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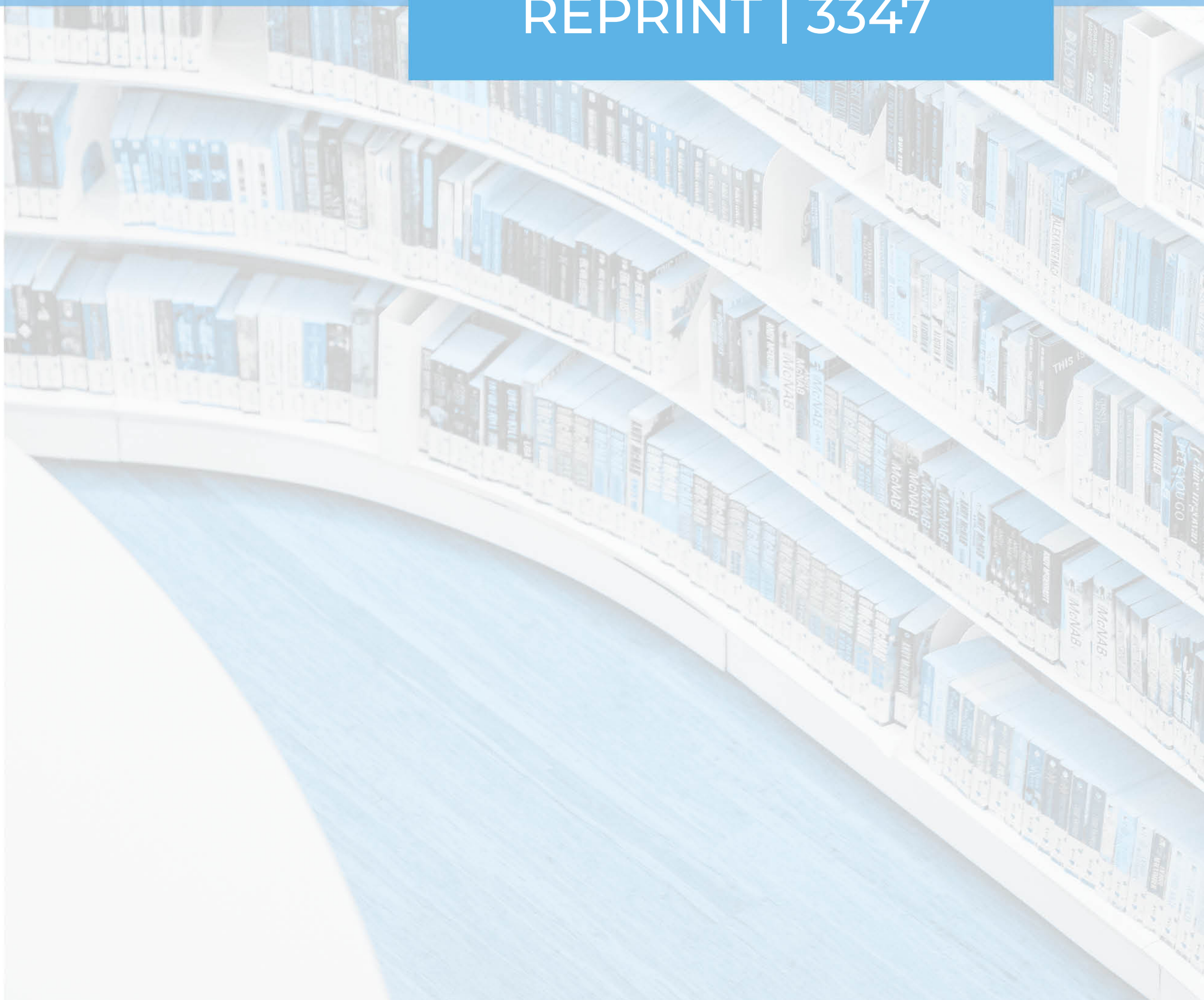

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Realized Covariance Models with Time-varying Parameters and Spillover Effects

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Abstract: A realized covariance model specifies a dynamic process for a conditional covariance matrix of daily asset returns as a function of past realized variances and covariances. We propose parsimonious parameterizations enabling a spillover effect in the conditional variance equations, and a specific nonlinear, time-varying, effect of the lagged realized covariance between each asset pair on the corresponding conditional covariance. We introduce these parameterizations in four classes of realized covariance models. In an application to the components of the Dow Jones index, we find that the extended models improve the fit of their less flexible scalar versions, and show a good out-of-sample forecast performance, in particular for short forecast horizons.

Key words: attenuation effect; forecasting; realized volatility; spillover effect; time-varying effect.

1 Introduction

The capability of storing and processing in continuous time huge amounts of data arising from the computerized functioning of financial markets has favored the use of these data. In particular, researchers have developed 'ex-post' measures of financial asset volatility based on such data, where the term volatility here refers to variances and covariances of financial returns relative to a reference period that we assimilate to a (calendar) day in the sequel. The measures are 'ex-post' since they can be computed or observed at the end of the reference day. Variances and covariances (or correlations) for a set of returns are used in risk management. Managers have always been interested in forecasting risk for implementing financial strategies. Correspondingly, researchers have developed 'realized covariance models', i.e., statistical models for time series of realized covariance matrices, which can be used to forecast volatilities.

A realized covariance model specifies a stochastic process for a time series of conditional covariance matrices, the term 'conditional covariance matrix' being a shortcut for 'conditional expectation of a covariance matrix'. Generally, a realized covariance model specifies the conditional covariance relative to day t as a parametric function of the realized covariance matrices of the previous days. The set of realized matrices observed before day t is included in the information set with respect to which the conditional expectation is defined.

More specifically, most realized covariance models include a constant matrix term that measures the base level around which the conditional covariances fluctuate from day to day, and one or several terms that capture the dynamics of the covariances; in one way or another, these terms capture the impact of past realized covariances on the current conditional covariances. For example, the diagonal BEKK-type model (Engle and Kroner, 1995)[1] specifies that each conditional covariance is a linear function of the previous day realized covariance and conditional covariance, implying that each conditional covariance is a moving average of all previous realized covariances. In a diagonal DCC (dynamic conditional correlation) type model (Engle, 2002), the same sort of dependence structure holds for the conditional correlations with respect to the past realized correlations. In multivariate HAR (heterogeneous autoregressive) models (Corsi, 2009), the conditional covariances (or correlations) are also moving averages of past realized covariances (or correlations). Each type of model defines in a specific way, more or less parsimoniously, the coefficients of the moving average polynomials. Scalar models impose the same lag polynomial coefficients for all the covariances or correlations. Non-scalar diagonal models do not impose this type of restriction, at the price of an inflation of the number of parameters, which prevents their estimation when the number of assets exceeds a handful. The diagonal structure

[1] BEKK comes from the first letter of the names of the authors that proposed the corresponding model in an unpublished paper, namely Baba, Engle, Kraft and Kroner. The model acronym has been kept as BEKK in the multivariate GARCH literature.

precludes "spillover" effects, e.g., the conditional variance of any asset cannot depend on the conditional variances of other assets. Obviously, a non-diagonal and non-scalar model is even much less parsimonious in parameters than its diagonal counterpart.

Another feature of the models mentioned above is that the dependence of the conditional covariances on the past realized covariances is specified to be linear in these variables, so that the marginal impacts of the previous day realized covariances on the current day conditional covariances are constant with respect to the level of the previous day realized covariances. One way of enabling a nonlinear dependence (hence, non constant marginal impacts) is to specify time-varying moving average (marginal) impact coefficients that depend on functions of the past realized covariances. This idea can also be combined with the main feature of the diagonal models, namely that the parameters are different between covariance equations. Both ideas are implemented by Bauwens and Otranto (2020, 2023), who proposed a new parameterization of the matrix of impact parameters, based on the Hadamard exponential (HE) operator (see, e.g., Reams, 1999), which provides flexible (i.e., specific and time–varying for each covariance) parameterizations by adding just one parameter with respect to a scalar model. In particular, Bauwens and Otranto (2023) consider both BEKK-type models of the conditional covariance matrix, and DCC-type models, adopting the two–step estimation procedure, but the focus is on modeling the covariance (correlation) part. In those models, the coefficients of the variance equations are fixed over time and the volatility of each asset does not depend on the volatility of the other assets, thus omitting the spillover effect typical in financial markets (Forbes and Rigobon, 2002).

We provide a general model framework, which nests baseline scalar models and their HE version, by including in realized covariance models specific and time–varying parameters in each conditional covariance (or correlation) equation, and specific and time-varying spillover effects in each conditional variance equation. This is done by extending the equations of the baseline scalar models, by introducing functions, depending on a very small number of parameters, to add flexibility to the scalar coefficients. In particular we extend the scalar BEKK, GARCH–DCC, MHAR (multivariate HAR) models and its version for 2-step estimation, the HAR–DRD (Oh and Patton, 2016). The result is a new family of realized covariance models, very flexible and general because the baseline models and their HE versions are obtained as subcases of the more general form that includes time-varying parameters and spillover effects; consequently, likelihood ratio (LR) statistics can be used to test the restrictions embedded in the less flexible models. Furthermore, the parameterization adopted is very parsimonious and guarantees the positive definiteness of the estimated covariance matrix, so it can potentially also be adopted for large covariance matrices (as in Bauwens and Otranto, 2023). We call this class of models 'flexible realized covariance' (FRC) models.

All the models are estimated by the method of quasi-maximum likelihood (QML) on a dataset relative to the components of the Dow Jones index. The models are compared both in terms of in-sample fit and out-of-sample forecasts using statistical and economic loss functions. Standard hypothesis tests show the statistical significance[2] of the parameters that introduce the spillover and time-varying specific effects, and a good forecasting performance of the extended models, in particular those with specific but time invariant parameters in the variance equations and time-varying specific

[2]We test the null hypothesis of nullity of a scalar parameter by comparing the ratio of its QML estimate to its robust standard error (defined in the supplementary material file) to the quantile of the standard Gaussian distribution corresponding to the chosen significance level.

HE parameterizations for the covariance (correlation) equations. The forecasting improvement due to the FRC specifications compared to their less flexible versions is predominant for short out-of-sample horizons.

The next section exposes the ideas of the flexible, yet parsimonious, parameterizations of the spillover and time-varying effects, independently of a specific model. In Section 3 the proposed parameterizations are included in four classes of models: BEKK, GARCH–DCC, MHAR, HAR–DRD, resulting in sixty-four possible models. Section 4 is devoted to the empirical illustration. Some remarks conclude the paper.

The list of stocks used in the empirical application and additional empirical results and analyses are included in a supplementary material file (referred to as SM), as well as the GAUSS codes to estimate the new models and data used in the empirical application.

2 Spillover effects and time-varying parameters

In this section, we explain, independently of any specific realized covariance (RC) model, the parsimonious parameterization we advocate of the term(s) capturing the impact of lagged realized covariances on the conditional covariances at time t. With respect to Bauwens and Otranto (2023), we add spillover effects in the conditional variance equations, i.e., a possible impact of the lagged realized variances of other assets on the conditional variance of a given asset, and we add an additional impact of the lagged own realized variance.

Let $\boldsymbol{C}_t$ be the $n \times n$ realized covariance matrix of day t, $\bar{\boldsymbol{C}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{C}_t$ the (sample) average realized covariance matrix, $\boldsymbol{P}_t$ the realized correlation matrix obtained from

$\boldsymbol{C}_t$, $\bar{\boldsymbol{P}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{P}_t$ the average realized correlation matrix, $\boldsymbol{S}_t$ the conditional covariance matrix, and $\boldsymbol{R}_t$ the conditional correlation matrix, i.e., the correlation matrix obtained from $\boldsymbol{S}_t$ (or estimated directly from a dynamic correlation model). At this stage, it is sufficient to define generically $\boldsymbol{S}_t$ as the conditional expectation of $\boldsymbol{C}_t$, i.e. $\boldsymbol{S}_t = \mathrm{E}(\boldsymbol{C}_t|\mathcal{I}_{t-1};\boldsymbol{\theta})$, where $\mathcal{I}_t = \{\boldsymbol{C}_t, \boldsymbol{C}_{t-1}, \dots\}$ is the conditioning information set and $\boldsymbol{\theta}$ a set of parameters including (among others) the specific coefficients that enable us to specify a non-constant marginal impact of lagged realized variances and covariances on $\boldsymbol{S}_t$; in the sequel, we refer to these specific parameters as 'impact coefficient(s)' or 'impact(s)'.

