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Micromechanics of rate-independent multi-phase composites. Application to Steel Fiber-Reinforced Concrete

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*Thèse présentée en vue de l'obtention du grade
de docteur en sciences appliquées*

Juillet 2006

Remerciements

Si ce travail a pu être mené à terme, c'est en partie grâce à l'aide et la contribution de plusieurs personnes à qui je souhaite exprimer mes remerciements les plus sincères et ma grande gratitude.

Je tiens à exprimer ma profonde gratitude à mes deux promoteurs les Professeurs Issam Doghri et Jean-François Thimus qui ont accepté de diriger mes travaux de recherche. Le Professeur Issam Doghri m'a initié à la *Computational Mechanics*. Il m'a aussi, par sa rigueur scientifique et son enseignement méthodique, éduqué à la recherche scientifique. A tout moment, il n'a pas hésité à replonger dans mes équations et mes lignes de codes pour discuter des problèmes rencontrés et essayer de les résoudre. Le Professeur Jean-François Thimus m'a accueilli à mon arrivée au sein de l'unité du Génie Civil. Sans son écoute, ses encouragements, et son aide scientifique portant sur les aspects expérimentaux, cette thèse n'aurait certainement pas eu la consistance qu'elle a aujourd'hui. Chers Professeurs, chers Issam et Jean-François je vous serai éternellement reconnaissant.

Je voudrais aussi exprimer mes remerciements les plus vifs et les plus sincères au Professeur Jean-François Remacle pour tout ce qu'il m'a apporté tant sur le plan humain que scientifique. Les nombreuses discussions que j'ai eues avec lui ont toujours été très captivantes et très fructueuses. Jean-François, ta porte a toujours été ouverte pour moi, je ne te remercierai jamais assez.

Mes remerciements vont aussi au Professeur Thomas Pardoën. Toujours disponible pour répondre aux questions, j'ai beaucoup apprécié sa lecture attentive de ma thèse, ses remarques enrichissantes et constructives ainsi que sa bonne humeur.

J'aimerais remercier les Professeurs Jean-Pierre Coyette, Djimédo Kondo, Jean Claude Samin et L.J. Bert Sluys pour m'avoir fait l'honneur d'accepter de faire partie de mon jury de thèse. Leur lecture attentive de ma thèse ainsi que leurs remarques pertinentes ont, sans aucun doute, contribué à l'amélioration du texte.

J'aimerais aussi adresser mes remerciements à tout le personnel du Laboratoire du Génie Civil, et en particulier à son directeur M. Guy Henriët et à Messieurs Alex Bertholet et Eric Dupuis pour l'aide inestimable qu'ils m'ont apportée lors de la partie

expérimentale de cette thèse.

Je tiens aussi à exprimer ma gratitude au Professeur François Dupret. J'ai énormément de respect pour ce grand monsieur tant pour ses qualités scientifiques qu'humaines.

Je voudrais aussi remercier le Bureau Fédéral pour les Affaires Scientifiques, Techniques et Culturelles pour son support financier partiel par le biais du projet PAIU P5/08.

Cher Christophe, je voudrais te remercier très chaleureusement et très fraternellement pour tout. Plus qu'un ami, tu es un frère pour moi. Olivier, voyager avec toi en Pologne pour le congrès ICTAM en 2004 fut un réel plaisir. Merci pour ta jovialité et pour ton aide si précieuse notamment lors des derniers moments de la rédaction de cette thèse. Je voudrais aussi vivement remercier Omar pour nos fructueuses et enrichissantes discussions et pour sa grande disponibilité. Un grand merci à Brigitte pour son écoute et ses encouragements.

Je voudrais remercier très chaleureusement : Laurent Delannay pour sa disponibilité et sa collaboration dans cette thèse. Bernie pour les excellents moments que nous avons partagés, moments lors desquels nous avons jeté les bases d'une nouvelle théorie des poutres bien particulière. Mon ami Patrick Conde, dit "Pedro", pour son soutien, sa bonne humeur et les nombreux matchs de volley que nous avons joués ensemble. Steeve pour son aide et ses conseils. Mourad, Christophe, Gaetan, tous les Laurent, Sébastien, Emilie, Céline, Nicolas, Quentin, Louis, Benoit, Sandra, Brieux, Olivier, Sveto, Cécile, Christine, Jonathan, ainsi que tous mes autres collègues et amis du CESAME et du Vinci, pour les magnifiques moments passés ensembles.

J'aimerais adresser mes remerciements à Roger Assaker, Laurent Adam et Laurent Tinel de chez e-Xstream engineering pour leur aide et les très enrichissantes discussions que nous avons eues.

Je souhaiterais aussi remercier très chaleureusement Mesdames Marie-Rose Decelle et Viviane Delmarcelle qui, par leur constante bonne humeur, leur amabilité et leur grande gentillesse ont, sans aucun doute, contribué au bon déroulement de cette thèse. Mes remerciements vont aussi à Madame Michèle Sergeant pour sa serviabilité et son efficacité.

Pour votre indispensable soutien, pour l'amour, l'affection et l'éducation que vous m'avez donnés, je tiens, très chère Maman et très cher Papa, à vous exprimer ma profonde gratitude et ma grande reconnaissance. C'est avec un grand plaisir et une immense fierté que je vous dédie cette thèse de doctorat. Très chers parents, mille merci.

Je tiens à remercier très chaleureusement Naziha Triba pour son indéfectible soutien, ses conseils avisés et sa contribution effective à la réalisation et l'achèvement de

ce travail. Naziha, je te suis profondément reconnaissant.

Pour son soutien lors des moments difficiles et de doute que j'ai traversés durant cette thèse, ainsi que pour sa gentillesse et sa générosité, je tiens à assurer Mademoiselle Nacera Sebaa de ma grande reconnaissance et de mon grand respect.

J'aimerais aussi dédier ce travail à mes frères et soeurs, Karim, Nadia, Samir, Lina et Naim, à mes grand-parents et en particulier à "Ma" Batoul, à Adlane et Nadjema, à ma petite nièce Kamélia, ainsi qu'à toute ma famille, tantes et oncles, cousines et cousins. Je souhaiterais particulièrement remercier Alilou et Gousseem, Hmidouche et Fatima, Himi et Saadia, et Mohamed et Louisa pour le cadre familial et le soutien qu'ils m'ont offert ici en Belgique.

En ce moment si particulier, j'ai une pensée pour feu tonton Amar sans qui mon parcours n'aurait jamais été ce qu'il est aujourd'hui.

Je remercie tout particulièrement Mademoiselle Nassima Djemouai et sa maman tata Dalila, qui nous a quitté récemment, pour leur générosité, gentillesse, simplicité ainsi que pour les inoubliables moments passés en leur compagnie.

Je souhaiterais, très sincèrement, exprimer ma grande gratitude à l'Université catholique de Louvain, grande institution au sein de laquelle j'ai pu m'épanouir en tant que personne et réaliser pleinement mes objectifs scientifiques.

Enfin, je remercie le tout Haut, Allah ou Dieu, pour m'avoir permis d'aller au terme de cette formidable entreprise.

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Chapter 1

Introduction

1.1 General introduction

Innovations in engineering design establish the need for new composite materials. Composites are created by combining two or more traditional materials to produce a compound product that outperforms its individual components. Although the cost of composite materials is higher than standard materials, this is offset by the substantial advantages their properties offer users such as light weight and strength. These serviceable properties have allowed composites to break into major markets in building trade, automotive construction, and aeronautics. Many types of composites exist on the market, among the most popular families of fiber- or inclusion-reinforced composite materials, one can cite:

- Metal Matrix Composites (MMC), reinforced for example with ceramic short fibers. Applications are: aero-engines to mainly improve the resistance at high-temperature applications.
- Polymer Matrix Composites (PMC) when reinforced with glass, carbon, ceramic or Kevlar fibers. PMC are used in boat hulls, aircraft wings, etc. to improve the stiffness and strength.
- Concrete Matrix Composites (CMC). CMC can be reinforced with steel fibers to improve strength and the ductility of the composite and can be used with rubber inclusions to improve impact resistance and acoustic isolation.



Figure 1.1: Concrete: A composite material

Because of the great potential of composite materials, the scientific community as well as the industry were interested in the study of this kind of materials. Experimental, modeling and numerical simulation aspects were investigated. Many theories and approaches were proposed to model composites behavior. In this work we have focused our interest on homogenization methods based on the mean-field approach.

The aims of homogenization methods is to provide constitutive laws richer than ones identified on the macro-level. Indeed, they can take into account information concerning the microstructure such as geometrical, mechanical, and thermal properties of each micro-constituent. They can be easily implemented thus providing a powerful numerical tool for composite materials' design with optimized properties.

In this thesis, we propose a micromechanical model based on an incremental linearization of constitutive laws using tangent operators to describe material mechanical characteristics. The proposed model uses the two-step Mori-Tanaka/Voigt (M-T/V) homogenization scheme and is generic enough to handle any rate-independent constitutive law. We point out several difficulties encountered in the homogenization of nonlinear composites, and more particularly those of the (tangent) nonuniform stiffness operators which induce problems in averaging procedures and Eshelby's tensor computation. The constitutive relationships of our homogenization model are time-discretized and robust algorithms are proposed enabling simulation within reasonable accuracy, computer time and memory, of several discriminating tests of various classes of composite materials.

Two constitutive models are selected to describe the behavior at micro-level. To handle the brittle behavior of materials we use Ju's energy-based model which accounts for damage evolution only when the material is submitted to tensile strains. The latter are extracted using spectral decomposition and positive projection tensors. Ju's model de-

uses a secant operator while the incremental formulation of homogenization schemes needs tangent moduli. The latter were approximated using a perturbation method. For materials displaying an elastoplastic behavior, we select the well-known J_2 plasticity model based on an incremental formulation.

Before investigating numerical simulations, we first carried out an experimental program where four-point bending tests were performed on both Plain concrete and Steel Fiber-reinforced Concrete specimens, respectively. The aims of this test program are primarily two:

- (1) To collect the needed data for model fitting and validation. For this purpose, four-point bending test data are collected under load-deflection curves, then transformed onto stress-strain curves through a simple and original method that we propose. This method is based on an incremental formulation able to handle the nonlinear behavior of materials.
- (2) To provide us a better understanding of the physics and mechanics which govern the behavior of the material in order to achieve an appropriate modeling.

Indeed, the experimental study led to the crucial finding that initial voids growth and microcracks coalescence due to damage process, and porosity due to fiber addition must not be confused even if both cause a decrease of the composite's stiffness. Assuming that concrete material including microcracks and initial voids represents one phase (the matrix) and fibers the other one (reinforcement phase), when those two phases are mixed in order to obtain SFRC, the initial porosity of the matrix phase increases in a significant manner. Motivated by those experimental observations, we concluded that the porosity increase should be regarded as a full phase and SFRC shall therefore be modeled as a three-phase composite. The first phase consists in concrete material, microcracks and initial voids, the second phase represents fibers while the third one represents the increase in porosity. In the last part of this thesis, the efficiency of our Mori-Tanaka- Voigt (M-T/V) nonlinear homogenization model is evaluated through numerical simulations performed on several composite materials: (i) Concrete-like Matrix Composite (CMC) and (ii) Elastoplastic-Matrix Composites (EMC). For CMC we consider Steel Fiber(or Particle)-Reinforced Concrete (SFRC or SPRC) while for EMC, Metal Matrix Composites (MMC) and Polymer Matrix Composites (PMC) are considered. Predicted estimates made using our model are validated against both experimental results and Finite Element computations. F.E. analyses are performed on two 2D Unit cells (UC1 and UC2) representing a 2D simplification of a composite with a periodic micro-structure. We also developed a two-scale approach in which we have integrated our M-T/V code to Abaqus F.E. software via its User Material (UMAT) option to enable the simulation of realistic structures with complex micro-structures.

1.2 About composite materials modeling

Over the two last decades, extensive research has been conducted on the modeling and numerical simulation of composite materials and many methods have been proposed and developed. Those methods are in general based on the concept of a representative volume element (RVE) which represents the neighborhood of a material point. The RVE must include a large number of micro-constituents and be statistically representative of the local continuum properties. In other words, the RVE must capture all micro-elements that have a first-order influence on the response at the macro-scale. The RVE must also be subjected to the correct boundary conditions in order to specify the micromechanical problem. Then, the link between micro-level and macro-level state quantities is established based on suitable micro-macro relationships, such as averaging procedures in the case of the mean-field approach for instance.

A first way to model the mechanical behavior of heterogeneous materials at the macro-scale, is the use of the so-called multilevel finite element method (Figure 1.2). In this method, numerical simulations are performed simultaneously on the various scales, see for instance Feyel and Chaboche [25] and Kouznetsova et al. [45]. The main advantage of this method is that detailed microstructure information is available for the composite during the loading stage, which provides the method a great accuracy. However, in cases of complex structures with complex microstructures the method may become very time and memory-consuming, which strongly limits its range of application.

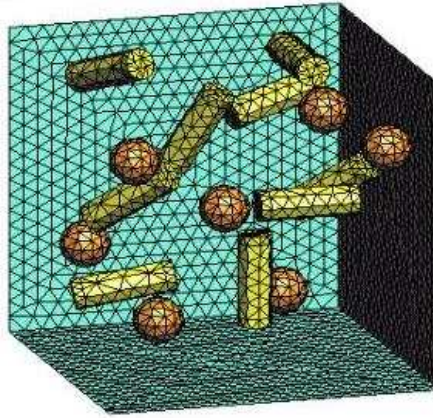


Figure 1.2: Direct F.E. computation method

Another alternative to predict the response of composites is provided by homogenization methods which can be mainly divided into two categories: (i) numerical homogenization methods and, (ii) analytical homogenization methods.

In numerical homogenization methods, the mechanical problem is solved making use of a numerical method: in most cases the finite element method is selected. A first step consists in selecting a constitutive model which is assumed to be able to describe the behavior of the composite at the macro-level. The parameters of the macro-model are then identified using numerical simulations performed on the heterogeneous RVE (which represents the microstructure). The main criticism of this method is that a constitutive model must be selected at the macro-level, which gives to the method an restrictive character.

Regarding analytical homogenization methods, a lot of work has been done on the topic. The first contributions may be due to Voigt and Reuss who assumed that strain and stress are the same for each constituent of the composite and the composite itself. The resulting overall mechanical properties, for instance the overall stiffness and compliance are expressed in terms of the average stiffness (Voigt) or compliance (Reuss) of each phase weighted by the volume fractions. Afterwards, more elaborate models were proposed leading to more accurate estimates. A first category of those models are based on Eshelby's results who resolved the problem of an ellipsoidal inclusion perfectly bonded in an infinitely extended matrix. To name but a few, dilute inclusion model, Mori-Tanaka model, Self-Consistent model, Lielens interpolative model,...etc. have been formulated in order to predict the response of various classes of composite materials. Those methods are derived within the framework of the meanfield approach and are based in linear elasticity, on the following main relationships:

- The first one is called *homogenization equation* written as:

$$\langle \sigma \rangle_\omega = \bar{C} : \langle \varepsilon \rangle_\omega \quad (1.1)$$

where $\langle \sigma \rangle_\omega$ and $\langle \varepsilon \rangle_\omega$ are the average stress and strain over the RVE ω and $\bar{C}$ is the homogenized elastic operator.

- The second one, called *localization equation*, relates the per-phase average strain $\langle \varepsilon \rangle_{\omega_r}$ to the macro average strain $\langle \varepsilon \rangle_\omega$ through the strain concentration tensor A as follows:

$$\langle \varepsilon \rangle_{\omega_r} = A : \langle \varepsilon \rangle_\omega \quad (1.2)$$

More details about the derivation of the so-called meanfield models will be given in sections 2 and 4.

Another alternative based on averaging procedures used for composite modeling, is provided by the Transformation Field Analysis. In this method, the constitutive equations of the composite and each constituent (phase) are written in terms of mechanical fields (terms due to loadings) and transformation fields (terms due to stress- or strain-free transformation). In linear elasticity this leads to:

$$\sigma_r = C_r^{el} : \varepsilon_r + \mu_r \quad (1.3)$$

where C_r^{el} is the elastic operator of the phase "r" and μ_r is a stress-free transformation strain in the phase "r". Then, the local (per-phase) strain (or stress) is related to the macro strain (or stress) through both strain (or stress) concentration tensors and transformation concentration factor tensors as follows:

$$\varepsilon_r = A : \varepsilon + \sum_{s=1}^N D_{rs} \mu_s \quad (1.4)$$

where D_{rs} are called transformation concentration tensors. They can be computed with analytical methods such as homogenization schemes (Mori-Tanaka, Self-Consistent) or with Finite Element analysis using Unit Cell models or Periodic Hexagonal Array model for instance, see Dvorak D. J. [23].

Other methods have been also proposed in the field. For fibrous composites with periodic structure or functionally graded composites for instance, the method of cells and its more elaborate version the generalized method of cells (GMS), were proposed since the 1960s, see [2] and [76]. Consider for instance a RVE of a unidirectional composite with its fibers directed in the x_1 direction. This RVE is also identified with a cell which consists in a number of subcells, each of which can have different material properties, see Figure 1.3. In the method of cells, the displacement function $u_i^{(\beta\gamma)}$ varies linearly in the subcell $(\beta\gamma)$ and can be written as:

$$u_i^{(\beta\gamma)}(x_2^\beta, x_3^\gamma) = w_i^{(\beta\gamma)}(X_2, X_3) + \Theta_i^{(\beta\gamma)} x_2^{(\beta)} + \Psi_i^{(\beta\gamma)} x_3^{(\gamma)} \quad (1.5)$$

where i is the coordinate direction ($i = 2, 3$), $w_i^{(\beta\gamma)}$ is the displacement at the center of the subcell $(\beta\gamma)$, and $\Theta_i^{(\beta\gamma)}$ and $\Psi_i^{(\beta\gamma)}$ represent gradients of the displacement across the subcell. In the subcell, the strains are related to the displacements with:

$$\varepsilon_{ij}^{(\beta\gamma)} = \frac{1}{2} \left(\partial_i u_j^{(\beta\gamma)} + \partial_j u_i^{(\beta\gamma)} \right) \quad (1.6)$$

and the average strain over the subcell surface S as:

$$\langle \varepsilon_{ij}^{(\beta\gamma)} \rangle_S = \frac{1}{S} \int_S \varepsilon_{ij}^{(\beta\gamma)} dS \quad (1.7)$$

The average strain in the subcell can be obtained in terms of the displacement variables. Traction continuity is assumed at the interface of two subcells. The displacements and tractions are assumed to be periodic at RVE's boundaries. By applying

the displacements continuity equations on an average basis between subcells, the average strains in the RVE can be expressed in terms of subcell strains. Note also that the average subcell stresses can be obtained from subcell strains using strain-stress relationships or continuity traction condition in the subcells. Thus, a relationship is obtained between the subcell stresses and the average strains in the RVE.

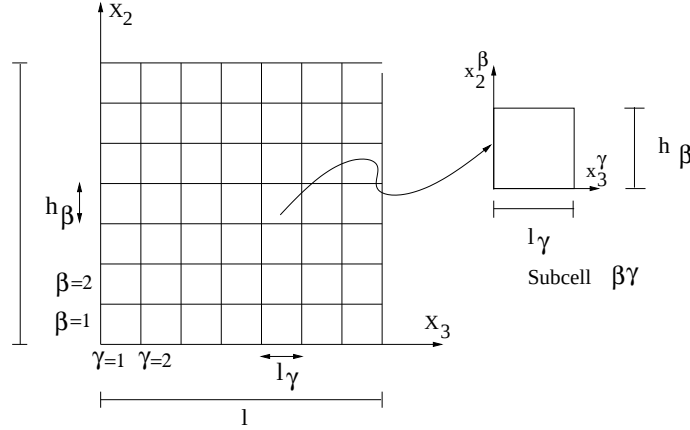


Figure 1.3: Cell Method: Repeating unit cell with subcells.

The Asymptotic Expansion Homogenization (AEH) techniques were developed for multiple scales problems since the 1970s. In this method, based on the so-called Periodic Microfield Approaches (PMA), the macroscale (Z) and microscale (z) are linked through the following relationship:

$$z_i = Z_i / \xi \quad (1.8)$$

where $\xi = l/L$ is a scaling parameter and, l and L are the characteristic lengths of micro and macro-scales respectively. In the considered unit cell (RVE) the displacement is expressed using an asymptotic expansion as follows:

$$u_i(Z, z, \xi) = u_i^{(0)} + \xi u_i^{(1)}(Z, z) + \xi^2 u_i^{(2)}(Z, z) + H.O.T., \quad (1.9)$$

where $u_i^{(0)}$ is a smooth term representing the macroscopic displacements, $u_i^{(1)}$ represents the oscillating term called the 1st order displacements perturbation due to the microstructure, $u_i^{(2)}$ the 2nd order displacements, and $H.O.T$ stands for Higher Order Terms. Then using the chain rule, and strain-displacement relationship, the strains can be expressed as follows:

$$\epsilon_{ij}(Z, z, \xi) = \epsilon_{ij}^1(Z, z) + \xi \epsilon_{ij}^2(Z, z) + H.O.T. \quad (1.10)$$

In the same manner as for strains the stresses are expanded. The equilibrium condition is then written as:

$$\left(\frac{\partial}{\partial Z_i} + \frac{1}{\xi} \frac{\partial}{\partial z_i} \right) \left[\sigma_{ij}^{(1)}(Z, z) + \xi \sigma_{ij}^{(2)}(Z, z) \right] + f_i(Z) = 0, \quad (1.11)$$

where f_i are the macroscopic body forces. The above equation leads to a hierarchical system of partial equations. The first one:

$$\frac{\partial}{\partial z_i} \sigma_{ij}^{(1)} = 0 \quad (1.12)$$

referred to as the micro-equation gives rise to a boundary value problem at the unit cell level. The second one is called the macro-equation and is written as:

$$\frac{\partial}{\partial Z_i} \sigma_{ij}^{(1)} + \frac{\partial}{\partial z_i} \sigma_{ij}^{(2)} + f_i(Z) = 0, \quad (1.13)$$

Averaging this equation over the unit cell volume, the microscopic and macroscopic fields can be related leading to following expression of the homogenized operator:

$$\bar{C}_{ijmn} = \frac{1}{V} \int_V C_{ijkl} \left(\delta_{km} \delta_{ln} + \frac{\partial}{\partial z_l} \chi_k^{mn} \right) dV \quad (1.14)$$

where V is the volume of the unit cell, δ is the Kronecker symbol, ($\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ otherwise), C is the elastic operator at the micro-level, and χ is the characteristic function (also called corrector) which describes the deformation modes of the unit cell. Note that χ can be obtained analytically or using F.E. method performed on an auxiliary problem. For more elaborate discussion on the method, see for instance [28] and [15]

When the composite behaves elastically its modeling using homogenization schemes does not present major problems. However when the microconstituents exhibit a non-linear behavior the problem becomes more complicated. Many routes have been proposed to overcome this problem. Within the framework of mean-field approach, all proposed methods are based on the concept of Linear Heterogeneous Comparison Material which is based on the following two ideas; (i) the linearization of the constitutive laws and (ii) the treatment of intra-phase heterogeneity. The main difference between all methods cited above is how they proceed for the linearization of the constitutive equations and how they tackle the problem of the intra-phase heterogeneity.

The pioneer works for nonlinear composites, were those of Hill [35] with his incremental formulation for elastoplastic polycrystals. Then, on the basis of Hill's formulation, Hutchinson [38] proposed a secant formulation for the case of steady-state creep behavior for which an incremental formulation is not required. Molinari [61] then developed the so-called tangent formulation which is a mix between the incremental and

secant formulations respectively. This method is based on a total deformation formulation (like a secant method) and used tangent operators to describe material behavior (like an incremental formulation). In 2000, Masson et al. [57] proposed the affine formulation which is an improved version of the tangent. In this method the problem is seen as a thermo-elastic one, and contrary to the tangent formulation of Molinari, the behavior at the macro-scale does not need to be known.

A generalization of the above formulations to cases of two or multi-phase composites consisting in a matrix phase in which inclusions are embedded have been proposed. To mention a few, the secant formulation was extended to elastoplastic composites by Tandon and Weng [102] and to visco-elastic composites by Friebe et al. [26]. Doghri and Ouair [21] proposed an incremental formulation for elastoplastic two-phase composite materials. An enhanced affine formulation was developed by Pierard and Doghri [73] for elasto-visco-plastic composites while Ouair et al. [68] extended the use of the incremental formulation to the case of brittle-matrix composites. In addition, variational approaches making use of effective potentials have also been developed, see Ponte Castaneda [78], and Ponte Castaneda and Suquet [80] for a general review. Those approaches provide rigorous results but are in general more difficult to derive and implement, and are restricted to some classes of material models.

In a general manner, it was observed that the analytical homogenization methods cited above (in nonlinear cases) suffer from the simplifying hypotheses made for the use of microconstituents mechanical properties (constitutive laws linearization and uniformization of operators) and to account for the geometrical arrangement of the microstructure. However, the benefits of these methods, contrary to the numerical homogenization ones, is that no assumptions have to be made for the constitutive law at the macro-scale in comparison with F.E. method for instance. The other advantage of homogenization methods is that they provide efficient macro stress-strain relationships which can be implemented in a commercial finite element software, enabling the simulation of the behavior of complex structures with complex microstructures under several discriminating loading tests.

1.3 Outline of the research

This thesis is organized in six chapters as follows:

- Chapter 2 recalls the main results of elastic two-phase composites homogenization based on the Eshelby's Equivalent Inclusion concept, within the framework of mean-field approach. Various homogenization schemes are reviewed and the problems of inclusions orientations and multi-phase composites are tackled. We

implemented several homogenization models and performed several numerical simulations. The predicted homogenization estimates were compared with finite element computations performed on unit cells.

- Chapter 3 presents the selected mechanical models to handle the behavior of composites microconstituents: (i) Ju's energy-based model for brittle materials such as concrete where damage is assumed to take place only under tensile loading and (ii) J_2 plasticity model for materials exhibiting elastoplastic behavior such as steel or polymer materials, respectively. The theories of the two models are detailed and a numerical algorithm is presented for each model.
- Chapter 4 deals with homogenization of nonlinear composites. We point out the crucial issues of the choice of the comparison reference material and the Eshelby's tensor computation. The various steps leading to the extension of the Mori-Tanaka scheme in nonlinear regime following an incremental formulation are detailed in particular. The final results for some homogenization models and the cases of multi-phase composites and inclusion orientation are also presented.
- Chapter 5 describes the experimental campaign which mainly consisted in fourth-point bending tests, performed on beams made of Plain Concrete and Steel Fiber-Reinforced Concrete materials. The acquired-results are presented under load-deflection curves and then transformed onto stress-strain curves using an original method which we have developed. The proposed method is based on an incremental formulation to be able to handle the nonlinear material behavior.
- Chapter 6 presents, under the light of Chapter 4 results, a micromechanical model and its corresponding algorithms for Steel Fiber-Reinforced Concrete in particular. Several physical and mechanical aspects such as porosity increase, voids growth, microcracks activation and fibers debonding are accounted for in the modeling. Several simulations using Abaqus F.E. software are performed on various unit cells. Our homogenization predictions are compared with F.E. and experimental results. We also present simulations performed on a realistic structure using a two-scale approach in which our homogenization code was integrated to Abaqus via its User MATerial (UMAT) option.
- Chapter 7 is devoted to the modeling and numerical simulations of a class of elasto-plastic composite materials: Metal and Polymer-Matrix Composites are both considered where the inclusions can be either solids or cavities. Computations using F.E. software are performed on various unit cells representing composites with periodic microstructure. The potential possibilities of our homogenization code are displayed through numerical simulations. Then our estimates are compared to F.E. results.
- Chapter 8 summarizes the main stages of the research and suggests some possible routes to be investigated in order to improve the present work.

Chapter 2

Homogenization of elastic composites

2.1 General Introduction

Innovations in engineering design exhibit the need for new composite materials with complex microstructures. Micromechanics' objective is to establish the bridge between the macroscopic properties of composites and their microconstituents' properties. Composite materials modeling has been the subject of the science community investigations since many decades ago. Based on scale-transition approaches many methods were developed, they can be classified into four categories.

- (1) Transformation field analysis or subcell method in which a unit cell or a representative volume element (RVE) is viewed as a number of 2D pixels or 3D voxels; see Dvorak [23].
- (2) Homogenization of RVE's or Unit cells using F.E. simulations.
- (3) Homogenization based on asymptotic expansion of the displacement field (it assumes a periodic microstructure and ends up with a unit cell problem to be solved by F.E.), see Jansson [39].
- (4) Homogenization based on mean-field approaches using averaged relationships to relate macro fields to micro ones: see for instance Lielens [54] and Pierard et al. [72].

In this thesis, we study the latter scale transition method. This choice is motivated by the fact when one is interested only in the effective properties and the per-phase averages of stress and strain, mean-field homogenization models provide by far the most cost-effective solution in terms of computer time and also ease-of-use. The models have been applied with success to a large variety of composites based on concrete, polymer, or metal matrices as we will see in chapters 6 and 7. The pioneer works on homogenization based on mean-field theory are due to Eshelby who resolved, in a rigorous manner, the problem of an ellipsoidal inclusion embedded in infinitely extended matrix by introducing the concept of the so-called eigenstrain. Then Hill formulated the dual problem by considering the suitable stress necessary called *polarization or constraint tensor* for homogenization. Then other researchers proposed various homogenization models based on Eshelby's or Hill's fundamental results. To cite but a few, Mori-Tanaka (M-T) model which is very accurate for composites with low to moderate volume fractions of inclusions, the Self-Consistent (SC) model [35] mainly used for composites with microstructure such as polycrystal, Christensen's Sphere model [14], the Double Inclusion (DI) model formulated by Nemat-Nasser [64] and more recently Lielens (L) model [54] used for composites with high volume fractions of inclusion where the problem is resolved assuming phases inversion. Other topics related to homogenization were also dealt with, one can cite the orientation problem which was investigated by Advani S. G and Tucker C. L. [3] using orientation tensors and the multi-phase composites problem where one of the proposed solution was the the multi-aggregate decomposition approach, see [54], [72] and [26] and references therein for instance.

In the following, we will focus our interest on the micromechanical modeling based on the Eshelby's equivalent inclusion problem of a class of composites consisting of a matrix phase with embedded inclusions. The developments are achieved within the framework of the mean-field theory. For the moment, only the elastic behavior is investigated.

