underline{x}^j$ then $\lim_{t\to\infty} x_t = \underline{x}$ and $x_t \leq \underline{x}$ for all $t \geq 1$.

(b) If $x_0^i + x_0^j \geq \overline{x}^i + \overline{x}^j$ then $\lim_{t\to\infty} x_t = \overline{x}$ and $x_t \geq \overline{x}$ for all $t \geq 1$.

(c) If $\hat{x}^i + \hat{x}^j \geq x_0^i + x_0^j \geq \check{x}^i + \check{x}^j$ then either $\lim_{t\to\infty} x_t = \check{x}$ and $x_t \geq \check{x}$ for all $t \geq 1$ or $\lim_{t\to\infty} x_t = \hat{x}$ and $x_t \leq \hat{x}$ for all $t \geq 1$.

Proof. *From proposition 4 (ii) and proposition 5, (a) and (b) follow. Suppose the condition under part (c) holds, from proposition 4 (ii) we obtain $\hat{x} \geq x_t \geq \check{x}$ for all $t \geq 1$. Proposition 5 establishes the result.* ■

For the consumption dependent time preference, Tohme and Larrosa (2007) have also studied the effects of strategic interaction by letting production function be concave and a priori assuming twice differentiability of value function. Instead, we characterize the equilibria by employing the tools provided by lattice theory and supermodular games.

Under various sources of endogeneity in the discount rate, it has been shown that optimal growth models have globally monotone capital sequences and that low levels of initial capital leads to a poverty trap (see Mantel, 1998; Stern, 2006; Le Van et. al., 2009). We obtain that under strategic interaction, non-monotonic capital sequence is also possible since while one agent accumulates, the other may deaccumulate capital stock at the first period of the game. After the first period, they both either accumulate or deaccumulate through the entire duration of the game and equilibrium dynamics are characterized by the monotonicity of capital stock. Hence, interaction between the rich and the poor may help the poor avoid poverty traps. Moreover, since we provide all of the results while abstracting from the curvature of the technology, the poor can theoretically push the rich to her lower steady state even under convex technology.

3.4 Concluding Remarks:

In this paper, we study a strategic growth model with two agents under open loop strategies. By employing the tools provided by lattice theory and supermodular games, we characterize the equilibria.

Our results differ from corresponding optimal growth model in several ways. We show that under strategic interaction, non-monotonic capital sequence may also be obtained. We also conclude that the rich can help the poor avoid poverty traps whereas the poor may push the rich to her lower steady state even under convex technology.

As a natural extension of our model, feedback strategies with time dependent share θ_t^i can be considered and the game can be solved for Markov perfect equilibria. It should be emphasized that the tools utilized here can be extended to other models that incorporate the supermodular game structure.

4 Appendix

4.1 Proof of Proposition 1

(i) Following from the existence of a maximum of $\boldsymbol{x}^j$ and a sustainable capital stock for i, It can be observed that there exists $A(x_0^i, \boldsymbol{x}^j) < \infty$ such that $0 \leq x_t^i \leq A(x_0^i, \boldsymbol{x}^j)\ \forall t$. With this bound and Tychonov theorem, $[0, A(x_0^i, \boldsymbol{x}^j)]^\infty$ is a compact topological space. Let's define $\Pi\left(x_0^i, \boldsymbol{x}^j\right)$ as the set of feasible sequences from x_0^i for $\boldsymbol{x}^j$, which is closed subset of $[0, A(x_0^i, \boldsymbol{x}^j)]^\infty$, hence compact. U^i is well defined since there exists a maximum sustainable capital stock and $\beta\left(\bar{x}\right) =$

$\beta_m < 1$. Moreover, U^i is continuous followed from continuity of u^i, f, β and the bound $\beta_m < 1$.It is now clear that the initial problem of agent i is equivalent to

$$\max\{U^i\left(\boldsymbol{x}^i, \boldsymbol{x}^j\right) : \boldsymbol{x}^i \in \Pi\left(x_0^i, \boldsymbol{x}^j\right)\}.$$

$\Pi\left(x_0^i, \boldsymbol{x}^j\right)$ is compact for the product topology defined on the space of sequences $\boldsymbol{x}^i$ and U^i is continuous for this product topology. Hence, an optimal path exists. Moreover, since u is increasing, the constraint will be binding so that $c_t^i = \theta^i f(x_t^i + x_t^j) - x_{t+1}^i, \forall t$.