In the rest of this section we explain how, in general, we can add spillover effects and time-varying impact coefficients to a generic baseline model; in Section 3 we present some specific models, whereby we enrich four of the most widespread baseline models with spillover and time-varying coefficients.

2.1 Spillover effect in conditional variances

The spillover term is active (with a lag of one period) on the diagonal elements of $\boldsymbol{S}_t$, and is parameterized by a diagonal matrix

$$\boldsymbol{\Theta}_t = \mathrm{diag}\{\vartheta_{ii,t}, i = 1, 2, \dots, n\} \tag{2.1}$$

where

$$\vartheta_{ii,t} = \vartheta \sum_{r=1}^{n} w_{ir,t} c_{rr,t}, \tag{2.2}$$

where ϑ is a positive parameter (see below for a discussion), $c_{rr,t}$ is the realized variance of asset r multiplied by $w_{ir,t}$, a weight extracted from $\boldsymbol{W}_t = \{w_{ir,t}\}$, a weighting matrix with zeroes on the main diagonal and rows adding up to 1. The weight $w_{ir,t}$, multiplied by ϑ, measures the spillover effect of the realized variance of asset r at time t on the conditional variance $s_{ii,t+1}$ of asset i at time $t+1$. The weight $w_{ii,t}$ is set to zero because the impact of $c_{ii,t}$ on $s_{ii,t+1}$ is introduced through a specific parameter, as explained in the next subsection.

For parsimony, the weights (at time t) are defined as functions of known variables at time t. In applications, we use several weight matrices:

$$\boldsymbol{W}_t = \{w_{ir,t}\} = \begin{cases} (\boldsymbol{H} \odot \boldsymbol{J}) \oslash \boldsymbol{G}_J \\ \left(\boldsymbol{H} \odot \bar{\boldsymbol{P}}\right) \oslash \boldsymbol{G}_{\bar{P}} \\ (\boldsymbol{H} \odot \boldsymbol{P}_t) \oslash \boldsymbol{G}_{\bar{P}_t} \\ (\boldsymbol{H} \odot \boldsymbol{R}_t) \oslash \boldsymbol{G}_{\bar{R}_t}, \end{cases} \tag{2.3}$$

where $\odot$ ($\oslash$) is the operator of element–by–element matrix product (division), $\boldsymbol{J}$ is a matrix of ones, $\boldsymbol{H}$ is a matrix with zeroes on the diagonal and ones elsewhere, and $\boldsymbol{G}_{X_t}$ is a matrix with n equal columns given by the vector containing the sum of each row of $\boldsymbol{H} \odot \boldsymbol{X}_t$ ($\boldsymbol{X}_t = \boldsymbol{J}, \bar{\boldsymbol{P}}, \boldsymbol{P}_t, \boldsymbol{R}_t$). More explicitly, if $x_{ir,t}$ is the element in position (i, r) of the matrix $\boldsymbol{X}_t$, the corresponding weight is

$$w_{ir,t} = \frac{x_{ir,t}}{\sum_{j \neq i} x_{ij,t}}. \tag{2.4}$$

Spillover effects are naturally linked to co–movements and, consequently, to the degree of correlation between the asset pairs (Forbes and Rigobon, 2002); the specifications of the $\boldsymbol{W}_t$ weight matrix in (2.3) are consistent with this property. When $\boldsymbol{X}_t = \boldsymbol{J}$, the weights are all equal to $1/(n-1)$. When $\boldsymbol{X}_t = \bar{\boldsymbol{P}}$, the weights are different for each asset pair and constant over time. If assets i and r are more positively correlated (on average in the sample) than assets i and s, $w_{ir,t}$ is larger than $w_{is,t}$ (for all t), corresponding to the expectation that the spillover effect is likely to be more important between more strongly correlated assets (e.g. for assets in the same economic sector). In the two other cases, the weight $w_{ir,t}$ depends on the time–varying (realized or conditional) correlations between assets i and r; the same idea as in the previous case prevails, but with the refinement of time-varying correlations.

Note that, except when the weights are equal to $1/(n-1)$, they can be negative; this occurs if some correlations (but not all) are negative. With a positive ϑ coefficient, if the correlation is negative between assets i and r, the lagged realized variance of asset r decreases the next conditional variance of asset i, and the lagged realized variance of asset i decreases the next conditional variance of asset r, but not necessarily with the same impact since $w_{ir,t}$ differs from $w_{ri,t}$.

Since the diagonal matrix $\mathbf{\Theta_t}$ of spillover effects is one of the terms that determine additively the conditional covariance matrix $\boldsymbol{S}_t$, we need it to be positive semidefinite; for this, a sufficient condition is that its diagonal elements $\vartheta \sum_{r=1}^{n} w_{ir,t} c_{rr,t}$ are nonnegative. The sign of the weighted sum multiplying ϑ depends on the signs and relative values of the weights $w_{ir,t}$, which are positive when they are equal, and can be of any sign in general when they are computed from correlations, as discussed above. Since correlations between stock returns are predominantly positive (see, e.g., (Engle, 2009)), assuming that the weighted sum is positive, the constraint $\vartheta \geq 0$ is a sufficient condition to ensure a positive semidefinite $\mathbf{\Theta_t}$ matrix.

In the models that we estimate, which include the spillover matrix, we always obtain a positive value of ϑ without having to impose the constraint $\vartheta \geq 0$. The fact that the estimate of ϑ in the models we use is positive means that the spillover effect is positive, which is a (not surprising) empirical result meaning that increasing variances of stock returns of other assets ($r \neq i$) tend to increase the conditional variance of the return of asset i.

It is worth noting that adding equation (2.2) to the baseline models presented in Section 3 does not change the autoregressive part of the models and, consequently, their usual stationarity constraints.

2.2 Time-varying impact of the realized covariance matrix

In most existing models, the impacts of the lagged realized variances and covariances contained in $\boldsymbol{C}_{t-1}$ on the conditional variances and covariances in $\boldsymbol{S}_t$ are limited to a linear impact of $c_{ij,t-1}$ on $s_{ij,t}$, thus excluding cross-impacts (a diagonality restriction). The parsimonious parameterization of the spillover effects presented in the previous subsection allows linear cross-impacts between variances. Keeping the diagonality restrictions for covariances, we parameterize the impact coefficients of $\boldsymbol{C}_{t-1}$ in the following nonlinear way:

$$a[\exp\odot(\phi(\boldsymbol{M}_{t-1} - \boldsymbol{J}))] + \beta\boldsymbol{V}_{t-1}, \tag{2.5}$$

where $a \in [0, 1)$, ϕ and β (unrestricted) are scalar parameters, $\exp\odot$ is the Hadamard exponential (HE) operator (element–by–element exponentiation), $\boldsymbol{M}_t$ is a positive definite matrix with all main diagonal elements equal to 1, and $\boldsymbol{V}_t$ is a diagonal matrix containing the realized variances (the diagonal elements of $\boldsymbol{C}_t$). The HE term is the parameterization proposed by Bauwens and Otranto (2020, 2023) to provide asset pair specific and possibly time varying coefficients for the covariances; indeed, the impact coefficient of $c_{ij,t-1}$ on $s_{ij,t}$ is equal to $a\exp(\phi m_{ij,t-1}))/\exp(\phi)$. This coefficient is equal to a if ϕ is equal to zero, and smaller than a if $\phi > 0$. If $\phi < 0$, the exponential function is greater than 1, but multiplying it by a, it is still possible to get a coefficient smaller than 1 if the value of a is small enough. In brief, the condition $\phi \geq 0$ is sufficient to have a non-explosive model; in the models we estimate, we always obtain a positive value of ϕ without imposing the condition. Bauwens and Otranto (2023) argue that the usual parameter constraints ($0 \leq a < 1$) can be adopted to obtain sufficient stationarity conditions with the HE parameterization.

In applications, we consider three choices of the $\boldsymbol{M}_t$ matrix, each being known at time $t+1$:

$$\boldsymbol{M}_t = \bar{\boldsymbol{P}} \textit{ or } \boldsymbol{P}_t \textit{ or } \boldsymbol{R}_t. \tag{2.6}$$

When $\boldsymbol{M}_t$ is chosen as $\bar{\boldsymbol{P}}$, the coefficients of the covariances are specific to each asset pair, but constant over time; with the other choices, they are specific to each asset pair through the lagged (realized or conditional) correlation coefficient between the two assets of each pair.

In (2.5), the variance impact coefficients are obtained by adding to the constant parameter a a time–varying part represented by β multiplied by the corresponding lagged variance. Note that, if β is negative (as usually found in practice), the variance coefficient is attenuated by higher values of the lagged realized volatility. This parameterization is consistent with the "attenuation" idea of Bollerslev et al. (2018), who parameterize the lagged realized variance impact coefficient as a linear function of the lagged bi–power variation (with an expected negative β), or, even more, of Cipollini et al. (2021), who replace the bi-power variation with the realized variance.

The parameterization (2.5), which provides specific and time–varying coefficients, combined with the spillover term (2.1), is very parsimonious since it adds only three scalar parameters (ϑ, ϕ and β) to the baseline specifications of several RC models, that use only the a scalar coefficient (see next section). Moreover, the same parameterization (2.5) provides the possibility to represent different effect of the lagged elements of the realized covariance matrix, including different parameterizations for elements on the diagonal (variances) and elements out of the diagonal (covariances). In fact the impact coefficient of the element $c_{ij,t-1}$ is given by:

$$a \exp(\phi m_{ij,t-1}))/\exp(\phi) + \beta v_{ij,t-1} = \begin{cases} a \exp(\phi m_{ij,t-1})/\exp(\phi) & \text{if } i \neq j, \\ a + \beta c_{ii,t-1} & \text{if } i = j. \end{cases} \tag{2.7}$$

Moreover the baseline specifications are nested within the proposed extended ones; in fact they are obtained by setting to zero the three new coefficients. Also the HE models of Bauwens and Otranto (2023) can be obtained from the new specification by constraining $\vartheta = \beta = 0$. In this sense the proposed models generalize the existing models for RC matrices, providing specific and time–varying coefficients for each asset in a very parsimonious way, and facilitating the comparison with baseline models by likelihood ratio (LR) tests.

We call this wide class of models the flexible realized covariance (FRC) models. To ease reading the next sections, where we include the proposed parameterization in four benchmark models, we list and define the used symbols in the SM.

3 Flexible RC models

The parameterizations proposed in the previous section can be easily included in existing models. We propose the extension of the BEKK and multivariate heterogenous auto-regressive (MHAR) models for conditional covariances, and the extension of the DCC and DRD models for conditional variances and correlations, adopting the two–step estimation procedure for the latter two.

For inference on the parameters of a model (denoted by $\boldsymbol{\theta}$), we use the QML method.

This requires an assumption on the distribution of the RC matrices $\boldsymbol{C}_t$. Following the conditional autoregressive Wishart (CAW) framework proposed by Golosnoy et al. (2012), Bauwens et al. (2012) have shown that the likelihood function based on the Wishart distribution is a quasi-likelihood function and provides a consistent estimator of $\boldsymbol{\theta}$ if the conditional expectation of $\boldsymbol{C}_t$ is correctly specified. In symbols, the assumption is written:

$$\boldsymbol{C}_t|\mathcal{I}_{t-1} \sim W_n\left(\nu, \boldsymbol{S}_t/\nu\right), \tag{3.1}$$

where $W_n(\nu, \boldsymbol{A})$ denotes a $n \times n$–variate central Wishart distribution with $\nu > n$ degrees of freedom and positive definite scale matrix $\boldsymbol{A}$. The parameterization adopted in (3.1) implies that $\mathrm{E}(\boldsymbol{C}_t|\mathcal{I}_{t-1};\boldsymbol{\theta}) = \boldsymbol{S}_t$. The matrix $\boldsymbol{S}_t$ is indexed by the parameter vector $\boldsymbol{\theta}$; ν does not affect the estimation of $\boldsymbol{\theta}$, so it can be excluded from the set of parameters to be estimated.

BEKK

The flexible extension of the BEKK RC model is written as

$$\boldsymbol{S}_t = c\,\bar{\boldsymbol{C}} + \boldsymbol{\Theta}_{t-1} + \left[a \exp\odot(\phi(\boldsymbol{M}_{t-1} - \boldsymbol{J})) + \beta \boldsymbol{V}_{t-1}\right] \odot \boldsymbol{C}_{t-1} + b\boldsymbol{S}_{t-1}, \tag{3.2}$$

where c is a positive parameter multiplying the sample average RC matrix. If $\vartheta = \phi = \beta = 0$, c is replaced by $1 - a - b$, but in the flexible version, this cannot be generalized while ensuring that the constant term matrix is positive definite.