The chapter is organized as follows. First, the concept of the Representative Volume Element (RVE) is introduced with notion of the prescribed boundary conditions and averaging procedure. Then the various steps leading to homogenization are detailed. In section 2.8 the concept of the Equivalent Inclusion is developed. In section 2.9 several homogenization schemes are reviewed. Those include the Dilute model (DM), Mori-Tanaka model (M-T), Self-Consistent model (SC), Double-Inclusion model (DI), Lielens model (L), and Voigt and Reuss models which constitute the upper and lower bounds respectively for the preceding models. Then, the problems of inclusions orientation and multi-phase composites respectively are treated following the concept of the aggregate decomposition. Finally, section 2.14 is devoted to numerical simulations where our predictions are validated against finite elements computations.

2.2 Representative volume element

The concept of the Representative Volume Element (RVE) was first introduced by Hill in 1963 and Hashin in 1964. Its aim is to relate the continuum quantities at the macro-level of a material point to the parameters that characterize its microstructure in terms of microconstituents' properties. In other words, the role of the RVE is to obtain the continuum constitutive properties in terms of microconstituents' properties. To this end, a RVE of a macro-material point must be statistically representative of the material neighborhood of that macro-material point. The volume V of the RVE must be large enough to contain many micro-constituents but small compared to any length scale over which the average stress or strain of the composite varies.

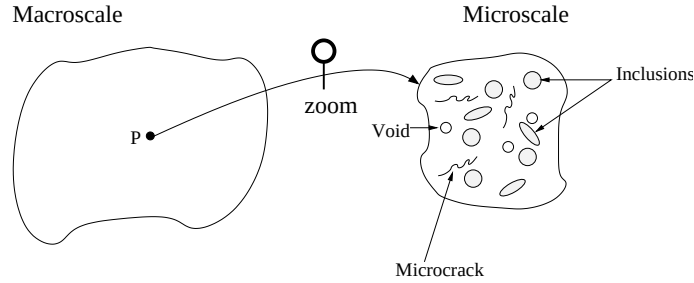


Figure 2.1: Multi-scale representation of material point

In Figure 2.1, one can see a schematic representation of a macro-material point and its complex micro-structure. Depending on the objective of the study and on the size of the continuum, RVE dimensions can considerably change. In general rule, the dimension of the RVE must be two orders of magnitude larger than the typical dimension of the greatest micro-constituent. For example, if the considered continuum medium or body is a powder-metallurgy a volume of a dimension of 100 microns would be sufficient as an RVE, whereas if the continuum medium which we want to characterize is an earth dam, the dimension of the RVE would be of order of tens meters.

2.2.1 Prescribed boundary conditions

Consider a RVE (ω) subjected to uniform boundary conditions, for instance, a uniform macro strain ϵ^0 corresponding to a uniform surface displacements u^0 such that:

$$u^0 = G^0 . x, \quad (2.1)$$

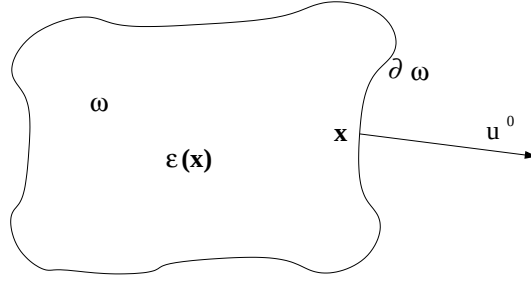


Figure 2.2: Prescribed boundary conditions: Uniform strain

and :

$$\varepsilon^0 = \frac{1}{2} (G^0 + (G^0)^T), \quad (2.2)$$

where G^0 is the macro-displacement gradient. Note that G^0 is not necessary symmetric. (Figure 2.2). The resulting strain and stress fields within the RVE are not necessary uniform. Consequently it is interesting to express those fields in term of average quantities. This will enable to completely describe the nonuniform strain and stress fields at the micro-level by the boundary conditions ε^0 . The average strain over the volume V of the RVE ω is given by:

$$\langle \varepsilon \rangle_\omega = \frac{1}{V} \int_\omega \varepsilon(x) dV. \quad (2.3)$$

Using strain-displacement relationship and Gauss theorem, the last equation is rewritten as:

$$\begin{aligned} \frac{1}{V} \int_\omega \varepsilon_{ij} dV &= \frac{1}{V} \int_\omega \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dV = \frac{1}{V} \int_{\partial\omega} \frac{1}{2} (u_i n_j + u_j n_i) ds \\ &= \frac{1}{V} \left\{ \frac{1}{2} \left[\varepsilon_{ik}^0 \int_{\partial\omega} x_k n_j ds + \varepsilon_{jk}^0 \int_{\partial\omega} x_k n_i ds \right] \right\}. \end{aligned} \quad (2.4)$$

From Gauss theorem we have:

$$\int_{\partial\omega} x_k n_i ds = \int_\omega \frac{\partial x_k}{\partial x_i} dV = V \delta_{kj}, \quad (2.5)$$

where δ_{kj} is Kronecker symbol, $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ otherwise. Substituting Eq.(2.5) into Eq.(2.4) then into Eq.(2.3) one obtains:

$$\langle \varepsilon \rangle_\omega = \varepsilon^0. \quad (2.6)$$

On the other hand if the RVE is subjected to the uniform macro stress σ^0 corresponding to a uniform boundary traction $t^0 = \sigma^0 \cdot n$ (figure 2.3), the nonuniform micro-stress

$\sigma(x)$ within the RVE can be related to σ^0 by averaging stress over the volume V of the RVE as follows:

$$\langle \sigma \rangle_\omega = \frac{1}{V} \int_\omega \sigma(x) dV. \quad (2.7)$$

Since the stress tensor is divergence-free ($\text{div} \sigma = 0$, the body forces are assumed to act at the macro level) and using Gauss theorem, the last equation is developed as follows:

$$\begin{aligned} \frac{1}{V} \int_\omega \sigma_{ij} dV &= \frac{1}{V} \int_{\partial\omega} \sigma_{ik} x_j n_k ds, \\ &= \frac{1}{V} \sigma_{ik}^0 \int_{\partial\omega} x_j n_k ds, \end{aligned} \quad (2.8)$$

since on the boundary surface $\partial\omega$ $\sigma \cdot n = \sigma^0 \cdot n$. Replacing Eq.(2.5) into Eq.(2.8) than into Eq.(2.7), leads to:

$$\langle \sigma \rangle_\omega = \sigma^0. \quad (2.9)$$

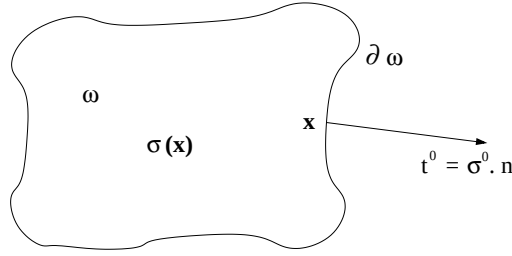


Figure 2.3: Prescribed boundary conditions: Uniform stress

2.2.2 Stress and strain averages

Consider a two-phase composite material consisting of a matrix phase ω_0 in which inclusions ω_1 having the same shape, material constituent and orientation are randomly embedded. Consider a RVE (ω) representing a material point of the above composite, Figure 2.4. The volume-average stress $\langle \sigma \rangle_\omega$ is defined as the average of the

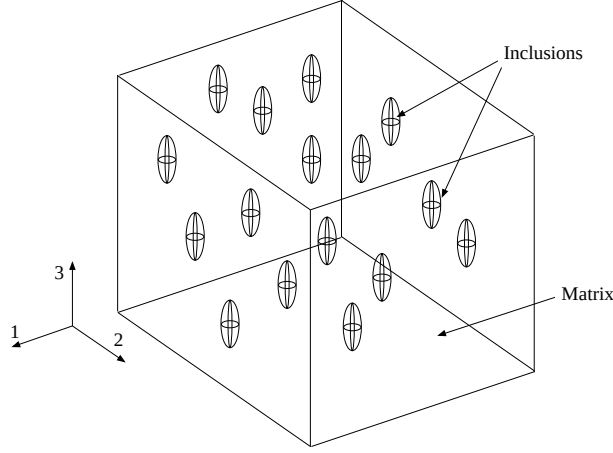


Figure 2.4: A 3D Representative Volume Element of a two-phase composite

pointwise stress $\sigma(x)$ over the volume V of the RVE:

$$\begin{aligned} \langle \sigma \rangle_{\omega} &\equiv \frac{1}{V} \int_{\omega} \sigma(x) dV \\ &= \frac{1}{V} \left[\underbrace{\int_{\omega_1} \sigma(x) dV_1}_{V_1 \langle \sigma \rangle_{\omega_1}} + \underbrace{\int_{V_0} \sigma(x) dV_0}_{V_0 \langle \sigma \rangle_{\omega_0}} \right], \end{aligned} \quad (2.10)$$

where V_0 and V_1 are the matrix and inclusion phases' volumes respectively. The RVE averaged macro stress is then written in terms of matrix and inclusion average stresses $\langle \sigma \rangle_{\omega_0}$ and $\langle \sigma \rangle_{\omega_1}$ respectively, one obtains:

$$\langle \sigma \rangle_{\omega} = v_1 \langle \sigma \rangle_{\omega_1} + v_0 \langle \sigma \rangle_{\omega_0}, \quad (2.11)$$

where $v_1 = V_1/V$ and $v_0 = V_0/V$ are the volume fractions of the inclusion and matrix phases respectively, with $v_1 + v_0 = 1$. In the same way, the RVE averaged macro strain is derived, the result is:

$$\begin{aligned} \langle \varepsilon \rangle_{\omega} &\equiv \frac{1}{V} \int_{\omega} \varepsilon(x) dV \\ &= \frac{1}{V} \left[\underbrace{\int_{\omega_1} \varepsilon(x) dV_1}_{V_1 \langle \varepsilon \rangle_{\omega_1}} + \underbrace{\int_{\omega_0} \varepsilon(x) dV_0}_{V_0 \langle \varepsilon \rangle_{\omega_0}} \right], \end{aligned} \quad (2.12)$$

which can be written as:

$$\langle \varepsilon \rangle_{\omega} = v_1 \langle \varepsilon \rangle_{\omega_1} + v_0 \langle \varepsilon \rangle_{\omega_0}, \quad (2.13)$$

where $\langle \varepsilon \rangle_{\omega_0}$ and $\langle \varepsilon \rangle_{\omega_1}$ are the average strains over the matrix and inclusion phases respectively.

2.3 Hill-Mandel condition

Let σ^* be a self-equilibrating stress field such as:

$$\text{div}(\sigma^*) = 0, \quad \sigma_{ij}^* = \sigma_{ji}^*, \quad (2.14)$$

and which satisfies the uniform boundary traction conditions on $\partial\omega$ and ε^* a kinematically admissible strain field which satisfy the linear boundary displacement conditions (see section 2.2.1). Under those conditions we have the following remarkable equality:

$$\langle \sigma^* : \varepsilon^* \rangle = \langle \sigma^* \rangle : \langle \varepsilon^* \rangle, \quad (2.15)$$

which is called Hill's macro-homogeneity condition or Hill-Mandel condition. This relationship is valid for materials of any constitutive properties, σ^* and ε^* must not be necessarily related, only equilibrium and compatibility are required. Hill's condition is very useful and plays a major role in homogenization theory.

2.4 Effective elastic moduli

In this section we derive the expression of effective elastic moduli for composite materials. First of all, define the strain concentration tensor $\mathbb{A}$ which relates the local strain ε to macro one $\bar{\varepsilon}$ such as:

$$\varepsilon = \mathbb{A} : \bar{\varepsilon}, \quad (2.16)$$

and define the tensor $\mathbb{B}$ which relates the local stress σ to macro one $\bar{\sigma}$ as:

$$\sigma = \mathbb{B} : \bar{\sigma}. \quad (2.17)$$

The two above equations are called *localization* relationships. Other concentration tensors will be defined next. Note that (see [8]):

$$\langle \mathbb{A} \rangle = I; \quad \text{and,} \quad \langle \mathbb{B} \rangle = I, \quad (2.18)$$

The effective elastic stiffness tensor is defined as the tensor which relates the strain and stress averages such as:

$$\begin{aligned} \bar{\sigma} = \langle \sigma \rangle &= \bar{C}^{el} : \langle \varepsilon \rangle, \\ &= \bar{C}^{el} : \bar{\varepsilon}, \end{aligned} \quad (2.19)$$

and the effective elastic compliance tensor $\bar{D}$ such as:

$$\begin{aligned}\bar{\varepsilon} = \langle \varepsilon \rangle &= \bar{D}^{el} : \langle \sigma \rangle, \\ &= \bar{D}^{el} : \bar{\sigma}.\end{aligned}\quad (2.20)$$

The local behavior is written in terms of the local elastic tensor C^{el} as:

$$\sigma = C^{el} : \varepsilon, \quad (2.21)$$

and using Eq.2.16 as:

$$\sigma = C^{el} : \mathbb{A} : \bar{\varepsilon}, \quad (2.22)$$

which averaged over the total volume gives:

$$\begin{aligned}\langle \sigma \rangle &= \langle C^{el} : \mathbb{A} : \bar{\varepsilon} \rangle, \\ &= \langle C^{el} : \mathbb{A} \rangle : \bar{\varepsilon}.\end{aligned}\quad (2.23)$$

Then, identification of Eq.2.23 with Eq.2.19 gives the expression of the effective elastic stiffness tensor. The result is:

$$\bar{C}^{el} = \langle C^{el} : \mathbb{A} \rangle. \quad (2.24)$$

The local behavior given by Eq.(2.21), can be expressed in terms of the local elastic compliance D^{el} as follows:

$$\varepsilon = D^{el} : \sigma, \quad (2.25)$$

and using Eq.2.17 as:

$$\varepsilon = D^{el} : \mathbb{B} : \bar{\sigma}, \quad (2.26)$$

which averaged over the total volume gives:

$$\begin{aligned}\langle \varepsilon \rangle &= \langle D^{el} : \mathbb{B} : \bar{\sigma} \rangle, \\ &= \langle D^{el} : \mathbb{B} \rangle : \bar{\sigma}.\end{aligned}\quad (2.27)$$

Identification of Eq.2.27 with Eq.2.20 gives the expression of the effective elastic compliance tensor:

$$\bar{D}^{el} = \langle D^{el} : \mathbb{B} \rangle. \quad (2.28)$$

It is interesting to note that if $\bar{C}^{el}$ and $\bar{D}^{el}$ are computed making the same boundary conditions choice (equivalence between the prescribed stress and prescribed strain conditions), one is then the inverse of the other one [8]. Otherwise, if different boundary conditions are used, we have according to Hill [36] and Nemat-Nasser [64] the following result:

$$\bar{C}^{el} : \bar{D}^{el} = I + O((d/l)^2), \quad (2.29)$$

where d is a characteristic size of the inhomogeneity and l the typical size of the RVE.

2.5 Strain and stress concentration tensors

Two strain concentration tensors are defined. The first one, designated by A , relates inclusion average strain to heterogeneous RVE average strain as follows:

$$\langle \varepsilon \rangle_{\omega_1} = A : \langle \varepsilon \rangle_{\omega} . \quad (2.30)$$

The second strain concentration tensor, which we will note $\tilde{A}$, relates inclusion and matrix phases average strain:

$$\langle \varepsilon \rangle_{\omega_1} = \tilde{A} : \langle \varepsilon \rangle_{\omega_0} . \quad (2.31)$$

$\tilde{A}$ and A are fourth-rank tensors, they generally depend on the constitutive properties of the matrix and inclusion phases and on the geometrical characteristics of the inclusions. Their expressions vary following the chosen homogenization scheme.

The strain concentration tensors $\tilde{A}$ and A can be linked as follows. Isolating $\langle \varepsilon \rangle_{\omega_0}$ in the Eq.(2.13) and using it in Eq.(2.31) one obtains:

$$\begin{aligned} \langle \varepsilon \rangle_{\omega_1} &= \tilde{A} : \frac{\langle \varepsilon \rangle_{\omega} - v_1 \langle \varepsilon \rangle_{\omega_1}}{1 - v_1}, \\ &= [v_1 I^4 + (1 - v_1) \tilde{A}^{-1}]^{-1} : \langle \varepsilon \rangle_{\omega}. \end{aligned} \quad (2.32)$$

then identification with Eq.(2.30) leads to:

$$A = [v_1 I^4 + (1 - v_1) \tilde{A}^{-1}]^{-1}. \quad (2.33)$$

It is worth to note that A (Eq.2.30) and $\tilde{A}$ (Eq.2.16) are different. The first one relates the inclusion average strain to composite average strain, while the second one relates the local inclusion strain to the composite average strain which is the prescribed farfield strain. $\mathbb{A}$. In the same way as for strain concentration tensors, stress concentration tensors can be defined. One can define B as:

$$\langle \sigma \rangle_{\omega_1} = B : \langle \sigma \rangle_{\omega}, \quad (2.34)$$

and $\tilde{B}$ as:

$$\langle \sigma \rangle_{\omega_1} = \tilde{B} : \langle \sigma \rangle_{\omega_0}. \quad (2.35)$$

2.6 Homogenized elastic operator

The aim of this section is to derive, in a simple way, the overall expression of the homogenized elastic operator for a two-phase composite. For this purpose, first select

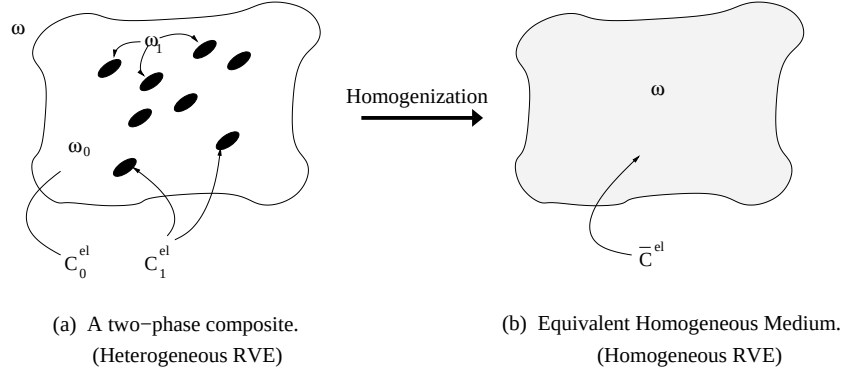


Figure 2.5: Homogenization procedure

an RVE (Figure 2.5-a) and consider its Equivalent Homogeneous Medium (EHM) which has the same mechanical properties everywhere in ω as displayed in Figure 2.5-b. In the EHM, the average stress and strain tensors are related through the fourth-rank tensor $\bar{C}^{el}$, called homogenized elastic operator, as follows:

$$\langle \sigma \rangle_{\omega} = \bar{C}^{el} : \langle \varepsilon \rangle_{\omega}. \quad (2.36)$$

Now, come back to the original RVE (Figure 2.5-a) and assume that each phase of this RVE obeys Hooke's law. Eq.(2.11) is then rewritten as:

$$\begin{aligned} \langle \sigma \rangle_{\omega} &= v_1 \langle C_1^{el} : \varepsilon \rangle_{\omega_1} + (1 - v_1) \langle C_0^{el} : \varepsilon \rangle_{\omega_0}, \\ &= v_1 C_1^{el} : \langle \varepsilon \rangle_{\omega_1} + (1 - v_1) C_0^{el} : \langle \varepsilon \rangle_{\omega_0}, \end{aligned} \quad (2.37)$$

where C_0^{el} and C_1^{el} are the uniform elastic operators of matrix and inclusion phases respectively. Replacing (2.30) into (2.13), the average strain in matrix phase is then given by:

$$\langle \varepsilon \rangle_{\omega_0} = \frac{\langle \varepsilon \rangle_{\omega} - v_1 A : \langle \varepsilon \rangle_{\omega}}{1 - v_1}. \quad (2.38)$$

Substituting Eqs.(2.38)-(2.30) into Eq.(2.37) and rearranging leads to:

$$\langle \sigma \rangle_{\omega} = \left[C_0^{el} + v_1 (C_1^{el} - C_0^{el}) : A \right] : \langle \varepsilon \rangle_{\omega}. \quad (2.39)$$

The mechanical equivalence imposes that the average stress in the original RVE (Figure 2.5-a, and Eq.2.39) must be equal to the average stress in the EHM (Figure 2.5-b and (Eq.2.36)). It follows that:

$$\left[C_0^{el} + v_1 (C_1^{el} - C_0^{el}) : A \right] : \langle \varepsilon \rangle_{\omega} = \bar{C}^{el} : \langle \varepsilon \rangle_{\omega}, \quad (2.40)$$

from which the expression of the homogenized elastic operator $\bar{C}^{el}$ of a two-phase composite is deduced:

$$\bar{C}^{el} = C_0^{el} + v_1(C_1^{el} - C_0^{el}) : A. \quad (2.41)$$

One can see from the above equation that the elastic homogenized operator $\bar{C}^{el}$ depends on the choice of the homogenization scheme through the strain concentration tensor A . In section 2.9 several models are proposed to determine A .

2.7 Eshelby's problem

Consider an infinitely extended unloaded domain ω consisting of a matrix material ω_0 in which is embedded a spheroidal inclusion ω_1 . Both of the matrix and inclusion are made of with the same constitutive material characterized by the elastic operator C^{el} . As displayed in Figure 2.6-a, stress and strain fields are zero in the matrix and inclusion respectively. Now imagine that the inclusion is extracted from the matrix and is subjected to a strain field $\varepsilon^*(x)$, Figure 2.6-b. In this case the inclusion can deform freely and the strain and stress are then:

$$\varepsilon_1 = \varepsilon^*, \quad (2.42)$$

$$\sigma_1 = 0. \quad (2.43)$$

However, because of the presence of surrounding matrix material (Figure 2.6-c), when the inclusion is subjected to the eigenstrain $\varepsilon^*(x)$ it can not deform freely. The Matrix material constrains the inclusion which transfers a part of $\varepsilon^*(x)$ to the matrix. The strain and stress fields in the matrix are:

$$\varepsilon_0 \neq 0, \quad (2.44)$$

$$\sigma_0 = C^{el} : \varepsilon_0, \quad (2.45)$$

and in the inclusion:

$$\varepsilon_1 \neq 0, \quad (2.46)$$

$$\sigma_1 = C^{el} : (\varepsilon_1 - \varepsilon^*). \quad (2.47)$$

The problem now is to relate the eigenstrain $\varepsilon^*(x)$ to the strain ε_1 inside the inclusion. In his paper [24], Eshelby showed that if:

- the inclusion is an ellipsoid, and
- ω_0 is a homogeneous and linearly elastic

then:

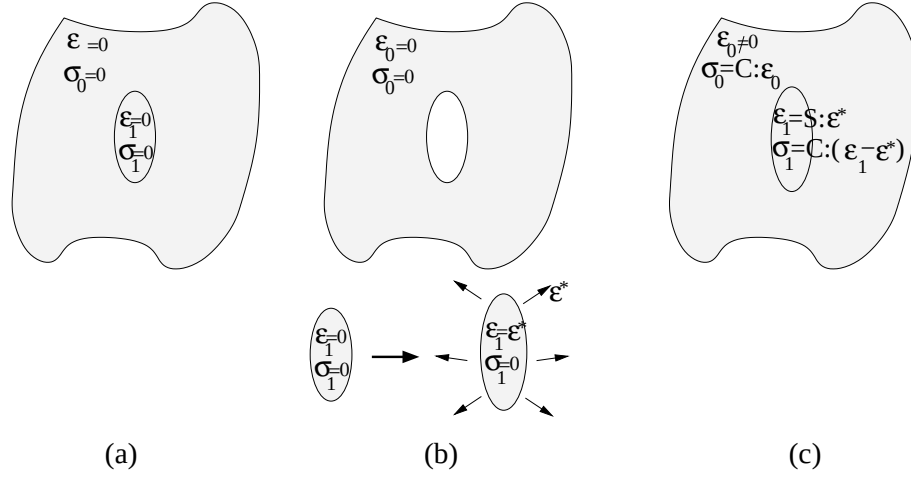


Figure 2.6: Eshelby's Homogeneous Problem

- ε^* is uniform in the inclusion and,
- the strain ε_1 in ω_1 is related to the eigenstrain ε^* through a fourth-rank tensor as:

$$\varepsilon_1 = S^{Es} : \varepsilon^*, \quad (2.48)$$

where S^{Es} is called Eshelby's tensor and has the following properties:

- S^{Es} has the following symmetries:

$$S_{ijkl}^{Es} = S_{jikl}^{Es} = S_{ijlk}^{Es}. \quad (2.49)$$

- when the matrix phase is isotropic and the inclusion is a spheroid, S^{Es} is completely defined in terms of the inclusion geometry through the aspect ratio Ar and the mechanical properties of matrix phase through its Poisson's ratio ν_0

Up to now, this result has a major impact on the micromechanical modeling and plays a key role in homogenization models proposed by researchers in the field. In Appendix C, the components of Eshelby's tensor are given for a spheroid embedded in an isotropic matrix or transversely isotropic matrix. For the simplicity of notation, the subscript "Es" will be omitted.

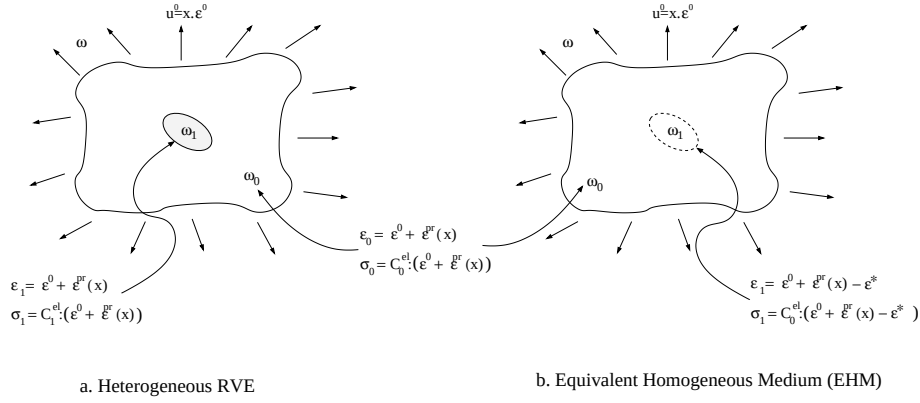


Figure 2.7: Eshelby's Isolated Inclusion Problem

2.8 Isolated Inclusion Problem

Consider a finite solid ω consisting of a matrix phase ω_0 with elastic operator C_0^{el} , and a single inclusion ω_1 of elastic operator C_1^{el} , Figure 2.7-a. Assume, for instance, that the solid ω is subjected to a far-field uniform strain ε^0 corresponding to linear surface displacements. If the inclusion were made of the same constitutive material as the matrix, the solid would be then totally homogeneous. In this case, the strain field and its corresponding stress field are uniform within both the two phases ω_1 and ω_0 respectively. Anywhere in the RVE (ω), the strain and stress fields are:

$$\begin{cases} \varepsilon(x) = \varepsilon^0, & \forall x \in \omega \\ \sigma(x) = C_0^{el} : \varepsilon^0, & \forall x \in \omega \end{cases} \quad (2.50)$$

However, the presence of an inclusion made of different material than the matrix disturbs the strain field and the stress field within the volume ω . Consequently, in the matrix phase, the strain and stress fields are then expressed as:

$$\begin{cases} \varepsilon(x) = \varepsilon^0 + \varepsilon_0^{pr}(x), & \forall x \in \omega_0 \\ \sigma(x) = C_0^{el} : (\varepsilon^0 + \varepsilon_0^{pr}(x)), & \forall x \in \omega_0 \end{cases} \quad (2.51)$$

while in the inclusion phase they are expressed as:

$$\begin{cases} \varepsilon(x) = \varepsilon^0 + \varepsilon_1^{pr}(x), & \forall x \in \omega_1 \\ \sigma(x) = C_1^{el} : (\varepsilon^0 + \varepsilon_1^{pr}(x)), & \forall x \in \omega_1 \end{cases} \quad (2.52)$$

where $\varepsilon_0^{pr}(x)$ and $\varepsilon_1^{pr}(x)$ represents the perturbation strain in the matrix and inclusion phases due to the presence of the inclusion (ω_1).

Now, our objective is to construct a homogeneous body which is, from the mechanical point of view, equivalent to the heterogeneous two-phase RVE. This homogeneous body is obtained by homogenizing the solid ω . For this purpose, we introduce a suitable strain field $\varepsilon^*(x)$, called eigenstrain, in the inclusion phase ω_1 in order to have a uniform stiffness everywhere in ω . The homogenizing strain $\varepsilon^*(x)$ is as follows:

$$\varepsilon^*(x) = \begin{cases} 0, & \forall x \in \omega_0 \\ \varepsilon^*(x), & \forall x \in \omega_1 \end{cases} \quad (2.53)$$

The strain field in both matrix and inclusion phases becomes:

$$\varepsilon(x) = \begin{cases} \varepsilon^0 + \varepsilon_0^{pr}(x), & \forall x \in \omega_0 \\ \varepsilon^0 + \varepsilon_1^{pr}(x) - \varepsilon^*(x), & \forall x \in \omega_1 \end{cases} \quad (2.54)$$

and taking advantage of Hooke's law, stress field is written as:

$$\sigma(x) = \begin{cases} C_0^{el} : (\varepsilon^0 + \varepsilon_0^{pr}(x)), & \forall x \in \omega_0 \\ C_0^{el} : (\varepsilon^0 + \varepsilon_1^{pr}(x) - \varepsilon^*(x)), & \forall x \in \omega_1 \end{cases} \quad (2.55)$$

From a mechanical point of view, stresses must be the same in the inclusion zone of original heterogeneous RVE and in the inclusion zone of the homogeneous equivalent medium. This condition is called *consistency condition* or *mechanical equivalence*. The fulfillment of the consistency condition allows to determine the eigenstrain ε^* necessary for the homogenization. Taking advantage of Eshelby's results given by Eq.(2.48), consistency condition is written as:

$$C_1^{el} : (\varepsilon^0 + \underbrace{S_{I,C_0^{el}} : \varepsilon^*}_{\varepsilon_1^{pr}(x)}) = C_0^{el} : (\varepsilon^0 + \underbrace{S_{I,C_0^{el}} : \varepsilon^* - \varepsilon^*}_{\varepsilon_1^{pr}(x)}), \quad (2.56)$$

which leads to:

$$\varepsilon^* = -(C_0^{el} - S_{I,C_0^{el}} : C_0^{el} + S_{I,C_1^{el}} : C_1^{el})^{-1} : (C_1^{el} - C_0^{el}) : \varepsilon^0, \quad (2.57)$$

$$= -[(C_0^{el})^{-1} : C_1^{el} - I^4]^{-1} + S_{I,C_0^{el}})^{-1} : \varepsilon^0. \quad (2.58)$$

For all homogenization schemes that we will derive below we will follow the same step as the above-cited problem.

