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Polyhedral Path-Complete Lyapunov Functions

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***Abstract*— Path-complete methods utilize a set of positive definite functions and a specially constructed graph in order to evaluate, among others, stability of switching systems. This tool is shown to be general, e.g., path-complete criteria are universal for linear switching systems and quadratic templates. In this work, we extend the approach to polyhedral Lyapunov functions, and introduce a simple parameterization that can be sufficient for stability analysis. Moreover, we indicate ways of obtaining less conservative stability criteria by partial graph extensions, all evaluated by solving Linear Programs (LPs).**

I. Introduction

Switching systems provide an accurate modeling framework for many processes, they are good approximations of complex, cyber-physical dynamics [1]–[3], but also pose major theoretical challenges [4]. In many applications, e.g. in dwell time and fault-detection settings [5]–[7], the switching pattern can be described by labeled directed graphs [8]–[10].

To put in perspective the developments of this work, we can categorize the existing stability methods in two groups, namely Lyapunov methods and reachability analysis. Common Lyapunov functions (CLFs) were first introduced for arbitrary switching systems, e.g., [11], [12]. Starting from quadratic Lyapunov functions, less conservative, however more complicated, parameterizations have been proposed, such as polynomials sum-of-squares [13], max-of-quadratics [14] etc. The emergence of Multiple Lyapunov functions [15], [16] provided a new relaxation for stability analysis, using, instead of one, a set of functions whose combination satisfies a Lyapunov decrease condition. *Path-Complete Lyapunov functions (PCLFs)* [17] formalize the above concept, by introducing labelled graphs whose edges express the Lyapunov inequalities between the pieces of the PCLF. These pieces are located in the nodes of the graph. Using properties of the graph, different hierarchies of stability criteria can be established [10], [18], [19]. Moreover, it was recently shown [20] that any PCLF can always be reinterpreted as a CLF expressed as a combination of minima and maxima of its pieces. In terms of computational complexity, evaluation of these stability criteria requires the solution of one or a series of semidefinite programs.

An appealing alternative is reachability analysis; in the context of stability analysis, the propagation of forward or backward reachability maps also provides stability criteria that are nonconservative [21]–[23]. When the involved sets are polyhedral, the set computations are linear, and break down to redundancy removal of set representations, which at the worst case is equivalent to solving a sequence of LPs. The downside of these approaches is the non-boundedness of the complexity of the members of the induced set sequence. Recent important papers address this issue by introducing special polyhedral templates, such as zonotopes [24], that provide approximate reachability analysis results, however particularly computationally efficient.

In this work we explore the use of polyhedral PCLFs, aiming to leverage the versatility of polytopic templates and at the same time bound the complexity of the resulting function. General polyhedral Lyapunov functions always exist for asymptotically stable switching systems [11], however, the underlying stability criteria correspond to nonlinear, nonconvex algebraic relations that cannot be solved efficiently. Taking this into account, we work with a simple zonotopic parameterization of polytopic sets for PCLFs that allows verification of the corresponding algebraic Lyapunov decrease conditions with an LP. Specifically, we consider polytopes with n pairs of parallel faces or polytopes that can be described as the (weighted) convex hull of n pairs of opposite vectors.

Although the suggested parameterization is simple and thus restricts the class of polyhedral functions we can use, the corresponding Common Lyapunov function can be quite complex, depending on the structure of the path-complete graph. This is shown by our main example, in which the stability of a family of arbitrary switching systems in n dimensions can be verified using as Lyapunov function the convex union of weighted infinity norms. Motivated by the above, we explore whether we can use in an opportunistic manner the path complete graph to implicitly add complexity in the description of the induced CLF. Starting from well known graph extensions, namely T-Lifts, P-Lifts and H-Lifts, see e.g., [10], [18]–[20], [25], we introduce *partial* extensions based on four elementary graph operations that preserve the path-completeness property of the graph. We report encouraging results by means of numerical examples.

In Section II, notation and preliminary notions are introduced. In Section III we develop the parameterizations for polyhedral Lyapunov PCLFs and formulate the Lyapunov decrease conditions. In Section IV we present an example that showcases the complexity of the resulting CLF induced by the chosen parameterization and chosen path-complete graph for a class of switching systems. In Section V we introduce graph extensions that preserve path-completeness, and propose, via an example, an algorithmic procedure for

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producing less conservative stability criteria using polyhedral PCLFs. Conclusions are drawn in Section VI.