(ii) First, we will prove that $x_t^i > 0$, $\forall t$. Assume the contrary. Take the smallest t such that $x_t^i = 0$ and call it n. Since $x_0^i > 0$, we have $x_{n-1}^i > 0$ for any value of n. Consider $\boldsymbol{x}^{i\prime}$such that $x_n^{i\prime} = \varepsilon$ for a sufficiently small ε, and $x_t^i = x_t^{i\prime}, \forall t \neq n$.

We have,

$$\begin{aligned}
&U^i\left(\boldsymbol{x}^{i\prime}, \boldsymbol{x}^j\right) - U^i\left(\boldsymbol{x}^i, \boldsymbol{x}^j\right) \geq \\
&\quad \left(\textstyle\prod_{s=1}^{n-1}\beta(x_s^i)\right) u^i(c_{n-1}^i - \varepsilon) + \left(\textstyle\prod_{s=1}^{n-1}\beta(x_s^i)\right) \beta(\varepsilon) u^i(\theta^i f(\varepsilon + x_n^j) - x_{n+1}^i) \\
&\qquad\qquad - \left(\textstyle\prod_{s=1}^{n-1}\beta(x_s^i)\right) u^i(c_{n-1}^i) \\
&\quad = \left(\prod_{s=1}^{n-1}\beta(x_s^i)\right)\left[u^i(c_{n-1}^i - \varepsilon) + \beta(\varepsilon)u^i(\theta^i f(\varepsilon + x_n^j) - x_{n+1}^i) - u^i(c_{n-1}^i)\right].
\end{aligned}$$

Recall that $\beta : \mathbb{R}_+ \to \mathbb{R}_{++}$, $\beta(0) > 0$. Therefore, from Inada Condition, for sufficiently small ε, $U^i\left(\boldsymbol{x}^{i\prime}, \boldsymbol{x}^j\right) - U^i\left(\boldsymbol{x}^i, \boldsymbol{x}^j\right) > 0$, which contradicts the optimality of $\boldsymbol{x}^i$. Hence, $x_t^i > 0$, $\forall t$.

Now, we will prove that $c_t^i > 0, \forall t$. Assume the contrary. Clearly zero consumption path after some period can never be optimal, because $x_t^i > 0$, $\forall t$, and positive capital will be accumulating forever with zero utility. Hence, there exists n such that $c_{n-1}^i = 0$, $c_n^i > 0$. Consider $\boldsymbol{x}^{i\prime}$such that $x_n^{i\prime} = x_n^i - \varepsilon$, for a sufficiently small ε, and $x_t^i = x_t^{i\prime}$, $\forall t \neq n$. By choosing ε such that $\theta^i\left[f(x_n^i + x_n^j) - f(x_n^i + x_n^j - \varepsilon)\right] < c_n^i$ and $\varepsilon < x_n^i$, we get $\mathbf{x}^{i\prime} \in \Pi\left(x_0^i, \boldsymbol{x}^j\right)$. Then, we have:

$$\begin{aligned}
&U^i\left(\boldsymbol{x}^{i\prime}, \boldsymbol{x}^j\right) - U^i\left(\boldsymbol{x}^i, \boldsymbol{x}^j\right) = \\
&\quad \left(\prod_{s=1}^{n-1}\beta(x_s^i)\right)\left[\begin{array}{r} u^i(\varepsilon) + \beta(x_n^i - \varepsilon)u^i(\theta^i f(x_n^i + x_n^j - \varepsilon) - x_{n+1}^i) \\ - \beta(x_n^i)u^i(\theta^i f(x_n^i + x_n^j) - x_{n+1}^i) \end{array}\right] \\
&\quad - \frac{1}{\beta(x_n^i)}\sum_{t=n+1}^{\infty}\left(\prod_{s=1}^{t}\beta(x_s^i)\right) u^i(\theta^i f(x_t^i + x_t^j) - x_{t+1}^i)\left[\beta(x_n^i) - \beta(x_n^i - \varepsilon)\right].
\end{aligned}$$

From Inada condition, along with assumptions on u, f, β, for sufficiently small ε, above expression becomes positive, leading to a contradiction. ■

4.2 Proof of Proposition 2

Since $S^{(x^i, x^j)} \subseteq \left[0, \theta^i f(\max(\bar{x}, (x_0^i + x_0^j)))\right] \times \left[0, \theta^j f(\max(\bar{x}, (x_0^i + x_0^j)))\right]$, we can conclude $S^{(x^i, x^j)}$ is a sublattice of $\mathbb{R}^2$. Payoff function of agent i, $P^{(x^i |\, \boldsymbol{x}^j)}(y^i)$, is supermodular in y^i on $S^{(x^i | \boldsymbol{x}^j)}$ for each y^j in $S^{(x^j | \boldsymbol{x}^i)}$ follows from the fact that $S^{(x^i \mid \boldsymbol{x}^j)}$ is a chain. Thus showing $P^{(x^i \mid \boldsymbol{x}^j)}(y^i)$ has increasing differences in $\left(y^i, y^j\right)$ where $\left(y^i, y^j\right) \in S^{(x^i, x^j)}$ will conclude our proof.

Let's make the induction hypothesis that the value function of the n period finite game,

$$V_n^i\left(x^i \mid \ \boldsymbol{x}^j\right) = \max_{\left\{x_{t+1}^i\right\}_{t=0}^n} \left\{\sum_{t=0}^n \left(\prod_{s=1}^t \beta(x_s^i)\right) u(\theta^i f(x_t^i + x_t^j) - x_{t+1}^i) \mid \ x_t^i \geq 0\right\},$$

has increasing differences in $\left(y^i, y^j\right)$ for any initial capital. We will show that the value function of $n+1$ period finite game,

$$V_{n+1}^i\left(x^i \mid \ \boldsymbol{x}^j\right) = \max_{y^i \in S^{(x^i, x^j)}} u(\theta^i f(x^i + x^j) - y^i) + \beta\left(y^i\right) V_n^i\left(y^i \mid \boldsymbol{x}^j\right),$$

has increasing differences in $\left(y^i, y^j\right)$ for any given level of the initial capital stock.

Let $y^i \geq \hat{y}^i, y^j \geq \hat{y}^j$ where $\left(y^i, y^j\right), \left(\hat{y}^i, \hat{y}^j\right), \left(y^i, \hat{y}^j\right), \left(\hat{y}^i, y^j\right)$ are feasible joint strategies. Since increasing diferences and supermodularity is the same notion on direct product of finite collection of chains, sum of supermodular functions is supermodular and supermodularity is preserved under maximization; showing $\beta\left(y^i\right) V_n^i\left(y^i \mid \boldsymbol{x}^j\right)$ is supermodular will be sufficient for our purpose. Thus, we need to show that

$$\beta\left(y^i\right) V_n^i\left(y^i \mid \boldsymbol{x}^j\right) + \beta\left(\hat{y}^i\right) V_n^i\left(\hat{y}^i \mid \hat{\boldsymbol{x}}^j\right) \geq \beta\left(y^i\right) V_n^i\left(y^i \mid \hat{\boldsymbol{x}}^j\right) + \beta\left(\hat{y}^i\right) V_n^i\left(\hat{y}^i \mid \boldsymbol{x}^j\right),$$

where $\boldsymbol{x}^j$ and $\hat{\boldsymbol{x}}^j$ differ only in y^j and $\hat{y}^j$. This follows from the induction hypothesis and the fact that $\beta(\cdot)$ is increasing in y^i. By means of the boundedness assumption on $\beta(\cdot)$, one can easily show that Bellman operator is a contraction mapping so that $V_n^i \to V$. Since supermodularity is preserved under pointwise convergence, V is supermodular. ∎