2.9 Some homogenization schemes

This section is devoted to the most popular homogenization schemes based on Eshelby's Equivalent Inclusion Problem. For each homogenization model studied, computation of strain concentration tensors, which constitute the signature of the considered homogenization scheme, is detailed. For all the models reviewed below, we consider a two phase composite consisting of a matrix phase and inclusions made of the

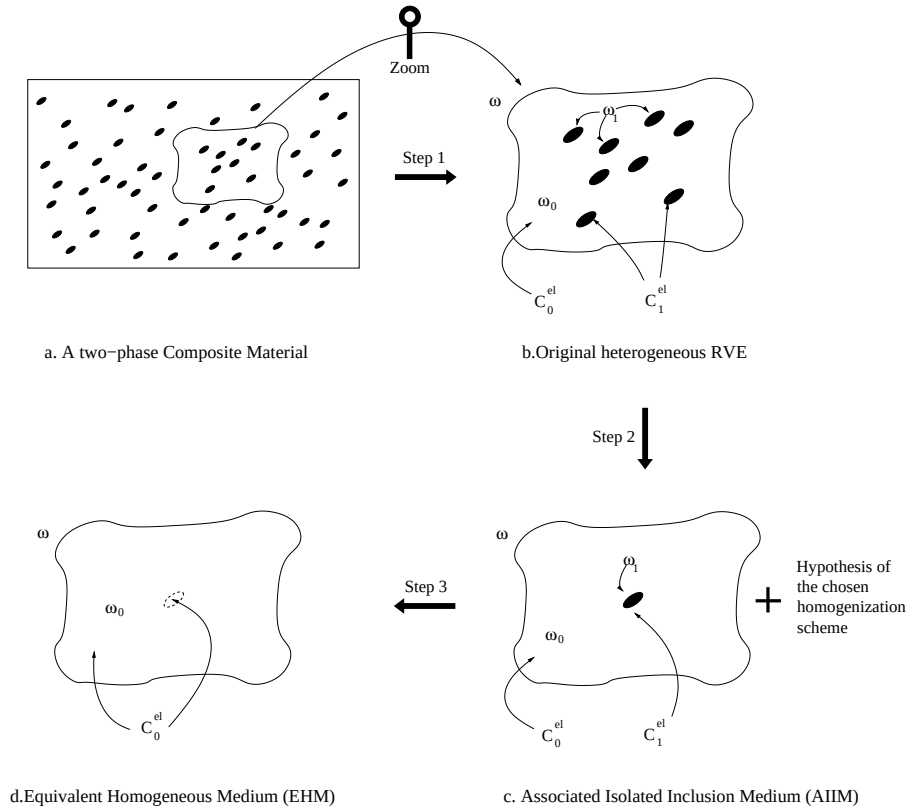


Figure 2.8: Homogenization process to obtain strain concentration tensors.

same constitutive material and similarly shaped and oriented, Figure 2.8-a. The first step consists in choosing an RVE (Figure 2.8-b) for the composite. Then we associate to this RVE a substitution RVE representing the Isolated Inclusion Medium (Associated Isolated Inclusion Medium, AIIM, Figure 2.8-c) according to the hypothesis made by the chosen homogenization scheme. In step 3, the mechanical equivalence between the AIIM and its Equivalent Homogeneous Medium (Figure 2.8-d) is established making use of Eshelby's results.

2.9.1 Dilute Inclusions model

Consider the RVE (ω) represented in Figure 2.9-a. The composite studied here consists in a matrix phase in which inclusions with a low volume fraction are embedded. In this case, the interaction between the inclusions can be neglected. According to this hypothesis, we consider a substitution RVE representing an Associated Isolated Inclusion Medium (AIIM) as displayed in Figure 2.9. In the matrix phase, the strain

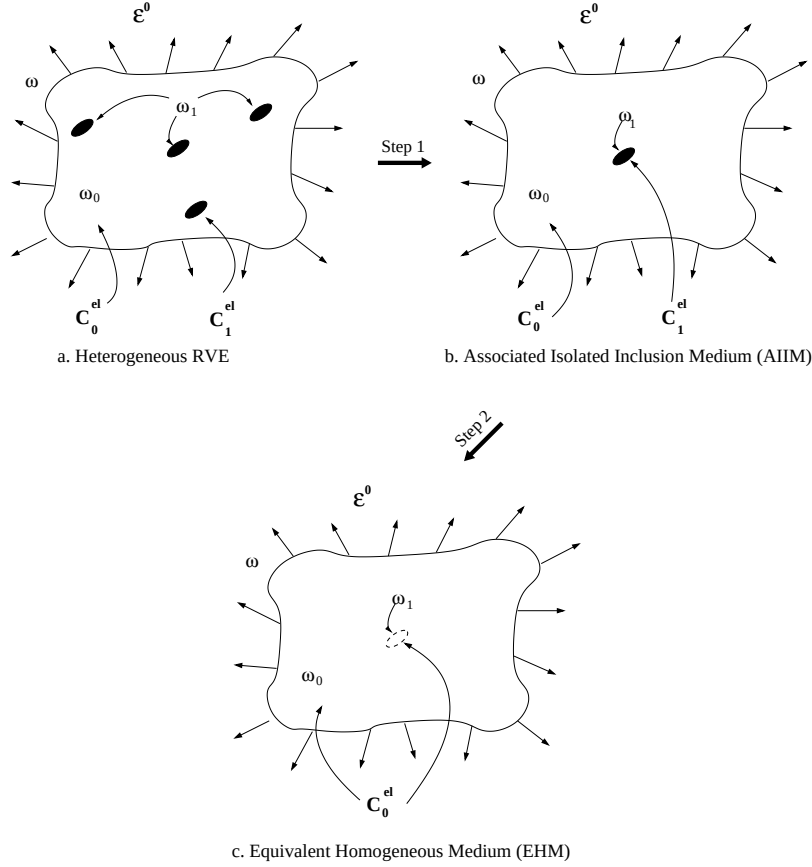


Figure 2.9: Schematic representation of Dilute model hypothesis

and stress fields are:

$$\varepsilon(x) = \varepsilon^0 + \varepsilon'(x), \quad (2.59)$$

$$\sigma(x) = C_0^{el} : (\varepsilon^0 + \varepsilon'(x)), \quad (2.60)$$

where $\varepsilon'(x)$ is the perturbation strain inside the matrix due to the presence of the inclusion. In the inclusion the strain and stress are:

$$\varepsilon(x) = \varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^*, \quad (2.61)$$

$$\sigma(x) = C_1^{el} : (\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^*), \quad (2.62)$$

while in the inclusion zone of the equivalent homogeneous medium they become:

$$\varepsilon(x) = \varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* - \varepsilon^*, \quad (2.63)$$

$$\sigma(x) = C_0^{el} : (\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* - \varepsilon^*). \quad (2.64)$$

The mechanical equivalence imposes that stresses are the same for the inclusion of the original RVE and for the inclusion zone of the equivalent homogeneous medium, so we have:

$$C_1^{el} : (\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^*) = C_0^{el} : (\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* - \varepsilon^*), \quad (2.65)$$

the eigenstrain strain ε^* necessary for the homogenization is then deduced as:

$$\varepsilon^* = -([C_0^{el}]^{-1} : C_1^{el} - I^4)^{-1} + S_{I,C_0^{el}})^{-1} : \varepsilon^0. \quad (2.66)$$

Substituting the value of ε^* into the inclusion strain expression (Eq2.61) and averaging over inclusion material leads to:

$$\begin{aligned} \langle \varepsilon \rangle_{\omega_1} &= \langle \varepsilon^0 - S_{I,C_0^{el}} : ([C_0^{el}]^{-1} : C_1^{el} - I^4)^{-1} + S_{I,C_0^{el}})^{-1} : \varepsilon^0 \rangle_{\omega_1}, \\ &= [I^4 - S_{I,C_0^{el}} : ([C_0^{el}]^{-1} : C_1^{el} - I^4)^{-1} + S_{I,C_0^{el}})^{-1}] : \varepsilon^0, \\ &= \left(I^4 + S_{I,C_0^{el}} : [(C_0^{el})^{-1} : C_1^{el} - I^4] \right)^{-1} : \underbrace{\varepsilon^0}_{\langle \varepsilon \rangle_{\omega}}, \end{aligned} \quad (2.67)$$

then, identification with Eq.(2.30) gives the expression of the strain concentration tensor A^{DM} for the dilute inclusion model (DM). One obtains:

$$A^{DM} = \left(I^4 + S_{I,C_0^{el}} : [(C_0^{el})^{-1} : C_1^{el} - I^4] \right)^{-1}. \quad (2.68)$$

Dealing with composites with low volume fraction of inclusions one can assume that $v_1 \rightarrow 0$, then from Eq.(2.33) it follows that:

$$\tilde{A}^{DM} = A^{DM}. \quad (2.69)$$

The expression of $\tilde{A}^{DM}$ can also be easily obtained as follows: since $v_1 \rightarrow 0$, Eq.(2.13) gives $\langle \varepsilon \rangle_{\omega} \approx \langle \varepsilon \rangle_{\omega_0}$, then substituting this result into Eq.(2.67) leads to:

$$\langle \varepsilon \rangle_{\omega_1} = \underbrace{\left(I^4 + S_{I,C_0^{el}} : [(C_0^{el})^{-1} : C_1^{el} - I^4] \right)^{-1}}_{\tilde{A}^{DM}} : \langle \varepsilon \rangle_{\omega_0}. \quad (2.70)$$

2.9.2 Mori Tanaka scheme

Mori-Tanaka (M-T) model is, without any doubt, the most used homogenization scheme for composites consisting of a matrix phase with inclusions embedded therein. It is mainly used for composites with moderate volume fractions of inclusions. Contrary to the dilute inclusion model, M-T model takes into account the interaction between inclusions. In the model proposed by Mori and Tanaka, the Associated Isolated Inclusion Medium (AIIM), representing the original heterogeneous RVE (Figure 2.10-a), is subjected to the far-field strain ε^0 and the nonuniform perturbation strain $\varepsilon''(x)$ resulting from the presence of the neighboring inclusions, as displayed in Figure 2.10-b. The perturbation strain $\varepsilon^{pr}(x)$ due to the presence of the inclusion I in the matrix is defined as follows:

$$\varepsilon^{pr}(x) = \begin{cases} S_{I,C_0^{el}} : \varepsilon^*, & \text{inside the isolated inclusion} \\ \varepsilon'(x), & \text{inside the matrix} \end{cases} \quad (2.71)$$

we also define

$$\varepsilon'''(x) = \varepsilon'(x) + \varepsilon''(x), \quad (2.72)$$

the resulting perturbation strain of all inclusions on the RVE. Of course, $\varepsilon'(x)$ is null inside the isolated inclusion.

In the matrix phase of the substitution RVE (figure 2.10-b), the strain and stress are:

$$\varepsilon(x) = \varepsilon^0 + \varepsilon'(x) + \varepsilon''(x) \equiv \varepsilon^0 + \varepsilon'''(x), \quad (2.73)$$

$$\sigma(x) = C_0^{el} : (\varepsilon^0 + \varepsilon'''(x)). \quad (2.74)$$

In the inclusion the strain and stress are:

$$\varepsilon(x) = \varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* + \varepsilon'''(x), \quad (2.75)$$

$$\sigma(x) = C_1^{el} : (\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* + \varepsilon'''(x)). \quad (2.76)$$

In the inclusion zone of the equivalent homogeneous medium, the strain and stress fields are written as:

$$\varepsilon(x) = \varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* + \varepsilon'''(x) - \varepsilon^*, \quad (2.77)$$

$$\sigma(x) = C_0^{el} : (\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* + \varepsilon'''(x) - \varepsilon^*). \quad (2.78)$$

Consistency condition imposes that:

$$C_1^{el} : (\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* + \varepsilon'''(x)) = C_0^{el} : (\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* + \varepsilon'''(x) - \varepsilon^*), \quad (2.79)$$

from the above equation, it clearly appears that the consistency condition cannot be exactly fulfilled due to the presence of the term $\varepsilon'''(x)$ representing the perturbing

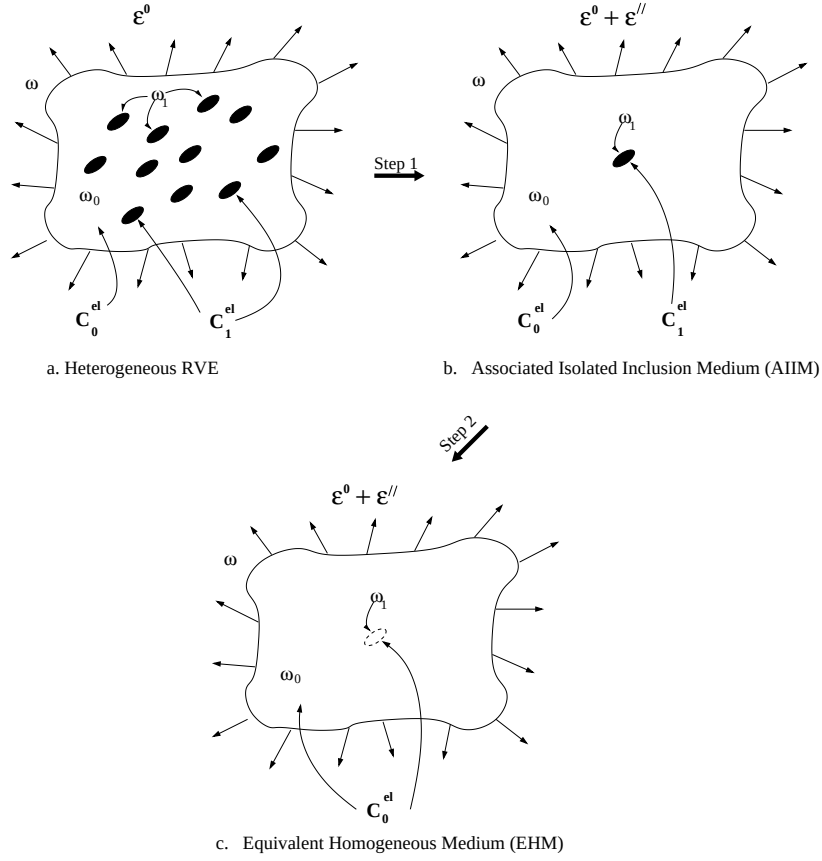


Figure 2.10: Schematic representation of Mori-Tanaka model hypothesis

effect of the surrounding inclusions. A workaround is to perform the average computation over inclusion material ω_1 (all inclusions). Consequently, the condition will be satisfied on the average inclusion material ω_1 . Eq.(2.79) then becomes:

$$C_1^{el} : \left(\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* + \langle \varepsilon'''(x) \rangle_{\omega_1} \right) = C_0^{el} : \left(\varepsilon^0 + S_{I,C_0^{el}} : \varepsilon^* + \langle \varepsilon'''(x) \rangle_{\omega_1} - \varepsilon^* \right). \quad (2.80)$$

Rearranging, we obtain the suitable strain “ ε^* ” for homogenization as:

$$\varepsilon^* = - \left(\left[(C_0^{el})^{-1} : C_1^{el} - I^4 \right]^{-1} + S_{I,C_0^{el}} \right)^{-1} : \left(\varepsilon^0 + \langle \varepsilon'''(x) \rangle_{\omega_1} \right), \quad (2.81)$$

replacing the last equation into Eq.(2.77) and averaging, the average strain in the inclusion is then found as:

$$\langle \varepsilon \rangle_{\omega_1} = \left(I^4 - S_{I, C_0^{el}} : \left([(C_0^{el})^{-1} : C_1^{el} - I^4]^{-1} + S_{I, C_0^{el}} \right)^{-1} \right) : (\varepsilon^0 + \langle \varepsilon'''(x) \rangle_{\omega_1}). \quad (2.82)$$

In their model, Mori and Tanaka assume that the disturbance strain $\varepsilon'''(x)$ due to the surrounding inclusions are on average the same for the matrix phase and for any inclusion, it follows that:

$$\langle \varepsilon'''(x) \rangle_{\omega_0} = \langle \varepsilon'''(x) \rangle_{\omega_1}, \quad (2.83)$$

and that:

$$\varepsilon^0 + \langle \varepsilon'''(x) \rangle_{\omega_1} = \varepsilon^0 + \langle \varepsilon'''(x) \rangle_{\omega_0} = \langle \varepsilon \rangle_{\omega_0}. \quad (2.84)$$

Eq.(2.82) is then rewritten as:

$$\langle \varepsilon \rangle_{\omega_1} = \tilde{A} : \langle \varepsilon \rangle_{\omega_0}, \quad (2.85)$$

where:

$$\tilde{A}^{MT} = \left(I^4 - S_{I, C_0^{el}} : \left([(C_0^{el})^{-1} : C_1^{el} - I^4]^{-1} + S_{I, C_0^{el}} \right)^{-1} \right), \quad (2.86)$$

which can be written as:

$$\tilde{A}^{MT} = \left(I^4 + S_{I, C_0^{el}} : [(C_0^{el})^{-1} : C_1^{el} - I^4] \right)^{-1}. \quad (2.87)$$

Using Eq.(2.33) one obtains expression of A^{MT} as follows:

$$A^{MT} = \left[(1 - v_1) \left\{ I^4 + S_{I, C_0^{el}} : [(C_0^{el})^{-1} : C_1^{el} - I^4] \right\} + v_1 I^4 \right]^{-1}. \quad (2.88)$$

Replacing A^{MT} into Eq.(2.41), the homogenized elastic operator of a two-phase composite following the Mori-Tanaka scheme is then obtained.

2.9.3 Self-Consistent model

In the Self-Consistent model (SC), to take advantage of the concept of the equivalent inclusion problem, the Associated Isolated Inclusion Medium representing the original RVE, consists in a single inclusion embedded in the matrix phase which has the yet-unknown mechanical properties of the composite, as displayed in Figure 2.11. This constitutes an implicit way to take into account the interaction between inclusions. Now analyze the different fields acting on the substitution RVE, Figure 2.11-b. In the matrix phase, strain and stress are:

$$\varepsilon(x) = \varepsilon^0 + \varepsilon'(x), \quad (2.89)$$

$$\sigma(x) = \bar{C}^{el} : (\varepsilon^0 + \varepsilon'(x)), \quad (2.90)$$

where $\varepsilon'(x)$ is the perturbation strain inside the matrix due to the presence of the inclusion. In the inclusion phase the strain and stress fields are:

$$\varepsilon(x) = \varepsilon^0 + S_{I, \bar{C}^{el}} : \varepsilon^*, \quad (2.91)$$

$$\sigma(x) = C_1^{el} : (\varepsilon^0 + S_{I, \bar{C}^{el}} : \varepsilon^*). \quad (2.92)$$

In the inclusion zone of the equivalent homogeneous medium (Figure 2.11-c), the strain and stress fields are written as:

$$\varepsilon(x) = \varepsilon^0 + S : \varepsilon^* - \varepsilon^*, \quad (2.93)$$

$$\sigma(x) = \bar{C}^{el} : (\varepsilon^0 + S_{I, \bar{C}^{el}} : \varepsilon^* - \varepsilon^*). \quad (2.94)$$

Consistency condition imposes for stresses given by Eqs.(2.92)-(2.94) to be equal, it

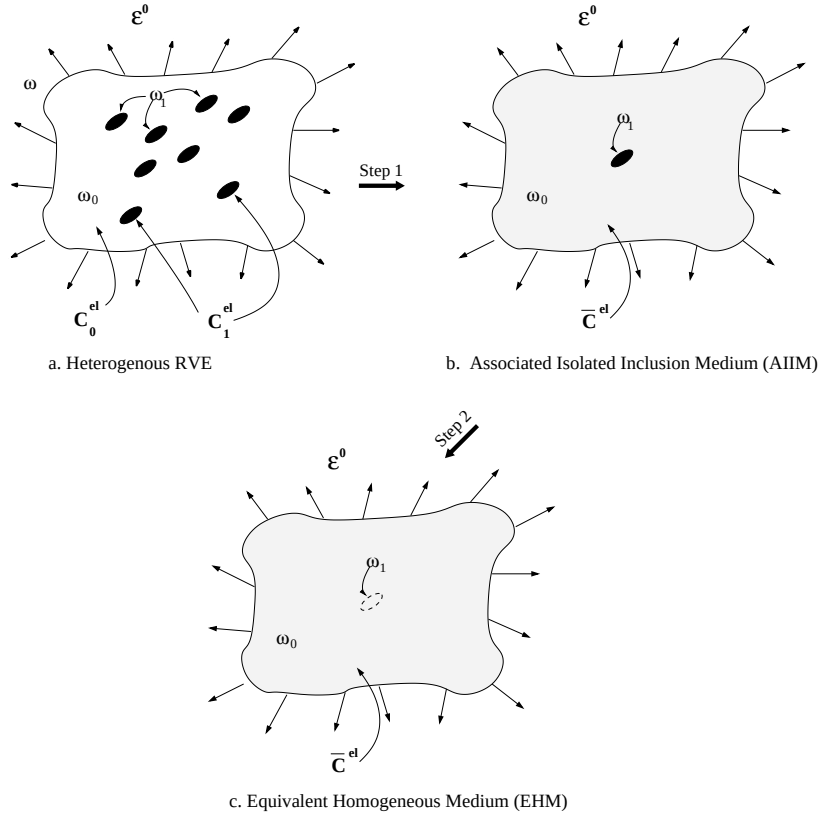


Figure 2.11: Schematic representation of Self-Consistent model hypothesis

follows that:

$$C_1^{el} : (\varepsilon^0 + S_{I, \bar{C}^{el}} : \varepsilon^*) = \bar{C}^{el} : (\varepsilon^0 + S_{I, \bar{C}^{el}} : \varepsilon^* - \varepsilon^*), \quad (2.95)$$

from which the eigenstrain ε^* necessary for homogenization is obtained. The result is:

$$\varepsilon^* = - \left([(\bar{C}^{el})^{-1} : C_1^{el} - I^4]^{-1} + S_{I, \bar{C}^{el}} \right)^{-1} : \varepsilon^0. \quad (2.96)$$

Replacing the last equation into Eq.(2.91) and averaging gives the average strain in the inclusion. The result is:

$$\langle \varepsilon \rangle_{\omega_1} = \left(I^4 + S_{I, \bar{C}^{el}} : [(\bar{C}^{el})^{-1} : C_1^{el} - I^4] \right)^{-1} : \underbrace{\varepsilon^0}_{\langle \varepsilon \rangle_{\omega}}, \quad (2.97)$$

then it remains to identify the last equation with Eq.(2.30) to obtain the expression of the strain concentration tensor A^{sc} for the Self-Consistent homogenization scheme. Then one obtains:

$$A^{SC} = \left(I^4 + S_{I, \bar{C}^{el}} : [(\bar{C}^{el})^{-1} : C_1^{el} - I^4] \right)^{-1}. \quad (2.98)$$

Replacing A^{SC} into Eq.(2.41), the homogenized elastic operator of a two-phase composite following the Self-Consistent scheme is then deduced.

In Eq.(4.31), $\bar{C}^{el}$ is of course unknown, however numerical solution can be obtained by an iterative scheme. One starts with an initial guess at $\bar{C}^{el}$ (provided by Voigt's or Reuss's models for instance, sections 2.9.6 and 2.9.7), evaluates A^{SC} and substitutes the result into Eq.(2.41) to get an improved value of $\bar{C}^{el}$. The procedure is repeated using this new value, and the iterations continue until convergence of $\bar{C}^{el}$.

2.9.4 Double-inclusion model

The double-inclusion (DI) model originally proposed by Hori and Nemat-Nasser (1993), provides a homogenization method more elaborate than the single equivalent inclusion one. For the case of two-phase composites, it relies on the geometrical configuration of an inclusion surrounded by a coating of the matrix which is embedded in an extended reference medium. The double-inclusion configuration appears to be a very good representation of the various interactions in the material. Indeed, it takes into account for the interaction between the inclusion and the matrix, as well as the interaction between the inclusions themselves. In addition, DI model provides a great flexibility by allowing several estimates depending on the choice of the mechanical properties of the reference medium. The model may include as special cases, the Mori-Tanaka and Self-Consistent-like estimates.

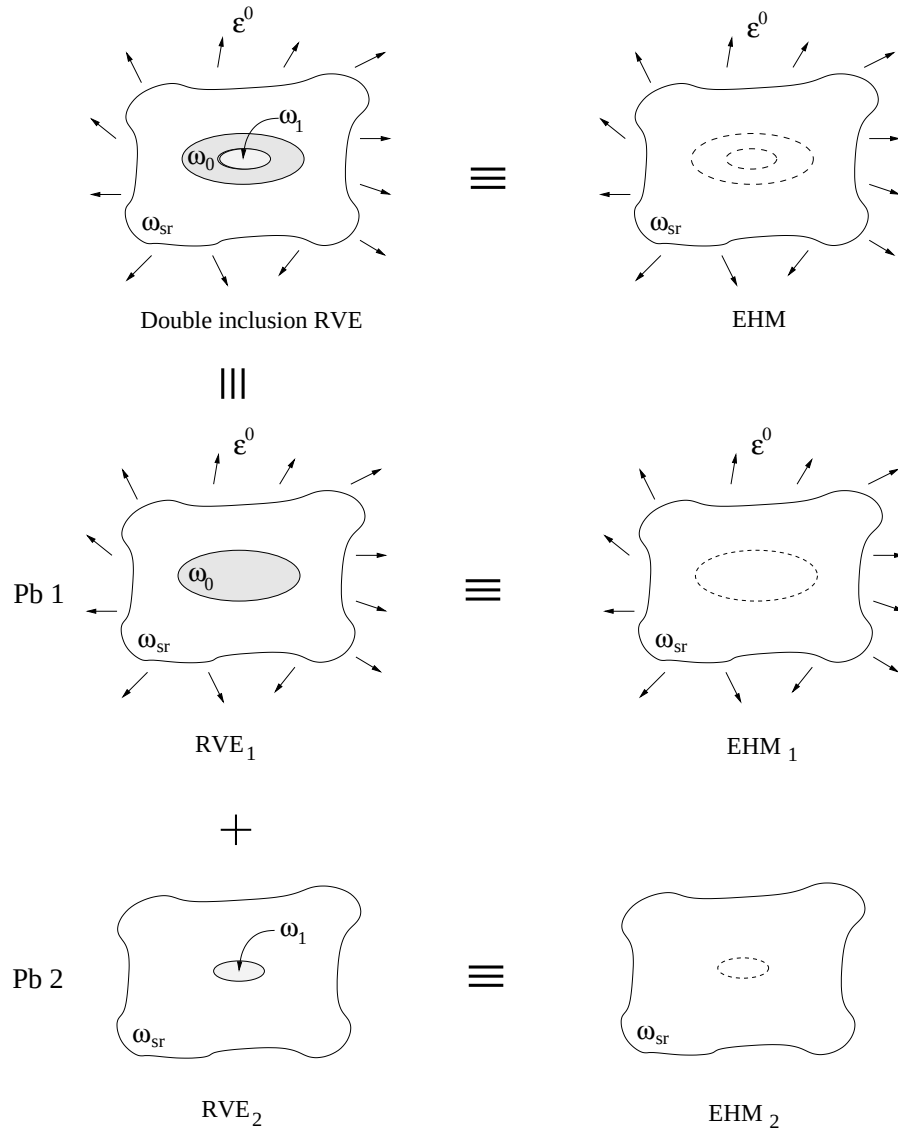


Figure 2.12: Schematic representation of Double-Inclusion model hypothesis

Consider the RVE represented in figure 2.12-a. It consists in a spheroidal inclusion ω_1 embedded in matrix material ω_0 of similar shape which is itself embedded in an extended reference material ω_{sr} . Note that the geometry of ω_0 must be similar to those of ω_1 in order to respect the symmetry of the composite [54]. Now, subject this RVE to a farfield strain ε^0 . The objective is to determine the suitable strains ε_0^* and ε_1^* in $\omega_0 - \omega_1$ and ω_1 respectively to obtain the Equivalent Homogeneous Medium with C_{sr}^{el} everywhere. Then, ε_0^* and ε_1^* being computed, one can compute the stress and strain fields acting on the original RVE. Let us start by considering the original RVE as the sum of two RVEs (RVE₁ and RVE₂) as displayed in Figure 2.12. Since we deal with linear elasticity, the problem is then resolved by superposing the solution of two simple problems Pb₁ and Pb₂ defined as follows:

1. In the first problem (Pb1) we assume that the original RVE is homogenized by subjecting both ω_0 and ω_1 to the same eigenstrain ε_0^* . This is equivalent to consider that ω_0 and ω_1 as a single inclusion ω_0 , as displayed in Figure 2.12-Pb1. The strain and stress fields are then:

- In the RVE₁, outside ω_0 :

$$\begin{aligned}\varepsilon(x) &= \varepsilon^0 + \varepsilon_0'(x), \\ \sigma(x) &= C_{sr}^{el} : (\varepsilon^0 + \varepsilon_0'(x)).\end{aligned}\quad (2.99)$$

- In the RVE₁, inside ω_0 :

$$\begin{aligned}\varepsilon(x) &= \varepsilon^0 + S_{I_0, C_{sr}^{el}} : \varepsilon_0^*, \\ \sigma(x) &= C_0^{el} : (\varepsilon^0 + S_{I_0, C_{sr}^{el}} : \varepsilon_0^*).\end{aligned}\quad (2.100)$$

- Inside ω_0 of the Equivalent Homogeneous Medium EHM₁ associated to RVE₁.

$$\begin{aligned}\varepsilon(x) &= \varepsilon^0 + S_{I_0, C_{sr}^{el}} : \varepsilon_0^* - \varepsilon_0^*, \\ \sigma(x) &= C_{sr}^{el} : (\varepsilon^0 + S_{I_0, C_{sr}^{el}} : \varepsilon_0^* - \varepsilon_0^*).\end{aligned}\quad (2.101)$$

2. In the second problem (Pb2, Figure 2.12-Pb2), the original RVE is homogenized by only subjecting ω_1 to the eigenstrain $\varepsilon_2^* = \varepsilon_1^* - \varepsilon_0^*$. Phases ω_0 and ω_{sr} are viewed in this case as a single phase ω_{sr} which does not undergoes any eigenstrain and any farfield strain. We must keep in mind that problems Pb1 and Pb2 are built such as when they are superposed they lead to the original problem to be resolved. So, for the Pb2, the strain and stress fields within the RVE₂ are:

- Outside ω_1 :

$$\begin{aligned}\varepsilon(x) &= \varepsilon_2'(x), \\ \sigma(x) &= C_{sr}^{el} : \varepsilon_2'(x).\end{aligned}\quad (2.102)$$

- Inside ω_1 :

$$\begin{aligned}\varepsilon(x) &= S_{I_1, C_{sr}^{el}} : \varepsilon_2^*, \\ \sigma(x) &= C_1^{el} : S_{I_1, C_{sr}^{el}} : \varepsilon_2^*.\end{aligned}\quad (2.103)$$

- Inside ω_1 zone of the Equivalent Homogeneous Medium EHM₂:

$$\begin{aligned}\varepsilon(x) &= S_{I_1, C_{sr}^{el}} : \varepsilon_2^* - \varepsilon_2', \\ \sigma(x) &= C_{sr}^{el} : (S_{I_1, C_{sr}^{el}} : \varepsilon_2^* - \varepsilon_2'),\end{aligned}\quad (2.104)$$

where $\varepsilon_2^* = \varepsilon_1^* - \varepsilon_0^*$ and $\varepsilon_2' = \varepsilon_1' - \varepsilon_0'$ in order to satisfy superposition condition as previously mentioned.