MHAR

An alternative to BEKK is the multivariate HAR model. We adopt a natural extension of the univariate HAR model of Corsi (2009), written as

$$\boldsymbol{S}_t = c\,\bar{\boldsymbol{C}} + \boldsymbol{\Theta}_{t-1} + [a_D \exp \odot(\phi(\boldsymbol{M}_{t-1} - \boldsymbol{J})) + \beta \boldsymbol{V}_{t-1}] \odot \boldsymbol{C}_{t-1} + \alpha_W \bar{\boldsymbol{C}}_{t-1:t-5} + \alpha_M \bar{\boldsymbol{C}}_{t-1:t-22}, \tag{3.3}$$

where $\bar{\boldsymbol{C}}_{r:s}$ is the mean of the RC matrices between time r and time s ($r < s$). The coefficients α_W and α_M represent respectively the weekly and monthly autoregressive effects on the conditional covariance at time t.

GARCH–DCC

Let us decompose the conditional expectation of the realized covariance matrix as

$$\boldsymbol{S}_t = \tilde{\boldsymbol{S}}_t^{1/2} \boldsymbol{R}_t \tilde{\boldsymbol{S}}_t^{1/2}, \tag{3.4}$$

where $\tilde{\boldsymbol{S}}_t = \text{diag}(s_{ii,t},\ i = 1, 2, \ldots, n)$ is the diagonal matrix of the conditional variances (the diagonal elements of $\boldsymbol{S}_t$). Under the Wishart distributional hypothesis, Bauwens et al. (2012) show that, analogously to the multivariate GARCH–DCC model under the Gaussian distribution hypothesis (Engle, 2002), the log-likelihood can be split in two parts that can be estimated separately: one relative to the variance part only (as the sum of the n univariate log–likelihoods of each asset; first step estimation) and the other relative to the correlation part (second step estimation, conditional on the results of the first step estimation). Working in the first step with univariate models, we specify n GARCH–type models as

$$\begin{aligned} s_{ii,t} &= \omega_i \bar{c}_{ii} + \vartheta_{i,t-1} + (a_i + \beta_i c_{ii,t-1}) c_{ii,t-1} + b_i s_{ii,t-1}, \\ \vartheta_{i,t} &= \vartheta_i \textstyle\sum_{r \neq i} w_{ir,t} c_{rr,t}. \end{aligned} \tag{3.5}$$

In this type of model, the weight matrix $\boldsymbol{W}_t$ cannot depend on $\boldsymbol{R}_t$ because the conditional correlation matrix is not estimated in the first step; as a consequence we can adopt only three of the four specifications in (2.3).

The model for the conditional correlation matrix is the same as the HE DCC model of Bauwens and Otranto (2023):

$$\begin{aligned} \boldsymbol{R}_t &= \tilde{\boldsymbol{Q}}_t^{-1/2}\boldsymbol{Q}_t\tilde{\boldsymbol{Q}}_t^{-1/2}, \\ \boldsymbol{Q}_t &= c\bar{\boldsymbol{P}} + a\exp\odot(\phi(\boldsymbol{M}_{t-1}-\boldsymbol{J}))\odot\left(\tilde{\boldsymbol{Q}}_{t-1}^{1/2}\boldsymbol{D}_{t-1}^{-1}\boldsymbol{C}_{t-1}\boldsymbol{D}_{t-1}^{-1}\tilde{\boldsymbol{Q}}_{t-1}^{1/2}\right) + b\boldsymbol{Q}_{t-1}, \\ \tilde{\boldsymbol{Q}}_t &= \mathrm{diag}(q_{ii,t},\ i=1,2,\ldots,n), \end{aligned} \tag{3.6}$$

where $\boldsymbol{D}_t$ is a diagonal matrix with the n diagonal elements equal to the square root of $s_{ii,t}$–see the first equation in (3.5). The number of parameters is $5n+4$: the five coefficients (ω_i, a_i, β_i, b_i, ϑ_i) of the n univariate GARCH–type models (3.5) and the four coefficients (c, a, ϕ, b) of the correlation part (3.6).

HAR–DRD

The HAR extension to the two–step estimation case was developed by Oh and Patton (2016), introducing the so–called DRD models. Based on the decomposition (3.4) of $\boldsymbol{S}_t$ into the variance part and the correlation part, the first step estimation consists in adopting n univariate HAR models (Corsi, 2009) as ($i=1,\ldots,n$):

$$\begin{aligned} s_{ii,t} &= \omega_i\bar{c}_{ii} + \vartheta_{i,t-1} + (a_{i,D}+\beta_i c_{ii,t-1})c_{ii,t-1} + a_{i,W}\bar{c}_{ii,t-1:t-5} + a_{i,M}\bar{c}_{ii,t-1_t-22}, \\ \vartheta_{i,t} &= \vartheta_i\textstyle\sum_{r\neq i} w_{ir,t}c_{rr,t}, \end{aligned} \tag{3.7}$$

where, similarly to the DCC case in (3.5), the possible specifications of the weight matrix $\boldsymbol{W}_t$ do not include $(\boldsymbol{H}\odot\boldsymbol{R}_t)\oslash\boldsymbol{G}_{\bar{R}_t}$.

The DRD conditional correlation is given by

$$\boldsymbol{R}_t = c\bar{\boldsymbol{P}} + \alpha_D\exp(\phi(\boldsymbol{M}_{t-1}-\boldsymbol{J}))\odot\boldsymbol{P}_{t-1} + \alpha_W\bar{\boldsymbol{P}}_{t-1:t-5} + \alpha_M\bar{\boldsymbol{P}}_{t-1:t-22}. \tag{3.8}$$

Unlike in the DCC case, the second step estimation does not depend on the estimates of the first step, because R_t is a correlation matrix by construction. The number of parameters is $6n+5$: the six coefficients (ω_i, $a_{i,D}$, β_i, $a_{i,W}$, $a_{i,M}$, ϑ_i) of the n univariate HAR–type models (3.7) and the five coefficients (c, α_D, ϕ, α_W, α_M) of the correlation part (3.8).

Model Set

In summary, we propose four families of models with a flexible parameterization. Let us denote by $Z = \bar{\boldsymbol{P}}, \boldsymbol{P}_t, \boldsymbol{R}_t$, the matrix $\boldsymbol{M}_t$ –see equation (2.6)– which drives the Hadamard exponential function; by $U = \bar{\boldsymbol{P}}, \boldsymbol{P}_t, \boldsymbol{R}_t, \boldsymbol{J}$, the matrix in the $\boldsymbol{W}_t$ function (2.3); by $V = \bar{\boldsymbol{P}}, \boldsymbol{P}_t, \boldsymbol{J}$, the matrix in the $\boldsymbol{W}_t$ function of the variance equations of the GARCH–DCC and HAR–DRD models. Moreover, $Z = 0$ indicates that $\phi = 0$; similarly, $\boldsymbol{U} = 0$ that $\theta = \beta = 0$, and $\boldsymbol{V} = 0$ that $\vartheta_i = \beta_i = 0$ for each i.

For example, BEKK–U–Z represents the FRC BEKK model (3.2) with $\boldsymbol{M}_t = Z$ and $\boldsymbol{W}_t = U$; BEKK–0–0 represents the simple BEKK model with scalar and constant coefficients; BEKK–0–Z represents the HE–BEKK model of Bauwens and Otranto (2023). Similarly we use the notations GARCH–DCC–V–Z for (3.5)-(3.6), MHAR–U–Z for (3.3) and HAR–DRD–V–Z for (3.7)-(3.8).

Combining all possible choices of U (V) and Z we obtain 64 different specifications: 16 BEKK–U–Z; 16 GARCH–DCC–V–Z; 16 MHAR–U–Z; 16 HAR–DRD–V–Z. The 16 combinations of $U - Z$ are: $0-0$, $0-\bar{\boldsymbol{P}}$, $0-\boldsymbol{P}_t$, $0-\boldsymbol{R}_t$, $\bar{\boldsymbol{P}}-\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{P}}-\boldsymbol{P}_t$, $\bar{\boldsymbol{P}}-\boldsymbol{R}_t$, $\boldsymbol{P}_t-\bar{\boldsymbol{P}}$, $\boldsymbol{P}_t-\boldsymbol{P}_t$, $\boldsymbol{P}_t-\boldsymbol{R}_t$, $\boldsymbol{R}_t-\bar{\boldsymbol{P}}$, $\boldsymbol{R}_t-\boldsymbol{P}_t$, $\boldsymbol{R}_t-\boldsymbol{R}_t$, $\boldsymbol{J}-\bar{\boldsymbol{P}}$, $\boldsymbol{J}-\boldsymbol{P}_t$, $\boldsymbol{J}-\boldsymbol{R}_t$. The 16 combinations of $V - Z$ are: $0-0$, $0-\bar{\boldsymbol{P}}$, $0-\boldsymbol{P}_t$, $0-\boldsymbol{R}_t$, $\bar{\boldsymbol{P}}-0$, $\bar{\boldsymbol{P}}-\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{P}}-\boldsymbol{P}_t$, $\bar{\boldsymbol{P}}-\boldsymbol{R}_t$, $\boldsymbol{P}_t-0$, $\boldsymbol{P}_t-\bar{\boldsymbol{P}}$, $\boldsymbol{P}_t-\boldsymbol{P}_t$, $\boldsymbol{P}_t-\boldsymbol{R}_t$, $\boldsymbol{J}-0$, $\boldsymbol{J}-\bar{\boldsymbol{P}}$, $\boldsymbol{J}-\boldsymbol{P}_t$, $\boldsymbol{J}-\boldsymbol{R}_t$.

4 Empirical application

We use the same data as Bauwens and Otranto (2023) and Bauwens and Xu (2023). This is a dataset of daily RC matrices for 29 stocks (listed in the SM file) of the Dow Jones Industrial Average index. The realized covariances are computed as the sum of the outer products of the 5–minute log–returns of each day[3] in the period from January 3, 2001 to April 16, 2018 ($T = 4319$ days). The first 3744 observations constitute the in–sample period (from January 2, 2001 to December 31, 2015), while the remaining 575 observations form the sample used for evaluating forecasts (section 4.3).

4.1 Estimation results for the in-sample period

We have estimated the 64 models defined in the previous section. Given the large number of models, we have selected, in each of the four families of models, the best model in terms of the Akaike (AIC), and Bayesian (BIC) information criteria (both criteria select the same models). The four selected models are BEKK–$\boldsymbol{J}$–$\boldsymbol{R}_t$, MHAR–$\bar{\boldsymbol{P}}$–$\boldsymbol{R}_t$, GARCH–DCC–$\bar{\boldsymbol{P}}$–$\boldsymbol{P}_t$, and HAR–DRD–$\bar{\boldsymbol{P}}$–$\boldsymbol{P}_t$. A first interesting result is the fact that the four selected models have a constant weight matrix for the spillover effects (the matrices selected for U and V are $\boldsymbol{J}$ or $\bar{\boldsymbol{P}}$), and specific, time–varying coefficients for the covariance (correlation) part ($\boldsymbol{P}_t$ or $\boldsymbol{R}_t$ chosen for Z).

Table 1 shows the LR statistics for restricting the FRC version to the simpler versions; in the table, $0-0$ refers to the corresponding baseline BEKK, GARCH–DCC, MHAR and HAR–DRD models, and $0 - \boldsymbol{M}_t$ to the corresponding HE with the same matrix

[3] The empirical analysis for the data computed from 1–minute log–returns is presented in the SM, highlighting a certain robustness of the results.

	$0-0$		$0-\boldsymbol{M}_t$	
BEKK–$\mathbf{J}$–$\mathbf{R_t}$	818.28	(3)	34.07	(2)
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	1510.79	(3)	64.15	(2)
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	326.49	(59)	326.41	(58)
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	320.89	(59)	289.60	(58)

Table 1: LR statistics to compare each row model to the same family models in columns 2 and 3. The number of restrictions to simplify the row model to the column model is indicated in parentheses. The LR statistics are obtained from the log-likelihood values reported in Tables 2 and 3.

in the matrix exponential function as the corresponding FRC model ($\boldsymbol{R}_t$ for BEKK and MHAR, $\boldsymbol{P}_t$ for DCC and DRD). The LR statistics are very large, hence the corresponding p–values are tiny (assuming a chi–square distribution with the degrees of freedom reported in the table), indicating that the FRC models clearly outperform the baseline and HE corresponding versions.