II. Preliminaries

Notation: A C-set $\mathcal{S} \subset \mathbb{R}^n$ is a convex compact set which contains the origin in its interior. Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$, or $\mathcal{G}$, be a labeled directed graph. We consider the sets of nodes and edges $\mathcal{V} := \{1, 2, ..., M\}$ and $\mathcal{E} = \{(s, d, \sigma) : s \in \mathcal{V}, d \in \mathcal{V}\}$, where σ is a finite sequence of N modes, i.e., $\sigma = \sigma_1 ... \sigma_k$, where $\sigma_i \in \{1, ..., N\}$. The set of outgoing nodes of a node $s \in \mathcal{V}$ is $\mathrm{Out}(s, \mathcal{G}) := \{d \in \mathcal{V} : (\exists \sigma : (s, d, \sigma) \in \mathcal{E})\}$, and the set of incoming nodes of a node $d \in \mathcal{V}$ is $\mathrm{In}(d, \mathcal{G}) := \{s \in \mathcal{V} : (\exists \sigma : (s, d, \sigma) \in \mathcal{E})\}$. The set of sequences of labels appearing in a path from a node $s \in \mathcal{V}$ to a node $d \in \mathcal{V}$ is denoted by $\sigma(s, d)$. We denote the norm and the absolute value of a vector (or matrix) x with $\|x\|$ and $|x|$ respectively. The vector with elements equal to one and zero is $\mathbf{1}$ and $\mathbf{0}$ respectively, and the identity matrix is denoted by I. The cardinality of a set $\mathcal{V}$ and the length of a sequence σ is denoted by $|\mathcal{V}|$ and $|\sigma|$ respectively.

Definition 1 (System) *Given a set of matrices $\mathcal{A} = \{A_1, ..., A_N\}$ and a labeled directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, the constrained switching system is described by the dynamics*

$$x(t+1) = A_{\sigma(t)} x(t), \tag{1}$$

$$z(t+1) \in \mathrm{Out}(z(t), \mathcal{G}(\mathcal{V}, \mathcal{E})), \tag{2}$$

$$(z(t), z(t+1), \sigma(t)) \in \mathcal{E}, \tag{3}$$

where $(x(0), z(0)) \in \mathbb{R}^n \times \mathcal{V}$, $t \geq 0$.

In the the above definition, the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is called the *switching constraints graph*. Stability of (1)–(3) is characterized by the *constrained joint spectral radius* [8] $\check{\rho}(\mathcal{A}, \mathcal{G}) = \lim_{t \to \infty} \check{\rho}_t(\mathcal{A}, \mathcal{G})$, where

$$\check{\rho}_t(\mathcal{A}, \mathcal{G}) := \max\{\|A_{\sigma(t-1)} \cdots A_{\sigma(0)}\|^{1/t} : z(0) \in \mathcal{V}, \\ z(t) \text{ satisfies (2)}, \sigma(t) \text{ satisfies (3)}, t = 0, ..., t-1\}.$$

It is well known that system (1)–(3) is stable if and only if $\check{\rho}(\mathcal{A}, \mathcal{G}) < 1$ [8, Corollary 2.8]. For arbitrary switching systems we have $\mathcal{V} = \{1\}$ and $\mathcal{E} = \{(1, 1, 1), ..., (1, 1, N)\}$, and the constrained joint spectral radius coincides with the notion of the joint spectral radius (JSR). For ease of exposition, in most cases we will focus on the arbitrary switching case, i.e,

$$x(t+1) = A_{\sigma(t)} x(t), \tag{4}$$

where $\sigma(t) \in \{1, ..., N\}$, $t \geq 0$. Roughly, a graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ is path-complete with respect to a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ if any sequence σ in $\mathcal{G}(\mathcal{V}, \mathcal{E})$ can be realized as part of a path in the graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$. The formal definition follows.

Definition 2 (Path-complete graph) *Given system* (1)–(3) *and the switching constraints graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, a graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ is* path-complete *if for any admissible sequence $\sigma = \sigma_1 \sigma_2 ... \sigma_k$ that appears in a path in $\mathcal{G}(\mathcal{V}, \mathcal{E})$, $k \geq 1$, there exist two nodes $s \in \mathcal{V}', d \in \mathcal{V}'$, a sequence $\sigma^\star = \sigma_1 \sigma \sigma_2$*[1]*, such that $\sigma^\star \in \sigma(s, d)$.*

Definition 3 (Path-Complete Lyapunov Function) *Consider the switching system* (4) *with the corresponding switching constraints graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Let $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ be a path-complete graph. The set of positive definite functions $V_i(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$, $i = 1, ..., |\mathcal{V}'|$, is called a Path-Complete Lyapunov Function (PCLF) if for all $(s, d, \sigma) \in \mathcal{E}'$, the following relations hold*

$$V_d(A_\sigma x) \leq \varepsilon^{|\sigma|} V_s(x), . \tag{5}$$

for some $\varepsilon \in [0, 1)$.

A PCLF can always be expressed as a Common Lyapunov Function. Its exact form depends on the path-complete graph, and it has been established in [20, Theorem III.8].