The strain and stress fields in the whole original heterogeneous RVE are then obtained by superposing the stress and strain fields of the above-defined problems. The result is:

- Inside the inclusion ω_1 , the strain and stress are given by superposition of Eqs.(2.100) and (2.103). One obtains:

$$\varepsilon(x) = \varepsilon^0 + S_{I_0, C_{sr}^{el}} : \varepsilon_0^* + S_{I_1, C_{sr}^{el}} : \varepsilon_2^*, \quad (2.105)$$

$$\sigma(x) = C_1^{el} : (\varepsilon^0 + S_{I_0, C_{sr}^{el}} : \varepsilon_0^* + S_{I_1, C_{sr}^{el}} : \varepsilon_2^*). \quad (2.106)$$

- Inside $\omega_0 - \omega_1$, the strain and stress are given by superposition of Eqs.(2.100) and (2.102). The result is:

$$\varepsilon(x) = \varepsilon^0 + \varepsilon_2' + S_{I_0, C_{sr}^{el}} : \varepsilon_0^*, \quad (2.107)$$

$$\sigma(x) = C_0^{el} : (\varepsilon^0 + \varepsilon_2' + S_{I_0, C_{sr}^{el}} : \varepsilon_0^*). \quad (2.108)$$

- Outside ω_0 and ω_1 the strain and stress are given by superposition of Eqs.(2.99) and (2.102) as:

$$\varepsilon(x) = \varepsilon^0 + \varepsilon_0' + \varepsilon_2', \quad (2.109)$$

$$\sigma(x) = C_{sr}^{el} : (\varepsilon^0 + \varepsilon_0' + \varepsilon_2'). \quad (2.110)$$

In the same way, the strain and stress in Equivalent Homogeneous Medium (EHM) of the RVE, can be obtain by superposing the strain and stress of the Equivalent Homogeneous Mediums EHM₁ and EHM₂ of the Pb₁ and Pb₂ respectively. The result is:

- Inside the inclusion ω_1 of EHM, the strain and stress are given by superposition of Eqs.(2.101)-(2.104) as:

$$\varepsilon(x) = \varepsilon^0 + S_{I_0, C_{sr}^{el}} : \varepsilon_0^* + S_{I_1, C_{sr}^{el}} : \varepsilon_2^* - \varepsilon_0^* - \varepsilon_2^*, \quad (2.111)$$

$$\sigma(x) = C_{sr}^{el} : (\varepsilon^0 + S_{I_0, C_{sr}^{el}} : \varepsilon_0^* + S_{I_1, C_{sr}^{el}} : \varepsilon_2^* - \varepsilon_0^* - \varepsilon_2^*). \quad (2.112)$$

- Inside $(\omega_0 - \omega_1)$ of the equivalent homogeneous medium, the strain and stress are given by superposition of Eqs.(2.101) and (2.102) as:

$$\varepsilon(x) = \varepsilon^0 + \varepsilon_2' + S_{I_0, C_{sr}^{el}} : \varepsilon_0^* - \varepsilon_0^*, \quad (2.113)$$

$$\sigma(x) = C_{sr}^{el} : (\varepsilon^0 + \varepsilon_2' + S_{I_0, C_{sr}^{el}} : \varepsilon_0^* - \varepsilon_0^*). \quad (2.114)$$

- Outside ω_0 and ω_1 , the strain and stress are given by superposition of Eqs.(2.99) and (2.102) as:

$$\varepsilon(x) = \varepsilon^0 + \varepsilon_2' + \varepsilon_2', \quad (2.115)$$

$$\sigma(x) = C_{sr}^{el} : (\varepsilon^0 + \varepsilon_2' + \varepsilon_2'). \quad (2.116)$$

Since the shape of the inclusion and the matrix are the same, it follows that $S_{I_0, C_{sr}^{el}} = S_{I_1, C_{sr}^{el}}$. Taking advantage of this result and of the fact that $\varepsilon_2^* = \varepsilon_1^* - \varepsilon_0^*$, the strain and stress fields in inclusion material ω_1 of the original RVE given by Eqs.(2.105) and (2.106) are rewritten as:

$$\varepsilon(x) = \varepsilon^0 + S_{I, C_{sr}^{el}} : \varepsilon_1^*, \quad (2.117)$$

$$\sigma(x) = C_1^{el} : (\varepsilon^0 + S_{I, C_{sr}^{el}} : \varepsilon_1^*), \quad (2.118)$$

while in the inclusion zone ω_1 of EHM, they (Eqs.2.111-2.112) are rewritten as:

$$\varepsilon(x) = \varepsilon^0 + S_{I_1, C_{sr}^{el}} : \varepsilon_1^* - \varepsilon_1^*, \quad (2.119)$$

$$\sigma(x) = C_{sr}^{el} : (\varepsilon^0 + S_{I_1, C_{sr}^{el}} : \varepsilon_1^* - \varepsilon_1^*). \quad (2.120)$$

Mechanical equivalence condition imposes that:

- stresses are the same in ω_1 zone for the original RVE and its equivalent homogeneous medium, so we have:

$$C_1^{el} : (\varepsilon^0 + S_{I_1, C_{sr}^{el}} : \varepsilon_1^*) = C_{sr}^{el} : (\varepsilon^0 + S_{I_1, C_{sr}^{el}} : \varepsilon_1^* - \varepsilon_1^*). \quad (2.121)$$

- stresses are the same in the $\omega_0 - \omega_1$ zone for the original RVE and its equivalent homogeneous medium, it follows that:

$$C_0^{el} : (\varepsilon^0 + \varepsilon_2' + S_{I_1, C_{sr}^{el}} : \varepsilon_0^*) = C_{sr}^{el} : (\varepsilon^0 + \varepsilon_2' + S_{I_1, C_{sr}^{el}} : \varepsilon_0^* - \varepsilon_0^*). \quad (2.122)$$

In the inclusion phase, the condition on ε_1^* to ensure the mechanical equivalence is exactly fulfilled and leads to:

$$\varepsilon_1^* = -\left([(C_{sr}^{el})^{-1} : C_1^{el} - I^4]^{-1} + S_{I_1, C_{sr}^{el}}\right)^{-1} : \varepsilon^0, \quad (2.123)$$

then replacing the last equation into Eq.(2.117) and averaging over ω_1 one obtains:

$$\langle \varepsilon \rangle_{\omega_1} = (I^4 + S_{I_1, C_{sr}^{el}} : [(C_{sr}^{el})^{-1} : C_1^{el} - I^4])^{-1} : \varepsilon^0. \quad (2.124)$$

On the other hand, in $\omega_0 - \omega_1$ material phase, due to the presence of ε_2' on Eq.(2.108), the condition on ε_0^* to ensure the mechanical equivalence cannot be exactly satisfied. A work around is to consider the average on $\omega_0 - \omega_1$. Mori and Tanaka shown that since $\omega_0 - \omega_1$ and ω_1 have the same shape, the average of ε_2' over $\omega_0 - \omega_1$ vanishes, see [101], [54]. Averaging over $\omega_0 - \omega_1$ mechanical condition (Eq.(2.123)) leads to:

$$\varepsilon_0^* = -\left([(C_{sr}^{el})^{-1} : C_0^{el} - I^4]^{-1} + S_{I_1, C_{sr}^{el}}\right)^{-1} : \varepsilon^0, \quad (2.125)$$

replacing the last equation into Eq.(2.119) and averaging over $\omega_0 - \omega_1$ leads to:

$$\langle \varepsilon \rangle_{\omega_0 - \omega_1} = (I^4 + S_{I_1, C_{sr}^{el}} : [(C_{sr}^{el})^{-1} : C_0^{el} - I^4])^{-1} : \varepsilon^0. \quad (2.126)$$

Isolating ε^0 and using it in Eq.(2.124) leads to:

$$\langle \varepsilon \rangle_{\omega_1} = \tilde{A}^{DI} : \langle \varepsilon \rangle_{\omega_0 - \omega_1}, \quad (2.127)$$

where

$$\tilde{A}^{DI} = (I^4 + S_{I_1, C_{sr}^{el}} : [(C_{sr}^{el})^{-1} : C_I^{el} - I^4])^{-1} : (I^4 + S_{I_1, C_{sr}^{el}} : [(C_{sr}^{el})^{-1} : C_m^{el} - I^4]). \quad (2.128)$$

The last equation suggests many possibilities to evaluate $\tilde{A}^{DI}$ since it depends on C_{sr}^{el} for which one can affect several constitutive material types.

1. The first one is to assume that the surrounding material ω_{sr} is the same as ω_0 material. This is choice leads to:

$$\tilde{A}^{DI} = \left(I^4 - S_{I_1, C_0^{el}} : ([(C_0^{el})^{-1} : C_1^{el} - I^4]^{-1} + S_{I_1, C_0^{el}})^{-1} \right), \quad (2.129)$$

which is equivalent to strain concentration tensor evaluated with Mori-Tanaka scheme, hence we have: ($\tilde{A}^{DI} = \tilde{A}^{MT}$).

2. the second choice is to suppose that the surrounding material ω_{sr} has the same mechanical properties than the composite one. This implies that:

$$\tilde{A}^{DI} = (I^4 + S_{I_1, \bar{C}^{el}} : [(\bar{C}^{el})^{-1} : C_1^{el} - I^4])^{-1} : (I^4 + S_{I_1, \bar{C}^{el}} : [(\bar{C}^{el})^{-1} : C_0^{el} - I^4]), \quad (2.130)$$

which, using Eq.(2.130), can be written as:

$$\tilde{A}^{DI} = A^{SC} : (A^{SC})^{-1}. \quad (2.131)$$

In this case, the double inclusion model can be viewed as a more elaborate form of Self Consistent scheme.

3. A third possible choice is to assume that the surrounding material ω_{sr} has the same mechanical properties as the inclusion material. Setting $C_{sr}^{el} = C_1^{el}$ Eq(2.128) is then rewritten as:

$$\tilde{A}^{DI} = (I^4 + S_{I_1, \bar{C}^{el}} : [(\bar{C}^{el})^{-1} : C_0^{el} - I^4]). \quad (2.132)$$

This choice corresponds to Mori-Tanaka prediction for a composite when the matrix and inclusion phases roles are swapped and assuming of course, the continuity of the inclusion phase. It may constitute an interesting option for composites with high volume fractions of inclusions.

4. Others possibilities are to affect to the reference material ω_{sr} the elastic operators predicted by Voigt or Reuss models (detailed below) for instance.

Applications of DI models are various. It can be used to model composites with coated inclusions or to model the so-called Interface Transition Zone in concrete material for instance.

2.9.5 Lielens model

Lielens [54] has developed a model based on the Double-Inclusion model results. In his model, he suggests at very high volume fractions, that the composite stiffness is much closer to the stiffness predicted by Mori-Tanaka model when the inclusion phase is taken as the continuous one, and at low concentrations that the composite stiffness is much closer to the stiffness predicted by Mori-Tanaka. This has led Lielens to propose a model that interpolates between the Mori-Tanaka and inverse Mori-Tanaka models, such that at low volume fractions the Mori-Tanaka model dominates and at high volume fractions the inverse Mori-Tanaka model dominates. This interpolation is performed on the strain concentration tensors $\tilde{A}_{low}$ and $\tilde{A}_{up}$ corresponding to concentration tensors for low and high concentrations of inclusions respectively. $\tilde{A}_{low}$ is the strain concentration predicted by the Double-Inclusion model when the reference material is assumed to be the equal to matrix material:

$$\tilde{A}_{low} = \left(I^4 + S_{I, C_0^{el}} : [(C_0^{el})^{-1} : C_1^{el} - I^4] \right)^{-1}, \quad (2.133)$$

while $\tilde{A}_{up}$ is the strain concentration predicted by the Double-Inclusion model when the reference material is replaced by inclusion material:

$$\tilde{A}_{up} = I^4 + S_{I, C_1^{el}} : [(C_1^{el})^{-1} : C_0^{el} - I^4]. \quad (2.134)$$

The interpolation formula proposed by Lielens for the strain concentration tensor $\tilde{A}^L$ is:

$$\tilde{A}^L = \left((1 - f(v_1))\tilde{A}_{low}^{-1} + f(v_1)\tilde{A}_{up}^{-1} \right)^{-1}, \quad (2.135)$$

where $f(v_1)$ is a function of the inclusion volume fraction fulfilling the following conditions:

$$0 \leq f(v_1) \leq 1, \quad \lim_{v_1 \rightarrow 0} f(v_1) = 0, \quad \lim_{v_1 \rightarrow 1} f(v_1) = 1.$$

To correctly construct $f(v_1)$, it is important to introduce the notion of the maximum packing concentration v_{maxP} . Consider a volume containing inclusions in concentration v_1 . If we increase v_1 until a certain value, the inclusions become in contact with each others and it will be no longer possible to add other inclusions in the volume without changing their shape. This maximum value of inclusions volume fraction is called maximum packing concentration. For example, for spherical inclusion of uniform size, $v_{maxP} \approx 0.75$. Beyond this value, one can consider that matrix and inclusions roles are swapped leading to what the author call *phase inversion*. The role of the function $f(v_1)$ is to make the tensor $\tilde{A}_{low}$ dominating in the case of low to moderate volume fraction and conversely, to return the tensor $\tilde{A}_{up}$ dominating in case of high inclusions volume fraction. According to this, for $f(v_1) = 0.5$ we expect that the volume fraction v_1 must be close to the maximum packing concentration v_{maxP} . Several choices can be made for the function f . The choice $f(v) = v_1$ leads, for $f(v_1) = 0.5$, to $v_1 = 0.5$ which is less than the expected value (close to v_{maxP}). The choice proposed by Lielens, which consists in a quadric form of f :

$$f(v_1) = \frac{v_1 + v_1^2}{2} \quad (2.136)$$

seems to be more appropriate since for $f(v_1) = 0.5$ we have $v_1 = 0.62$.

2.9.6 Voigt model

Voigt model (with Reuss model) is one of the most simple homogenization schemes. It assumes a uniform strain in the composite, which means that strains are the same in each phase of the composite. We assume the existence of an equivalent homogeneous body in which the average stress is related to the average strain through $\bar{C}^{el}$ as:

$$\langle \sigma \rangle_\omega = \bar{C}^{el} : \langle \varepsilon \rangle_\omega \quad (2.137)$$

in the real two-phase composite the average stress is:

$$\langle \sigma \rangle_\omega = v_1 \langle \sigma \rangle_{\omega_1} + (1 - v_1) \langle \sigma \rangle_{\omega_0}, \quad (2.138)$$

which is rewritten by assuming that each phase obeys Hooke's law as:

$$\langle \sigma \rangle_\omega = v_1 C_1^{el} : \langle \varepsilon \rangle_{\omega_1} + (1 - v_1) C_0^{el} : \langle \varepsilon \rangle_{\omega_0}. \quad (2.139)$$

Under Voigt's hypothesis, the strains everywhere in the composite are the same, thus we have:

$$\langle \varepsilon \rangle_\omega = \langle \varepsilon \rangle_{\omega_1} = \langle \varepsilon \rangle_{\omega_0}, \quad (2.140)$$

and Eq.(2.139) is then rewritten as:

$$\langle \sigma \rangle_\omega = (v_1 C_1^{el} + (1 - v_1) C_0^{el}) : \langle \varepsilon \rangle_\omega. \quad (2.141)$$

To satisfy the mechanical equivalent condition, the average stress in the real two-phase composite (Eq.(2.137)) must equal to the average stress in the equivalent homogeneous solid (Eq.(2.141)), it follows that:

$$\bar{C}^{el} : \langle \varepsilon \rangle_\omega = (v_1 C_1^{el} + (1 - v_1) C_0^{el}) : \langle \varepsilon \rangle_\omega, \quad (2.142)$$

from which the homogenized elastic operator $\bar{C}^{el}$ is deduced:

$$\bar{C}^{el} = v_1 C_1^{el} + (1 - v_1) C_0^{el}. \quad (2.143)$$

This result can be directly found using Eq.(2.41) where, from Voigt's hypothesis (Eq.(2.140)) and with the help of Eq.(2.30), the strain concentration tensor A is equal to:

$$A^{Voigt} = I^4. \quad (2.144)$$

Substitution of $A^{Voigt} = I^4$ into Eq.(2.41) leads to the expression of the homogenized

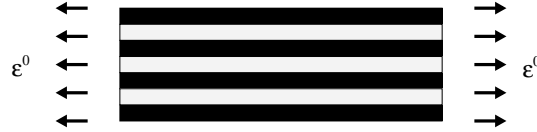


Figure 2.13: Idealized schematic representation of the RVE under Voigt hypothesis

elastic operator given by Eq.(2.143).

Voigt's assumption (Eq.(2.140)) suggests that the composite can be viewed as parallel bars made of matrix and inclusion materials subjected to a uniform longitudinal (axial) strain as displayed in Figure 2.13. The main criticism of this model is that it does not account enough for the microstructure information such as inclusions' shape and orientation. For isotropic constituents (matrix and inclusions), Voigt's scheme always predicts isotropic mechanical properties for the composite. In general, estimates made with this model are very stiff except for the above specified case (Figure 2.13).

2.9.7 Reuss model

Consider a two-phase RVE and its EHM. The average strain-stress relationship in the homogeneous equivalent medium is defined as:

$$\langle \varepsilon \rangle_\omega = \bar{D}^{el} : \langle \sigma \rangle_\omega, \quad (2.145)$$

where $\bar{D}^{el}$ represents the elastic compliance of the homogenized composite. In the heterogeneous RVE, the average strain is written as:

$$\langle \varepsilon \rangle_\omega = v_1 \langle \varepsilon \rangle_{\omega_1} + (1 - v_1) \langle \varepsilon \rangle_{\omega_0}, \quad (2.146)$$

assuming that each phase obeys Hooke's law, the last equation is written as:

$$\langle \varepsilon \rangle_\omega = v_1 D_1^{el} : \langle \sigma \rangle_{\omega_1} + (1 - v_1) D_0^{el} : \langle \sigma \rangle_{\omega_0}, \quad (2.147)$$

where D_1^{el} and D_0^{el} are the inclusion and matrix elastic compliances respectively. The mechanical equivalence imposes that:

$$\bar{D}^{el} : \langle \sigma \rangle_\omega = v_1 D_1^{el} : \langle \sigma \rangle_{\omega_1} + (1 - v_1) D_0^{el} : \langle \sigma \rangle_{\omega_0}. \quad (2.148)$$

Reuss hypothesis sets that stresses are the same everywhere in the composite, it follows that:

$$\langle \sigma \rangle_\omega = \langle \sigma \rangle_{\omega_1} = \langle \sigma \rangle_{\omega_0}, \quad (2.149)$$

which, with the help of Eq.(2.148), gives the homogenized elastic compliance $\bar{D}^{el}$:

$$\bar{D}^{el} = v_1 D_1^{el} + (1 - v_1) D_0^{el}, \quad (2.150)$$

then, taking advantage of the fact that: $C^{el} = (D^{el})^{-1}$, Eq.(2.150) is rewritten as:

$$\bar{C}^{el} = \left[v_1 (C_1^{el})^{-1} + (1 - v_1) (C_0^{el})^{-1} \right]^{-1}, \quad (2.151)$$

where $\bar{C}^{el}$ is the homogenized elastic stiffness predicted using Reuss scheme.

It should be pointed out that under Reuss's assumption (Eq.(2.149)), the composite looks like parallel bars made of matrix and inclusion materials undergoing a transverse stress as displayed in figure 2.14.

Reuss's model suffers from the same limitations as that for Voigt's model. The microstructure is not enough taken into account in this model which often leads to predictions that are underestimated. Note that Voigt's and Reuss's estimates constitute upper and lower bounds for shear and bulk moduli (see [112]). When dealing with composites such as polycrystals, it was however observed, that Voigt's and Reuss's models together with SC scheme may constitute appropriate models (except for metals concerning Reuss's model).

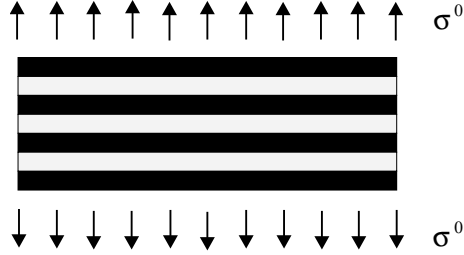


Figure 2.14: Idealized schematic representation of the RVE under Reuss hypothesis

2.10 Hill's procedure for homogenization

Hill [35] has proposed an alternative approach to resolve the isolated inclusion problem. Hereafter we develop the key idea of the method, for more further discussion see [8] and references therein.

2.10.1 Hill's tensors

In this section we derive expressions of the most important Hill's tensors and establish the link between those tensors and Eshelby's tensor. For this purpose, consider the same isolated inclusion problem as those of section 2.7. According to Eq.(2.156), the stress in the inclusion is rewritten as:

$$\begin{aligned}\sigma(x) &= C_0^{el} : (\varepsilon_1 - \varepsilon^*), \\ &= C_0^{el} : \varepsilon_1 + \sigma^*,\end{aligned}\tag{2.152}$$

where:

$$\sigma^* = -C_0^{el} : \varepsilon^*,\tag{2.153}$$

is called *Polarization* tensor. This tensor characterizes the response of the inclusion when it undergoes freely the eigenstrain ε^* . Combining the last equation together with Eq.(2.48) leads to:

$$\varepsilon_1 = -P : \sigma^*,\tag{2.154}$$

where P is called Hill's tensor and is related to Eshelby's tensor $S_{I,C_0^{el}}$ as follows:

$$P = S_{I,C_0^{el}} : (C_0^{el})^{-1}.\tag{2.155}$$

Hill's tensor P depends only on the mechanical properties of the matrix phase and of the inclusion's geometry. It has more interesting properties than those of Eshelby's

tensor $S_{I,C_0^{el}}$ since it is symmetric by construction and positive definite. P will be denoted $P_{I,C_0^{el}}$ in the following.

The other interesting tensor defined in Hill's approach is the Hill's constraint tensor denoted by C^* . This tensor characterizes the response of the surrounding matrix medium when the inclusion undergoes the eigenstrain ε^* or any other strains. Consider that

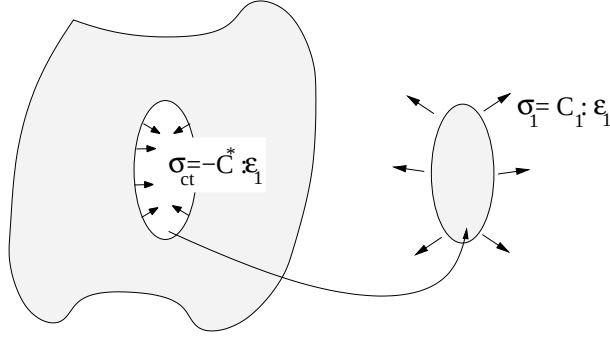


Figure 2.15: Schematic interpretation of Hill's constraint tensor

the inclusion has a stiffness C_1^{el} and that the matrix phase has a stiffness C_0^{el} . As displayed in Figure 2.15, according with Eshelby's result, the strain and stress fields in the inclusion are:

$$\varepsilon_1 = S_{I,C_0^{el}} : \varepsilon^*, \quad (2.156)$$

$$\sigma_1 = C_1^{el} : \varepsilon_1, \quad (2.157)$$

$$= C_0^{el} : (S_{I,C_0^{el}} : \varepsilon^* - \varepsilon^*). \quad (2.158)$$

The surrounding medium will be opposed to the inclusion deformation by exerting on the latter one a stress σ_{ct} which has the following expression:

$$\begin{aligned} \sigma_{ct} &= -C^* : \varepsilon_1, \\ &= -C^* : S_{I,C_0^{el}} : \varepsilon^*. \end{aligned} \quad (2.159)$$

At equilibrium $\sigma_{ct} = \sigma_1$, so we have:

$$-C^* : S_{I,C_0^{el}} : \varepsilon^* = C_0^{el} : (S_{I,C_0^{el}} : \varepsilon^* - \varepsilon^*), \quad (2.160)$$

rearranging and using Eq.(2.155) leads to:

$$C^* = P_{I,C_0^{el}}^{-1} - C_0^{el}. \quad (2.161)$$

The tensor C^* depends only on the mechanical properties of the matrix and on the geometrical characteristics of the inclusion. It is symmetric and definite positive.

2.10.2 Mori-Tanaka scheme using Hill's tensors

The goal of this section is to express strain concentration tensors and homogenized elastic operator expressions for Mori-Tanaka scheme, in terms of Hill's tensors. Accordingly, the expression of the strain concentration tensor $\tilde{A}$ (Eq.2.86) which relates inclusion and matrix strain averages respectively, is first rewritten as:

$$\tilde{A}^{MT} = \left(I^4 + S_{I,C_0^{el}} : (C_0^{el})^{-1} : [C_1^{el} - C_0^{el}] \right)^{-1}, \quad (2.162)$$

then in terms of Hill's tensor $P_{I,C_0^{el}}$ (using Eq.(2.155)) as:

$$\tilde{A}^{MT} = \left(I^4 + P_{I,C_0^{el}} : [C_1^{el} - C_0^{el}] \right)^{-1}, \quad (2.163)$$

then in terms of Hill's constraint tensor C^* (using Eq.2.161) as:

$$\begin{aligned} \tilde{A}^{MT} &= \left[I^4 + (C^* + C_0^{el})^{-1} : (C_1^{el} - C_0^{el}) \right]^{-1}, \\ &= \left[\underbrace{(C^* + C_0^{el})^{-1} : (C^* + C_0^{el})}_{I^4} + (C^* + C_0^{el})^{-1} : (C_1^{el} - C_0^{el}) \right]^{-1}, \\ &= (C^* + C_1^{el})^{-1} : (C^* + C_0^{el}). \end{aligned} \quad (2.164)$$

Using (Eq.2.88), strain concentration tensor A^{MT} is then rewritten as:

$$A^{MT} = \left[(1 - \nu_1) \left\{ (C^* + C_1^{el})^{-1} : (C^* + C_0^{el}) \right\} + \nu_1 I^4 \right]^{-1}. \quad (2.165)$$

It suffices to replace the last equation in Eq.(2.41) to obtain the expression of the homogenized elastic operator written in terms of Hill's constraint tensor, for a two-phase composite following Mori-Tanaka scheme. It should be worth to note that in comparison with the homogenized elastic operator written in terms of Eshelby's tensor, those written in terms of Hill's tensors involve only symmetric tensors.

2.11 Bounds

Models derived in previous sections provide estimates for overall elastic (and inelastic, see chapter 4) properties " $\bar{C}^{el}$ " of composites. Homogenization methods also provide for those properties ($\bar{C}^{el}$), rigorous bounds obtained using appropriate variational principles. Hereafter, we recall in short the most popular bounds.

2.11.1 Hill's bounds

Very simple bounds, called Hill's bounds, are obtained using the potential energy and the complementary energy together with uniform strain and stress trial functions. The upper bound denoted by $\bar{C}_{Hup}^{el}$, corresponds to Voigt's estimate, while the lower one $\bar{C}_{Hlow}^{el}$, corresponds to Reuss's estimate. Hill's bounds are expressed as follows:

$$\underbrace{\left[\sum_{(r)} v_r C_r^{el} \right]^{-1}}_{\bar{C}_{Hlow}^{el}} \leq \bar{C}^{el} \leq \underbrace{\sum_{r=0}^N v_r C_r^{el}}_{\bar{C}_{Hup}^{el}}. \quad (2.166)$$

Here, the notation " $\leq$ " compares for instance the two fourth-order tensors $\bar{C}_{Hlow}^{el}$ and $\bar{C}^{el}$ and means that $\bar{C}_{Hlow}^{el}$ is softer than $\bar{C}^{el}$. This notation is defined as follows:

$$\bar{C}_{Hlow}^{el} \leq \bar{C}^{el} \iff a : \bar{C}_{Hlow}^{el} : a \leq a : \bar{C}^{el} : a; \quad (2.167)$$

where a is a second-order symmetric tensor ($a_{ij} = a_{ji}$).

The main criticism that can be made for those bounds is the same that for Voigt and Reuss models. Hill's bounds do not take into account geometrical information of the microstructure and thus are considered as large bounds.

2.11.2 Hashin-Shtrikman bounds

Hashin and Shtrikman [33] proposed more elaborate and much tighter bounds obtained from a variational formulation. Written in terms of the polarization tensor (Eq.2.153) and in combination with the Hill-Mandel condition (Eq.2.15), the variational formulation provides the expression of the composite complementary energy which, when it is maximized, gives the Hashin-Shtrikman upper bound denoted $\bar{C}_{HSup}^{el}$ and, when it is minimized, provides the Hashin-Shtrikman lower bound denoted $\bar{C}_{HSlow}^{el}$. For a two phase composite consisting of a matrix phase reinforced inclusions, the Hashin-Shtrikman lower bound $\bar{C}_{HSlow}^{el}$ can be written in terms of Eshelby's tensor $S_{I,C_0^{el}}$ as:

$$\bar{C}_{HSlow}^{el} = C_0^{el} + v_1 \left[\left(C_1^{el} - C_0^{el} \right)^{-1} + (1 - v_1) S_{I,C_0^{el}} : C_0^{el} \right]^{-1}. \quad (2.168)$$

The upper bound $\bar{C}_{HSup}^{el}$ can be obtained by swapping matrix and inclusions roles. The result is:

$$\bar{C}_{HSup}^{el} = C_1^{el} + (1 - v_1) \left[\left(C_0^{el} - C_1^{el} \right)^{-1} + v_1 S_{I,C_1^{el}} : C_1^{el} \right]^{-1}. \quad (2.169)$$

During the last two decades, the research in the field was active and improved bounds were derived to take into account the composite nonlinear behavior. Among the most important developments for inelastic composites, one can cite those of Talbot and Willis [100] and those of Ponte Castañeda [78], see reviews by Suquet [98], and by Willis [109].