Tables 2 and 3 report the QML estimation results of the parameters of these twelve models. For the variance equations of GARCH–DCC and HAR-DRD, we report the average of the 29 estimates of each parameter (indicated with an upper bar) and their standard deviation. The averages of the ϑ and β estimates are much greater (in absolute value) than the corresponding values of the BEKK and MHAR models. This is due to the heterogeneity of the estimated parameters of the variance equations of the GARCH-DCC and HAR-DRD models, which is not allowed in the BEKK and MHAR models; for example, in the GARCH-DCC-$\bar{\boldsymbol{P}}$-$\boldsymbol{P}_t$ case, the estimate of ϑ ranges between 0.829 for CVX and 18.705 for TRV, the estimate of β between -83.599 for PFE and -0.461 for GS.

BEKK											
$U-Z$	ϑ	β	c	a	b		ϕ	Log–Lik	AIC	TIC	BIC
$0-0$			0.009	0.138	0.855			-93518.3	49.971	49.975	49.976
			(0.001)	(0.008)	(0.007)						
$0-\boldsymbol{R}_t$			0.014	0.177	0.787		0.389	-93126.2	49.762	49.765	49.769
			(0.001)	(0.010)	(0.012)		(0.017)				
$\boldsymbol{J}-\boldsymbol{R}_t$	0.004	-1.904	0.013	0.175	0.786		0.358	-93109.1	49.754	49.756	49.764
	(0.001)	(0.939)	(0.001)	(0.009)	(0.012)		(0.016)				

MHAR											
$U-Z$	ϑ	β	c	α_D	α_W	α_M	ϕ	Log–Lik	AIC	TIC	BIC
$0-0$			0.066	0.122	0.314	0.509		-93426.3	49.923	49.925	49.929
			(0.004)	(0.005)	(0.015)	(0.012)					
$0-\boldsymbol{R}_t$			0.058	0.232	0.210	0.407	1.758	-92703.0	49.537	49.539	49.545
			(0.004)	(0.009)	(0.010)	(0.012)	(0.050)				
$\bar{\boldsymbol{P}}-\boldsymbol{R}_t$	0.028	0.011	0.056	0.205	0.210	0.398	1.519	-92670.9	49.521	49.522	49.532
	(0.008)	(0.007)	(0.004)	(0.007)	(0.012)	(0.018)	(0.053)				

Table 2: Estimates and robust standard errors in parentheses of the baseline $(0-0)$, HE $(0-\boldsymbol{R}_t)$, and selected FRC models for the in-sample period (3744 observations). Robust standard errors are explained in the SM file.

In Tables 2 and 3 three model choice criteria are reported: AIC, BIC and the robust version of AIC, TIC (Takeuchi, 1976), to take into account that inference is based on a quasi-likelihood. When the model is correctly specified, TIC is equal to AIC. Comparing the models within each family, the FRC version has the best fit for each criterion, except for TIC and BIC in the GARCH-DCC family, where the best fitting model is the HE version. This exception is in line with the fact that in the first step estimation of the 29 univariate $GARCH-\bar{\boldsymbol{P}}$ models, 14 ϑ_i coefficients and 13 β_i coefficients in equation (3.5) are not statistically significant.

Ranking the twelve models globally, the best one is the flexible GARCH-DCC according to AIC, and the flexible MHAR according to TIC and to BIC. Note that the three HAR-DRD models have the largest AIC (with one exception), TIC (with one exception), and BIC, among the twelve models.

GARCH–DCC

$V-Z$	$\bar{\omega}$	$\bar{a}$	$\bar{b}$	$\bar{\vartheta}$	$\bar{\beta}$	c	a	b	ϕ	Log–Lik	AIC	TIC	BIC
$0-0$	0.022	0.364	0.617			0.000	0.052	0.928		-92695.0	49.578	49.587	49.728
	(0.007)	(0.045)	(0.044)			(0.000)	(0.004)	(0.006)					
$0-\boldsymbol{P}_t$	0.022	0.364	0.617			0.000	0.056	0.926	0.051	-92691.0	49.576	49.585	49.728
	(0.007	(0.045)	(0.044)			(0.000)	(0.004)	(0.006)	(0.020)				
$\bar{\boldsymbol{P}}-\boldsymbol{P}_t$	0.017	0.325	0.583	0.075	-24.096	0.000	0.054	0.928	0.047	-92531.8	49.520	52.552	49.761
	(0.008)	(0.046)	(0.049)	(0.042)	(23.743)	(0.000)	(0.004)	(0.006)	(0.017)				

HAR–DRD

$V-Z$	$\bar{\omega}$	$\bar{a}_D$	$\bar{a}_W$	$\bar{a}_M$	$\bar{\vartheta}$	$\bar{\beta}$	c	α_D	α_W	α_M	ϕ	Log–Lik	AIC	TIC	BIC
$0-0$	0.048	0.354	0.383	0.214			0.384	0.028	0.107	0.609		-93500.1	50.024	50.018	50.224
	(0.014)	(0.043)	(0.060)	(0.049)			(0.012)	(0.001)	(0.006)	(0.022)					
$0-\boldsymbol{P}_t$	0.048	0.354	0.383	0.214			0.309	0.057	0.093	0.556	1.352	-93484.5	50.016	50.009	50.218
	(0.014)	(0.043)	(0.060)	(0.049)			(0.023)	(0.002)	(0.006)	(0.038)	(0.087)				
$\bar{\boldsymbol{P}}-\boldsymbol{P}_t$	0.041	0.252	0.345	0.196	0.144	0.114	0.309	0.057	0.093	0.556	1.352	-93339.7	49.939	49.953	50.140
	(0.017)	(0.065)	(0.058)	(0.056)	(0.063)	(11.023)	(0.023)	(0.002)	(0.006)	(0.038)	(0.087)				

Table 3: Estimates and robust standard errors in parentheses of the baseline $(0-0)$, HE $(0-\boldsymbol{P}_t)$, and selected FRC models for the in-sample period (3744 observations). The parameters with an upper bar refer to the first step estimation of the models and the reported values are the averages of the 29 corresponding estimates of the univariate models; the corresponding value in parentheses is the standard deviation of the 29 estimates. Robust standard errors are explained in the SM file.

4.2 Specific time–varying coefficients

The main characteristics of the FRC models are parsimony, because the number of additional coefficients with respect to the baseline specifications is very small, and flexibility, because, despite the parsimonious representation, specific and time–varying coefficients are obtained for each element of the covariance (correlation) matrices.

Focusing on the BEKK–$\boldsymbol{J}$–$\boldsymbol{R}_t$ case to analyze the time-varying effects, from (3.2) and (2.2)-(2.4), the sum of the constant and spillover terms of the variance equation of the i-th asset on day t is given by

$$c\,\bar{c}_{ii} + \vartheta \sum_{r=1, r\neq i}^{n-1} \frac{c_{rr,t-1}}{n-1}. \tag{4.1}$$

From (2.7), the coefficient of the lagged realized variance $c_{ii,t-1}$ is given by

$$a + \beta c_{ii,t-1}, \tag{4.2}$$

so that it consists of a constant part and a time-varying part depending on the lagged variance itself. This introduces a dependence of the conditional variance on the lagged squared realized variance. As shown by Cipollini et al. (2021), this nonlinear dependence captures the attenuation effect of the lagged variance, which Bollerslev et al. (2018) originally introduced with a linear function of the bi-power variation, aiming to reduce the effect of large lagged jumps in the realized variance (with an expected negative β). As shown by Cipollini et al. (2021), there is a strong positive correlation between realized volatility and bi-power variation, so using the former is practical when, like in our dataset, the bi-power variation is not available.

Finally, again from (2.7), the time–varying coefficient of the lagged realized covariance $c_{ij,t-1}$ between assets i and j is $a \exp(\phi r_{ij,t-1})/\exp(\phi)$. In Figure 1, we illustrate the estimated time–varying coefficients of the UTX variance equation of the BEKK-$\boldsymbol{J}$-$\boldsymbol{R}_t$ model. The top graph displays the time–varying coefficient (4.1), with the constant term (the horizontal line) and the spillover effect from the variances of the other 28 assets on the variance of UTX; notice the sharp increases following the collapse of Lehman Brothers in September 2008 and other shocks (such as the 9/11 attack of 2001, the stock market downturn of 2002, the flash crash in May 2010).

The middle graph shows the estimated values of the coefficient (4.2) that attenuates the baseline impact coefficient a (the horizontal line) of the lagged realized variance of UTX ($c_{ii,t-1}$) on its conditional variance. This attenuation effect appears clearly in opposition to the spillover one, but of less strong value.

The bottom graph of Figure 1 shows the time-varying coefficient $a\exp(\phi r_{ij,t-1})/\exp(\phi)$ measuring the impact of the lagged realized covariance between UTX and VZ on the corresponding conditional covariance; in this case, the coefficient fluctuates between 0.124 and 0.175 (the estimate of a) and increases in correspondence with high volatility periods (as suggested by Bauwens and Otranto, 2016).

Variation of the spillover effect across stocks

As previously underlined, an advantage of the FRC models is the possibility of having *specific* time–varying coefficients for each element of the realized covariance matrix, i.e. *different* spillover and attenuation effects for each asset. In Figure 2 we summarize, using Box-plots, the percentage of conditional volatility due to the spillover effect for each of the 29 assets over time during the in–sample estimation period. Each Box–plot shows the distribution of the spillover percentage effect on the estimated conditional volatility of an asset in the estimation period. The spillover effects vary

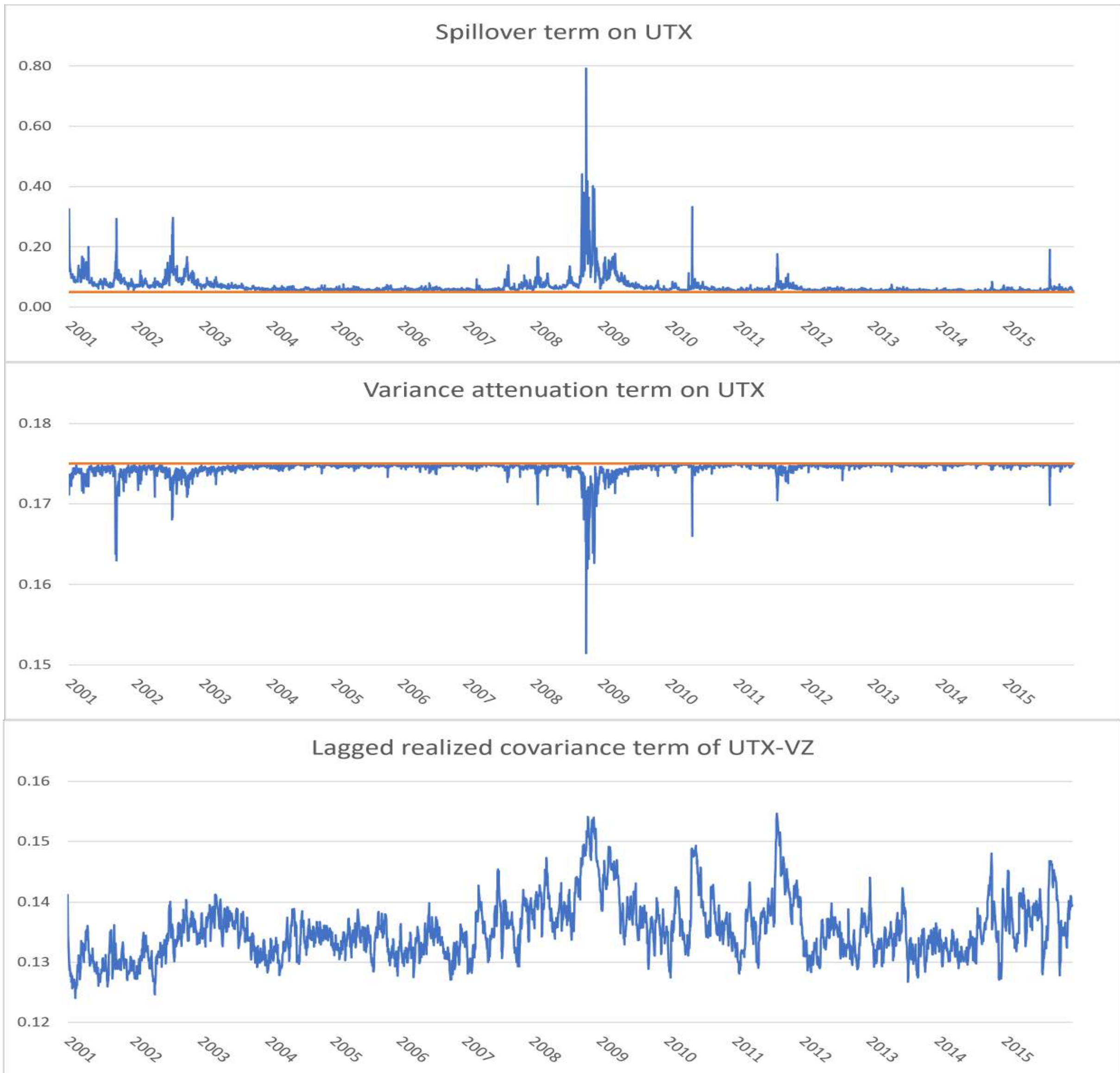

Figure 1: Time–varying coefficients estimated by $BEKK - \boldsymbol{J} - \boldsymbol{R}_t$: intercept plus spillover term ($c\,\bar{c}_{ii} + \vartheta_{ii,t-1}$), variance attenuation term ($\beta c_{ii,t-1}$), and coefficient of lagged covariance ($a \exp(\phi(r_{ij,t-1} - 1))$, for assets $i = UTX$ and $j = VZ$.