III. Parameterizations for Polyhedral PCLFs

We consider polytopes that are $\mathcal{C}$-sets, of the form

$$\mathcal{S} = \{x \in \mathbb{R}^n : Gx \leq w\}, \tag{6}$$

with $G \in \mathbb{R}^{m \times n}$. Since $\mathcal{S}$ is compact and contains the origin in its interior, we have $m \geq n+1$, $\mathrm{rank}(G) = n$, and $w > 0$. A dual equivalent representation of $\mathcal{S}$ is

$$\mathcal{S} = \{Vy : c^\top y \leq 1, y \geq 0\}. \tag{7}$$

Similarly as (6), $V \in \mathbb{R}^{n \times q}$, $q \geq n+1$, $\mathrm{rank}(V) = n$ and $c > 0$. We note that (6) and (7) are equivalent [26, Chap. 1]. Specifically, (6) relates to the Halfspace representation of polytopes, while (7) is the Vertex representation[2]. The polytopic sets can be written as the sublevel sets of their Minkowski functionals, namely,

$$\mathcal{S} = \{x \in \mathbb{R}^n : V(x) \leq 1\},$$

with

$$V(x) = \max_{i=1,..,m} \left\{ \frac{(Gx)_i}{w_i} \right\}, \tag{8}$$

or

$$V(x) = \min_{y \geq 0} \{c^\top y : x = Vy\}. \tag{9}$$

The following result allows to evaluate (5) using a set of algebraic relations. It is presented without a proof as it can be derived by a straightforward adaptation of classical results, see, e.g., [27, Chap. 4].

Proposition 1 *Consider two $\mathcal{C}$-polytopic sets $\mathcal{S}_d \subset \mathbb{R}^n$, $\mathcal{S}_s \subset \mathbb{R}^n$ described in* (6), (7) *with matrices $G_s \in \mathbb{R}^{m_s \times n}$, $G_d \in \mathbb{R}^{m_d \times n}$, $V_s \in \mathbb{R}^{n \times q_s}$, $V_d \in \mathbb{R}^{n \times q_d}$ and vectors w_s, w_d, c_s, c_d. Consider $\varepsilon > 0$, a matrix set $\mathcal{A} = \{A_1, ..., A_N\}$, the sequence $\sigma = \sigma_1 ... \sigma_k$, $\sigma_i \in \{1, ..., \mathbb{N}\}$ and the matrix $A_\sigma \in \mathbb{R}^{n \times n}$. The following are equivalent.*

- (i) *$V_d(A_\sigma x) \leq \varepsilon^{|\sigma|} V_s(x)$.*
- (ii) *There exists a nonnegative matrix $H \in \mathbb{R}^{m_d \times m_s}$ such that $G_d A_\sigma = H G_s$ and $H w_s \leq \varepsilon^{|\sigma|} w_d$.*

[1] σ_1, σ_2 can be chosen to be the empty word.

[2] When $c = \mathbf{1}$, relation (7) is the convex hull of the columns of V.

(iii) *There exists a nonnegative matrix $P \in \mathbb{R}^{q_d \times q_s}$ such that $A_\sigma V_s = V_d P$ and $c_d^\top P \leq \varepsilon^{|\sigma|} c_s^\top$.*

The algebraic relations in Proposition 1 are difficult to solve due to the bilinear matrix equation appearing in both cases (ii) and (iii). To overcome this, we restrict ourselves to symmetric polytopes with n pairs of parallel faces or n pairs of opposite vertices, that can be written in the form

$$\mathcal{S} = \{x \in \mathbb{R}^n : |Gx| \leq w\}, \tag{10}$$

$$\mathcal{S} = \{Vy : c^\top |y| \leq 1\}, \tag{11}$$

where $G \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{n \times n}$. In this case, the following Corollary of Proposition 1 can be stated.

Corollary 1 *Consider two $\mathcal{C}$-polytopic sets $\mathcal{S}_d \subset \mathbb{R}^n$, $\mathcal{S}_s \subset \mathbb{R}^n$ described in* (10)*,* (11) *with matrices $G_s, G_d \in \mathbb{R}^{n \times n}$, $V_s, V_d \in \mathbb{R}^{n \times q_d}$ and vectors w_s, w_d, c_s, c_d. Consider $\varepsilon > 0$, a matrix set $\mathcal{A} = \{A_1, ..., A_N\}$, the sequence $\sigma = \sigma_1 ... \sigma_k$, $\sigma_i \in \{1, ..., \mathbb{N}\}$ and the matrix $A_\sigma \in \mathbb{R}^{n \times n}$. The following are equivalent.*

$$(i) \qquad V_d(A_\sigma x) \leq \varepsilon^{|\sigma|} V_s(x). \tag{12}$$

$$(ii) \qquad |G_d A_\sigma G_s^{-1}| w_s \leq \varepsilon^{|\sigma|} w_d. \tag{13}$$

$$(iii) \qquad c_d^\top |V_d^{-1} A_\sigma V_s| \leq \varepsilon^{|\sigma|} c_s^\top. \tag{14}$$