4.3 Proof of Proposition 3

Set $S^{(x^i, x^j)}$ of feasible joint strategies is nonempty and compact and the payoff function $P^{(x^i \mid \boldsymbol{x}^j)}(y^i)$ is upper semicontinuous in y^i on $S^{(x^i \mid \boldsymbol{x}^j)}$ for each y^j in $S^{(x^j \mid \boldsymbol{x}^i)}$ and i. Results follow from Theorem 1. ∎

4.4 Proof of Proposition 4

i) X is a partially ordered set and since f is strictly increasing, $S^{(x^i, x^j)}$ is increasing in $\left(x^i, x^j\right)$ on X. In order to utilize Theorem 2, it is sufficient to show that the payoff function $P^{(x^i \mid \boldsymbol{x}^j)}(y^i)$ has increasing differences in $(y^i, (x^i, x^j))$ on $\left(\cup_{(x^i, x^j) \in X} S^{(x^i, x^j)}\right) \times X$.

Let us first show that $P^{(x^i \mid \boldsymbol{x}^j)}(y^i)$ has (strictly) increasing differences in (y^i, x^i) while keeping $\tilde{x}^j$ constant. Let $y^i \geq \hat{y}^i$, and $x^i \geq \hat{x}^i$. We need to show that:

$$u(\theta^i f(x_t^i + \tilde{x}_t^j) - y^i) + \beta\left(y^i\right) V^i\left(y^i \mid \boldsymbol{x}^j\right) + u(\theta^i f(\hat{x}_t^i + \tilde{x}_t^j) - \hat{y}^i) + \beta\left(\hat{y}^i\right) V^i\left(y^i \mid \boldsymbol{x}^j\right) \geq$$

$$u(\theta^i f(\hat{x}_t^i+\tilde{x}_t^j)-y^i)+\beta\left(y^i\right)V^i\left(y^i\mid \boldsymbol{x}^j\right)+u(\theta^i f(x_t^i+\tilde{x}_t^j)-\hat{y}^i)+\beta\left(\hat{y}^i\right)V^i\left(\hat{y}^i\mid \boldsymbol{x}^j\right).$$

Since we know that $\theta^i f(x^i+\tilde{x}^j)-y^i>\theta^i f(\hat{x}^i+\tilde{x}^j)-y^i$,

$$u(\theta^i f(x_t^i+\tilde{x}_t^j)-\hat{y}^i)-u(\theta^i f(x_t^i+\tilde{x}_t^j)-y^i)<u(\theta^i f(\hat{x}_t^i+\tilde{x}_t^j)-\hat{y}^i)-u(\theta^i f(\hat{x}_t^i+\tilde{x}_t^j)-y^i)$$

follows from the strict concavity of u. Accordingly, the fact that $P^{(x^i\mid \boldsymbol{x}^j)}(y^i)$ has (strictly) increasing differences in (y^i,x^j) while keeping $\tilde{x}^i$ constant can be shown in the same way.

ii) $\hat{X}$ is a chain and by definition we have $S^x=S^{(x^i,x^j)}$, $P^{(x\mid \boldsymbol{x}^j)}=P^{(x^i\mid \boldsymbol{x}^j)}$. $(N,S^x,\{P^{(x\mid \boldsymbol{x}^j)}:i\in N\})$ is a collection supermodular games parameterized by x in $\hat{X}$. $S^{(x)}$ is increasing in x on $\hat{X}$ follows from strictly increasing f and $P^{(x\mid \boldsymbol{x}^j)}(y^i)$ has increasing differences in (y^i,x) follows from the strict concavity of u. Then the result follows from Theorem 2. ■