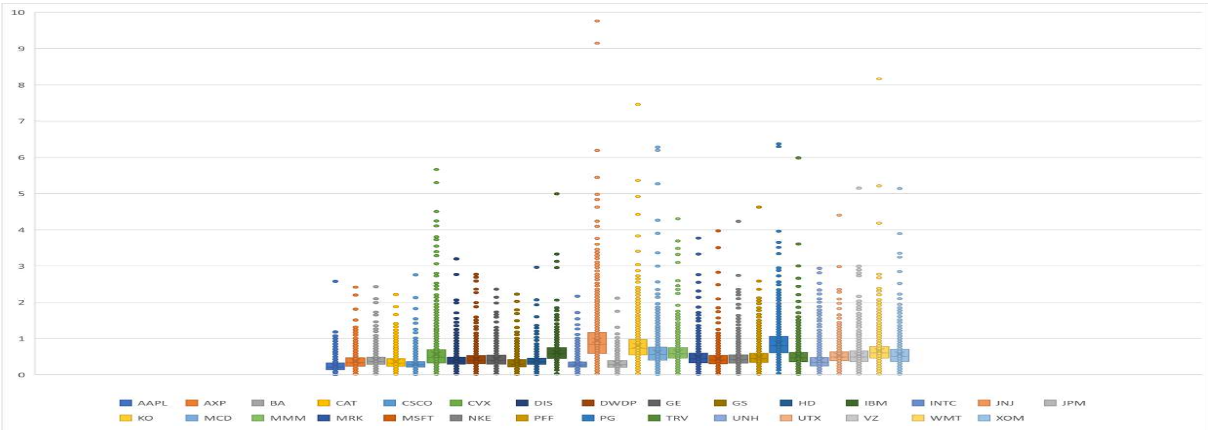

Figure 2: Model $BEKK - \boldsymbol{J} - \boldsymbol{R}_t$: Box plot of the percentage of spillover effects of each asset on the corresponding conditional volatility. Each box plot refers to the assets in alphabetical order (from left to right).

across assets: although the median is between zero and one percent, the differences are more important in the right tails, which reveal an asymmetric behavior, characterized by several peaks. The highest values and highest variability occur for the JNJ and PG stocks. On two days, the spillover effect of JNJ is between 9 and 10 percent.

To check in which periods the spillover effects are most intense, we provide in Table 4 the number of days per year (if any) where the spillover effect accounts for more than 3% of the conditional variance: this occurs for 18 of the 29 assets. Once again, JNJ

Asset	2001	2005	2006	2007	2008	2010	2015	Total
CVX	17	0	0	0	0	2	0	19
DIS	0	0	0	0	0	1	0	1
HD	0	0	0	0	0	1	0	1
IBM	0	1	0	0	0	2	0	3
JNJ	2	1	1	6	22	1	1	34
KO	1	0	0	0	6	2	0	9
MCD	1	0	0	0	3	2	0	6
MMM	1	0	0	0	3	1	0	5
MRK	1	0	0	0	0	1	0	2
MSFT	0	0	0	0	0	2	0	2
NKE	0	0	0	0	0	1	0	1
PFF	0	0	0	0	0	1	0	1
PG	1	0	0	0	4	1	0	6
TRV	1	0	0	0	0	1	1	3
UTX	0	0	0	0	0	1	0	1
VZ	0	0	0	0	0	1	0	1
WMT	0	0	0	0	2	2	0	4
XOM	3	0	0	0	0	1	0	4

Table 4: Model $BEKK - \boldsymbol{J} - \boldsymbol{R}_t$: Number of days per year in which the spillover effects account for more than 3% of the conditional variance of 18 assets (for the other eleven assets, the spillover effects are less than 3%). For the missing years, the number of days is equal to zero for each asset.

is the asset with the highest number of cases (34 days), mainly concentrated during the subprime mortgage crisis of 2007–2008. CVX has 19 cases, 17 of them during the 2001 bursting of the dot-com bubble. Interestingly, all 18 assets present one or two high spillover effects during the 2010 flash crash. Figure S1 of the SM file shows the minimum, first quartile, median, third quartile, and maximum of the terms (4.1). Most of them have a value around 0.1, with peaks between 0.8 and 0.9.

A similar analysis of the attenuation term is presented in the SM file.

4.3 Forecast evaluations

The forecasting performance of each model is evaluated with four loss functions:[4]

- Quasi–Likelihood:

$$\text{QLIK} = \frac{1}{T_h}\sum_{t=1}^{T_h}\left\{\ln|\hat{\boldsymbol{S}}_t| + \text{trace}\left(\hat{\boldsymbol{S}}_t^{-1}\boldsymbol{C}_t\right)\right\}, \tag{4.3}$$

  where $\hat{\boldsymbol{S}}_t$ is the model forecast of $\boldsymbol{S}_t$ and and T_h is the number of forecasts.

- Frobenius Norm:

$$\text{FN} = \frac{1}{T_h}\sum_{t=1}^{T_h}\text{trace}\left[\left(\hat{\boldsymbol{S}}_t - \boldsymbol{C}_t\right)'\left(\hat{\boldsymbol{S}}_t - \boldsymbol{C}_t\right)\right]. \tag{4.4}$$

- Global Minimum Variance (GMV) portfolio with estimated covariance:

$$\text{GMV}_S = \frac{1}{T_h}\sum_{t=1}^{T_h}\hat{\boldsymbol{v}}_{\boldsymbol{t}}{}'\hat{\boldsymbol{S}}_t\hat{\boldsymbol{v}}_{\boldsymbol{t}}, \tag{4.5}$$

[4]The univariate versions of the QLIK and Frobenius Norm (corresponding to MSE) statistical losses are consistent in the Hansen and Lunde (2006) sense; Laurent et al. (2013) give an analogous multivariate result.

where $\hat{\boldsymbol{v}}_t = \sqrt{n}\hat{\boldsymbol{S}}_t^{-1}\boldsymbol{j}_n/(\boldsymbol{j}_n'\hat{\boldsymbol{S}}_t^{-1}\boldsymbol{j}_n)$ (with $\boldsymbol{j}_n$ a vector of ones) is the global minimum variance weight vector associated to $\hat{\boldsymbol{S}}_t$. This loss function does not use any data of the forecast period.

- Sample empirical variance of the GMV portfolio, using the observed returns:

$$GMV_V = \mathrm{Var}(\hat{\boldsymbol{v}}_t{}'\boldsymbol{r}_t), \tag{4.6}$$

where we apply the same weights as for (4.5) to the observed returns $\boldsymbol{r}_t$, obtaining the portfolio return at time t, $\hat{\boldsymbol{v}}_t{}'\boldsymbol{r}_t$, for $t = 1, 2, \ldots, T_h$, and we compute their empirical variance.

In Table 5, we show, for the last 1000 observations of the in–sample period, the value of the four loss functions and the models entering the model confidence set (Hansen et al., 2011) at the 95% confidence level, denoted by MCS. To perform the tests of the procedure that delivers the MCS, we adopt the semi-quadratic test statistic with 10,000 replications of the bootstrap procedure of Hansen et al. (2003) to obtain the variance of the differences between the loss functions of each pair of models.

Each FRC model has a smaller loss value than the corresponding basic version, except HAR-DRD for GMV$_S$. MHAR–$\bar{\boldsymbol{P}}$–$\boldsymbol{R}_t$ has the smallest QLIK loss, GARCH–DCC–$\bar{\boldsymbol{P}}$–$\boldsymbol{P}_t$ has the smallest FN loss, BEKK–$\boldsymbol{J}$–$\boldsymbol{R}_t$ has the minimum GMV$_S$ loss, while MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$ has the minimum GMV$_V$ loss.

Considering the compositions of the MCS, the FN loss function does not discriminate between the twelve models. For QLIK, three of the four FRC models are in the MCS, accompanied by the baseline HAR-DRD (the flexible HAR-DRD is also in). For GMV$_S$, BEKK–$\mathbf{J}$–$\mathbf{R_t}$ is alone in the MCS. For GMV$_V$, the baseline and flexible versions of BEKK and MHAR, plus GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$, constitute the MCS. In

	QLIK	FN	GMV_S	GMV_V
BEKK–0–0	31.22	**1041.48**	12.92	**12.18**
MHAR–0–0	31.16	**1026.10**	12.93	**12.14**
GARCH–DCC–0-0	30.91	**1003.44**	12.01	12.69
HAR–DRD–0–0	**30.81**	**1001.75**	11.31	12.78
BEKK–$\mathbf{J}$–$\mathbf{R_t}$	31.05	**1010.90**	*10.31*	**12.10**
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	*30.78*	**1022.77**	10.62	*12.04*
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	**30.81**	*984.28*	11.78	**12.41**
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	**30.75**	**995.54**	11.43	12.56

Table 5: Loss function values for in-sample one-step ahead forecasts (last 1000 observations), and MCS compositions. In italic the lowest value of the loss function indicated in column; in bold the models entering the 95% confidence level MCS for each loss function.

the SM file, we analyze the behavior of the squared errors between the RV matrix and the fitted covariance matrix.

For out-of-sample evaluations, we consider h–step–ahead forecasts, for $h = 1$, 5, and 22. We keep the estimated parameters fixed during 25 days to compute the out-of-sample forecasts (instead of re-estimating after each new observation), and then re–estimate them after adding the new 25 observations, and so on until the end of the available sample. The number of forecasts (T_h) is 575 for $h = 1$, 571 for $h = 5$, and 554 for $h = 22$.

The results are shown in Table 6. Considering the QLIK loss, GARCH–DCC–$\bar{\boldsymbol{P}}$–$\boldsymbol{P}_t$ is in the three MCS, together with MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$ at horizon 1, BEKK–$\mathbf{J}$–$\mathbf{R_t}$, BEKK–0–0, and GARCH–DCC–0-0 at horizon 5, and GARCH–DCC–0-0 at horizon 22.

	QLIK	FN	GMV_S	GMV_V
	1–step ahead (575 forecasts)			
BEKK–0–0	31.27	**1310.66**	12.17	**9.79**
MHAR–0–0	31.19	**1239.05**	12.18	**9.99**
GARCH–DCC–0-0	30.89	**1218.24**	11.22	**9.85**
HAR–DRD–0–0	32.84	**1298.29**	**9.87**	11.13
BEKK–$\mathbf{J}$–$\mathbf{R_t}$	31.11	**1288.69**	*9.83*	**9.72**
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	**30.81**	**1298.53**	**10.04**	**9.87**
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	*30.80*	*1209.36*	11.43	*9.57*
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	32.84	**1291.41**	10.06	11.01
	5–step ahead (571 forecasts)			
BEKK–0–0	**32.58**	**1732.31**	13.10	*9.97*
MHAR–0–0	33.41	**1742.17**	15.89	**10.16**
GARCH–DCC–0-0	*32.48*	**1666.88**	10.07	**10.18**
HAR–DRD–0–0	33.57	*1618.36*	12.39	11.57
BEKK–$\mathbf{J}$–$\mathbf{R_t}$	**32.77**	**1669.45**	*9.41*	**10.02**
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	32.98	**1628.52**	11.02	**10.09**
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	**32.51**	**1675.66**	**9.77**	**10.24**
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	34.22	**1625.63**	11.36	11.18
	22–step ahead (554 forecasts)			
BEKK–0–0	34.01	1913.79	16.54	**9.58**
MHAR–0–0	42.92	4009.99	32.30	*9.55*
GARCH–DCC–0-0	*32.17*	*1622.02*	11.55	10.22
HAR–DRD–0–0	34.58	1710.44	15.98	10.59
$BEKK - \boldsymbol{J} - \boldsymbol{R}_t$	34.27	1694.99	*7.89*	**9.97**
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	39.58	1742.92	15.60	**9.59**
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	**32.23**	**1630.98**	10.83	10.42
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	34.69	**1632.18**	13.57	10.18

Table 6: Loss function values for out–of–sample h-step ahead forecasts, and MCS compositions. In italic the lowest value of the loss function indicated in column; in bold the models entering the 95% confidence level MCS for each loss function.