2.12 Inclusions orientation treatment

Up to now, we dealt with composites containing aligned inclusions, however in many industrial applications that is not the case. Among composites containing misaligned inclusions, one can cite Steel Fiber-Reinforced Concrete (SFRC). SFRC composite is manufactured by adding fibers in the concrete paste then mixing the whole for a few minutes. It results a random orientation of fibers within the concrete matrix. Orientation is one of the dominant structural features of composites. They are stiffer and stronger in the direction of the greatest orientation and weaker and more compliant in the direction of least orientation. The orientation influences also the thermal expansion and electrical conductivity. Assuming that the spheroidal inclusion is rigid and uniform in length, its orientation can be represented by using unit vector p directed along the axis of revolution of the spheroid, as shown in figure 2.16. The components of p are related to the angles θ and ϕ as:

$$\begin{cases} p_1 = \sin\theta \cos\phi \\ p_2 = \sin\theta \sin\phi \\ p_3 = \cos\theta \end{cases}$$

The set of all possible orientations of p corresponds to the unit sphere and is denoted by:

$$\oint dp = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\theta d\theta d\phi \quad (2.170)$$

An equivalent way to describe the orientation state at a point in space is the use of a probability distribution function ψ . This function is defined so that the probability to find a fiber between angles θ_1 and $\theta_1 + d\theta$, and, ϕ_1 and $\phi_1 + d\phi$ is given by:

$$P(\theta_1 \leq \theta \leq \theta_1 + d\theta, \phi_1 \leq \phi \leq \phi_1 + d\phi) = \psi(\theta_1, \phi_1) \sin\theta_1 d\theta d\phi. \quad (2.171)$$

Two conditions must be fulfilled by ψ . The first one is that an inclusion with an orientation (θ, ϕ) is indistinguishable from an inclusion with an orientation $(\pi + \theta, \pi + \phi)$. It follows that:

$$\psi(\theta, \phi) = \psi(\pi + \theta, \pi + \phi), \quad (2.172)$$

which can be written as:

$$\psi(p) = \psi(-p). \quad (2.173)$$

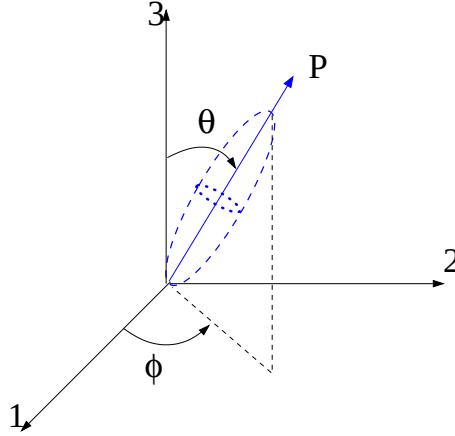


Figure 2.16: Spheroidal inclusion orientation

This also means that two ends of a spheroidal inclusion are not distinguishable. The second condition is: since every inclusion has some orientation it follows that ψ must be normalized such that:

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \psi(\theta, \phi) \sin\theta d\theta d\phi = \oint \psi(p) dp = 1. \quad (2.174)$$

With those definitions one can now treat the problem of composites with misaligned inclusions similar in shape and constitutive material. The aggregates decomposition approach, which is summarized in Figure 2.17, is used to resolve this problem. In a first step, the total volume $V(\Omega)$ of the RVE (Ω) is divided into a set of infinitesimal pseudo-grains Ω_p . Each pseudo-grain Ω_p is constructed such as it contains a matrix material in the same concentration v_0 as in Ω and inclusions sharing the same orientation p in a concentration $1 - v_0$. The volume conservation of inclusions imposes that:

$$(1 - v_0)dV(\Omega_p) = v_1 V(\Omega) \psi(p) dp \quad (2.175)$$

which leads to:

$$\frac{dV(\Omega_p)}{V(\Omega)} = \psi(p) dp, \quad (2.176)$$

The concept of the aggregate decomposition suggests two stages of averaging over the RVE Ω . The first one is performed on each pseudo-grain, while the second one is performed over all pseudo-grains. The average over the entire volume $V(\Omega)$ of a given field μ is then written as:

$$\langle \mu \rangle_{\Omega} = \frac{1}{V(\Omega)} \int_{\Omega} \langle \mu \rangle_{\Omega_p} dV(\Omega_p) \quad (2.177)$$

then using Eq.(2.176), Eq.(2.177) becomes:

$$\langle \mu \rangle_{\Omega} = \oint \langle \mu \rangle_{\Omega_p} \psi(p) d(p) \quad (2.178)$$

which can be written under a more compact form as:

$$\langle \mu \rangle_{\Omega} = \left\langle \langle \mu \rangle_{\Omega_p} \right\rangle_{\psi} \quad (2.179)$$

$$\langle \mu \rangle_{\Omega} = \underbrace{\left\langle \underbrace{\langle \mu \rangle_{\Omega_p}}_{1^{st} \text{ homogenization}} \right\rangle_{\psi}}_{2^{nd} \text{ homogenization}} \quad (2.180)$$

The first homogenization step is performed on each pseudo-grain Ω_p (figure 2.17)

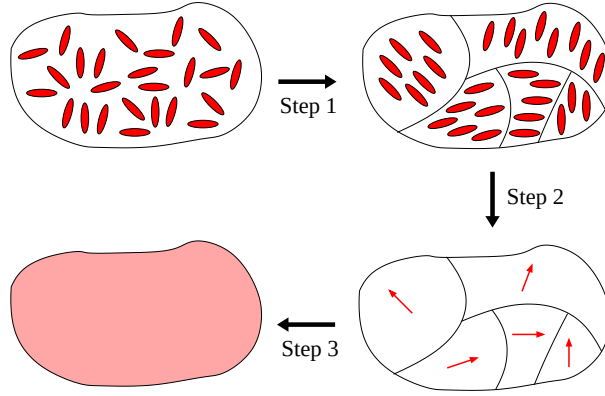


Figure 2.17: Aggregate decomposition approach for two-phase composite with misaligned inclusions

which is itself considered as an RVE consisting of a matrix phase with a volume fraction v_0 containing inclusions sharing a same orientation p and with $v_1 = 1 - v_0$. The pseudo-grain Ω_p is homogenized using one of the homogenization models detailed in section 2.9. For example, for a particular pseudo-grain Ω_p , strain concentration tensors A and $\tilde{A}$ are such as:

$$\begin{aligned} \langle \varepsilon \rangle_{\Omega_p}^{\omega_1} &= A_p : \langle \varepsilon \rangle_{\Omega_p} \\ \langle \varepsilon \rangle_{\Omega_p}^{\omega_0} &= \tilde{A}_p : \langle \varepsilon \rangle_{\Omega_p}^{\omega_0} \end{aligned} \quad (2.181)$$

where $\langle \varepsilon \rangle_{\Omega_p}^{\omega_1}$ and $\langle \varepsilon \rangle_{\Omega_p}^{\omega_0}$ represents the average strains over the inclusion and matrix phases of the pseudo-grain. Note that the tensors A_p and $\tilde{A}_p$ can be related with the

help of Eq.(2.33).

A_p and $\tilde{A}_p$ being computed it remains to evaluate the homogenized elastic operator $\bar{C}_p^{el}$ of the pseudo-grain Ω_p using the following relationship:

$$\bar{C}_p^{el} = C_0^{el} + v_1(C_1^{el} - C_0^{el}) : A_p \quad (2.182)$$

For example, if one wants to homogenize the pseudo-grain Ω_p using Mori-Tanaka scheme, it suffices to replace A_p in Eq.(2.182) by A_p^{MT} given by Eq.(2.88).

As above-mentioned, in the second homogenization stage which corresponds to step 3 in Figure 2.17, the set of all pseudo-grains is homogenized. Several hypotheses can be made to account for the distribution of strain between the pseudo-grains.

1. As a first assumption we assume that the strains are the same everywhere in the RVE Ω . It follows that the average strains are the same for each pseudo-grain Ω_p characterized by the orientation p . This corresponds to Voigt's assumption which is written as:

$$\langle \varepsilon \rangle_\Omega = \langle \varepsilon \rangle_{\Omega_p} \quad \forall p \quad (2.183)$$

The average stress in the RVE Ω is:

$$\langle \sigma \rangle_\Omega = \left\langle \langle \sigma \rangle_{\Omega_p} \right\rangle_\psi = \left\langle \bar{C}_p^{el} : \langle \varepsilon \rangle_{\Omega_p} \right\rangle_\psi$$

which is written using Eq.(2.183) as:

$$\langle \sigma \rangle_\Omega = \left\langle \bar{C}_p^{el} \right\rangle_\psi : \langle \varepsilon \rangle_\Omega \quad (2.184)$$

then by identification with Eq.(2.36) the expression of the overall elastic operator $\bar{C}^{el}$ of the RVE Ω is deduced. The result is:

$$\bar{C}^{el} = \left\langle \bar{C}_p^{el} \right\rangle_\psi \quad (2.185)$$

2. A second assumption is to suppose that stresses are the same everywhere in the RVE Ω and hence the average stresses are equal for each pseudo-grain Ω_p characterized by the orientation p . This corresponds to Reuss's assumption which is written as:

$$\langle \sigma \rangle_\Omega = \langle \sigma \rangle_{\Omega_p} \quad \forall p \quad (2.186)$$

The averaged strain over the RVE Ω is:

$$\langle \varepsilon \rangle_\Omega = \left\langle \langle \varepsilon \rangle_{\Omega_p} \right\rangle_\psi = \left\langle \left(\bar{C}_p^{el} \right)^{-1} : \langle \sigma \rangle_{\Omega_p} \right\rangle_\psi \quad (2.187)$$

which using Eq.(2.186) takes the following form:

$$\langle \varepsilon \rangle_{\Omega} = \left\langle \left(\bar{C}_p^{el} \right)^{-1} \right\rangle_{\psi} : \langle \sigma \rangle_{\Omega} \quad (2.188)$$

it follows, by identification with Eq.(2.36), the expression of homogenized elastic operator of the RVE Ω :

$$\bar{C}^{el} = \left\langle \left(\bar{C}_p^{el} \right)^{-1} \right\rangle_{\psi}^{-1} \quad (2.189)$$

3. A third hypothesis is to assume that the average strain in the matrix phase is the same for each pseudo-grain. Starting from the average strain in the RVE Ω we have:

$$\begin{aligned} \langle \varepsilon \rangle_{\Omega} &= \left\langle \langle \varepsilon \rangle_{\Omega_p} \right\rangle_{\psi} = \left\langle v_1 \langle \varepsilon \rangle_{\Omega_p^{\omega_1}} + (1 - v_1) \langle \varepsilon \rangle_{\Omega_p^{\omega_0}} \right\rangle_{\psi} \\ &= \left\langle \left(v_1 \tilde{A}_p + (1 - v_1) I^4 \right) : \langle \varepsilon \rangle_{\Omega_p^{\omega_0}} \right\rangle_{\psi} \\ &= \left\langle v_1 \tilde{A}_p + (1 - v_1) I^4 \right\rangle_{\psi_i} : \langle \varepsilon \rangle_{\Omega^{\omega_0}} \\ &= \left(v_1 \left\langle \tilde{A}_p \right\rangle_{\psi} + (1 - v_1) I^4 \right) : \langle \varepsilon \rangle_{\Omega^{\omega_0}} \end{aligned} \quad (2.190)$$

on the other hand the RVE average stress $\langle \sigma \rangle_{\Omega}$ is written as:

$$\begin{aligned} \langle \sigma \rangle_{\Omega} &= \left\langle \langle \sigma \rangle_{\Omega_p} \right\rangle_{\psi} = \left\langle v_1 \langle \sigma \rangle_{\Omega_p^{\omega_1}} + (1 - v_1) \langle \sigma \rangle_{\Omega_p^{\omega_0}} \right\rangle_{\psi} \\ &= \left\langle v_1 (C_1^{el} : \tilde{A}_p + (1 - v_1) C_0^{el}) : \langle \varepsilon \rangle_{\Omega_p^{\omega_0}} \right\rangle_{\psi} \\ &= \left\langle v_1 C_1^{el} : \tilde{A}_p + (1 - v_1) C_0^{el} \right\rangle_{\psi} : \langle \varepsilon \rangle_{\Omega^{\omega_0}} \\ &= \left(v_1 \left\langle C_1^{el} : \tilde{A}_p \right\rangle_{\psi} + (1 - v_1) C_0^{el} \right) : \langle \varepsilon \rangle_{\Omega^{\omega_0}} \end{aligned} \quad (2.191)$$

isolating $\langle \varepsilon \rangle_{\Omega^{\omega_0}}$ from Eq.(2.190) and using it into Eq.(2.191), then by identification with Eq.(2.36) one obtains the expression of the homogenized elastic operator $\bar{C}^{el}$. So we have:

$$\bar{C}^{el} = \left(v_1 \left\langle C_1^{el} : \tilde{A}_p \right\rangle_{\psi} + (1 - v_1) C_0^{el} \right) : \left(v_1 \left\langle \tilde{A}_p \right\rangle_{\psi} + (1 - v_1) I^4 \right)^{-1} \quad (2.192)$$

2.12.1 Averaging using orientation tensors

Averaging using Eq.(2.178) can be a very cumbersome and time-consuming operation . In some cases (for instance, composites with transversely isotropic stiffness), an efficient approach is the use a set of orientation state tensors. It has the advantage to characterize the inclusion-orientation effect in an average way. The orientation tensors a_i are defined as the i^{th} order moment of the distribution function $\psi(p)$. They are obtained by taking the dyadic products of the vector p and then integrating with the distribution function over all possible directions. For example, the second and fourth-rank orientation tensors are defined as:

$$a_{ij} = \oint p_i p_j \psi(p) dp, \quad (2.193)$$

$$a_{ijkl} = \oint p_i p_j p_k p_l \psi(p) dp, \quad (2.194)$$

and can be written under more compact form as:

$$a_{ij} = \langle p_i \otimes p_j \rangle_{\psi(p)} \quad (2.195)$$

$$a_{ijkl} = \langle p_i \otimes p_j \otimes p_k \otimes p_l \rangle_{\psi(p)}, \quad (2.196)$$

where $\otimes$ denotes the dyadic product. a_{ij} is symmetric and its trace is equal to unity, and $a_{11}, a_{22}, a_{33} \geq 0$. a_{ijkl} has the following properties: it possesses all symmetries, $a_{ijij} \geq 0$, and $a_{ijil} = a_{ij}$

When each pseudo-grain's stiffness is transversely isotropic with p axis, the overall stiffness can be written as:

$$\begin{aligned} \bar{C}_{ijkl}^{el}(p) &= B_1(p_i p_j p_k p_l) \\ &+ B_2(p_i p_j \delta_{ij} + p_k p_l \delta_{ij}) \\ &+ B_3(p_i p_k \delta_{jl} + p_i p_l \delta_{jk} + p_j p_l \delta_{ik} + p_j p_k \delta_{il}) \\ &+ B_4(\delta_{ij} \delta_{kl}) + B_5(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \end{aligned} \quad (2.197)$$

where $B_i, i = 1...5$ are scalar constants related to the five invariants of the transversely isotropic elastic tensor as follow:

$$\begin{aligned} B_1 &= \bar{C}_{1111} + \bar{C}_{2222} - 2\bar{C}_{1122} - 2\bar{C}_{1313} \\ B_2 &= \bar{C}_{1122} - \bar{C}_{2233} \\ B_3 &= \frac{1}{2}(\bar{C}_{1313} - \bar{C}_{2233} - \bar{C}_{2222}) \\ B_4 &= \bar{C}_{2233} \\ B_5 &= \frac{1}{2}(\bar{C}_{2222} - \bar{C}_{2233}). \end{aligned} \quad (2.198)$$

Averaging Eq.(2.197) over all possible orientation leads then to:

$$\begin{aligned} \langle \bar{C}_{ijkl}^{el}(p) \rangle_\psi &= B_1 a_{ijkl} + B_2 (a_{ij} \delta_{kl} + a_{kl} \delta_{ij}) \\ &+ B_3 (a_{ik} \delta_{jl} + a_{il} \delta_{jk} + a_{jl} \delta_{ik} + a_{jk} \delta_{il}) \\ &+ B_4 (\delta_{ij} \delta_{kl}) + B_5 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \end{aligned} \quad (2.199)$$

2.12.2 Closure approximations

The weakness of the use of orientation tensors is that the prediction of many material properties always required the acknowledge of higher-order orientation tensors. For example, the prediction of the elastic operator of a composite using orientation tensors necessitate the computation of the four-order tensor a_{ijkl} as displayed by (2.199). The aim of closure approximations is to evaluate higher-order orientation tensors from the lower-order ones. In case of the elastic tensor estimate, the close approximations will be used to approximate the fourth-rank tensor a_4 in terms of the second-order tensor a_2 . Many closure approximation formulas can be found in the literature ([3], [4], [16]), the most popular of them are:

Linear closure approximation

In 3D case, the linear approximation gives:

$$\begin{aligned} \hat{a}_{ijkl} = & - \frac{1}{35} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ & + \frac{1}{7} (a_{ij} \delta_{kl} + a_{ik} \delta_{jl} + a_{il} \delta_{jk} + a_{kl} \delta_{ij} + a_{jl} \delta_{ik} + a_{jk} \delta_{il}). \end{aligned} \quad (2.200)$$

Note that for an random distribution of orientation, the linear closure approximation gives an exact estimation of the fourth-order orientation tensor a_4 , see [3].

Quadratic closure approximation

The quadratic closure approximation is obtained by taking the dyadic product of the second-order tensor by itself. The result is:

$$\tilde{a}_{ijkl} = a_{ij} a_{kl}. \quad (2.201)$$

For perfectly aligned inclusion, the quadratic closure formula is exact, see [3].

Hybrid closure approximation

Hybrid closure approximation is a mix between quadratic and linear formulas pondered by a factor which represents a measure of the orientation. Its aim is to provide an approximation which is accurate enough over the entire possible orientations. The hybrid closure formula is written as:

$$\bar{q}_{ijkl} = (1 - f)\hat{a}_{ijkl} + f\tilde{a}_{ijkl}, \quad (2.202)$$

where f is a scalar measure of the orientation. If $f = 0$ this implies that $\bar{\mathbf{q}} = \hat{\mathbf{a}}_4$, inclusions are assumed to be randomly oriented, while if $f = 1$ they are assumed to be perfectly aligned and $\bar{\mathbf{q}} = \tilde{\mathbf{a}}_4$. Several forms of f were proposed, in [4] the authors suggest:

$$f = 1 - N_d \det a_2, \quad (2.203)$$

with $N_d = 4$ in the case of planar orientation, and $N_d = 27$ in the case of three-dimensional orientation.

It should be interesting to note, for both linear and hybrid closure approximations respectively, that estimated fourth-rank orientation tensors $\bar{\mathbf{q}}$ and $\hat{\mathbf{a}}_4$ loose certain properties that should have a fourth-rank orientation tensor. For instance, $\bar{\mathbf{q}}$ and $\hat{\mathbf{a}}_4$ do not possess the full symmetry.

2.13 Multi-phase composite materials

In some industrial applications, composite materials are manufactured by using a combination of more than two materials. For example, along speedways sections passing by urban zones, walls are built by adding to the concrete mixture steel fibers to improve mechanical properties of the wall and rubber particles in order to improve its sound-proofing. Many researchers were interested in the modeling of such composites and several methods were proposed to tackle this problem. In this section, we focus our interest on the aggregate decomposition approach based on the mean-field homogenization method, see for instance [54], and [72]. The key idea of this method can be summarized as follows: first we gather together in one pseudo-grain all inclusions sharing the same shape, orientation and material (figure 2.18 step 1). Next, each pseudo-grain is homogenized using one of the homogenization schemes cited in section 2.9 (figure 2.18 step 2). Finally, the set of the homogenized pseudo-grains forming our RVE is itself homogenized (figure 2.18 step 3). Let us now investigate this approach in some details. Consider a multi-phase composite made of a matrix phase in which different types of inclusions are embedded. ω_i represents the inclusion phase of the pseudo-grain Ω_i , and ω_0 the matrix phase. Consequently we have $\Omega_i = \omega_i \cup \omega_0$. Each pseudo-grain constituting our RVE is considered as a two-phase composite con-

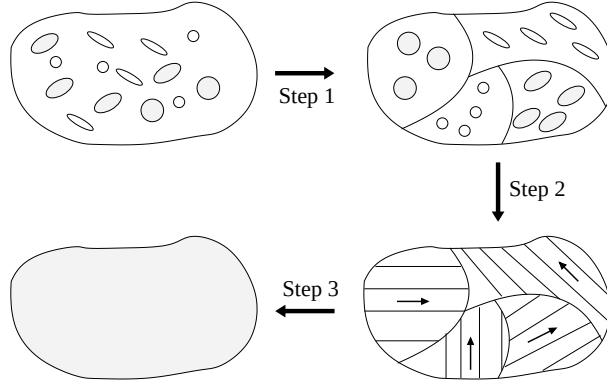


Figure 2.18: Aggregate decomposition approach for a multi-phase RVE

taining a matrix phase in concentration v_0 and inclusions in concentration $(1 - v_0)$. Note that the inclusions can be cavities.

In order to respect the concentration of the different types of inclusions present in the multi-phase, each pseudo-grain Ω_i (containing inclusions of type i) must have a volume:

$$\frac{dV(\Omega_{i,p})}{V(\Omega)} = \frac{v_i}{1 - v_0} \psi_i(p) dp, \quad (2.204)$$

where $\psi_i(p)$ represents a probability distribution function to find an inclusion whose orientation is between p and dp . It follows that $\psi_i(p)$ must satisfy the following conditions:

$$\psi_i(p) = \psi_i(-p), \quad \oint \psi_i(p) dp = 1, \quad (2.205)$$

as viewed in section 2.12.

For each pseudo-grain $\Omega_{i,p}$, a suitable homogenization scheme for a two phase composite gives the macro elastic homogenized operator $\bar{C}_{\Omega_{i,p}}^{el}$ and the macro stress $\langle \sigma \rangle_{\Omega_{i,p}}$ (Figure 2.18 step 2). Then, the set of pseudo-grains is homogenized according to the desired scheme (Figure 2.18 step 3). According to this, the average stress over Ω can be expressed as the sum of the average stress of each pseudo-grain weighted by its volume fraction (Eq.2.204)

$$\langle \sigma \rangle_{\Omega} = \sum_{i=1}^N \frac{v_i}{1 - v_0} \oint \langle \sigma \rangle_{\Omega_{i,p}} \psi_i(p) dp, \quad (2.206)$$

$$\bar{C}^{el} : \langle \varepsilon \rangle_{\Omega} = \sum_{i=1}^N \frac{v_i}{1 - v_0} \oint \bar{C}_{\Omega_{i,p}}^{el} : \langle \varepsilon \rangle_{\Omega_{i,p}} \psi_i(p) dp, \quad (2.207)$$

which can be written in more compact form as:

$$\langle \sigma \rangle_{\Omega} = \bar{C}^{el} : \langle \varepsilon \rangle_{\Omega} = \left\langle \bar{C}_{\Omega_i,p}^{el} : \langle \varepsilon \rangle_{\Omega_i,p} \right\rangle_{i,\psi_i}. \quad (2.208)$$

Several possibilities can be considered to obtain the expression of the overall elastic operator $\bar{C}^{el}$ of the multi-phase RVE Ω .

- The first possibility studied hereafter is the use of Voigt's assumption which states that the strains in the composite are uniform, and consequently, the average strains are the same for each pseudo-grain. So we have:

$$\langle \varepsilon \rangle_{\Omega} = \langle \varepsilon \rangle_{\Omega_{1,p}} = \langle \varepsilon \rangle_{\Omega_{2,p}} = \dots = \langle \varepsilon \rangle_{\Omega_{N,p}}. \quad (2.209)$$

Eq.(2.208) is then rewritten as:

$$\begin{aligned} \bar{C}^{el} : \langle \varepsilon \rangle_{\Omega} &= \left\langle \bar{C}_{\Omega_i,p}^{el} : \langle \varepsilon \rangle_{\Omega} \right\rangle_{i,\psi_i}, \\ &= \left\langle \bar{C}_{\Omega_i,p}^{el} \right\rangle_{i,\psi} : \langle \varepsilon \rangle_{\Omega}, \end{aligned} \quad (2.210)$$

from which is deduced the expression of the homogenized elastic operator $\bar{C}^{el}$ for a multi-phase composite following the two-step homogenization approach with Voigt scheme as the second homogenization step:

$$\bar{C}^{el} = \left\langle \bar{C}_{\Omega_i,p}^{el} \right\rangle_{i,\psi}. \quad (2.211)$$

For a fixed first-step homogenization scheme, the estimate provided with this scheme is considered as the upper bound.

- A second possibility is the use of Reuss's model for the second homogenization. The average strain $\langle \varepsilon \rangle_{\Omega}$ in the multi-phase RVE Ω is given by:

$$\begin{aligned} \langle \varepsilon \rangle_{\Omega} &= \left\langle \langle \varepsilon \rangle_{\Omega_i,p} \right\rangle_{i,\psi_i} = \left\langle \left(\bar{C}_{i,p}^{el} \right)^{-1} : \langle \sigma \rangle_{\Omega_i,p} \right\rangle_{i,\psi_i}, \\ &= \left\langle \left(\bar{C}_{i,p}^{el} \right)^{-1} \right\rangle_{\psi_i} : \langle \sigma \rangle_{\Omega}, \end{aligned} \quad (2.212)$$

which can be rewritten as:

$$\langle \varepsilon \rangle_{\Omega} = \left\langle \left(\bar{C}_{i,p}^{el} \right)^{-1} \right\rangle_{\psi_i} : \langle \sigma \rangle_{\Omega}. \quad (2.213)$$

since Reuss hypothesis assumes that stresses are the same for all constituents of the RVE, it results that:

$$\langle \sigma \rangle_{\Omega} = \langle \sigma \rangle_{\Omega_{1,p}} = \langle \sigma \rangle_{\Omega_{2,p}} = \dots = \langle \sigma \rangle_{\Omega_{N,p}}. \quad (2.214)$$

Inversion of Eq.(2.213) then identification with Eq.(2.208) gives the expression of the homogenized elastic operator $\bar{C}^{el}$ for a multi-phase composite following the two-step homogenization approach with Reuss scheme as second homogenization step. One obtains:

$$\bar{C}^{el} = \left\langle \left(\bar{C}_p^{el} \right)^{-1} \right\rangle_{i, \psi_i}^{-1}. \quad (2.215)$$

For a fixed first-step homogenization scheme, the estimate provided with this scheme is considered as the lower bound.

- Another possibility is to assume that the average strains $\langle \varepsilon \rangle_{\Omega_{i,p}^{\omega_0}}$ of the matrix phase are the same for each pseudo-grain:

$$\langle \varepsilon \rangle_{\Omega_{1,p}^{\omega_0}} = \langle \varepsilon \rangle_{\Omega_{2,p}^{\omega_0}} = \dots = \langle \varepsilon \rangle_{\Omega_{N,p}^{\omega_0}} = \langle \varepsilon \rangle_{\Omega^{\omega_0}}. \quad (2.216)$$

Starting from the average strain of RVE Ω , one can write:

$$\begin{aligned} \langle \varepsilon \rangle_{\Omega} &= \left\langle \langle \varepsilon \rangle_{\Omega_p} \right\rangle_{i, \psi_i} = \left\langle (1 - v_0) \langle \varepsilon \rangle_{\Omega_{i,p}^{\omega_1}} + v_0 \langle \varepsilon \rangle_{\Omega_{i,p}^{\omega_0}} \right\rangle_{i, \psi_i}, \\ &= \left\langle ((1 - v_0) \tilde{A}_{i,p} + v_0 I^4) : \langle \varepsilon \rangle_{\Omega_{i,p}^{\omega_0}} \right\rangle_{i, \psi_i}, \\ &= \langle (1 - v_0) \tilde{A}_{i,p} + v_0 I^4 \rangle_{i, \psi_i} : \langle \varepsilon \rangle_{\Omega^{\omega_0}}, \\ &= \left((1 - v_0) \left\langle \tilde{A}_{i,p} \right\rangle_{i, \psi_i} + v_0 I^4 \right) : \langle \varepsilon \rangle_{\Omega^{\omega_0}}. \end{aligned} \quad (2.217)$$

the average stress $\langle \sigma \rangle_{\Omega}$ in Ω is written as:

$$\begin{aligned} \langle \sigma \rangle_{\Omega} &= \left\langle \langle \sigma \rangle_{\Omega_p} \right\rangle_{i, \psi_i} = \left\langle (1 - v_0) \langle \sigma \rangle_{\Omega_{i,p}^{\omega_1}} + v_0 \langle \sigma \rangle_{\Omega_{i,p}^{\omega_0}} \right\rangle_{i, \psi_i}, \\ &= \left\langle (1 - v_0) (C_i^{el} : \tilde{A}_{i,p} + v_0 C_0^{el}) : \langle \varepsilon \rangle_{\Omega_{i,p}^{\omega_0}} \right\rangle_{i, \psi_i}, \\ &= \langle (1 - v_0) C_i^{el} : \tilde{A}_{i,p} + v_0 C_0^{el} \rangle_{i, \psi_i} : \langle \varepsilon \rangle_{\Omega^{\omega_0}}, \\ &= \left((1 - v_0) \left\langle C_i^{el} : \tilde{A}_{i,p} \right\rangle_{i, \psi_i} + v_0 C_0^{el} \right) : \langle \varepsilon \rangle_{\Omega^{\omega_0}}. \end{aligned} \quad (2.218)$$

Isolating $\langle \varepsilon \rangle_{\Omega^{\omega_0}}$ from Eq.(4.59) and using it into Eq.(4.60), then by identification with $\langle \sigma \rangle_{\Omega} = \bar{C}^{el} : \langle \varepsilon \rangle_{\Omega}$ one obtain:

$$\bar{C}^{el} = \left((1 - v_0) \left\langle C_i^{el} : \tilde{A}_{i,p} \right\rangle_{i, \psi_i} + v_0 C_0^{el} \right) : \left((1 - v_0) \left\langle \tilde{A}_{i,p} \right\rangle_{i, \psi_i} + v_0 I^4 \right)^{-1}. \quad (2.219)$$

For a fixed first-step homogenization scheme, the first possibility which consists in choosing Voigt's hypothesis for the second-step homogenization scheme, leads to estimate (Eq.2.210) considered as the upper bound. The second possibility provides estimate (Eq.2.215) considered as the lower bound. It is worth to note that in some cases for example if Mori-Tanaka is chosen as the first-step homogenization scheme, it was observed that predicted overall elastic operator obtained with the third two-step homogenization scheme (Eq.2.219) becomes non-symmetric and estimates may violate the upper bound, see [72].

2.14 Numerical simulations

The accuracy of homogenization models developed above in the elasticity regime, is evaluated through series of numerical simulations. In all numerical simulations presented in this section, we consider three constitutive elastic materials:

1. Concrete material with Young's modulus $E = 36900 \text{ MPa}$ and Poisson's ratio $\nu = 0.19$.
2. Steel material with Young's modulus $E = 210000 \text{ MPa}$ and Poisson's ratio $\nu = 0.3$
3. Polymer material with Young's modulus $E = 6000 \text{ MPa}$ and Poisson's ratio $\nu = 0.3$.