4.5 Differentiability of Value Function

We will show the differentiability of value function at three steps. First we will provide a monotonicity result for the best response correspondance of an agent. Then we will prove that left and right derivatives of value functions is exist. By using these results, finally we will show that value function is differentiable at almost everywhere.

`Step 1:`

The following proposition, stating that the best response correspondance of an agent is increasing in its initial capital, will be crucial in proving the differentiability of value function.

Proposition 8 *Best response correspondance* $\mu_i(x^i\mid \mathbf{x}^j)$ *is increasing in* x^i *for any given* y^j *on* $S^{(x^j\mid \boldsymbol{x}^i)}$.

Proof. For any given $\boldsymbol{x}^j$ where $y^j\in\boldsymbol{x}^j$, problem of agent i corresponds to single agent problem. Let $X=[0,\max(x_0,\bar{x})]$ and $T=\left[0,\max\left(x_0^i,\bar{x}\right)\right]$. $S^{(x^i\mid \boldsymbol{x}^j)}=\left[0,\theta^i f(x^i+x^j)\right]$ is a subset of X for each $x^i\in T$. $S^{(x^i\mid \boldsymbol{x}^j)}$is increasing in x^i on T. $P^{(x^i\mid \boldsymbol{x}^j)}$ is supermodular in y^i on $S^{(x^i\mid \boldsymbol{x}^j)}$ for each x^i follows from the fact that $S^{(x^i\mid \boldsymbol{x}^j)}$ is a chain. Moreover we've already shown $P^{(x^i\mid \boldsymbol{x}^j)}(y^i)$ has srictly increasing differences in (y^i,x^i). The result then follows from Theorem 3. ■

`Step 2:`

Next lemma proves the existence of left and right derivatives of V^i. We will denote the left and right derivatives of V^i at a point x^i by $V^{i\prime}(x_-^i\mid \boldsymbol{x}^j)$ and $V^{i\prime}(x_+^i\mid \boldsymbol{x}^j)$, respectively, and the left and right limits of $\mu(x^i\mid x^j)$ at a point x^i by $\mu(x_-^i\mid x^j)$ and $\mu(x_+^i\mid x^j)$, respectively.

Lemma 3 *The value function* V^i *has finite left and right derivatives almost everywhere on* $[0,\max(\bar{x},x_0^i)]$. *Moreover,*

$$\begin{aligned}V^{i\prime}(x_-^i\mid \boldsymbol{x}^j)&=u'(\theta f(x^i+x^j)-\underline{\mu}_i\left(x^i\mid \boldsymbol{x}^j\right))\theta f'(x^i+x^j)\leq\\&\qquad u'(\theta f(x^i+x^j)-\overline{\mu}_i\left(x^i\mid \boldsymbol{x}^j\right))\theta f'(x^i+x^j)=V^{i\prime}(x_+^i\mid \boldsymbol{x}^j),\end{aligned}$$

where $\underline{\mu}_i\left(x^i\mid \boldsymbol{x}^j\right)=min\left\{\mu\left(x^i\mid \boldsymbol{x}^j\right)\right\}$ *and* $\overline{\mu}_i\left(x^i\mid \boldsymbol{x}^j\right)=\max\left\{\mu\left(x^i\mid \boldsymbol{x}^j\right)\right\}$.