Considering the FN loss, all models belong to the MCS for horizons 1 and 5, due to the large variability of the loss values caused by several peaks in the dynamics of realized variances and covariances; for the 22–step–ahead case, the MCS includes two FRC models (GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$ and HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$) and the baseline GARCH–DCC–0-0 model.

For the GMV_S loss, BEKK–$\mathbf{J}$–$\mathbf{R_t}$ has the smallest loss at each horizon and thus is in the three MCS, accompanied by HAR–DRD–0–0 and MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$ at horizon 1, and by GARCH–DCC–$\bar{\boldsymbol{P}}$–$\boldsymbol{P}_t$ at horizon 5. For the GMV_V loss, the same six models are included in the MCS for horizons 1 and 5 (only both HAR-DRD are not included), while at horizon 22 all correlation models (GARCH-DCC and HAR-DRD) are not in the MCS.

As the forecasting horizon increases, we observe that the baseline models improve their performances relatively to the flexible versions, since they have the smallest values for three loss functions at horizons 5 and 22, instead of none for horizon 1.[5]

Nevertheless, flexible models are included in each MCS, meaning that the differences between the loss function values of the included models are not statistically significant. There is no combination of loss function and horizon, for which the MCS includes only baseline models, but there are four combinations for which the MCS includes only flexible models (QLIK-horizon 1, and GMV_S for the three horizons). This result is not trivial: it is well known that simpler models perform better than more complex models in out-of-sample forecasts. In particular, Hansen (2010) and Hansen and Timmerman

[5]Except for MHAR–0–0, which has very high loss function values (apart for GMV_V); this model presents very bad forecasts in March 2016 (a mini recession period) and in the last forecasting period, which falls in early 2018 (considered the worst year for the US financial market since 2008).

(2020) show analytically that there is a strongly negative relationship between in- and out-of-sample performance. In out-of-sample analysis, models with more parameters tend to perform worse than more parsimonious models. Several studies on forecast volatility have empirically verified this property (see, e.g., Gallo and Otranto, 2015, 2018; Xie and Yu, 2015). The fact that FRC models enter the best set, according to the MCS procedure, showing prediction performances not significantly different from the simpler models, is a good result.

5 Concluding remarks

The availability of big datasets of realized covariance matrices has favored the development of multivariate models which are congruent with the characteristics of the covariance matrix (first of all the positive definiteness). These constraints and the curse of dimensionality problem, which implies an enormous increase in the number of unknown parameters as the size of the matrix increases, have been mainly solved by adopting scalar or diagonal parameterizations. They introduce parsimonious models at the cost of excluding important features, such as the reciprocal influence among variances (spillover effect), and the imposition of strong constraints, such as fixed and common coefficients for all elements of the covariance matrix.

The extensions we add to baseline models have several interesting features. Firstly, we add the spillover effect in conditional variance equations, and a specific nonlinear, time-varying, impact of the lagged realized covariance between each asset pair on the corresponding conditional covariance. This lagged impact effect, for the diagonal elements of the matrix, is consistent with the idea of attenuation effect developed by Bollerslev et al. (2018). Secondly, these effects are introduced sparingly by adding a few parameters to the baseline models. This implies that they can also be used for high–dimensional matrices, as shown, for the HE case, by Bauwens and Otranto (2023). Thirdly, the proposed family of models is flexible: the parameterizations of Sections 2.1 and 2.2 can be inserted into scalar baseline models, and even non-scalar models of small dimension. We introduced them in conditional covariance models (BEKK and MHAR) and in two–step variance–correlation models (GARCH–DCC and HAR–DRD), but it is possible to add them to any GARCH or other model with a lagged term of the realized covariance (correlation) matrix (for an overview, see Bauwens et al., 2006). For example, the extension to the Markov Switching case is possible starting from the HE representation for RSDC models (Pelletier, 2006) proposed by Bauwens and Otranto (2020). The extensions to include additive jumps (as, for example, in Andersen et al., 2007) or asymmetric effects (as in Glosten et al., 1993) are trivial. Alternative parameterizations, for example, based on the Cholesky decomposition (Tsay, 2014, ch.7) could be also adopted.[6] Finally, the corresponding baseline models and the corresponding nonlinear HE specification proposed in Bauwens and Otranto (2020, 2023) can be obtained from our models by setting some coefficients equal to zero, so that they are nested in the FRC models and can be compared by LR tests. In other words, the proposed FRC models can be considered a very general class of models.

We have applied the proposed models to a dataset of 29 assets. The results show the statistical significance of the parameters that introduce the spillover and time–varying specific effects; this happens even if the difference in the time–varying coefficients is

[6] The Cholesky factorization has the advantage to guarantee the positive definiteness of the fitted covariance matrix, but the results depend on the ordering of the assets, which is in general arbitrary (see, for example, Scaffidi Domianello and Otranto, 2023).

small. Furthermore, the MCS results show a good forecasting performance of the FRC models, especially for short horizons. An interesting result, which deserves further analysis in future applied work, is that the best performing models have specific but time invariant parameters in the variance equations and time-varying specific HE parameterizations for the covariance (correlation) equations.

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Declaration of conflicting interests

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Supplementary Material: Realized Covariance Models with Time-varying Parameters and Spillover Effects

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This document contains an explanation of the notion of 'robust' standard error used in the paper, a list of matrices and operators used in the paper, the list of stocks used in the empirical application, and additional empirical results and analyses.

Robust standard errors for QML estimation

Robust standard errors are computed by using the so-called 'sandwich' matrix which is the asymptotic variance-covariance matrix of the QML estimator when the latter is asymptotically Gaussian, i.e. (under appropriate conditions) if $\hat{\boldsymbol{\theta}}$ is the QML estimator of $\boldsymbol{\theta}$, T the sample size, $\boldsymbol{I}$ the expected value of the outer product of the first derivative vector of the log-likelihood contribution $f_t(\boldsymbol{\theta})$ of observation t, $-\boldsymbol{J}$ the expected value of the second derivative matrix of the same log-likelihood contribution, then $\sqrt{T}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \sim N(0, \boldsymbol{J}^{-1}\boldsymbol{I}\boldsymbol{J}^{-1})$, where $\boldsymbol{J}^{-1}\boldsymbol{I}\boldsymbol{J}^{-1}$ is the so-called 'sandwich' matrix. Both $\boldsymbol{I}$ and $\boldsymbol{J}$ are estimated consistently by the corresponding sample averages evaluated at the QML estimate, i.e. $\boldsymbol{I} = E(\boldsymbol{s}_t(\boldsymbol{\theta})\boldsymbol{s}_t'(\boldsymbol{\theta}))$ is estimated by $\sum_{t=1}^{T} \boldsymbol{s}_t(\hat{\boldsymbol{\theta}})s_t'(\hat{\boldsymbol{\theta}}/T$, where $\boldsymbol{s}_t(\hat{\boldsymbol{\theta}}) = [\partial f_t(\boldsymbol{\theta})/\partial(\boldsymbol{\theta})]_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$, and $\boldsymbol{J} = -E(\boldsymbol{H}_t(\boldsymbol{\theta}))$ is estimated by $-\sum_{t=1}^{T} \boldsymbol{H}_t(\hat{\boldsymbol{\theta}})/T$ where $\boldsymbol{H}_t(\hat{\boldsymbol{\theta}}) = [\partial^2 f_t(\boldsymbol{\theta})/\partial(\boldsymbol{\theta})\partial(\boldsymbol{\theta}')]_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$. The robust standard error of a scalar element of $\hat{\boldsymbol{\theta}}$ is the square root of the corresponding diagonal element of the estimated sandwich matrix. We refer to White (1982) for more details.

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Definitions of matrices and operators used in the paper

$\boldsymbol{C}_t$	realized covariance matrix
$\bar{\boldsymbol{C}}$	average of sample realized covariance matrices
$\boldsymbol{H}$	matrix of zeroes on the diagonal, ones elsewhere
$\boldsymbol{I}$	identity matrix
$\boldsymbol{J}$	matrix of ones
$\boldsymbol{M}_t$	$\bar{\boldsymbol{P}}$, $\boldsymbol{P}_t$ or $\boldsymbol{R}_t$
$\boldsymbol{P}_t$	realized correlation matrix
$\bar{\boldsymbol{P}}$	average of sample realized correlation matrices
$\boldsymbol{R}_t$	conditional correlation matrix
$\boldsymbol{S}_t$	conditional covariance matrix
$\tilde{\boldsymbol{S}}_t$	diagonal matrix of the conditional variances
$\boldsymbol{V}_t$	diagonal matrix of the realized variances
$\boldsymbol{W}_t$	matrix of weights
$\boldsymbol{G}_J$	matrix with n equal columns given by the vector of sums of each row of $\boldsymbol{H} \odot \boldsymbol{J}$
$\boldsymbol{G}_{\bar{P}}$	matrix with n equal columns given by the vector of sums of each row of $\boldsymbol{H} \odot \bar{\boldsymbol{P}}$
$\boldsymbol{G}_{\bar{P}_t}$	matrix with n equal columns given by the vector of sums of each row of $\boldsymbol{H} \odot \boldsymbol{P}_t$
$\boldsymbol{G}_{\bar{R}_t}$	matrix with n equal columns given by the vector of sums of each row of $\boldsymbol{H} \odot \boldsymbol{R}_t$
$\odot$	element–by–element matrix product
$\oslash$	element–by–element matrix division
$\exp\odot$	element–by–element matrix exponential
$\mathcal{I}_t$	information set at time t

All matrices are of dimension $n \times n$.

List of stocks used in the empirical application

The stock names and tickers are: Apple Inc.(AAPL), American Express Company (AXP), The Boeing Company (BA), Caterpillar Inc. (CAT), Cisco Systems, Inc. (CSCO), Chevron Corporation (CVX), The Walt Disney Company (DIS), DowDuPont Inc. (DWDP), General Electric Company (GE), The Goldman Sachs Group, Inc. (GS), Home Depot Inc.(HD), International Business Machines Corporation (IBM), Intel Corporation(INTC), Johnson & Johnson (JNJ), JPMorgan Chase & Co. (JPM), The Coca-Cola Company (KO), McDonald's Corporation (MCD), 3M Company (MMM), Merck &Co., Inc. (MRK), Microsoft Corporation (MSFT), NIKE, Inc. (NKE), Pfizer Inc.(PFE), The Procter & Gamble Company (PG), The Travelers Companies, Inc.(TRV), United Health Group Incorporated (UNH), United Technologies Corporation (UTX), Verizon Communications Inc. (VZ), Walmart Inc. (WMT), Exxon Mobil Corporation (XOM).

Variation of the time-varying coefficients of the $BEKK - \boldsymbol{J} - \boldsymbol{R}_t$ model across stocks (supplement to Section 4.2)

The time-varying coefficient is the sum of the constant and spillover terms of the variance equation, see eq. (4.1) of the paper. Figure S1 shows the 5–number summary (minimum, first quartile, median, third quartile, maximum) of these coefficients. Most of them have a value around 0.1, with peaks between 0.8 and 0.9.

Variation of the attenuation effect of the $BEKK - \boldsymbol{J} - \boldsymbol{R}_t$ model across stocks (supplement to Section 4.2)

Figure S2 shows the Box plots of the percentage of volatility (reduced in case of negative sign) due to the attenuation effect. In the period considered, this effect was around zero, with some negative peaks. Some assets are more sensitive to the attenuation effect; in particular GS presents a volatility reduction of more than 30% on one day and between 10 and 20 percent on five other days. MRK also shows a single reduction of more than 25%. Reductions exceeding 3% affect all assets excluding JNJ and MSFT (see Table S1); JPM, GE and GS show the most frequent reductions, concentrated in particular in 2008, where we have the highest peaks of realized volatility in the time span considered.