Compared to the conditions (ii), (iii) of Proposition 1 which are very difficult to solve, relations (13), (14) can be verified by solving a linear program for a given choice of the matrices G, V. In addition, we can use conditions (13), (14) to have a unified description of the PCLF decrease conditions of (5). To do so, given a path-complete graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and a set of polytopes of the form (10), we first define the matrices $T_1, T_2 \in \mathbb{R}^{n|\mathcal{E}| \times n|\mathcal{V}|}$, that are described block-wise as follows

$$T_{1,k,l} = \begin{cases} |G_j A_\sigma G_i^{-1}|, & l = i \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

$$T_{2,k,l} = \begin{cases} \varepsilon I, & l = j \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

for $k = 1, ..., |\mathcal{E}|$, and k corresponds to the edge $(i, j, \sigma) \in \mathcal{E}$. The following two Propositions group the relations of Corollary 1 for all decrease conditions induced by the path-complete graph.

Proposition 2 *Consider the system* (4)*, a path-complete graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, a set of polyhedral sets $\{\mathcal{S}_i\}_{i \in \{1, ..., |\mathcal{V}|\}}$ of the form* (10)*, and let $w = \begin{bmatrix} w_1^\top & ... & w_{|\mathcal{V}|}^\top \end{bmatrix}^\top$. The set $\{\mathcal{S}_i\}_{i \in \{1, ..., |\mathcal{V}|\}}$ is a PCLF if and only if $(T_1 - T_2) w \leq 0$.*

Taking the vertex representation (11), we can define analogously the matrices $L_1, L_2 \in \mathbb{R}^{n|V| \times n|E|}$ with

$$L_{1,k,l} = \begin{cases} |V_j^{-1} A_\sigma V_i|, & l = j \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

$$L_{2,k,l} = \begin{cases} \varepsilon I, & l = i \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

for $k = 1, ..., |\mathcal{E}|$ and k corresponds to the edge $(i, j, \sigma) \in \mathcal{E}$.

Proposition 3 *Consider the system* (4)*, a path-complete graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, a set of polyhedral sets $\{\mathcal{S}_i\}_{i \in \{1, ..., |\mathcal{V}|\}}$ of the form* (11)*, and let $c = \begin{bmatrix} c_1^\top & ... & c_{|\mathcal{V}|}^\top \end{bmatrix}^\top$. The set $\{\mathcal{S}_i\}_{i \in \{1, ..., |\mathcal{V}|\}}$ is a PCLF if and only if $c^\top (L_1 - L_2) \leq 0$.*

IV. Main Example

Propositions 2 and 3 offer an appealing algebraic equivalent of evaluating whether a set of polyhedral functions is a PCLF. Nevertheless, since parameterizations (10), (11) can describe only a subset of polytopes, we wish to explore how conservative they are for given choices of the matrices G, V in (10), (11). To do so, we consider an n-dimensional switching system consisting of n modes. The important element in this example is the appropriate choice of the path-complete graph. Let us consider the set of n $n \times n$ matrices $\mathcal{A}_1 = \varepsilon \left\{ \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, ..., \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} \right\}$, with $\varepsilon < 1$.

Our goal is to analyze stability for the arbitrary switching system (4) with matrices taking values from $\mathcal{A}_1$. We choose the simplest parameterization (10) with $G = I$. The corresponding Minkowski functional is $V(x) = \max_{l=1,...,n} \left\{ \frac{|x|_l}{w_l} \right\}$. Moreover, we consider the fully connected graph $\mathcal{G}_1$, consisting of n nodes (Figure 2, left). In detail, each node i is connected to a node j by an edge labeled by j, i.e., $(i, j, j) \in \mathcal{E}$ for all $i = 1, .., n$, $j = 1, ..., n$. We consider n sets $\mathcal{S}_i = \{x : |x| \leq w_i\}$, $w_i \in \mathbb{R}^n$, and correspondingly $V_i(x) = \max_{l=1,...,n} \left\{ \frac{|x|_l}{w_{i,l}} \right\}$, $i = 1, ..., l$.