Simulations are performed on composites consisting of a matrix phase made of concrete in which steel or polymer inclusions are embedded.

2.14.1 Volume fraction influence

As a first example we wish to study the influence of the volume fraction of inclusions on the overall mechanical properties E of two-phase concrete-like matrix composites. Two composites are considered: (i) Concrete matrix Composite reinforced with Steel Spherical inclusions denoted SSCC, and (ii) Concrete matrix Composite with Polymer Spheres denoted PSCC. The overall elastic modulus E of those two composites are estimated with several homogenization schemes and the predicted results are compared with finite elements calculations carried out using Abaqus software. For finite elements calculations we consider a composite with a periodic micro-structure. Figure 2.19 illustrates the different steps of the transition from a 3D representation of the two-phase composite to a 2D axisymmetric unit cell model. The prescribed boundary conditions are as follows: Zero displacements are imposed in the radial direction to the

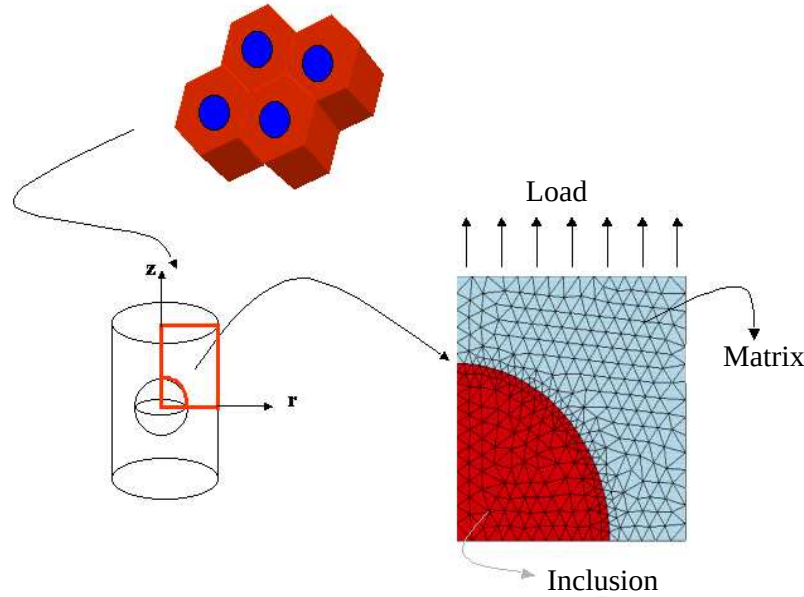


Figure 2.19: 2D Unit cell representation of a two-phase composite with periodic microstructure under axisymmetric conditions.

left vertical side and in the vertical direction to the bottom horizontal side. The right vertical side is constrained to have uniform radial displacement. Uniform vertical displacement is imposed on the top horizontal side. Figures 2.20 and 2.21 illustrate the evolution of the homogenized elastic modulus for SSCC and PSCC specimen versus inclusion volume fraction. One can see a good agreement between the homogenized elastic modulus predicted by Mori-Tanaka model and F.E. calculations. It can be seen also that Voigt's and Reuss's predictions constitute the upper and lower bounds respectively.

2.14.2 Aspect ratio influence

In this example, the influence of inclusions aspect ratio on the mechanical properties of the composite consisting of a concrete matrix with steel inclusions embedded therein,

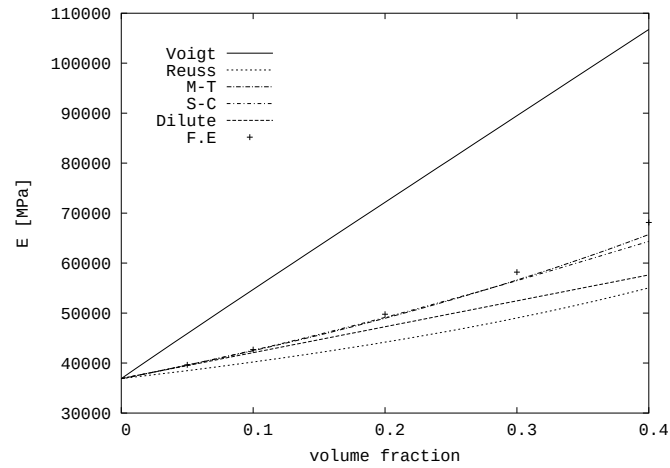


Figure 2.20: Overall Young modulus of SSCC versus inclusions volume fractions: Homogenization schemes estimates against unit cell F.E computation.

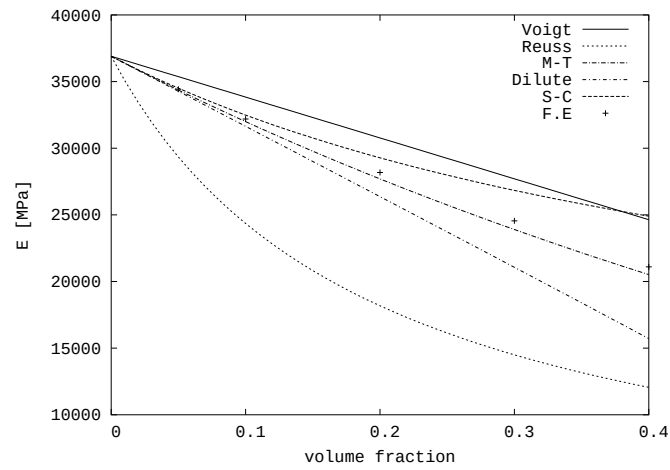


Figure 2.21: Overall Young modulus of PSCC versus inclusions volume fractions: Homogenization schemes estimates against unit cell F.E computation

is emphasized. Several values of aspect ratio of inclusion are considered, $Ar = 1$ for spherical inclusions, $Ar = 5$ and 10 for short fibers, and $Ar = 100$ for long fibers. Of course, the fibers are assumed aligned. Predictions were made using Mori-Tanaka homogenization scheme and results are displayed in Figure 2.22. Voigt's and Reuss's

predictions, which do not account for inclusions aspect ratio, are displayed in the same figure. One can see that composite softens with decreasing values of Ar . Note that for values of Ar higher than 100, Mori-Tanaka and Voigt predictions are very close (superposed). This means that M-T scheme regards discontinuous fibers with Ar higher than 100 as long continuous fibers.

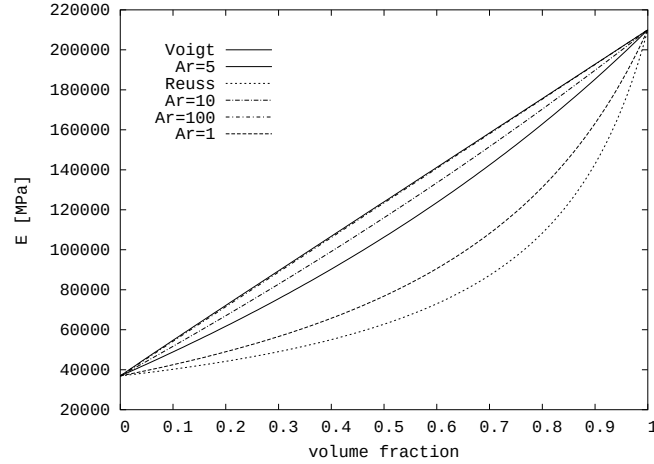


Figure 2.22: Influence of reinforcement shape on homogenized elastic modulus of a steel inclusion-reinforced concrete composite.

2.14.3 Inclusions orientation influence

In the following, the influence of the orientation of inclusions on the homogenized longitudinal elastic modulus is studied. The mechanical properties of the considered composite are the same as that for the previous example. Two cases of orientation were investigated: (i) inclusions aligned with the longitudinal axis and (ii) inclusions randomly oriented in space. In the first set of simulations which are displayed in Figure 2.23, the inclusions of the composite have an $Ar = 60$ while in the second set of simulations (Figure 2.24) their $Ar = 10$. Are also plotted in those figures prediction for composite with spherical inclusions $Ar = 1$ and predictions obtained using Voigt and Reuss models. Figures 2.23 and 2.24 show that responses of composites with randomly oriented fibers having $Ar = 10$ and $Ar = 60$ respectively are very close to those of composite with spherical inclusions $Ar = 1$, which is not affected by orientation. Under the lights of those results, one can conclude that for composites with randomly oriented fibers, the aspect ratio does not influence the response of the composite in a

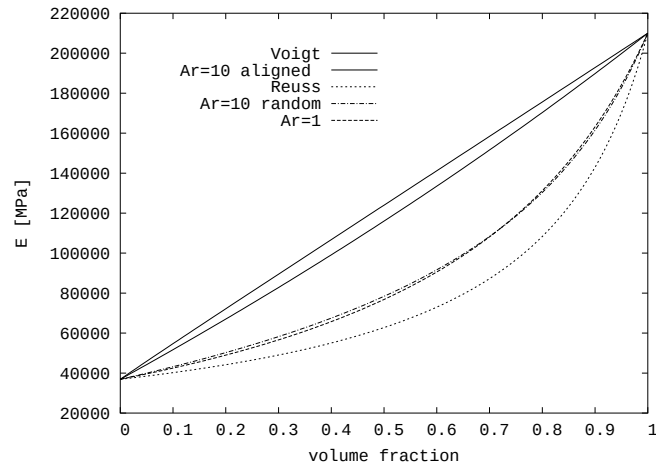


Figure 2.23: Orientation influence on the predicted Young modulus of Steel Fiber-Reinforced Concrete with $A_r = 10$

significant manner, in particular in ranges of a very low and very high volume fractions of fibers.

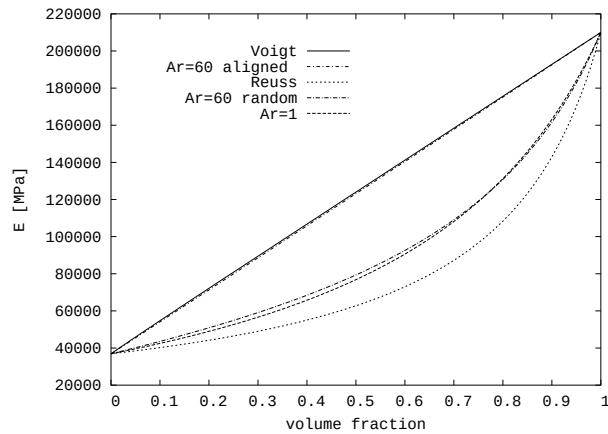


Figure 2.24: Orientation influence on the predicted Young modulus of Steel Fiber-Reinforced Concrete with $A_r = 60$

2.14.4 Composite Poisson's ratio estimate

The prediction of Poisson's ratio ν of composite materials in elastic regime is also possible using homogenization schemes. In Figure 2.25 the evolution of Poisson's ratio of concrete matrix reinforced with spherical steel inclusions ($A_r = 1$) versus the volume fraction of the reinforcement, is displayed.

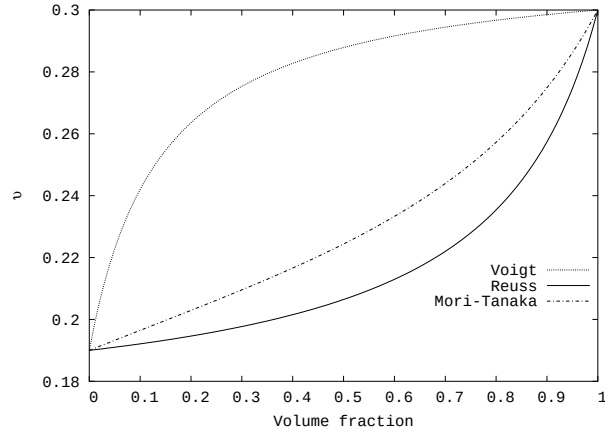


Figure 2.25: Evolution of Poisson's ratio versus reinforcement volume fraction.

One can notice that Mori-Tanaka estimate is located between Voigt and Reuss predictions respectively. It seems that Voigt and Reuss predictions also constitute an upper and lower bounds for Poisson's ratio estimate.

Chapter 3

Micro-constitutive models

3.1 General Introduction

Before we investigate the homogenization problem of inelastic composite materials, it is worth to introduce the nonlinear constitutive laws that will be used to model the behavior of composite constituents at the micro-level. The knowledge of constitutive equations which govern the behavior of materials is an important step in order to achieve structural design, structural computation,... etc. This chapter provides a quick survey of mechanical modeling of two classes of materials. The first class gathers together materials describing a brittle damage behavior such as concrete, rocks, ceramics, etc. The model that we will use for this first family of materials is Ju's damage energy-based model. The second family regroups materials obeying elasto-plastic behavior such as metals or polymers for instance. Their behavior is modeled making use of J_2 p elasto-plasticity theory. We limit our study to the case of small strain, isothermal, and time-independent conditions.

This chapter is divided in two main parts; one devoted to damage modeling and one to elasto-plastic behavior modeling. In the first part, we first recall the most popular Continuum Damage Mechanics theories. Then the concept of damage is introduced through a uniaxial tensile loading test. Those general concepts being introduced, we tackle the derivation of Ju's model. We address a particular attention to the spectral decomposition and the thermomechanical framework used in this model. Then we derive the secant elasto-damaged operator and finally we propose an algorithm which enables to integrate Ju's damage model in a numerical code. In the second part of the

chapter, which is dedicated to elastoplastic modeling, we first introduce the concept of strain irreversibility through a uniaxial tensile loading test. A one-dimensional model is developed and extended to the three-dimensional case. Then some useful results used in J_2 p theory are presented and the continuum tangent operator is derived. Several hardening laws are proposed. Finally we present the return mapping algorithm used for J_2 p model, and we derive the expression of the consistent tangent operator, also called *algorithmic*, resulting from time-discretization of the constitutive laws of J_2 p model.

3.2 Damage behavior modeling

Damage modeling has been the subject of many investigations in the science community. To describe a such process, many models based on various concepts were proposed. There are at least three categories of models:

- (1) Statistical models
- (2) Micro mechanical models
- (3) Continuum models

In this thesis we focus our interest on models based on Continuum Damage Mechanics (CDM) which belong to the last category. The CDM concept was initiated by L. M. Kachanov in 1958 [42] and Rabotnov in 1969 [82] when they studied creep of metals. Hult, Lemaitre, Chaboche, Leckie,...etc. generalized this concept to other loading cases and applied it to various kinds of materials. In the phenomenological approach of CDM two concepts can be used, the first one is damage effect variables introduced by Kachanov, while the second one is the effective stress-effective strain proposed by Lemaitre-Chaboche.

CDM was developed within a thermodynamical framework. First, Chaboche [11] in 1977 dealt with the case of the isotropic damage. Then, generalization of this work to anisotropic cases was achieved with the works of many researchers. Krajcinovic and Fonseka [43] in 1981 incorporated damage anisotropy by the means of vectors, Cordebois and Sidoroff in 1979 [17] used second-order tensor while Lemaitre and Chaboche in 1985 [50], Simo and Ju in 1987 [90] and Ju in 1989 [40] introduced this anisotropy with the help of fourth-rank tensors.

3.2.1 Most popular CDM theories

There are three typical CDM theories. Hereafter, we give the basic ideas used in each theory, for a more elaborate discussion the reader is referred to [12] and references therein.

CDM model with scalar damage

The first Unilateral Damage Model with Scalar variables was proposed by Ladevèze and Lemaitre in 1984 [46] and reviewed by Mazars and Pijaudier-Cabot in 1989 [59]. In this theory, damage is described by two scalar variables. One, D_t , for tensile loading cases and one, denoted D_c for compression tests. The unilateral condition is introduced by the decomposition of the stress tensor σ into a positive part σ^+ and a negative one σ^- . The model derives from a thermodynamical potential which is written in terms of Young's modulus E , Poisson's ratio ν , D_t and D_c , and negative and positive parts of stress. Damage evolution is governed by damage loading surface (derived from a dissipative potential) which is expressed under the form of a doubles criterion, to take into account the tension and compression cases. The stress tensor decomposition is performed in the principal stress system as follows:

$$\sigma_- = - \begin{pmatrix} \langle -\sigma_1 \rangle & 0 & 0 \\ 0 & \langle -\sigma_2 \rangle & 0 \\ 0 & 0 & \langle -\sigma_3 \rangle \end{pmatrix}; \quad \sigma_+ = \begin{pmatrix} \langle \sigma_1 \rangle & 0 & 0 \\ 0 & \langle \sigma_2 \rangle & 0 \\ 0 & 0 & \langle \sigma_3 \rangle \end{pmatrix} \quad (3.1)$$

where $\langle \cdot \rangle$ refers to the Mac-Cauley bracket defined by $\langle x \rangle = x$ for $x \geq 0$ and $\langle x \rangle = 0$ otherwise. The thermodynamic potential from which the constitutive law is derived, is defined as follows:

$$\begin{aligned} \Psi^* = & \frac{1}{2} \left\{ \frac{1}{E(1-D_t)} [(1+\nu)\sigma_+ : \sigma_+ - \nu(tr\sigma)_+^2] \right. \\ & \left. + \frac{1}{E(1-D_c)} [(1+\nu)\sigma_- : \sigma_- - \nu(tr\sigma)_-^2] \right\}, \end{aligned} \quad (3.2)$$

where:

$$(tr \sigma)_+ = \langle tr \sigma \rangle; \quad \text{and} \quad (tr \sigma)_- = - \langle -tr \sigma \rangle, \quad (3.3)$$

with:

$$\sigma = \sigma_- + \sigma_+. \quad (3.4)$$

The strain-stress relationship for this model is then written as:

$$\begin{aligned} \varepsilon = \frac{\partial \Psi^*}{\partial \sigma} = & \frac{1}{E(1-D_t)} [(1+\nu)\sigma_+ - \nu(tr\sigma)_+ \mathbf{1}] \\ & + \frac{1}{E(1-D_c)} [(1+\nu)\sigma_- - \nu(tr\sigma)_- \mathbf{1}], \end{aligned} \quad (3.5)$$

where 1 is the unit second-rank tensor.

CDM model with second-rank damage tensors

In the second CDM, the damage is described by second order tensors representation. In the model proposed by Ramtani in 1990 [83] for instance, the unilateral effect is accommodated through a strain tensor decomposition defined in the same way as for the above-mentioned stress decomposition. The proposed free energy is written in terms of Lamé's coefficients λ and μ , positive and negative strains, damage second order tensors d^* and d , and scalar damage δ . The role of d^* , called *transmitted damage*, is to take into account microcracks that are parallel to the compressive stress direction, while d called *direct damage* accounted for microcracks parallel to the maximum tensile stress. The scalar damage δ introduces volumetric damaging effect. The thermodynamical forces associated to damage variables are consequently second order tensors Y and Y^* , and the scalar Y_δ . Three independent damage loading surfaces are also used. The first one is associated to d^* , the second one to d and the third one to $Tr\varepsilon$. The thermodynamic potential function is defined by:

$$\begin{aligned}\Psi &= \mu[(1-d)^{1/2} \cdot \langle \varepsilon \rangle_+] : [(1-d)^{1/2} \cdot \langle \varepsilon \rangle_+] \\ &+ \mu[(1-d^*)^{1/2} \cdot \langle \varepsilon \rangle_-] : [(1-d^*)^{1/2} \cdot \langle \varepsilon \rangle_-] \\ &+ \frac{1}{2}\lambda(1-\delta)(tr \varepsilon)^2;\end{aligned}\tag{3.6}$$

from which the stress-strain relationship is derived as:

$$\begin{aligned}\sigma &= 2\mu(1-d)^{1/2} \cdot \langle \varepsilon \rangle_+ \cdot (1-d)^{1/2} \\ &+ 2\mu(1-d^*)^{1/2} \cdot \langle \varepsilon \rangle_- \cdot (1-d^*)^{1/2} \\ &+ \lambda(1-\delta)(tr \varepsilon)1.\end{aligned}\tag{3.7}$$

CDM model with fourth-rank damage tensors

The third CDM theory was developed by Simo and Ju in 1987 [90]. In this theory, which we will see in more detail next in this chapter, damage is taken into account through a fourth rank tensor. The original idea of this model is that Ju and Simo used the secant stiffness operator as internal variable to represent damage. The unilateral effect is taken into account through the spectral decomposition of the strain tensor.

CDM Models evaluation

Each one of models cited above is discussed hereafter briefly, according to the following properties. For a more elaborate discussion see [12] and references therein.

(1) Capability of a model to reproduce anisotropy of damage

- Models with scalar damage: In those models, damage is always isotropic
- Models with second-order tensor damage: Those models are the only ones able to reproduce anisotropic damage evolution under any loading cases.
- Models with fourth-order tensor damage: Not sufficiently, the damage induced in the case where the principal strains are all positive is isotropic.

(2) Damaged operator symmetry

- Models with scalar damage: Always satisfied.
- Models with second-order tensor damage: In cases of complex loading, the models can lead to nonsymmetric damaged operator.
- Models with fourth-order tensor damage: Always satisfied.

(3) Stress continuity when the unilateral condition takes place

- Models with scalar damage: Always obtained.
- Models with second-order tensor damage: Always obtained.
- Models with fourth-order tensor damage: In some complex loading cases and under certain conditions, the stress-strain response predicted may exhibit discontinuity (see section 3.2.5).

3.2.2 Damage concept

The basic ideas of brittle damage concept can be understood by studying the response of concrete-like material specimen submitted to uniaxial tensile loading and unloading test. The stress-strain curve of a such test is displayed in figure 3.1. The path OA is a straight line of slope E , Young's modulus of the material. Along this line, the material exhibits an elastic behavior. Stress σ and strain ε are proportional and are related as follows:

$$\sigma = E\varepsilon. \quad (3.8)$$

Note that in this range, total strain is equal to elastic one: $\varepsilon = \varepsilon^e$. Up to now, the material did not suffer from any damage. Along the path AB , the response becomes

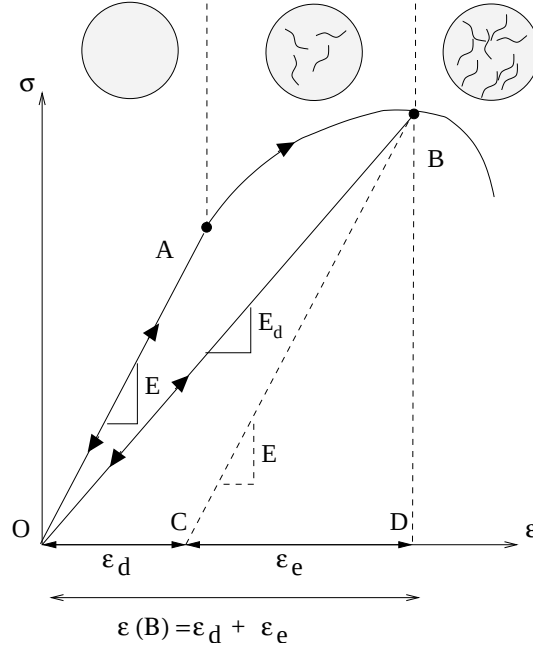


Figure 3.1: Stress-strain curve of concrete-like material specimen under uniaxial tensile loading and unloading test.

nonlinear, stress and strain are not proportional and damage takes place. Indeed, voids growth and microcracks activation processes can be observed within the sample. The state of damage in the material can be characterized by means of a damage criterion (damage surface loading) g with the following functional form:

$$g \equiv \tilde{G} - r \leq 0; \quad (3.9)$$

where $\tilde{G}$ is a damage energy release rate and r is a damage threshold. This criterion can be viewed as a competition between the available energy for damage " $\tilde{G}$ " and the suitable energy for damage " r ". As long as $g \leq 0$, damage does not evolve. If we unload anywhere along AB , the response will be along the straight line BO with a slope E_d called damaged elastic modulus. If we reload, the response will be along OB and damage will evolve again only when point B is reached. Beyond this point, the material softens and microcracks nucleate into a macrocrack leading to the total rupture of the sample. At point B the strain $\epsilon(B)$ is equal to $\overline{OD}$. This strain can be divided into an elastic part $\epsilon^e(B)$ and a part due to damage $\epsilon^d(B)$. If starting from point B we fictitiously unload the sample following a straight line of a slope equal to initial elastic modulus E , the response will cross the strain axis at point C . The elastic

strain $\varepsilon^e(B)$ is then equal to $(\overline{CD})$ while the strain due to damage $\varepsilon^d(B)$ is equal to $(\overline{OC})$.

It should be pointed out that when dealing with material exhibiting elasto-damaged (brittle) behavior the tensile test response does not display a permanent strain (as it is the case with elasto-plastic material). However the effect of the irreversibility in brittle damage process is perceptible on the irreversible alteration of the stiffness resulting from material degradation.

3.2.3 Spectral decomposition

For multi-axial stress states, the evolution of the pattern of microcracks in brittle materials such as concrete or rocks is governed primarily by the principal strains or stresses. When such materials undergo tensile fields, microcracks are propagated in planes which are orthogonal to the principal tensile stresses (Krajcinovic, 1996). In Ju's model, damage evolution is governed by principal tensile strains. To accommodate the microcrack opening/closing unilateral effect in the model, a spectral decomposition of the strain tensor is performed. First, we start by computing the eigenvalues ε_i and eigenvectors e_i (forming an orthonormal and direct basis) of the total strain tensor ε . Therefore, the following spectral decomposition holds:

$$\varepsilon = \sum_{i=1}^3 \varepsilon_i e_i \otimes e_i, \quad e_i \cdot e_j = \delta_{ij}, \quad (3.10)$$

where δ_{ij} designates Kronecker's symbol. Next, the following positive projection operators are defined:

$$Q^+(\varepsilon) = \sum_{i=1}^3 \hat{H}(\varepsilon_i) e_i \otimes e_i \quad (3.11)$$

$$P_{ijkl}^+(Q^+) = \frac{1}{2} (Q_{ik}^+ Q_{jl}^+ + Q_{il}^+ Q_{jk}^+), \quad (3.12)$$

where Q^+ and P^+ are symmetric second- and fourth-rank tensors respectively, and $\hat{H}(\varepsilon_i)$ represents the Heaviside ramp function: $\hat{H}(\varepsilon_i) = 1$ if $\varepsilon_i > 0$ and $\hat{H}(\varepsilon_i) = 0$ otherwise. The aim of the Heaviside function, as one can expect, is to eliminate the negative eigenvalues representing the negative (compressive) principal strains which we assume that they do not operate in damage process. The positive or tensile part of the strain tensor can now be defined as follows:

$$\varepsilon^+ (\varepsilon) = P^+ : \varepsilon. \quad (3.13)$$

Note that this relation is nonlinear, because P^+ depends on ε . It is easily checked that if all three principal strains ε_i are tensile, then $Q_{ij}^+ = \delta_{ij}$, $P^+ = I$ the identity fourth-

rank tensor, and $\varepsilon^+ = \varepsilon$. However, if all ε_i are compressive then $\varepsilon^+ = 0$ and no damage will be predicted in the model described hereafter.

3.2.4 Thermodynamic framework for Ju's model

Ju's unilateral damage model was developed within the thermodynamical framework of Continuum Damage Mechanics. In his theory, Ju postulated the existence of a total damaged potential from which the constitutive equation can be derived. In the so-called uncoupled formulation, Ψ is written as the sum of elastic damaged potential Ψ_{ed} and plastic damaged potential Ψ_{pd} as:

$$\Psi(\varepsilon, q, C^d) = \Psi_{ed}(\varepsilon, C^d) + \Psi_{pd}(q, C^d), \quad (3.14)$$

where ε is the elastic-damage strain tensor, the damaged stiffness C^d and the set of plastic variables q represents the state variables. In the present study, we only consider $\Psi_{ed}(\varepsilon, C^d)$. This choice is motivated by the fact that dealing with brittle material such as concrete or rock, only the elastic-damaged potential is needed. According to thermodynamical considerations, $\Psi_{ed}(\varepsilon, C^d)$ must be convex. A quadratic form is chosen, so we have:

$$\Psi \equiv \Psi_{ed}(\varepsilon, C^d) = \frac{1}{2} \varepsilon : C^d : \varepsilon. \quad (3.15)$$

Assuming isothermal condition $\Delta T = 0$, Clausius-Duhem inequality can be written under the following form:

$$\sigma : \dot{\varepsilon} - \dot{\Psi} \geq 0. \quad (3.16)$$

Time derivative of the total free energy potential Ψ leads to:

$$\begin{aligned} \dot{\Psi} &= \frac{\partial \Psi}{\partial \varepsilon} : \dot{\varepsilon} + \frac{\partial \Psi}{\partial C^d} :: \dot{C}^d, \\ &= C^d : \varepsilon : \dot{\varepsilon} + \frac{1}{2} \varepsilon \otimes \varepsilon :: \dot{C}^d. \end{aligned} \quad (3.17)$$

Replacing Eq.3.17 into Eq.3.16 and rearranging, Clausius-Duhem inequality (Eq.3.16) becomes:

$$[\sigma - C^d : \varepsilon] : \dot{\varepsilon} - \frac{1}{2} (\varepsilon \otimes \varepsilon :: \dot{C}^d) \geq 0. \quad (3.18)$$

Assuming, for example, an elastic transformation which does not induce damage evolution ($\dot{C}^d = 0$) the previous inequality then implies that:

$$\sigma = C^d : \varepsilon \quad (3.19)$$

which is the stress-strain relationship. This equality being satisfied, it follows the expression of damage dissipation $\mathbb{D}^d$ deduced from (3.16) as:

$$\mathbb{D}^d = -\frac{1}{2}(\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} :: \dot{\mathbf{C}}^d) \geq 0; \quad (3.20)$$

and which can be represented in a schematic manner as displayed in figure 3.2.

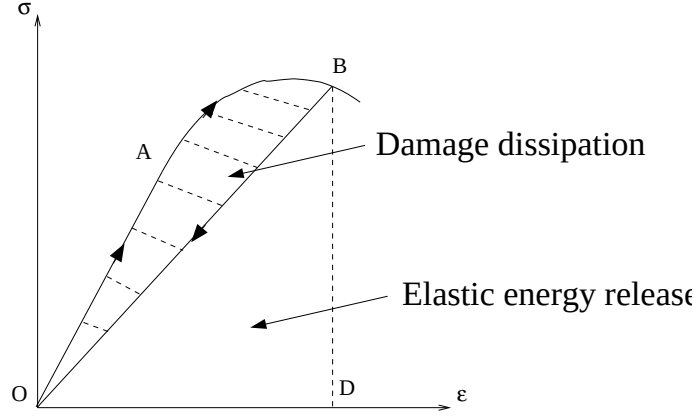


Figure 3.2: Energy balance for brittle material.