Proof. For a sufficiently small $\alpha > 0$, $\mu_i\left(x^i + \alpha \mid \boldsymbol{x}^j\right) \leq \theta f(x^i + x^j)$ and $\mu_i\left(x^i \mid \boldsymbol{x}^j\right) \leq \theta f(x^i + \alpha + x^j)$ follow from the facts that optimal investment will be interior and f is increasing. In accordance with these,

$$\begin{aligned} V^i\left(x^i \mid \boldsymbol{x}^j\right) &= u\left[\theta f(x^i + x^j) - \mu_i\left(x^i \mid \boldsymbol{x}^j\right)\right] + \beta[\mu_i\left(x^i \mid \boldsymbol{x}^j\right)]V(\mu_i\left(x^i \mid \boldsymbol{x}^j\right) \mid \boldsymbol{x}^j) \\ &\geq u\left[\theta f(x^i + x^j) - \mu_i\left(x^i + \alpha \mid \boldsymbol{x}^j\right)\right] \\ &\qquad + \beta[\mu_i\left(x^i + \alpha \mid \boldsymbol{x}^j\right)]V(\mu_i\left(x^i + \alpha \mid \boldsymbol{x}^j\right) \mid \boldsymbol{x}^j), \end{aligned}$$

and

$$\begin{aligned} V^i\left(x^i + \alpha \mid \boldsymbol{x}^j\right) &= u\left[\theta f(x^i + \alpha + x^j) - \mu_i\left(x^i + \alpha \mid \boldsymbol{x}^j\right)\right] \\ &\qquad + \beta[\mu_i\left(x^i + \alpha \mid \boldsymbol{x}^j\right)]V(\mu_i\left(x^i + \alpha \mid \boldsymbol{x}^j\right) \mid \boldsymbol{x}^j) \\ &\geq u\left[\theta f(x^i + \alpha + x^j) - \mu_i\left(x^i \mid \boldsymbol{x}^j\right)\right] + \beta[\mu_i\left(x^i \mid \boldsymbol{x}^j\right)]V(\mu_i\left(x^i \mid \boldsymbol{x}^j\right) \mid \boldsymbol{x}^j). \end{aligned}$$

By using the two inequalities above and replacing μ_i with $\overline{\mu}_i$, we obtain:

$$\begin{aligned} &u\left[\theta f(x^i + \alpha + x^j) - \overline{\mu}_i\left(x^i \mid \boldsymbol{x}^j\right)\right] - u\left[\theta f(x^i + x^j) - \overline{\mu}_i\left(x^i \mid \boldsymbol{x}^j\right)\right] \leq \\ &\qquad V^i\left(x^i + \alpha \mid \boldsymbol{x}^j\right) - V^i\left(x^i \mid \boldsymbol{x}^j\right) \leq \\ &u\left[\theta f(x^i + \alpha + x^j) - \overline{\mu}_i\left(x^i + \alpha \mid \boldsymbol{x}^j\right)\right] - u\left[\theta f(x^i + x^j) - \overline{\mu}_i\left(x^i + \alpha \mid \boldsymbol{x}^j\right)\right]. \end{aligned}$$

Dividing the inequalities by $\alpha > 0$, and taking the limit as α tends to zero we get

$$u'(\overline{h}(x^i))\theta f'(x^i + x^j) \leq \lim_{\alpha \to 0^+} \frac{V^i\left(x^i + \alpha \mid \boldsymbol{x}^j\right) - V^i(x \mid \boldsymbol{x}^j)}{\alpha} \leq u'(\overline{h}(x^i_+))\theta f'(x^i_+ + x^j),$$

where $\overline{h}(x^i) = \theta f(x^i + x^j) - \overline{\mu}_i\left(x^i \mid \boldsymbol{x}^j\right)$. Since $\overline{\mu}$ is right continuous, $\overline{\mu}(x^i \mid x^j) = \overline{\mu}(x^i_+ \mid x^j)$. Hence, we have:

$$V'^i(x^i_+ \mid \boldsymbol{x}^j) = u'(\overline{h}(x^i))\theta f'(x^i + x^j) = u'(\overline{h}({x_+}^i))\theta f'(x^i + x^j).$$