We proceed to show that the set $\{V_i(x)\}_{i=1,...,n}$ is a PCLF for the system, by setting

$$w_{i,k} = \begin{cases} 1, & k = i \\ a, & \text{otherwise}, \end{cases}$$

for some positive constant a. To do so, we choose an arbitrary edge (i, j, j) of $\mathcal{G}_1$ and show that $V_j(A_j x) \leq V_i(x)$, or, equivalently, $A_j \mathcal{S}_i \subseteq \mathcal{S}_j$. The set $\mathcal{S}_i$ consists of 2^n vertices of the form $\begin{bmatrix} \pm a & ... & \pm a & \pm 1 & \pm a & ... & \pm a \end{bmatrix}^\top$. To verify the set inclusion, it must hold that the points $\begin{bmatrix} 0 & ... & 0 & (\pm 1 + \sum_{i=1}^{n-1} (\pm a)) & 0 & ... & 0 \end{bmatrix} \in \mathcal{S}_j$, or, $(\pm 1 + \sum_{i=1}^{n-1} (\pm a)) \leq 1$. The latter relation holds for any a satisfying $a < \frac{1 - \varepsilon}{\varepsilon(n-1)}$. Consequently, a PCLF function has been found, and, e.g., from [20, Corollary III.3], the function $V(x) = \min\{V_1(x), ..., V_n(x)\}$ is a CLF for the system. The geometric insight of this choice is shown in Figure 1 for $n = 3$. The sublevel set of the CLF is the union of the three sets $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ shown on the left for $a = 0.1$, which is invariant. On the right, the convex hull of $\mathcal{S}$ is shown, which is also invariant due to the linearity of the dynamics of the system.

The alternative vertex representation parameterization (11) can also be used efficiently to verify stability. Let us consider the following set of $n \times n$ matrices $\mathcal{A}_2 = \varepsilon \left\{ \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}, ..., \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \right\}$, where $\varepsilon < 1$. We notice

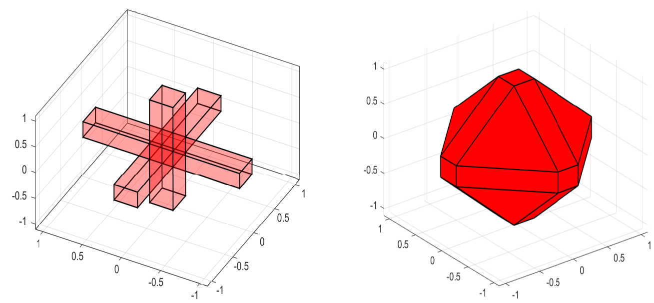

Fig. 1: Main Example, left: The sublevel sets of the PCLF using the parameterization (10) and the path-complete graph $\mathcal{G}_1$ for $n = 3$. Right, the convex union of the three sets $\mathcal{S}_i$, $i = 1, 2, 3$, which is the sublevel set of the resulting CLF.

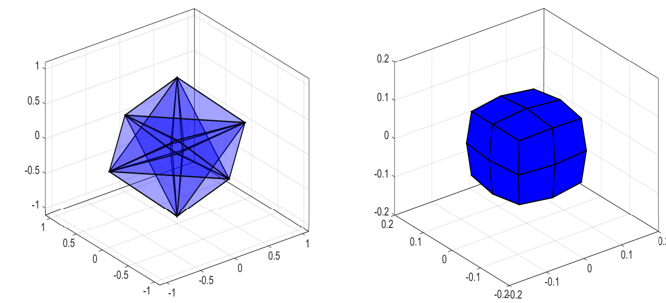

Fig. 3: Main Example, Left: the sublevel sets of the PCLF using the parameterization (11) and the path-complete graph $\mathcal{G}_2$. Right: their intersection, which is the resulting CLF (notice the different scales in the axes).

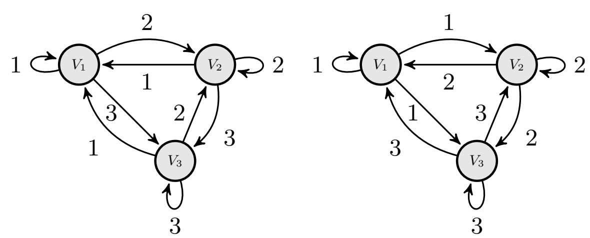

Fig. 2: Main Example, left: the path-complete graph $\mathcal{G}_1$ for $n = 3$. The corresponding common Lyapunov function is $V(x) = \min\{V_1(x), V_2(x), V_3(x)\}$. Right: the path-complete graph $\mathcal{G}_2$. The corresponding common Lyapunov function is $V(x) = \max\{V_1(x), V_2(x), V_3(x)\}$.