On the other hand we define Y as:

$$-Y \equiv \frac{\partial \Psi}{\partial \mathbf{C}^d} = \frac{1}{2} \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}. \quad (3.21)$$

Eq.3.20 is then rewritten as:

$$\mathbb{D}^d = -(-Y) :: \dot{\mathbf{C}}^d \geq 0, \quad (3.22)$$

where from the above inequality, one can conclude that $-Y$ is the thermodynamic force conjugate to the damaged secant stiffness tensor $\mathbf{C}^d$.

3.2.5 Secant elasto-damaged operator derivation

According to the second principle of thermodynamics, the evolution of the state variables is governed by the dissipative potential [50]. In the same way as for associative plasticity, the damage loading surface is chosen to be the dissipative potential. The evolution of the state variable tensor $\mathbf{C}^d$ is then given by:

$$\dot{\mathbf{C}}^d = -\dot{\mu} \frac{\partial g}{\partial (-Y)}, \quad (3.23)$$

where:

$$\dot{\mu} \geq 0 \quad g \leq 0 \quad \dot{\mu}g = 0, \quad (3.24)$$

are the Kuhn-Tucker relations. Eq.3.23 means that the damaged operator C^d is the operator that maximizes the damaged energy dissipation. g is defined in terms of Y as:

$$g \equiv \tilde{G}(-Y, a, c, p, \dots) - r \leq 0, \quad (3.25)$$

where a is the aggregate size, c the microcrack size, and p the porosity. However, to properly introduce Y into the function $\tilde{G}$, it is essential to define the characteristic damage measure ξ such as:

$$\frac{\partial \xi}{\partial(-Y)} \equiv P^+ : C^{el} : P^+, \quad (3.26)$$

which leads, by neglecting $\frac{\partial P^+}{\partial \varepsilon} : \varepsilon$ ahead P^+ , to

$$\xi \equiv \frac{1}{2} \varepsilon^+ : C^{el} : \varepsilon^+. \quad (3.27)$$

The damage loading surface (Eq.3.25) is then rewritten in terms of ξ as:

$$g \equiv \tilde{G}(\xi, a, s, p, \dots) - r \leq 0, \quad (3.28)$$

and damage stiffness operator evolution $\dot{C}^d$ as:

$$\dot{C}^d = -\dot{\mu} \frac{\partial \tilde{G}}{\partial \xi} \frac{\partial \xi}{\partial(-Y)}. \quad (3.29)$$

On the other hand, we define the damage evolution function H and damage threshold r such as:

$$H = \frac{\partial \tilde{G}}{\partial \xi}, \quad (3.30)$$

$$\dot{r} = \dot{\mu} H. \quad (3.31)$$

From Kuhn-Tucker conditions (Eqs.3.24) several cases of behavior can be considered. If $g < 0$, the damage criterion is not fulfilled and according to condition given by Eq.(3.24)-c, we have $\dot{\mu} = 0$ and hence following Eq.3.29 it results that $\dot{C}^d = 0$. In this case damage do not evolve. However, if $\dot{\mu} > 0$ Eq.(3.24)-c implies that $g = 0$ which means that damage is taking place.

$$g = 0 \Rightarrow \dot{g} = 0, \quad (3.32)$$

it follows that:

$$\dot{g} = \underbrace{\frac{\partial \tilde{G}}{\partial \xi}}_H \dot{\xi} - \underbrace{\dot{r}}_{\dot{\mu} H} = 0, \quad (3.33)$$

from which one can conclude that:

$$\dot{\mu} = \dot{\xi}. \quad (3.34)$$

Substituting Eqs.3.26-3.30-3.34 into Eq.3.29 results in the following expression of the damaged stiffness operator evolution $\dot{C}^d$:

$$\dot{C}^d = -\dot{\xi} H(\xi) P^+ : C^{el} : P^+. \quad (3.35)$$

This equation indicates that only principal tensile strains acting in their associated directions contribute to damage evolution by micro-crack growth.

Concerning the negative points, Ju's model may exhibit a stress discontinuity in the stress-strain response under complex multi-axial loadings, as pointed out by Chaboche in [12]. Assume for instance plane strain conditions, the elastic operator is then written as:

$$C^{el} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix} \quad (3.36)$$

Assume also that the initial damage takes place when all principal strains are positive. In this case $P^+ = I$, then according to Eq.3.35, damage evolution operator is written as:

$$\dot{C}^d = -\dot{\xi} H(\xi) C^{el}, \quad (3.37)$$

and the damage secant operator as:

$$C^d = (1 - \underbrace{\dot{\xi} H(\xi)}_d) C^{el}, \quad (3.38)$$

which corresponds to an isotropic damage operator, where the damage variable is reduced to the scalar d . Using Eq.3.36, Eq.3.38 is then rewritten as:

$$C^d = (1 - d) \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix} \quad (3.39)$$

In the principal strain system, one can define three regions in which the stresses are written as:

(1) for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, region 1

$$\sigma_{11} = (1 - d)(\lambda + 2\mu)\varepsilon_1 + (1 - d)\lambda\varepsilon_2 \quad (3.40)$$

$$\sigma_{22} = (1 - d)\lambda\varepsilon_1 + (1 - d)(\lambda + 2\mu)\varepsilon_2 \quad (3.41)$$

(2) for $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$, region 2

$$\sigma_{11} = (\lambda + 2\mu)\varepsilon_1 + \lambda\varepsilon_2 \quad (3.42)$$

$$\sigma_{22} = \lambda\varepsilon_1 + (1-d)(\lambda + 2\mu)\varepsilon_2 \quad (3.43)$$

(3) for $\varepsilon_1 < 0$ and $\varepsilon_2 < 0$, region 3

$$\sigma_{11} = (\lambda + 2\mu)\varepsilon_1 + \lambda\varepsilon_2 \quad (3.44)$$

$$\sigma_{22} = \lambda\varepsilon_1 + (1-d)(\lambda + 2\mu)\varepsilon_2 \quad (3.45)$$

Examination of the case where $\varepsilon_1 = 0$ and $\varepsilon_2 \neq 0$, which corresponds to the transition point between regions 1 and 2 respectively, indicates a stress discontinuity. Indeed, from Eq.3.40 of the zone 1 we have:

$$\sigma_{11} = (1-d)\lambda\varepsilon_2, \quad (3.46)$$

while from Eq.3.42 of the zone 2 we have:

$$\sigma_{11} = \lambda\varepsilon_2. \quad (3.47)$$

3.2.6 Numerical algorithm

From a numerical point of view, consider a time interval $[t_n, t_{n+1}]$. The data are history variables at t_n and the strain ε_{n+1} at time t_{n+1} . Ju's model was implemented with the algorithm described hereafter.

- Spectral decomposition of ε_{n+1} -Eq.3.10
- Second-rank positive projector $Q^+(\varepsilon_{n+1})$ -Eq.3.11
- Fourth-rank positive projector $P^+(Q^+_{n+1})$ -Eq.3.12
- Tensile strain tensor $\varepsilon^+(\varepsilon_{n+1})$ -Eq.3.13
- Characteristic damage measure $\xi(\varepsilon^+_{n+1})$ -Eq.3.27
- Damage function $\tilde{G}(\xi_{n+1})$ -Eq.3.30
- If $\tilde{G}(\xi_{n+1}) - r_n \leq 0$, damage did not evolve: $C^d_{n+1} = C^d_n$,
- else, damage evolution: $r_{n+1} = \tilde{G}(\xi_{n+1})$ and C^d_{n+1} is obtained by an implicit backward-Euler time-discretization of Eq.3.35, i.e.

$$C^d_{n+1} = C^d_n - (\xi_{n+1} - \xi_n)H_{n+1}P^+_{n+1} : C^{el} : P^+_{n+1}$$

- Stress update: $\sigma_{n+1} = C^d_{n+1} : \varepsilon_{n+1}$

3.2.7 Tangent damage operator computation

Ju's model defines a secant operator C^d while the incremental formulation of homogenization schemes we developed in Chapter 4 needs tangent moduli C^{dtg} . The latter were approximated using a perturbation method. By definition, tangent damage operator is written as:

$$C^{dtg} = \frac{\partial \sigma}{\partial \varepsilon}. \quad (3.48)$$

First, we store the second-order tensors σ and ε in vectors with 6 components and the fourth-rank tensor C^{dtg} in a 6 by 6 matrix. Perturbation of the s^{th} component of vector $[\sigma]$ will give the s^{th} column of $[C^{dtg}]$. Let $[\varepsilon_s^{per}]$ is the strain vector which has its s^{th} component perturbed such as:

$$\varepsilon_s^{per}[i] = (1 + he)\varepsilon[i], \quad i = 1, \dots, 6; \quad (3.49)$$

where h is a scalar designating the perturbation factor and e is a scalar parameter defined such as $e = 1$ if $i = s$ and $e = 0$ otherwise. For example if $s = 3$ then $[\varepsilon_3^{per}]$ is the strain vector which has its third component perturbed. Thus, perturbation of the 6 components of the strain vector $[\varepsilon]$ will gives 6 vectors $[\varepsilon_1^{per}]$, $[\varepsilon_2^{per}]$, $[\varepsilon_3^{per}]$, $[\varepsilon_4^{per}]$, $[\varepsilon_5^{per}]$, $[\varepsilon_6^{per}]$ denoted by $[\varepsilon_s^{per}]$ with $s = 1, \dots, 6$. The perturbed stress vector $[\sigma_s^{per}]$ is then computed using the corresponding perturbed strain vector $[\varepsilon_s^{per}]$ such as:

$$[\sigma_s^{per}] = [C^{dper}][\varepsilon_s^{per}]; \quad (3.50)$$

where $[C^{dper}]$ is the perturbed secant damage matrix computed with the corresponding $[\varepsilon_s^{per}]$. It remains now to compute the s^{th} column of $[C^{dtg}]$ corresponding to the perturbation of s^{th} component of $[\sigma_s^{per}]$ according to:

$$C^{dtg}[i][s] = \frac{1}{h} [\sigma_s^{per}[i] - \sigma[i]]; \quad i = 1, \dots, 6. \quad (3.51)$$

3.3 Elastic-plastic behavior modeling

This section deals with the mechanical modeling of materials describing elasto-plastic behavior. The first works on the topic were conducted by Tresca (1864) with his memoir on the maximal shear stress. Then the first isotropic flow law was formulated by Saint-Venant and Levy in 1871. Many others researchers investigated the problem and a lot of work has been achieved, but the most significant contributions were certainly those of Hill and Bishop in 1950 who reformulated Taylor's model within the rigorous framework of the mathematical theory of the plasticity. The elasto-plastic model described hereafter is based on the J_2p theory developed in the so called incremental formulation (see also Doghri [19]) which enables the achievement of several discriminating loading cases.

This section is organized as follows: First we recall the basis of elasto-plastic model, accordingly to J_2 plasticity theory, by considering the one-dimensional case then three-dimensional one. Then the elasto-plastic continuum tangent operator is derived in the incremental formulation of the model. The resolution of elasto-plastic constitutive equation is proposed by the means of the return mapping algorithm. Finally, the consistent algorithmic tangent operator is computed and compared to the continuum one.

3.3.1 One-dimensional model

In the same manner than for damage in previous section, for a correct understanding of how the model works, let us consider a metallic specimen submitted to uniaxial tensile test. In Figure 3.3 is represented the mechanical response of the tested sample. Along

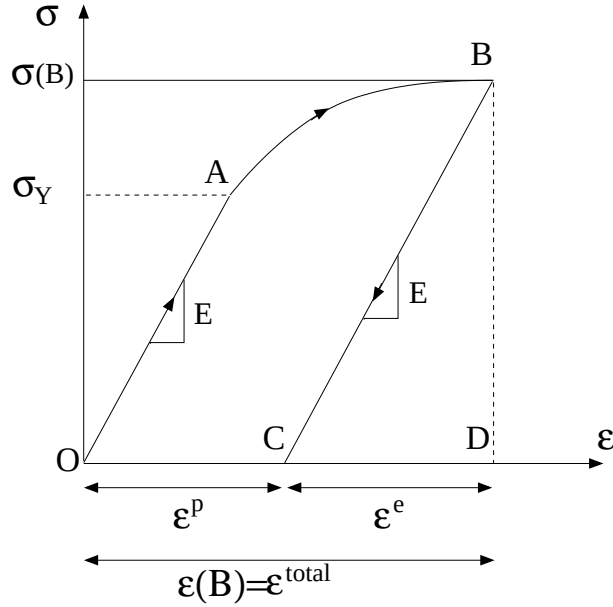


Figure 3.3: Stress-strain curve of elasto-plastic material specimen under uniaxial tensile loading and unloading test.

path OA , one can see that the response is linear elastic. Stress-strain relationship is written as:

$$\sigma = E \varepsilon = E \varepsilon^e, \quad (3.52)$$

where E is Young's modulus. Note that for all load changes (loading/unloading) within the range OA the loading path is reversible. Beyond point A , called yield point, the response become nonlinear, stress and strain are no longer proportional. Total strain $\varepsilon(B)$ at point B can be evaluated by an orthogonal projection of stress-strain curve on strain axis, hence we have: $\varepsilon(B) = \overline{OD}$. Now, if starting from point B the sample is unloaded, the response will follow a straight line of slope E . However, at zero stress corresponds a non zero strain (point C): there is an irreversible or permanent strain called plastic strain $\varepsilon^p \equiv \varepsilon(C) = \overline{OC}$. Hence we conclude that strain at point B can be divided into an elastic part $\varepsilon^e = \overline{CD}$ and a plastic one $\varepsilon^p = \overline{OC}$. If from point C the sample is reloaded, the load-point will follow the linear elastic unloading path until it reaches the initial loading path.

Along branch AB , stress can be expressed as:

$$\sigma = \sigma_Y + \underbrace{R(p)}_{\geq 0}, \quad (3.53)$$

where σ_Y is initial yield stress, $R(p)$ is hardening stress function, and p the accumulated plastic strain which is written as:

$$p(t) = \int_0^t |\dot{\varepsilon}^p(\tau)| d\tau, \quad (3.54)$$

with $\dot{\varepsilon}^p$ the plastic strain rate. Rearranging Eq.3.53 and assuming that response in compression is the same as that in tension, the following yield criterion can be set:

$$|\sigma| - \sigma_Y - R(p) \leq 0. \quad (3.55)$$

In summary, the following relations hold:

- for any point located on branches OA and BC

$$\varepsilon = \varepsilon^e, \quad \sigma = E\varepsilon^e. \quad (3.56)$$

- for any point located on branch AB :

$$\varepsilon = \varepsilon^e + \varepsilon^p, \quad \sigma = \sigma_Y + R(p), \quad \sigma = E\varepsilon^e. \quad (3.57)$$

3.3.2 Generalization to three-dimensional model

It is clearly more interesting when dealing with solids and structures simulation, to have a 3D model at one's disposal. In what follows the basic ideas developed for one-dimensional model are extended to formulate 3D elasto-plastic theory. Total, elastic and plastic strains are written in terms of tensors accordingly to:

$$\varepsilon = \varepsilon^e + \varepsilon^p. \quad (3.58)$$

Stress-strain relationship of Eq. (3.52) is rewritten as:

$$\sigma = C^{el} : \varepsilon^e, \quad (3.59)$$

where σ and ε^e are stress and strain second rank tensors respectively, and C^{el} is the elastic operator.

The yield criterion, becomes:

$$f(\sigma, R) \equiv \sigma_{eq} - R(p) - \sigma_Y \leq 0, \quad (3.60)$$

where σ_{eq} is the von Mises equivalent stress defined by:

$$\sigma_{eq} \equiv \sqrt{\frac{3}{2} s : s}, \quad (3.61)$$

where s , the deviatoric part of σ , is expressed in terms of the second-order identity tensor 1 as:

$$s = \sigma - \frac{1}{3}(\text{tr} \sigma)1. \quad (3.62)$$

The accumulated plastic strain p takes the following form:

$$p(t) = \int_0^t \dot{p}(\tau) d\tau, \quad (3.63)$$

where $\dot{p}$ is accumulated plastic strain rate defined by:

$$\dot{p} = \sqrt{\frac{2}{3} \dot{\varepsilon}^p : \dot{\varepsilon}^p}. \quad (3.64)$$

The yield function given by Eq.3.60 is not sufficient in the case of 3D model, we need another relationship which governs the evolution of the plastic strain in loading and unloading cases. This is done with the help of the plastic flow rule called also normality rule. Dealing with associative plasticity, the normality rule is derived from yield function as:

$$\dot{\varepsilon}^p = \dot{\gamma} \frac{\partial f}{\partial \sigma}; \quad \text{with } \dot{\gamma} \geq 0; \quad (3.65)$$

where $\dot{\gamma}$ is called the plastic multiplier and its sign depends on the following loading/unloading and elastic/plastic conditions (see Figure 3.4):

$$\dot{\gamma} > 0 \quad \text{if } f = 0 \text{ (plasticity) and } \dot{f} > 0 \text{ (yielding)} \quad (3.66)$$

$$\dot{\gamma} = 0 \quad \text{if } f < 0 \text{ (elasticity) or } f = 0 \text{ and } \dot{f} < 0 \text{ (unloading)} \quad (3.67)$$

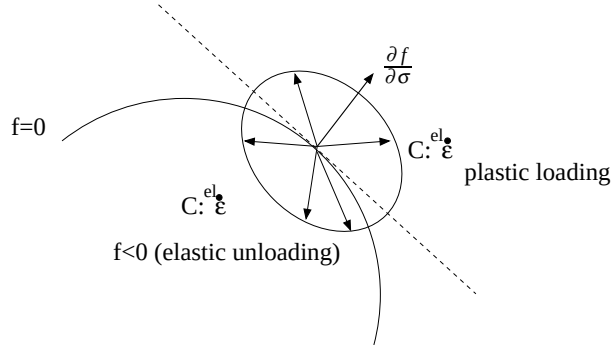


Figure 3.4: Geometrical interpretation of loading/unloading and elastic/plastic conditions

3.3.3 Some useful results

In this section, we give some results which will be very useful in the remaining developments of the model. The first one concerns both the accumulative plastic strain rate $\dot{p}$ and the plastic multiplier $\dot{\gamma}$. Starting from Eq.(3.64) we have:

$$\begin{aligned} \dot{p}^2 &= \frac{2}{3} \dot{\epsilon}^p : \dot{\epsilon}^p, \\ &= \frac{2}{3} \left(\dot{\gamma} \frac{\partial f}{\partial \sigma} \right) : \left(\dot{\gamma} \frac{\partial f}{\partial \sigma} \right). \end{aligned} \quad (3.68)$$

Eq.(3.60) leads to:

$$\begin{aligned} \frac{\partial f}{\partial \sigma} &= \frac{\partial \sigma_{eq}}{\partial \sigma} \\ &= \frac{\partial}{\partial \sigma} \left(\frac{3}{2} S : S \right)^{1/2} \\ &= \left(\frac{3}{2} S : S \right)^{-1/2} \frac{3}{2} S : \frac{\partial S}{\partial \sigma}, \end{aligned} \quad (3.69)$$

then, writing S under the following form:

$$\begin{aligned} S &= \sigma - \frac{1}{3} \underbrace{1}_{1:\sigma} tr \sigma \\ &= \underbrace{\left[I - \frac{1}{3} 1 \otimes 1 \right]}_{I^{dev}} : \sigma, \end{aligned} \quad (3.70)$$

it follows that:

$$\frac{\partial S}{\partial \sigma} = I^{dev}. \quad (3.71)$$

Substituting Eqs.(3.61)-(3.71) into Eq.(3.69) gives:

$$\frac{\partial f}{\partial \sigma} = \underbrace{\left(\frac{3}{2}S:S\right)^{-1/2}}_{(\sigma_{eq})^{-1}} \frac{3}{2} \underbrace{S:I^{dev}}_S \quad (3.72)$$

$$= \frac{3}{2} \frac{S}{\sigma_{eq}}, \quad (3.73)$$

then, the following result is obtained:

$$\begin{aligned} \frac{\partial f}{\partial \sigma} : \frac{\partial f}{\partial \sigma} &= \left(\frac{3}{2} \frac{S}{\sigma_{eq}}\right) : \left(\frac{3}{2} \frac{S}{\sigma_{eq}}\right), \\ &= \left(\frac{3}{2} \frac{1}{\sigma_{eq}}\right)^2 \underbrace{S:S}_{\frac{2}{3}\sigma_{eq}^2}, \\ \frac{\partial f}{\partial \sigma} : \frac{\partial f}{\partial \sigma} &= \frac{3}{2}. \end{aligned} \quad (3.74)$$

Substituting this result in Eq.3.68 leads to:

$$\dot{p}^2 = \dot{\gamma}^2, \quad (3.75)$$

$\dot{p}$ and $\dot{\gamma}$ being always positive or nil, it follows that:

$$\dot{p} = \dot{\gamma}. \quad (3.76)$$

The other result concerns the incompressibility of the plastic flow $\dot{\epsilon}^p$ which is dictated by experimental observations. Indeed, plastic flow of metal occurs without volume change. Substitution of Eq.(3.71) into Eq.(3.65) gives the following result:

$$\begin{aligned} \dot{\epsilon}^p &= \dot{\gamma} \frac{\partial f}{\partial \sigma} = \dot{\gamma} \frac{3}{2} \frac{S}{\sigma_{eq}}, \\ \Rightarrow tr \dot{\epsilon}^p &= \dot{\gamma} \frac{3}{2} \frac{1}{\sigma_{eq}} \underbrace{tr S}_{=0}, \\ \Rightarrow tr \dot{\epsilon}^p &= 0, \end{aligned} \quad (3.77)$$

which is in total accordance with the experimental observations.

3.3.4 Tangent operator derivation

The aim of this section is to compute, in the general case, the elasto-plastic “continuum” tangent operator C^{ep} which relates stress and strain tensors rate as follows:

$\dot{\sigma} = C^{ep} : \dot{\varepsilon}$. To compute C^{ep} , the model is formulated in terms of rate. Time derivative of Eq.(3.59) gives:

$$\dot{\sigma} = C^{el} : \dot{\varepsilon}^e = C^{el} : (\dot{\varepsilon} - \dot{\varepsilon}^p), \quad (3.78)$$

substituting Eq.(3.75) into Eq.(3.65) then into Eq.(3.59) on obtain:

$$\dot{\sigma} = C^{el} : \left(\dot{\varepsilon} - \dot{p} \frac{\partial f}{\partial \sigma} \right). \quad (3.79)$$

Time derivative of the yield function $f(\sigma, R(p))$ given by Eq.(3.60) leads to:

$$\begin{aligned} \dot{f} = \underbrace{\frac{\partial \sigma_{eq}}{\partial \sigma}}_{\frac{\partial f}{\partial \sigma}} : \dot{\sigma} - \frac{dR}{dp} \dot{p} &= \frac{\partial f}{\partial \sigma} : C^{el} : \dot{\varepsilon} - \dot{p} \frac{\partial f}{\partial \sigma} : C^{el} : \frac{\partial f}{\partial \sigma} - \frac{dR}{dp} \dot{p} \\ &= \frac{\partial f}{\partial \sigma} : C^{el} : \dot{\varepsilon} - h \dot{p}, \end{aligned} \quad (3.80)$$

with

$$h = \frac{\partial f}{\partial \sigma} : C^{el} : \frac{\partial f}{\partial \sigma} + \frac{dR}{dp}. \quad (3.81)$$

Depending on the loading path, two events can be considered:

- if $f = 0$ and $\dot{f} \leq 0$ (elastic unloading) it follows from Eq.(3.67) that $\dot{\gamma} = 0$ then from Eq.(3.75) that $\dot{p} = \dot{\gamma} = 0$ (no evolution of plastic strain), hence from Eqs.(3.67)-(3.80) we have:

$$\dot{f} = \frac{\partial f}{\partial \sigma} : C^{el} : \dot{\varepsilon} < 0; \quad (\text{elastic unloading, Figure 3.4}) \quad (3.82)$$

- if $f = 0$ and $\dot{f} = 0$ (plastic loading) then $\dot{\gamma} (= \dot{p}) > 0$, and from Eq.(3.80) one can write:

$$\dot{p} = \frac{1}{h} \frac{\partial f}{\partial \sigma} : C^{el} : \dot{\varepsilon} > 0; \quad (\text{plastic loading, Figure 3.4}) \quad (3.83)$$

Assuming that h is always positive, the distinction between elastic unloading $\dot{f} < 0$ and plastic loading $\dot{f} = 0$ cases become easy. In fact it suffices to study the sign of the cosine of the “angle” between the normal to the yield surface $\frac{\partial f}{\partial \sigma}$ and $C^{el} : \dot{\varepsilon}$, Figure 3.4.

Substituting Eq.(3.83) into Eq., stress-strain relationship is written as:

$$\begin{aligned} \dot{\sigma} &= C^{el} : \left(\dot{\varepsilon} - \frac{\partial f}{\partial \sigma} \frac{1}{h} \frac{\partial f}{\partial \sigma} : C^{el} : \dot{\varepsilon} \right) \\ &= \underbrace{\left\{ C^{el} - \frac{1}{h} \left(C^{el} : \frac{\partial f}{\partial \sigma} \right) \otimes \left(C^{el} : \frac{\partial f}{\partial \sigma} \right) \right\}}_{C^{ep}} : \dot{\varepsilon}, \end{aligned} \quad (3.84)$$

from which the expression of the continuum tangent operator is deduced.

3.3.5 Hardening stress functions

Depending on the sample constitutive material, the hardening stress function $R(p)$ can have a various forms. The most commonly used forms are:

- linear hardening: $R(p) = kp$;
- power law hardening $R(p) = kp^m$;
- exponential law hardening $R(p) = R_\infty[1 - e^{-mp}]$;

where k and m are material parameters, and R_∞ is the hardening stress at saturation.

3.3.6 Return mapping algorithm

Knowing the value of the variables at time t_n and those of $\Delta\epsilon$, the aim of the algorithm detailed hereafter, is to find the value of the variables at time t_{n+1} . Starting from the stress increment $\Delta\sigma = \sigma_{n+1} - \sigma_n$, one can write:

$$\begin{aligned}\sigma_{n+1} = \sigma_n + \Delta\sigma &= C^{el} : (\epsilon_n - \epsilon_n^p) + C^{el} : (\Delta\epsilon - \Delta\epsilon^p) \\ &= \underbrace{C^{el} : (\epsilon_n - \epsilon_n^p + \Delta\epsilon)}_{\text{known}} - \underbrace{C^{el} : \Delta\epsilon^p}_{\text{unknown}}\end{aligned}\quad (3.85)$$

The first phase in the return mapping algorithm is called elastic predictor, it is assumed that the first increment is elastic. Under this hypothesis, it follows that:

$$\Delta\epsilon^p = 0 \quad \text{and}; \quad (3.86)$$

$$\sigma_{n+1}^{trial} = \sigma_n + C^{el} : \Delta\epsilon, \quad (3.87)$$

which is rewritten as:

$$\sigma_{n+1}^{trial} = C^{el} : (\epsilon_{n+1} - \epsilon_n^p). \quad (3.88)$$

Now, check the yield condition. If:

$$f_{n+1}^{trial} \equiv (\sigma_{eq})_{n+1}^{trial} - \sigma_Y - R(\underbrace{p_{n+1}^{trial}}_{=p_n}) \leq 0, \quad (3.89)$$

the increment is then really elastic and the elastic predictor is the solution of the problem. It follows:

$$\sigma_{n+1} = \sigma_{n+1}^{trial}; \quad \varepsilon_{n+1}^p = \varepsilon_n^p; \quad p_{n+1} = p_n. \quad (3.90)$$

However, if $f_{n+1}^{trial} > 0$ the assumption is not true. During this increment, the plasticity has taken place and stresses must be reevaluated such that the yield criterion is satisfied. This is what we call the plastic corrector step.

Assuming that we are dealing with isotropic case, and since $\Delta\varepsilon^p$ is deviatoric, combining Eqs.3.85-3.88 leads to:

$$\sigma_{n+1} = \sigma_{n+1}^{trial} - 2G\Delta\varepsilon^p. \quad (3.91)$$

Plastic strain and accumulative plastic rate equations are time discretized according to the backward Euler scheme as:

$$\Delta\varepsilon^p \approx \dot{\varepsilon}_{n+1}^p \Delta t; \quad \Delta p \approx \dot{p}_{n+1} \Delta t, \quad (3.92)$$

then, with the help of the normality rule (Eq.3.65) and Eq.3.76, one can write:

$$\begin{aligned} \Delta\varepsilon^p &\approx \dot{p}_{n+1} \frac{\partial f}{\partial \sigma} \Big|_{n+1} \Delta t \\ &\approx \Delta p \frac{\partial f}{\partial \sigma} \Big|_{n+1} \\ \Delta\varepsilon^p &\approx \Delta p N_{n+1}. \end{aligned} \quad (3.93)$$

Replacing Eq.(3.93) into is Eq.(3.91) gives:

$$\sigma_{n+1} = \sigma_{n+1}^{trial} - 2G\Delta p N_{n+1}. \quad (3.94)$$

Resolution of the last equation will be done by computing the trace and deviatoric parts of σ_{n+1} . The trace of σ_{n+1} is given by:

$$\begin{aligned} tr\sigma_{n+1} &= tr\sigma_{n+1}^{trial} - 2G(\Delta p) \underbrace{trN_{n+1}}_{=0} \\ &= 3Ktr(\varepsilon_{n+1}), \end{aligned} \quad (3.95)$$

which is a known quantity. Now, it only remains to compute the deviatoric part $(S)_{n+1}$ of σ_{n+1} , so we have:

$$S_{n+1} = S_{n+1}^{trial} - 2G\Delta p N_{n+1}. \quad (3.96)$$

From now onwards, the subscript (n+1) will be omitted for simplicity of notation. We first recall that:

$$N \equiv \frac{\partial f}{\partial \sigma} = \frac{3}{2} \frac{S}{\sigma_{eq}}; \quad \text{and that } N:N = \frac{3}{2}. \quad (3.97)$$

Using the last equation, Eq.(3.96) is rewritten as:

$$\frac{2}{3}N\sigma_{eq} = \frac{2}{3}N^{trial}\sigma_{eq}^{trial} - 2G\Delta pN, \quad (3.98)$$

which can be rearranged as:

$$\underbrace{\left(\frac{2}{3}\sigma_{eq} + 2G\Delta p\right)}_{>0} N = \underbrace{\frac{2}{3}\sigma_{eq}^{trial} N^{trial}}_{known}. \quad (3.99)$$

The above equation implies that:

$$\frac{2}{3}\sigma_{eq} + 2G\Delta p = \frac{2}{3}\sigma_{eq}^{trial}, \quad (3.100)$$

and that:

$$N = N^{trial}. \quad (3.101)$$

The stress-solution must be such that $f = 0$, so we have:

$$\sigma_{eq} = \sigma_Y + R(p), \quad (3.102)$$

then, replacing Eq.(3.102) into Eq.(3.100) and rearranging, we have:

$$k(p) \equiv \sigma_Y + R(p) + 3G\Delta p - \sigma_{eq}^{trial} = 0, \quad (3.103)$$

which is a scalar equation with p as a sole unknown. This nonlinear equation (since $R(p)$ can be nonlinear) is resolved iteratively using Newton-Raphson method. At each iteration (r), we have:

$$k(p^{(r)}) + \underbrace{\frac{dk}{dp}(p^{(r)})}_{R'(p^{(r)})+3G} (p^{(r+1)} - p^{(r)}) = 0. \quad (3.104)$$

At iteration ' $r = 0$ ', we set $p_{n+1}^{(0)} = p_n$ and $k(p^{(0)}) = R(p_n) + \sigma_Y - \sigma_{eq}^{trial}$. The solution is assumed to be converged when the convergence criterion is satisfied:

$$\frac{|k(p^{(r)})|}{\sigma_Y} \leq \text{tolerance e.g. } 10^{-6}. \quad (3.105)$$

Note that this value of tolerance has been motivated by numerical experiments. After convergence we have $p(= p_{n+1})$, the solution at $t = t_{n+1}$ is then computed:

- $\sigma_{eq} = \sigma_Y + R(p)$;
- $N = N^{trial}$;

- $S = \frac{2}{3}\sigma_{eq}N$;
- $\sigma = S + \frac{1}{3}(tr\sigma^{trial}1)$;
- $\Delta\varepsilon^p = N\Delta p$;
- $\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\varepsilon^p$.