If we write $V^i(x^i \mid x^j)$, $V^i(x^i - \alpha \mid x^j)$ and repeat the same manipulations with $\underline{\mu}$, we get $V'^i(x^i_- \mid x^j) = u'(\underline{h}(x^i))\theta f'(x^i + x^j) = u'(\underline{h}(x^i_-))\theta f'(x^i + x^j)$. Since $\underline{h}(x^i) \geq \overline{h}(x^i)$ and u is a concave function,

$$V'^i(x^{i-} \mid \boldsymbol{x}^j) = u'(\underline{h}(x^i))\theta f'(x^i + x^j) \leq u'(\overline{h}(x^i))\theta f'(x^i + x^j) = V'^i(x^{i+} \mid \boldsymbol{x}^j)$$

concludes the proof. ■

`Step 3:`

Lemma 1 *V^i is differentiable almost everywhere with*

$$V^{i\prime}(x^i \mid \boldsymbol{x}^j) = u'(\theta f(x^i + x^j) - \mu_i\left(x^i \mid \boldsymbol{x}^j\right))]\theta f'(x^i + x^j)$$

where $\mu_i(x^i \mid \mathbf{x}^j)$ is the best response correspondance of agent i.

Proof. Since the best response functions, $\overline{\mu}(x^i \mid \boldsymbol{x}^j)$ and$\underline{\mu}(x^i \mid \boldsymbol{x}^j)$ are increasing, they are continuous almost everywhere. We will prove that the set of points $x^i > 0$ where $\overline{\mu}$ is continuous equals to the set of points where $\underline{\mu}$ is continous.

Moreover, these best response functions coincide on this set. First, let $k > 0$ be a point where $\overline{\mu}$ is continuous. Let $\check{x}^i_n < x^i, \hat{x}^i_n > x^i \; \forall \; n$ where $\check{x}^i_n \to x^i$ and $\hat{x}^i_n \to x^i$. We have,

$$\underline{\mu}(\check{x}^i_n \mid \boldsymbol{x}^j) < \overline{\mu}(\check{x}^i_n \mid \boldsymbol{x}^j) < \overline{\mu}(x^i \mid \boldsymbol{x}^j) < \underline{\mu}(\hat{x}^i_n \mid \boldsymbol{x}^j) < \overline{\mu}(\hat{x}^i_n \mid \boldsymbol{x}^j),$$

implying that

$$\lim_{\check{x}^i_n \to x^i} \underline{\mu}(\check{x}^i_n \mid \boldsymbol{x}^j) = \lim_{\hat{x}^i_n \to x^i} \underline{\mu}(\hat{x}^i_n \mid \boldsymbol{x}^j) = \overline{\mu}(x^i \mid \boldsymbol{x}^j).$$

Since we have:

$$\underline{\mu}(\check{x}^i_n \mid \boldsymbol{x}^j) < \underline{\mu}(x^i_n \mid \boldsymbol{x}^j) < \underline{\mu}(\hat{x}^i_n \mid \boldsymbol{x}^j),$$

the function $\underline{\mu}(x^i \mid \boldsymbol{x}^j)$ is continuous at x^i and $\overline{\mu}(x^i \mid \boldsymbol{x}^j) = \underline{\mu}(x^i \mid \boldsymbol{x}^j)$. Similarly, we prove that $\overline{\mu}(x^i \mid \boldsymbol{x}^j)$ is continuous where $\underline{\mu}(x^i \mid \boldsymbol{x}^j)$ is continuous and at this point $\overline{\mu}(x^i \mid \boldsymbol{x}^j) = \underline{\mu}(x^i \mid \boldsymbol{x}^j)$. Hence, we can conclude that the best response correspondance $\mu(x^i \mid \boldsymbol{x}^j)$ is single valued almost everywhere, or equivalently, the value function is differentiable almost everywhere. ■

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