$\mathcal{A}_2 = \mathcal{A}_1^\top$. We choose the simplest parameterization (11) with $V = I$. The Minkowski functional for this set would be $V(x) = \min\{c^\top |x|\}$. In addition, we consider the fully connected graph $\mathcal{G}_2$ (Figure 2, right). In detail, the set of edges is $\mathcal{E} = \{(i, j, i) : i = 1, ..., n, j = 1, ..., n\}$. By considering n sets $\mathcal{S}_i = \{x : c_i^\top |x| \leq 1\}$, $c_i \in \mathbb{R}^n$, and $V_i(x) = \min\{c_i^\top |x|\}$, one can show the set $\{V_i(x)\}_{i=1,...,n}$ is a PCLF, setting

$$c_{i,k} = \begin{cases} 1, & k = i \\ a, & \text{otherwise}, \end{cases}$$

for any a satisfying $a > \frac{\varepsilon(n-1)}{1-\varepsilon}$. Consequently, a PCLF function has been found. Moreover the function $V(x) = \max\{V_1(x), ..., V_n(x)\}$ [20, Corollary III.3] is a CLF for the system. The three sets $\mathcal{S}_i$, $i = 1, 2, 3$ are shown in Figure 3 for $n = 3$, while their intersection, which is invariant as it corresponds to the sublevel set of the CLF for this system, is shown in for $a = 0.1$.

V. Partial Lifts preserving path-completeness

In the previous Section it was illustrated that stability can be deduced for simple PCLF parameterizations, as long as the path-complete graph is suitable. In this section we are exploring how to induce complexity in the PCLF by extending a path-complete graph while preserving its properties. The motivation is to use such extensions to guide a recursive procedure that can be used for stability analysis. For the following four procedures we consider $\mathcal{G}(\mathcal{V}, \mathcal{E})$ to be the switching constraints graph and $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ be a path-complete graph related to $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Moreover, we consider an arbitrary edge $(s, d, \sigma) \in \mathcal{E}'$.

Definition 4 (Partial T-Lift) *The partial T-lift with respect to the edge (s, d, σ) is denoted by $\mathcal{G}_T(\mathcal{E}_T, \mathcal{V}_T)$. The procedure is outlined below.*

1. *Set $\mathcal{E}_T = \mathcal{E}'$, $\mathcal{V}_T = \mathcal{V}'$.*
2. *Let $\mathcal{E}_d \subset \mathcal{E}'$ be the set of edges in $\mathcal{G}'$ with source node the node d, i.e., $\mathcal{E}_d = \{(d, d_i, \sigma_i) \in \mathcal{E}'\}$.*
3. *Set $\mathcal{E}_T = \mathcal{E}_T \setminus \{s, d, \sigma\}$.*
4. *For each edge (d, d_i, σ_i) in $\mathcal{E}_d$, set $\mathcal{E}_T = \mathcal{E}_T \cup \{(s, d_i, \sigma\sigma_i)\}$.*

Definition 5 (Partial T*-Lift) *The partial T*-lift with respect to the edge (s, d, σ) is denoted by $\mathcal{G}_{T^\star}(\mathcal{E}_{T^\star}, \mathcal{V}_{T^\star})$. The procedure is outlined below.*

1. *Set $\mathcal{E}_{T^\star} = \mathcal{E}'$, $\mathcal{V}_{T^\star} = \mathcal{V}'$.*
2. *Let $\mathcal{E}_s \subset \mathcal{E}'$ be the set of edges in $\mathcal{G}'$ with destination node the node s, i.e., $\mathcal{E}_s = \{(s_i, s, \sigma_i) \in \mathcal{E}'\}$.*
3. *Set $\mathcal{E}_{T^\star} = \mathcal{E}_{T^\star} \setminus \{s, d, \sigma\}$.*
4. *For each edge $(s_i, s, \sigma_i) \in \mathcal{E}_s$, set $\mathcal{E}_{T^\star} = \mathcal{E}_{T^\star} \cup \{(s, d_i, \sigma_i\sigma)\}$.*

Definition 6 (Partial P-Lift) *We assume $\mathrm{In}(d, \mathcal{G}') \geq 2$. The partial P-lift with respect to the edge (s, d, σ) is denoted by $\mathcal{G}_P(\mathcal{E}_P, \mathcal{V}_P)$. The procedure is outlined below.*

1. *Introduce a new node d'. Set $\mathcal{V}_P = \mathcal{V}' \cup \{d'\}$.*
2. *Set $\mathcal{E}_P = \mathcal{E}' \cup \{(s, d', \sigma)\} \setminus \{(s, d, \sigma)\}$.*
3. *Let $\mathcal{E}_d \subset \mathcal{E}'$ be the set of edges in $\mathcal{G}'$ with source node the node d, i.e., $\mathcal{E}_d = \{(d, d_i, \sigma_i) \in \mathcal{E}'\}$.*
4. *For each edge $(d, d_i, \sigma_i) \in \mathcal{E}_d$, set $\mathcal{E}_P = \mathcal{E}_P \cup \{(d', d_i, \sigma_i)\}$.*