Figure S3 shows the 5–number summary of the lagged variance coefficients, defined in equation (4.2) of the paper, of the 29 assets and the 406 lagged covariance coefficients defined in equation (4.3). The upper panel is consistent with Figure S2: the attenuation effect is null in most cases, so that the coefficient is then constant and approximately equal to the estimated a (=0.175); the variations are in correspondence with the first quarter of the distribution, with the minimum value varying as a function of the sporadic attenuation effects, and the largest variation for GS. The lower panel shows the 5-number summary of the time–varying coefficients of the lagged covariances. Most of them vary in the range 0.11–0.16. The only notable exceptions are the pairs INTC–CSCO, which ranges in 0.128–0.164, and XOM–CVX, in 0.131–0.168.

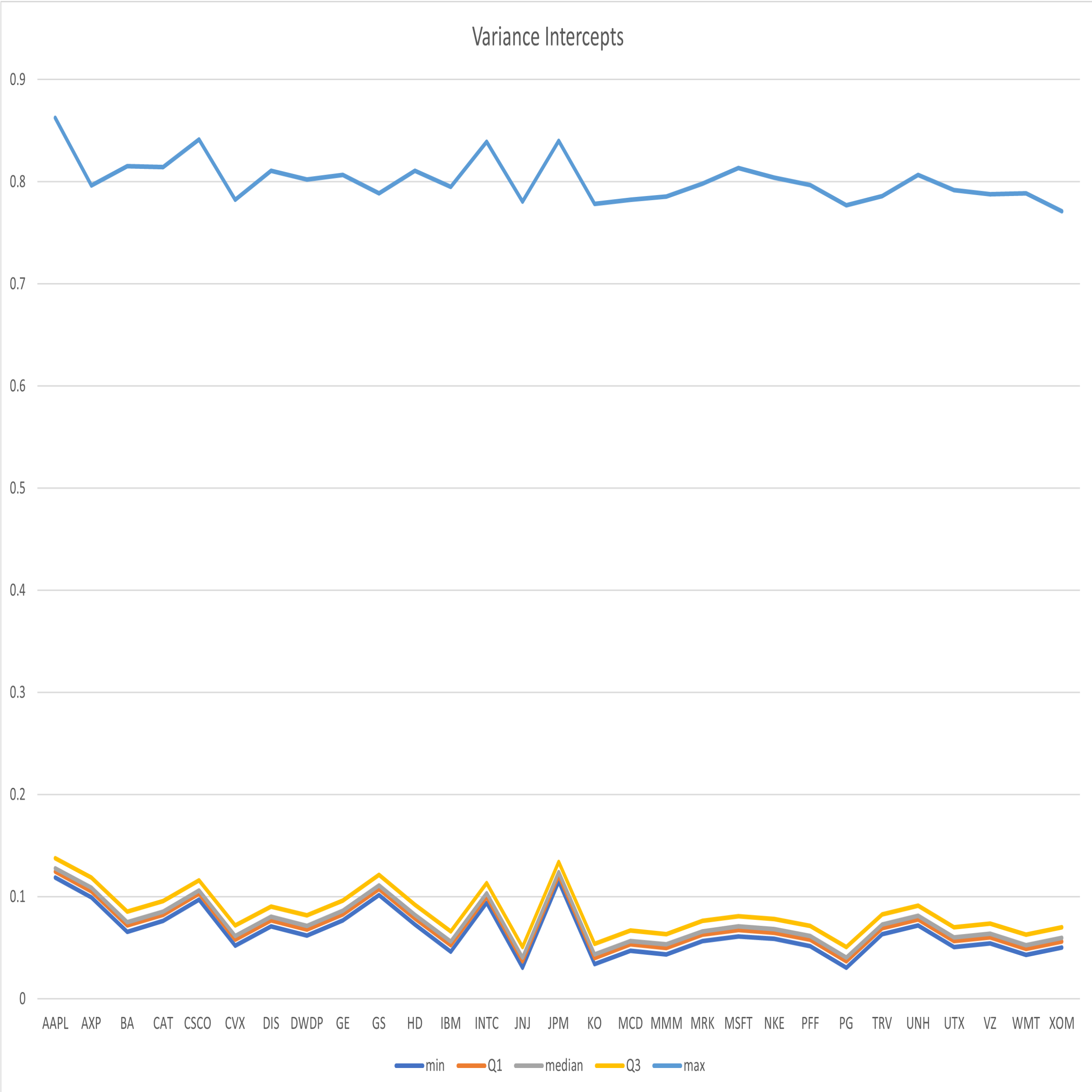

Figure S1: Model $BEKK-\boldsymbol{J}-\boldsymbol{R}_t$: 5–number summary (minimum, first quartile, median, third quartile, and maximum) of the time–varying intercepts of the realized variance of the 29 assets.

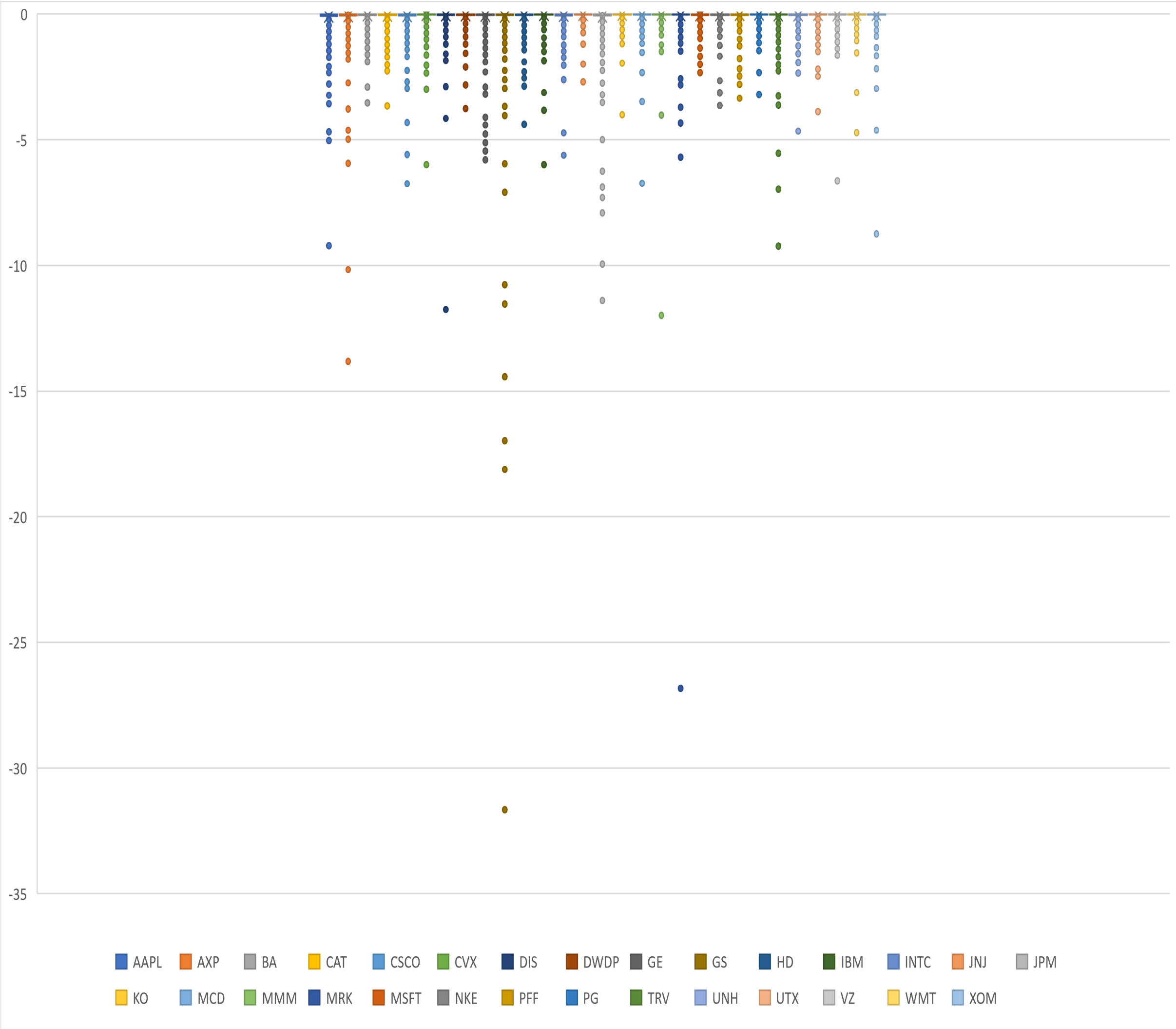

Figure S2: Model $BEKK - \boldsymbol{J} - \boldsymbol{R}_t$: Box plot of the percentage of attenuation effects of each asset on the corresponding conditional volatility

Asset	2001	2002	2005	2007	2008	2009	2010	2015	Total
AAPL	2	1	0	1	3	0	1	0	8
AXP	2	0	0	0	5	0	0	0	7
BA	0	0	0	0	0	0	1	0	1
CAT	0	0	0	0	1	0	0	0	1
CSCO	2	0	0	0	0	0	1	0	3
CVX	0	0	0	0	1	0	0	0	1
DIS	1	0	0	0	1	0	0	0	2
DWDP	0	0	0	0	1	0	0	0	1
GE	1	0	0	0	6	2	1	0	10
GS	1	0	0	0	8	0	1	0	10
HD	0	0	0	0	1	0	0	0	1
IBM	2	0	0	0	1	0	0	0	3
INTC	1	0	0	0	0	0	1	0	2
JPM	1	4	0	0	7	2	0	0	14
KO	0	0	0	0	1	0	0	0	1
MCD	0	1	0	0	1	0	0	0	2
MMM	0	0	0	0	1	0	1	0	2
MRK	1	0	2	0	2	0	1	0	6
NKE	0	0	0	0	1	0	1	1	3
PFF	0	0	0	0	1	0	0	0	1
PG	0	0	0	0	1	0	0	0	1
TRV	0	0	0	0	5	0	0	0	5
UNH	0	0	0	0	1	0	0	0	1
UTX	0	0	0	0	1	0	0	0	1
VZ	0	0	0	0	1	0	0	0	1
WMT	0	0	0	0	1	0	0	1	2
XOM	0	0	0	0	1	0	1	0	2

Table S1: Model $BEKK - \boldsymbol{J} - \boldsymbol{R}_t$: Number of days per year in which the attenuation effect reduces by more than 3% the conditional variance of each asset (for two assets, the attenuation effects are less than 3% in absolute value). For the missing years, the number of days is equal to zero for each asset.

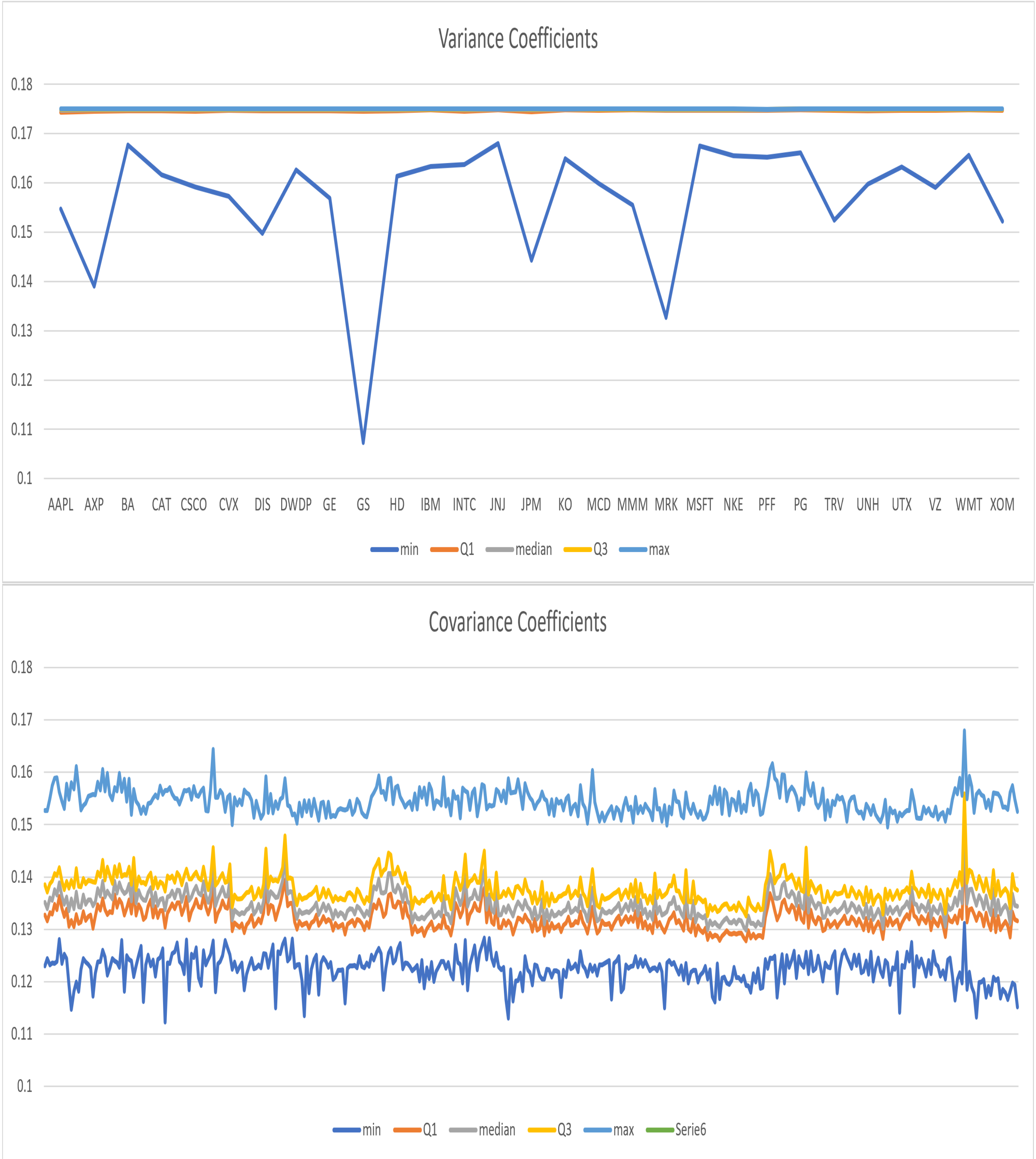

Figure S3: Model $BEKK - \boldsymbol{J} - \boldsymbol{R}_t$: 5–number summary of the time–varying coefficients of the lagged realized variance of the 29 assets (upper panel), and of the lagged 406 realized covariances (bottom panel)