Definition 7 (Partial P*-Lift) *We assume* $\mathrm{Out}(s,\mathcal{G}') \geq 2$. *The partial P*-lift with respect to the edge* (s,d,σ) *is denoted by* $\mathcal{G}_{P^\star}(\mathcal{E}_{P^\star},\mathcal{V}_{P^\star})$. *The procedure is outlined below.*

1. *Introduce a new node* s'. *Set* $\mathcal{V}_{P^\star} = \mathcal{V}' \cup \{s'\}$.
2. *Set* $\mathcal{E}_{P^\star} = \mathcal{E}' \cup \{(s',d,\sigma)\} \setminus \{(s,d,\sigma)\}$.
3. *Let* $\mathcal{E}_s \subset \mathcal{E}'$ *be the set of edges in* $\mathcal{G}'$ *with destination node the node s, i.e.,* $\mathcal{E}_s = \{(s_i,s,\sigma_i) \in \mathcal{E}'\}$.
4. *For each edge* $(s_i,s,\sigma_i) \in \mathcal{E}_s$, *set* $\mathcal{E}_{P^\star} = \mathcal{E}_{P^\star} \cup \{(s_i,s',\sigma_i)\}$.

Example 1 *We consider the graph* $\mathcal{G}_3$ *shown in Figure 4 which we assume is a path-complete graph related to a switching constraints graph* $\mathcal{G}$ *with labels* $\{1,2,3,4,5\}$. *We illustrate all four partial lifts for the edge* $(1,2,23)$ *in Figures 5 and 6.*

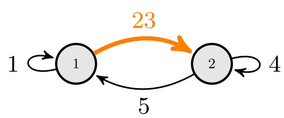

Fig. 4: Example 1, graph $\mathcal{G}_3$. The edge to be lifted is depicted with orange.

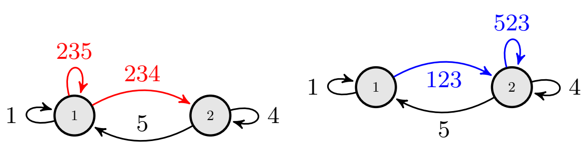

Fig. 5: Example 1, above: The partial T-lift, with the added edges depicted in red. The edge removed is $(1,2,23)$. Below, the T*-lift of $\mathcal{G}_3$ for the edge $(1,2,23)$, with the new introduced edges depicted in blue.

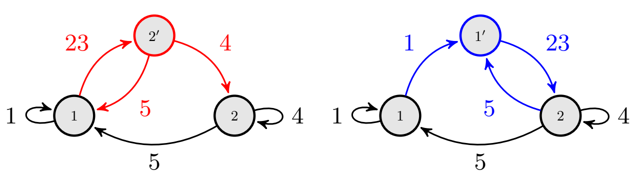

Fig. 6: Example 1, above: the partial P-lift with respect to the edge $(1,2,23)$. The added node $(2')$ and edges are depicted in red. Below, the P*-lift of $\mathcal{G}_3$ for the edge $(1,2,23)$. The added node $(1')$ and edges are depicted in blue.

By construction, the path-completeness property of any partial Lift defined above of any edge of $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ is preserved. Consequently, less conservative stability criteria can be formulated, as stated formally below.

Theorem 1 *Consider a switching system* (1)–(3)*, and a path complete graph* $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ *related to switching constraints graph* $\mathcal{G}(\mathcal{V},\mathcal{E})$. *Consider a graph* $\bar{\mathcal{G}}(\bar{\mathcal{V}},\bar{\mathcal{E}})$ *that is generated by applying a partial T-Lift (resp. partial T*, P, P* Lift) of* $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$. *Then, if there exists a PCLF for the system with path-complete graph* $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$*, there exists a PCLF for the system with path-complete graph* $\bar{\mathcal{G}}(\bar{\mathcal{V}},\bar{\mathcal{E}})$.

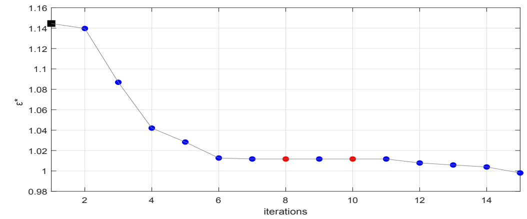

Fig. 7: Example 2, The optimal value of ε with respect to the iterations of the algorithm. Stability is proven after 15 steps. The partial liftings corresponding to T/T* partial lifts are shown in blue, whereas the partial P/P* lifts are shown in red.

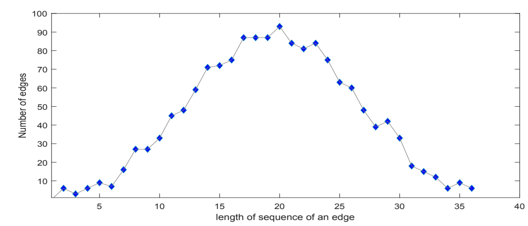

Fig. 8: Example 2, the distribution of the length of the switching sequences labeling the edges of the resulting path-complete graph that proves stability.

Proof Given any set of functions $\{V_i'(x)\}_{i=1,\dots,|\mathcal{V}'|}$ which constitute a PCLF with the graph $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$, we can always construct a set of functions $\{\bar{V}_i(x)\}_{i=1,\dots,|\bar{\mathcal{V}}|}$ which are a PCLF for the system with $\bar{\mathcal{G}}(\bar{\mathcal{V}},\bar{\mathcal{E}})$, by assigning to each node in $\bar{\mathcal{G}}(\bar{\mathcal{V}},\bar{\mathcal{E}})$ the function of their corresponding node in $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$. ■

Theorem 1 allows the development of algorithmic procedures for stability analysis, by searching iteratively for less conservative stability certificates. Such a possible approach is illustrated in the following example.

Example 2 *We consider an arbitrary switching system* (1)–(3)*, consisting of two matrices, i.e.,* $\mathcal{A} = \{A_1, A_2\}$*, where* $A_1 = \begin{bmatrix} 0.095 & -0.2375 & 0.2375 \\ -1.9 & 0 & 0 \\ 0 & 0 & 0.475 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.4753 & 0 & 0 \\ 0 & 0 & 0.4753 \\ 0 & -1.9012 & 0 \end{bmatrix}$, *and the switching constraints graph* $\mathcal{G}(\mathcal{V},\mathcal{E})$*, with* $\mathcal{V} = \{1\}$, $\mathcal{E} = \{(1,1,1),(1,1,2)\}$. *We consider the initial path-complete graph* $\mathcal{G}'(\mathcal{V}',\mathcal{E}') = \mathcal{G}(\mathcal{V},\mathcal{E})$. *In order to show that the system is asymptotically stable, we iteratively lift the path-complete graph, solving at each iteration the corresponding algebraic conditions. In particular, at each iteration, for a path-complete graph* $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$*, by setting* $G_i = I$, $i = 1,\dots,|\mathcal{V}'|$*, we first solve the program*[3]

$$\min_{\varepsilon, w_1,\dots,w_{\mathcal{V}'}} \varepsilon$$

[3]This problem can be reduced to a sequence of LPs using bisection.

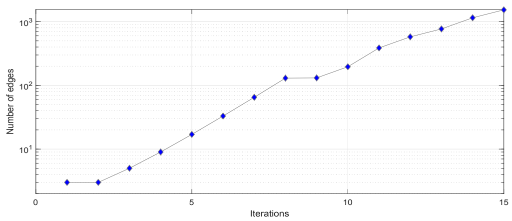

Fig. 9: Example 2, the number of edges of the path-complete graph of each iteration.

subject to (13)*, for all* $(s, d, \sigma) \in \mathcal{E}'$*. Next, we identify the edges* $(s, d, \sigma) \in \mathcal{E}'$ *that correspond to active constraints* (13) *of the problem, in the sense that at least one element of the vector inequality* (13) *holds with equality. For each such edge and each partial T/T*/P/P* lift, we solve the optimization problem again, finally choosing the lift with the smallest corresponding optimal value* $\varepsilon*$*. For the specific example, stability is verified at the* 15*th iteration. In Figure 7, the optimal values of* $\varepsilon*$ *for each step of the algorithm are shown. In Figure 8, the different lengths of the labels of the final path-complete graph are shown. Finally, in Figure 9, the number of edges for each path-complete graph of the algorithm is shown, with the final path-complete graph consisting of* 1533 *edges.*

VI. Conclusions

In this work we proposed an opportunistic way to build multiple Lyapunov functions by relying on the notion of the path-complete Lyapunov function. This concept provides us with a clear combinatorial tool (the path-complete graph), allowing to 'lift' the stability criterion in a smart way. As shown, our approach allows for a versatile construction of the Lyapunov criterion, which can leverage insight from a previously solved Optimization Program and build a new, ad-hoc, optimization program tailored for the specific problem studied. We have exemplified this strategy with polyhedral Lyapunov functions and in particular a simple parameterization that allows for efficient solution of the corresponding algebraic Lyapunov decrease conditions. Nevertheless, similar ideas can be applied to any path-complete Lyapunov function, for instance the often-used quadratic Lyapunov functions. In the future, we plan to convert this opportunistic strategy into a well-defined procedure allowing to build Lyapunov criteria in a smart way, with possible extensions to nonlinear hybrid systems and systems with control inputs.

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