Additional results on model fitting (supplement to Section 4.3)

Table S2 summarizes the behavior of the squared errors between the RV matrix and the fitted covariance matrix. Some errors are very large due to the several jumps in the returns, which cause a large variability of the RV series; consequently the model confidence set based on the FN loss function, which is an average of the squared errors, includes all the models (see Table 5 of the paper). Considering the five-number summary of the squared errors for the 8 competing models and excluding the maximum values (which correspond to outliers), we see that the FRC models have lower values of the minimum and 3 quartiles than the corresponding baseline models; the only exception is the minimum for the GARCH-DCC models.

	min	Q1	Q2	Q3	max
$BEKK-0-0$	17.01	152.42	384.55	1445.94	7.76E+06
$BEKK-\boldsymbol{J}-\boldsymbol{R}_t$	15.48	109.63	307.31	1349.51	8.36E+06
$MHAR-0-0$	17.94	152.75	374.93	1424.91	7.69E+06
$MHAR-\bar{\boldsymbol{P}}-\boldsymbol{R}_t$	14.33	112.54	309.26	1322.74	8.54E+06
$GARCH-DCC-0-0$	13.81	147.08	359.08	1329.91	7.21E+06
$GARCH-DCC-\bar{\boldsymbol{P}}-\boldsymbol{P}_t$	15.02	133.76	332.94	1273.99	7.44E+06
$HAR-DRD-0-0$	15.13	140.6	348.77	1329.44	7.75E+06
$HAR-DRD-\bar{\boldsymbol{P}}-\boldsymbol{P}_t$	12.97	127.71	333.97	1280.16	7.64E+06

Table S2: Five-number summary of the squared errors of each model for the full in-sample period.

Results for data set based on 1-minute frequency interval

For the empirical results reported in the paper, we used a data set of realized covariances based on log-returns sampled at the 5-minute frequency to avoid the so-called Epps effect (a bias towards zero in the covariation statistics, due to the greater difficulty in obtaining sufficient synchronized observations). To check the robustness of the results to the intradaily data frequency, we analyze the in- and out-of-sample performance of the FRC model for the data set based on log-returns sampled at the 1-minute frequency. The results are in Tables S3-S7, which correspond to Tables 1, 2, 3, 5 and 6 of the paper for the 5-minute case.

The results show no substantial differences in terms of outperforming models for the in-sample comparisons: LR tests (Table S3) always favor FRC models, as well as (in Tables S4-S5) AIC, TIC and BIC (except the DCC-GARCH case for the latter). In-sample (Table S6) and out-of-sample (Table S7) forecasting comparisons again show the presence of FRC models in the MCS, excluding two cases in the 22-step ahead forecasts ($QLIK$ and GMV_V).

	$0-0$		$0-\boldsymbol{M}_t$	
BEKK–$\mathbf{J}$–$\mathbf{R_t}$	336.64	(3)	41.76	(2)
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	658.14	(3)	33.42	(2)
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	183.44	(59)	180.72	(58)
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	190.30	(59)	176.00	(58)

Table S3: LR statistics to compare each row model to the same family models in columns 2 and 3. The number of restrictions to simplify the row model to the column model is indicated in parentheses.

BEKK											
$U-Z$	ϑ	β	c	a	b		ϕ	Log–Lik	AIC	TIC	BIC
$0-0$			0.011	0.274	0.855			-106810.6	57.074	57.077	57.079
			(0.001)	(0.014)	(0.012)						
$0-\boldsymbol{R}_t$			0.014	0.278	0.695		0.255	-106663.2	56.996	56.999	57.002
			(0.001)	(0.017)	(0.017)		(0.019)				
$\boldsymbol{J}-\boldsymbol{R}_t$	0.008	-3.129	0.013	0.273	0.692		0.219	-106642.3	56.985	56.988	56.995
	(0.005)	(4.491)	(0.002)	(0.032)	(0.040)		(0.048)				

MHAR											
$U-Z$	ϑ	β	c	α_D	α_W	α_M	ϕ	Log–Lik	AIC	TIC	BIC
$0-0$			0.039	0.230	0.421	0.323		-106732.2	57.032	57.035	57.039
			(0.003)	(0.012)	(0.022)	(0.018)					
$0-\boldsymbol{R}_t$			0.038	0.288	0.328	0.294	1.066	-106419.9	56.866	56.869	56.874
			(0.003)	(0.015)	(0.017)	(0.017)	(0.040)				
$\bar{\boldsymbol{P}}-\boldsymbol{R}_t$	0.025	0.016	0.036	0.264	0.326	0.290	0.905	-106403.2	56.858	56.860	56.870
	(0.005)	(0.017)	(0.003)	(0.012)	(0.016)	(0.017)	(0.059)				

Table S4: Estimates and robust standard errors in parentheses of the baseline $(0-0)$, HE $(0-\boldsymbol{R}_t)$, and selected FRC models for the in-sample period (3744 observations).

GARCH–DCC

$V-Z$	$\bar{\omega}$	$\bar{a}$	$\bar{b}$	$\bar{\vartheta}$	$\bar{\beta}$	c	a	b	ϕ	Log–Lik	AIC	TIC	BIC
hline $0-0$	0.021	0.436	0.547			0.000	0.095	0.879		-106383.4	56.892	56.886	57.042
	(0.005)	(0.043)	(0.040)			(0.000)	(0.007)	(0.009)					
$0-\boldsymbol{P}_t$	0.021	0.436	0.547			0.000	0.098	0.878	0.030	-106382.0	56.892	56.885	57.043
	(0.005)	(0.043)	(0.040)			(0.000)	(0.007)	(0.009)	(0.012)				
$\bar{\boldsymbol{P}}-\boldsymbol{P}_t$	0.018	0.385	0.525	0.067	-5.835	0.000	0.098	0.880	0.032	-106291.6	56.875	56.835	57.122
	(0.008)	(0.051)	(0.042)	(0.041)	(13.006)	(0.006)	(0.007)	(0.009)	(0.024)				

HAR–DRD

$V-Z$	$\bar{\omega}$	$\bar{a}_D$	$\bar{a}_W$	$\bar{a}_M$	$\bar{\vartheta}$	$\bar{\beta}$	c	α_D	α_W	α_M	ϕ	Log–Lik	AIC	TIC	BIC
$0-0$	0.039	0.416	0.376	0.172			0.171	0.046	0.163	0.609		-106855.4	57.160	57.143	57.360
	(0.010)	(0.043)	(0.058)	(0.050)			(0.030)	(0.005)	(0.021)	(0.068)					
$0-\boldsymbol{P}_t$	0.039	0.416	0.376	0.172			0.137	0.068	0.151	0.462	0.678	-106848.2	57.157	57.139	57.358
	(0.010)	(0.043)	(0.058)	(0.050)			(0.006)	(0.002)	(0.007)	(0.011)	(0.056)				
$\bar{\boldsymbol{P}}-\boldsymbol{P}_t$	0.035	0.321	0.351	0.161	0.113	-0.588	0.137	0.068	0.151	0.462	0.678	-106760.2	57.111	57.098	57.316
	(0.014)	(0.073)	(0.056)	(0.054)	(0.065)	(6.431)	(0.006)	(0.002)	(0.007)	(0.011)	(0.056)				

Table S5: Estimates and robust standard errors in parentheses of the baseline $(0-0)$, HE $(0-\boldsymbol{P}_t)$, and selected FRC models for the in-sample period (3744 observations). The parameters with an upper bar refer to the first step estimation of the models and the reported values are the averages of the 29 corresponding estimates of the univariate models; the corresponding value in parentheses is the standard deviation of the 29 estimates. Robust standard errors are explained in the SM file.

	QLIK	FN	GMV_S	GMV_V
BEKK–0–0	37.52	500.13	10.44	**12.46**
MHAR–0–0	37.42	487.69	10.33	**12.39**
GARCH–DCC–0-0	37.28	480.78	9.93	**12.74**
HAR–DRD–0–0	37.18	476.28	9.57	**12.76**
BEKK–$\mathbf{J}$–$\mathbf{R_t}$	37.42	481.76	**9.02**	**12.63**
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	37.22	486.46	**8.90**	**12.62**
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	37.27	**473.23**	9.89	**12.78**
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	**37.14**	**472.22**	9.56	**12.67**

Table S6: Loss function values for in-sample one-step ahead forecasts (last 1000 observations), and MCS compositions. In blue the lowest value of the loss function indicated in column; in bold the models entering the 95% confidence level MCS for each loss function.

	QLIK	FN	GMV_S	GMV_V
	1–step ahead (575 forecasts)			
BEKK–0–0	36.96	**1051.54**	10.04	**10.08**
MHAR–0–0	42.69	1318.18	19.40	**10.38**
GARCH–DCC–0-0	39.09	1346.55	7.39	11.00
HAR–DRD–0–0	36.57	**1006.31**	9.24	**10.21**
BEKK–$\mathbf{J}$–$\mathbf{R_t}$	36.85	**1019.91**	8.74	**10.23**
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	41.02	1252.24	15.84	**10.53**
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	39.78	1350.36	**6.66**	11.02
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	**36.55**	**996.83**	9.24	**10.23**
	5–step ahead (571 forecasts)			
BEKK–0–0	**38.83**	1628.30	11.29	**10.29**
MHAR–0–0	39.87	1507.46	14.14	**10.28**
GARCH–DCC–0-0	39.06	**1431.03**	7.45	**11.02**
HAR–DRD–0–0	44.52	1574.85	17.08	**10.55**
BEKK–$\mathbf{J}$–$\mathbf{R_t}$	38.94	**1470.47**	8.59	**10.63**
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	39.47	**1396.09**	9.99	**10.58**
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	39.98	**1445.19**	**6.61**	11.19
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	**38.67**	**1381.18**	10.06	**10.59**
	22–step ahead (554 forecasts)			
BEKK–0–0	41.36	1855.57	15.94	**9.50**
MHAR–0–0	45.94	2471.58	23.12	**9.43**
GARCH–DCC–0-0	**38.72**	**1416.48**	8.81	10.90
HAR–DRD–0–0	45.07	1699.46	18.16	10.28
BEKK–$\mathbf{J}$–$\mathbf{R_t}$	41.00	1464.99	**7.60**	10.55
MHAR–$\bar{\mathbf{P}}$–$\mathbf{R_t}$	44.24	1502.11	11.93	10.17
GARCH–DCC–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	39.16	**1420.59**	**7.75**	10.86
HAR–DRD–$\bar{\mathbf{P}}$–$\mathbf{P_t}$	40.31	**1464.05**	11.57	10.21

Table S7: Loss function values for out–of–sample h-step ahead forecasts, and MCS compositions. In blue the lowest value of the loss function indicated in column; in bold the models entering the 95% confidence level MCS for each loss function.