3.3.7 Consistent tangent operator

One way to derive the expression of the so-called consistent or “algorithmic” tangent operator (C^{alg}), is to time-discretize constitutive equations of the elasto-plastic model derived above. For more elaborate discussions the readers may consult [19]. In what follows, the objective is to compute C^{alg} such as:

$$\delta\sigma_{n+1} = C^{alg} : \delta\varepsilon_{n+1}. \quad (3.106)$$

Differentiation of stress-strain relationship, plastic flow rule and the yield surface function gives (subscript “n+1” will omitted for simplicity of notation):

$$\delta\sigma = C^{el} : (\delta\varepsilon - \delta\varepsilon^p), \quad (3.107)$$

$$\delta\varepsilon^p = N\delta p + (\Delta p)\frac{\partial N}{\partial\sigma} : \delta\sigma, \quad (3.108)$$

$$N : \delta\sigma - \frac{dR}{dp}\delta p = 0. \quad (3.109)$$

Substituting Eq.(3.108) into Eq.(3.107) then into Eq.(3.109), and with help of Eq.(3.81) one obtains:

$$\delta p = \frac{1}{h}N : C^{el} \left[\delta\varepsilon - (\Delta p)\frac{\partial N}{\partial\sigma} : \delta\sigma \right]. \quad (3.110)$$

Combination of Eqs.(3.107)-(3.108)- and rearranging gives:

$$\left[I + (\Delta p)C^{el} : \frac{\partial N}{\partial\sigma} - \frac{\Delta p}{h}(C^{el} : N) \otimes \left(N : C^{el} : \frac{\partial N}{\partial\sigma} \right) \right] : \delta\sigma = C^{ep} : \delta\varepsilon. \quad (3.111)$$

It is easy to check that:

$$N : \frac{\partial N}{\partial\sigma} = 0, \quad (3.112)$$

and dealing with isotropic elasticity that:

$$C^{el} : \frac{\partial N}{\partial\sigma} = 2G\frac{\partial N}{\partial\sigma}. \quad (3.113)$$

Using the two last equations, Eq.(3.111) takes the following simple form:

$$\left(I + 2G(\Delta p) \frac{\partial N}{\partial \sigma} \right) : \delta \sigma = C^{ep} : \delta \varepsilon, \quad (3.114)$$

which can be casted as (see [19]):

$$\delta \sigma = \left[C^{ep} - (2G)^2 \xi \frac{\partial N}{\partial \sigma} \right] : \delta \varepsilon, \quad (3.115)$$

with the help of:

$$\left(I + 2G(\Delta p) \frac{\partial N}{\partial \sigma} \right)^{-1} = I - 2G\xi \frac{\partial N}{\partial \sigma}, \quad (3.116)$$

where ξ is a dimensionless coefficient given by:

$$\xi = \frac{\Delta p}{1 + \frac{3G\Delta p}{\sigma_{eq}}}. \quad (3.117)$$

Then, from Eq.3.115, then expression of the algorithmic tangent operator is deduced:

$$C^{alg} = C^{ep} - (2G)^2 \xi \frac{\partial N}{\partial \sigma}. \quad (3.118)$$

It clearly appears from the above equation that the algorithmic tangent operator C^{alg} is close to the consistent one C^{ep} when $\Delta p \rightarrow 0$, otherwise the two operators are quite different. In fact, the linearization of the constitutive equations in order to compute C^{alg} , introduces additional terms which do not appear in C^{ep} computation.

In comparison with other elastoplastic models such as the secant formulation which it is limited to proportional monotonic loading cases, the incremental one has the advantage that it allows to simulate cyclic loading and unloading, tension and compression tests. However, in some cases the the return mapping algorithm may display some convergence problems such as when dealing with composite materials with constituents exhibiting a strong nonlinear behaviors. The other criticism from which the elastoplastic model derived in this section suffers, is that the hardening is always isotropic.

Chapter 4

Homogenization of inelastic composites

4.1 General Introduction

In chapter 2 we dealt with homogenization of composite materials displaying an elastic behavior. However in many industrial applications, the truly observed behavior of composites is mainly nonlinear. In fact at least one of the constituents of those composites may exhibit an inelastic behavior depending on industrial application goal or/and the loading level reached. To cite but few, Concrete-like Matrix Composites (CMC) may exhibit an elastic-damage behavior when they are submitted to tensile loading. Another example is Metal Matrix Composites (MMC) used in aero-engines. At high temperatures, those composites may display an viscoelastoplastic behavior. While Polymer Matrix Composites (PMC) may display an elastoplastic response when they experience a certain level of loadings.

Within the framework of homogenization methods based on the mean-field approach, several formulations were proposed to tackle the problem of inelastic composites. The key idea of those formulations is based of the linearization of the the constitutive laws at both micro and macro scales, and each formulation differs from the other in the manner with which the linearization is achieved. The most popular linearization are presented hereafter.

4.1.1 The incremental formulation

The incremental formulation, credited to Hill [35], is considered as the starting point of many nonlinear homogenization theories of composites and it still inspires many researchers in the field. The key idea of this formulation is based on an incremental linearization of the phases' constitutive relations. In his work, Hill investigated the elastoplastic behavior of polycrystals using the self-consistent scheme based on the use of instantaneous (tangent) moduli at both macro and micro scales within an incremental, step by step procedure. In Hill's incremental formulation, the effective and per-phase constitutive laws are linearized as follows:

$$\dot{\sigma}(x, t) = C^{tg}(\varepsilon(x, t)) : \dot{\varepsilon}(x, t), \quad (4.1)$$

where C^{tg} is the instantaneous tangent operator, x the position and t the time which refers to the loading increment. More recently, on the basis of Hill's incremental formulation, Doghri and Ouaar [21], Doghri and Friebel [20] proposed a formulation for two-phase elastoplastic composites using two constitutive models at the micro-scale: J_2p theory and Chaboche's model for cyclic plasticity; and several homogenization schemes (Mori-Tanaka model and Double inclusion model). Ouaar et al [68] also extend Hill's formulation for the case of elastic-damaged behavior of concrete-like matrix composites where the matrix damage is described with Ju's model which is based on Continuum Damage Mechanics.

4.1.2 The secant formulation

This formulation was first proposed by Hutchinson in 1976 [38] who extended Hill's formulation to steady-state creep case. In his work, Hutchinson first used the tangent creep compliances instead of the elastoplastic ones, then showed, for power-law viscous materials, that an incremental treatment is no longer necessary since Hill's incremental formulation may be integrated into a 'total deformation formulation' by using 'secant' creep compliances. In fact, by neglecting the elastic part of the behavior, the secant creep compliance $D^{cr.sct}$ is simply related to tangent one $D^{cr.tgt}$ as: $D^{cr.sct} = nD^{cr.tgt}$. Berveiller and Zaoui in 1979 [6] then proposed an other formulation for more general types of composites making use of other secant modulus tensors. More recently, Tandon and Weng [102] studied the elastoplastic behavior of composites consisting of elastic inclusions embedded in an elastoplastic matrix making use of Mori-Tanaka scheme based on a total formulation in which the mechanical properties of the composite are described by secant elastoplastic moduli. For linear viscoelastic multi-phase composites, Friebel et al [26] used the Laplace-Carsson transform to obtain a problem similar to a linear elastic one. In the secant formulation, the lin-

earization of effective and per-phase constitutive laws has the following form:

$$\sigma(x, t) = C^{sec}(\varepsilon(x, t)) : \varepsilon(x, t), \quad (4.2)$$

where C^{sec} denotes the secant operator. Note that this formulation is limited to monotonic and proportional loadings which strongly reduce its application fields.

4.1.3 Tangent and affine formulations

The tangent formulation which must not be confused with the incremental one, was first proposed by Molinari [61], in 1987. It is a mixture of the incremental and secant formulations. Indeed, it makes use of the tangent operators as the incremental approach does and resolves the nonlinear problem directly (not step by step) as a secant approach. In this method the linearization of a nonlinear behavior $\sigma = f(\varepsilon)$ is as follows:

$$\sigma(x, t) = C(x) : \varepsilon(x, t) + \tau(x, t), \quad (4.3)$$

with:

$$C(x) = \frac{df}{d\varepsilon}(\varepsilon(x)), \quad (4.4)$$

and where τ is the σ_0 ordinate at the origin as displayed in Figure 4.2. In this method however, the shape of the overall behavior of the composite needs to be known. This is the reason why Molinari assumed a power-law behavior at both micro and macro scale with a same exponent, which strongly limits the application of the method.

The affine formulation is quite similar to the tangent one concerning the manner with which it proceeds for the linearization of the local constitutive laws. However, the former sees that the linearized constitutive equations are similar to those of a thermo-elastic problem, and then the homogenization procedure can be achieved as for the linear thermo-elastic problem. This gives to the method a large range of applications contrary to the tangent formulation, since for the affine formulation the analytical shape of the constitutive law at the macro-level does not need to be known. In the case of viscoplasticity for instance, the affine formulation gives the following local constitutive equations:

$$\sigma_r = C_r^{vp} : \dot{\varepsilon}_r + \tau_r, \quad (4.5)$$

where r designates the phase. Then homogenization procedure leads to the following constitutive relationships at macro-scale:

$$\bar{\sigma} = \bar{C}^{vp} : \bar{\dot{\varepsilon}} + \bar{\tau}, \quad (4.6)$$

with the following homogenized mechanical properties:

$$\tilde{C}_r^{vp} = \langle C_r^{vp} : \mathbb{A} \rangle, \quad \text{and} \quad \bar{\tau} = \langle \mathbb{A}^t : \tau \rangle, \quad (4.7)$$

where $\mathbb{A}$ represents the localization tensor (Eq.2.16) called also the strain concentration tensor. Affine formulation was improved and extended to handle the modeling of two-phase composites. In the formulation proposed by Pierard and Doghri [73], the total strain rate $\dot{\varepsilon}$ is written as:

$$\dot{\varepsilon}(t) = \underbrace{S : \dot{\sigma}(t)}_{\dot{\varepsilon}^{el}(t)} + \dot{\varepsilon}^{in}(\sigma(t), p(t)), \quad (4.8)$$

where S is the elastic compliance operator, $\dot{\sigma}(t)$ the stress rate, $p(t)$ the accumulated plastic strain, and $\dot{\varepsilon}^{el}(t)$ and $\dot{\varepsilon}^{in}(t)$ the elastic and inelastic strain rates respectively. After a linearization around a time $\tau \in [t_n, t_{n+1}]$, Eq.(4.8) is written in the Laplace-Carson domain “ s ” as:

$$\varepsilon^*(s) = S^*(\tau, s) : \sigma^*(s) + \varepsilon^{0*}(\tau, s), \quad (4.9)$$

or equivalently as:

$$\sigma^*(s) = C^*(\tau, s) : \varepsilon^*(s) - \underbrace{C^*(\tau, s) : \varepsilon^{0*}(\tau, s)}_{\sigma^{0*}}, \quad \text{with : } C^*(\tau, s) = [S^*(\tau, s)]^{-1}, \quad (4.10)$$

which looks like as a thermo-elastic problem. The homogenization procedure is achieved in the Laplace-Carson domain such as for a thermo-elastic problem, the homogenized fields are then written in the time domain using the numerical Laplace inversion.

The main criticism that can be made for formulations cited above is that, all are developed within the framework of the mean-field approach which does not account for the intra-phase strain or stress fluctuations. Indeed, per-phase reference strain and stress are assumed to be the per-phase average of strain and stress. This is the reason why, they all can lead to estimates that are too stiff, and can even, in some cases, violate Hashin-Shtrikman estimates [30]. This argument will be discussed in chapters 6 and 7. During the last decade, solutions were proposed to overcome this problem. They consist in the derivation of the effective strain or stress from energy approaches which allow taking into account strain or stress fluctuations within the considered phase. A possibility, proposed by Parton and Buryachenko [70], Qui and Weng [81], Suquet [97], Suquet and Castaneda [95] is the use of second order moment of strain per phase as the effective strain per phase.

4.1.4 Variational approaches

In recent years, there has been increasing interest in the variational approaches to model composites displaying a nonlinear behavior (Ponte Castanęda [77], Suquet [94]). One of the advantages of formulation based on variational approaches is their

capability to provide both estimates and bounds. In those approaches, the local constitutive response is defined by:

$$\sigma = \frac{\partial w}{\partial \varepsilon}(x, \varepsilon), \quad (4.11)$$

where

$$w(x, \varepsilon) = \sum_{r=1}^N \chi^r(x) w^r(\varepsilon), \quad (4.12)$$

is the potential at macro-scale and $\chi^r(x)$ characteristic functions describing the position. They are equal to 1 if the position vector x is inside the phase r and zero otherwise.

More recently a second order variational formulation making use of the Taylor formula to express phase potentials w^r was developed by Ponte Castaneda [79]. The local potential is expressed as follows:

$$w_T^r(\varepsilon) = w^r(\varepsilon^r) + \sigma^r(\varepsilon^r) : (\varepsilon(x) - \varepsilon^r) + \frac{1}{2} (\varepsilon(x) - \varepsilon^r) : C_0^r : (\varepsilon(x) - \varepsilon^r), \quad (4.13)$$

where σ^r denotes the first derivative of the phase potential w^r :

$$\sigma^r(\varepsilon^r) = \frac{\partial w^r}{\partial \varepsilon}(\varepsilon^r), \quad (4.14)$$

and C_0^r represents the tangent operator defined as:

$$C_0^r = \frac{\partial^2 w^r}{\partial \varepsilon \partial \varepsilon}(\varepsilon^r), \quad (4.15)$$

Other variational formulations were proposed, someones use secant operators instead of tangent ones, others use the second-order moment of strains which for the considered phase “ r ” is expressed as:

$$\langle \varepsilon \otimes \varepsilon \rangle_r = \frac{1}{v_f} \bar{\varepsilon} : \frac{\partial \bar{C}}{\partial C_r}, \quad (4.16)$$

where $\bar{C}$ and C_r are the effective and per-phase stiffnesses respectively.

In this thesis, we propose an incremental formulation which enables the deformation behavior modeling of inelastic composites consisting of a matrix phase with inclusions embedded therein. The proposed formulation is developed within the mean-field approach framework and is based on Hill’s incremental formulation. At each step of our model derivation, we point out the major encountered difficulties, propose solution to overcome those problems and highlight the hidden assumptions of the proposed solutions. The Eshelby’s tensor computation and the mechanical characterization (stiffness operators) of composite constituents are investigated with a particular care.

This chapter is outlined as follows: in section 2 we introduce the *Linear Heterogeneous Comparison Material* concept then in section 3 we propose an incremental formulation for rate-independent inelastic two-phase composites. The calculations are detailed for Mori-Tanaka homogenization scheme while only final results are presented for the other schemes. In this section the Eshelby's tensor computations is also examined. Then the problem of nonlinear multi-phase composites is dealt with in section 4.

4.2 Linear Heterogeneous Comparison Material concept

When one deals with homogenization of inelastic composites, two problems arise: The nonlinear behavior of composite's micro-constituents and the intra-phase heterogeneity induced by this nonlinearity. Solutions were proposed to overcome those two problems and were gathered together under the concept of '*Linear Heterogeneous Comparison Material*' (LHCM). This concept, without therefore naming it LHCM, was first used by Hill in 1965 [35] then introduced in systematic manner by Tandon and Weng [102] and by Ponte Castañeda [77]. The main objective of this concept is to

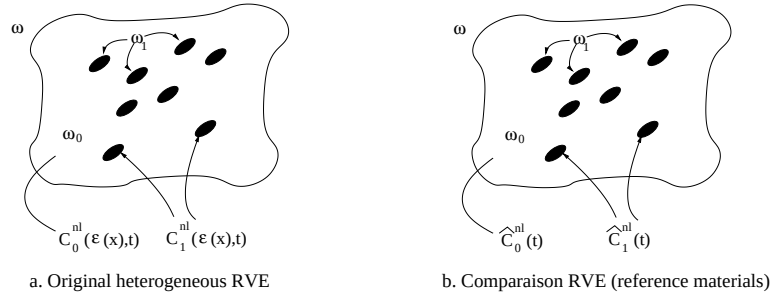


Figure 4.1: Linear Heterogeneous Comparison Material concept

provide a framework which makes possible homogenization and averaging procedures for composites displaying a nonlinear behavior. The key idea of this concept consists in replacing the original heterogeneous RVE with mechanical properties that vary in space and time by an RVE which has mechanical properties varying only with time, Figure (4.1). In other words, that amounts to consider the nonlinear composite as a composite which has, at each fixed time, a per phase uniform linearized mechanical characteristics.

Linearization and uniformization are the necessary processes to achieve homogenization procedure for nonlinear composites:

- Concerning the linearization process, several choices can be made as displayed in figure (4.2). A first possibility, proposed by Hill [35], is a linearization using an incremental formulation which consists in relating the stress increment to the strain one through the tangent operator as presented in section 4.1.1. Others choices are possible, the most popular of them being the secant, tangent, and affine linearization procedures.
- Phase heterogeneity: each phase displaying a nonlinear behavior has mechanical characteristics (stiffness operators) that vary in space as well as in time. This space variation of stiffness operators, induces an intra-phase heterogeneity that renders averaging procedures complex. A good deal is to assume that, at a given fixed time, stiffness operators are uniform in space for the considered phase. Also here, there are several possibilities, for example, for each phase, Hill has considered as uniform stiffness operators the instantaneous tangent operators computed with the corresponding phase average strain.

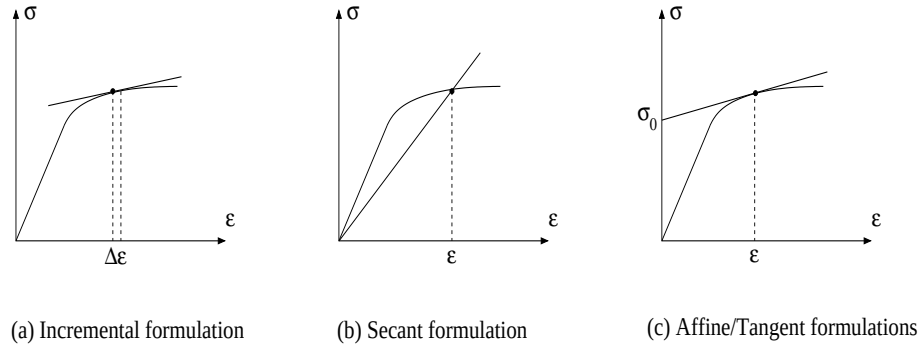


Figure 4.2: Linearization schemes

4.3 An incremental formulation for nonlinear two-phase composites

Within the framework of mean-field approaches, homogenization procedure for nonlinear two-phase composites is based, as previously presented, on the linearization of

the constitutive laws and the per-phase uniformization of the mechanical properties. Those two conditions being satisfied, the nonlinear homogenization problem becomes similar to an elastic or a thermo-elastic homogenization problem, according to the choice of the linearization process. In this thesis we propose a framework for non-linear two-phase composites modeling based on the Hill's incremental linearization process.

In general, the homogenization of a two-phase composite can be conceived in two stages. The first homogenization is achieved to obtain the expression of the overall stiffness operator of the composite (Figure 4.3) while the second one is performed in order to derive the expression of strain concentration tensors according to the chosen homogenization scheme (Figure 4.4).

4.3.1 Basic results

In the following, we will rewrite in a rate form, the most important relationships needed to develop our incremental formulation for homogenization of two-phase inelastic composites.

Stress and strain rates averages

The RVE averaged macro stress rate can be obtained by averaging the stress rate field over the two phases, the final result is:

$$\langle \dot{\sigma} \rangle_{\omega} = v_1 \langle \dot{\sigma} \rangle_{\omega_1} + (1 - v_1) \langle \dot{\sigma} \rangle_{\omega_0}, \quad (4.17)$$

where ω , ω_0 , and ω_1 designate the RVE, the matrix and the inclusion phases respectively and v_1 is the inclusions volume fraction. Likewise, the RVE average strain rate $\langle \dot{\epsilon} \rangle_{\omega}$ is related to the matrix and inclusions strain rates averages $\langle \dot{\epsilon} \rangle_{\omega_0}$ and $\langle \dot{\epsilon} \rangle_{\omega_1}$ respectively through the following expression:

$$\langle \dot{\epsilon} \rangle_{\omega} = v_1 \langle \dot{\epsilon} \rangle_{\omega_1} + (1 - v_1) \langle \dot{\epsilon} \rangle_{\omega_0}. \quad (4.18)$$

Strain concentration tensors definition

We define two strain concentration tensors. The first one, A , relates the inclusions average strain rate to the one of the composite:

$$\langle \dot{\epsilon} \rangle_{\omega_1} = A : \langle \dot{\epsilon} \rangle_{\omega}, \quad (4.19)$$

while the second, noted $\tilde{A}$, relates the inclusion and matrix average strain rates as:

$$\langle \dot{\epsilon} \rangle_{\omega_1} = \tilde{A} : \langle \dot{\epsilon} \rangle_{\omega_0} . \quad (4.20)$$

The two tensors $\tilde{A}$ and A can be related with:

$$A = [v_1 I^4 + (1 - v_1) \tilde{A}^{-1}]^{-1} . \quad (4.21)$$

4.3.2 Nonlinear homogenized tangent operator

Contrary to linear elasticity, the tangent operator is not uniform anymore in the case of nonlinear behavior and depends on the strain or stress states. This non uniformity of the tangent operator renders, as we will see, homogenization and averaging procedures more complex. Consider RVE (ω) representing a two-phase composite made of a matrix phase ω_0 in which aligned inclusions (phase ω_1) having a similar shape and constitutive material are embedded, Figure 4.3a-b.

Assuming that each phase obeys to a nonlinear behavior, the stress-strain relationship at the micro-level is linearized in terms of rate as follow:

$$\dot{\sigma}(x, t) = C_r^{tg}(\epsilon(x, t), t) : \dot{\epsilon}(x, t), \quad (4.22)$$

where C_r^{tg} with $r = 0, 1$ are the per-phase nonuniform anisotropic tangent operators. They vary in space as well as in time. Replacing Eq.(4.22) into Eq.(4.17) leads to:

$$\langle \dot{\sigma} \rangle_{\omega} = v_1 \langle C_1^{tg}(\epsilon(x, t), t) : \dot{\epsilon}(x, t) \rangle_{\omega_1} + (1 - v_1) \langle C_0^{tg}(\epsilon(x, t), t) : \dot{\epsilon}(x, t) \rangle_{\omega_0} . \quad (4.23)$$

At this stage, it clearly appears from Eq.(4.23), that averaging procedure becomes unrealizable in consequence to the non-uniformity of the operators C_r^{tg} . To avoid those problems, a solution is to consider, for each phase constituting the RVE, a so-called reference material such that its stiffness operator $\hat{C}_r^{tg}(t)$ is uniform in space and varies only with time. In other words, in place of the original RVE, we consider a linear heterogeneous comparison material which is made of constituents (phases) that have uniform linearized mechanical properties varying only with time, as displayed in figure 4.3-c. Thus, in each phase of the LHCM, the stress rate is then written as:

$$\dot{\sigma}(x, t) = \hat{C}_r^{tg}(t) : \dot{\epsilon}(x, t), \quad \forall x \in \omega_r . \quad (4.24)$$

There are many ways to render uniform the operators of the so-called reference materials. In the present work, we proceed as follows. The constitutive box of the real material of each phase is called with the average strains and strain rates in that phase as arguments, and the moduli that the constitutive box computes are taken as the reference moduli for that phase. This is a generalization of the approach of Tandon [102] from secant to incremental formulation.

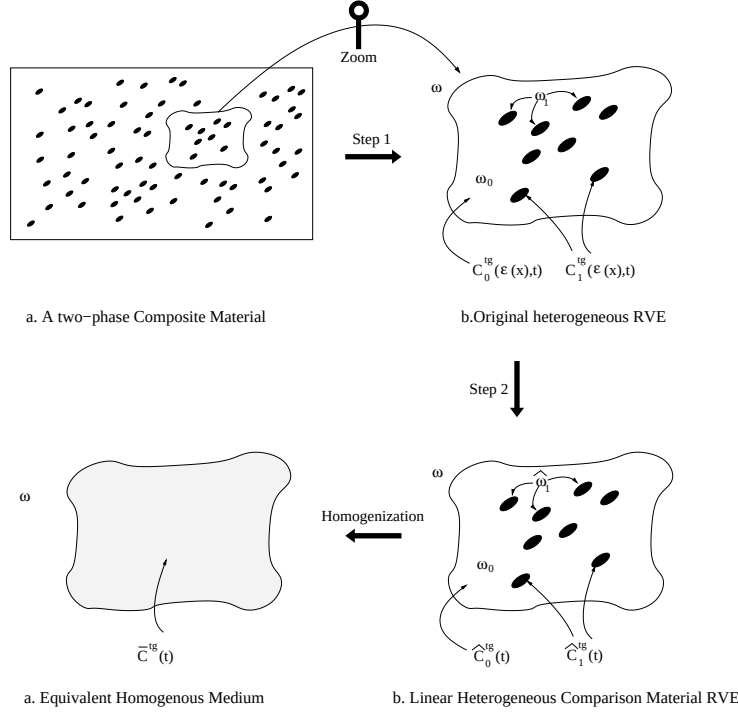


Figure 4.3: Homogenization procedure to obtain the expression of the overall tangent operator for a nonlinear two-phase composite.

Nevertheless, it should be pointed out that it is not necessary to replace the original material of the inclusion phase by a reference material since the nonlinear operator of the real material can be assumed uniform according to the scale separation condition made in the micromechanics theory.

Substitution of Eq.(4.24) into Eq.(4.23) leads to:

$$\begin{aligned}
 \langle \dot{\sigma} \rangle_{\omega} &= v_1 \langle \hat{C}_1^{tg}(t) : \dot{\epsilon}(x, t) \rangle_{\omega_1} + (1 - v_1) \langle \hat{C}_0^{tg}(t) : \dot{\epsilon}(x, t) \rangle_{\omega_0}, \\
 &= v_1 \hat{C}_1^{tg}(t) : \langle \dot{\epsilon}(x, t) \rangle_{\omega_1} + (1 - v_1) \hat{C}_0^{tg}(t) : \langle \dot{\epsilon}(x, t) \rangle_{\omega_0}, \quad (4.25)
 \end{aligned}$$

then, making use of Eqs.(4.18) and (4.19), Eq.(4.25) is rewritten as:

$$\langle \dot{\sigma} \rangle_{\omega} = \left[\hat{C}_0^{tg}(t) + v_1 (\hat{C}_1^{tg}(t) - \hat{C}_0^{tg}(t)) : A \right] : \langle \dot{\epsilon} \rangle_{\omega}. \quad (4.26)$$

On the other hand, in the Equivalent Homogeneous Medium (Figure 4.3-d), the linearized average stress-strain rate relationship is written as:

$$\langle \dot{\sigma} \rangle_{\omega} = \bar{C}^{tg} : \langle \dot{\epsilon} \rangle_{\omega} . \quad (4.27)$$

By design, average stresses in the Homogeneous Equivalent Medium and in the Associated Isolated Inclusion Medium must be the same medium ω_{EHM} is assumed to be equivalent to the medium ω (and also to ω) in terms of average stress rate. So, identification of Eq.(4.26) with Eq.(4.27) leads to:

$$\bar{C}^{tg} = \hat{C}_0^{tg}(t) + v_1(\hat{C}_1^{tg}(t) - \hat{C}_0^{tg}(t)) : A, \quad (4.28)$$

where $\bar{C}^{tg}$ represents the overall homogenized tangent (stiffness) operator of a nonlinear two-phase composite.

4.3.3 Mori-Tanaka homogenization scheme

Evaluation of the strain concentration tensor constitutes a major stage in the homogenization process. In Figure 4.4 are displayed the various steps needed to obtain strain concentration tensors expressions in the homogenization process of nonlinear composites. First of all, consider a two-phase composite made of a matrix and inclusion materials displaying both a nonlinear behavior (Figure 4.4-a). The first step (step 1) of this homogenization procedure consists in choosing a Representative Volume Element RVE of this composite (figure 4.4-b). In a second step, we replace the original nonlinear RVE with a Heterogeneous Linear Comparison Material which has a linearized constitutive law and per phase uniform mechanical properties. In the step 4, the LHCM is replaced by an RVE representing the Isolated Inclusion Problem, Figure 4.4-d. This transition is achieved according to the chosen homogenization scheme hypothesis, which tries to reproduce the effect of the surrounding inclusions on both the isolated inclusion and matrix material. Finally in step 5, the necessary eigenstrain for homogenization and which allows strain concentration tensors computation, is determined by imposing mechanical equivalence (consistency condition) between the Associated Isolated Inclusion Medium (Figure 4.4-d) and its Equivalent Homogeneous Medium (Figure 4.4-e).

With this in mind, the objective is the derivation of strain concentration tensor predicted by Mori-Tanaka homogenization model for an inelastic two-phase composite. We recall that the original RVE representing the inelastic composite is replaced by a Heterogeneous Linear Comparison Material HLCM (Figure F.1-a) with spatially uniform mechanical properties and were linearized following an incremental formulation. The behavior at the micro-level is described using Eq.(4.24). Now the problem looks like an elastic one and can be solved by following the same steps as in linear elasticity.

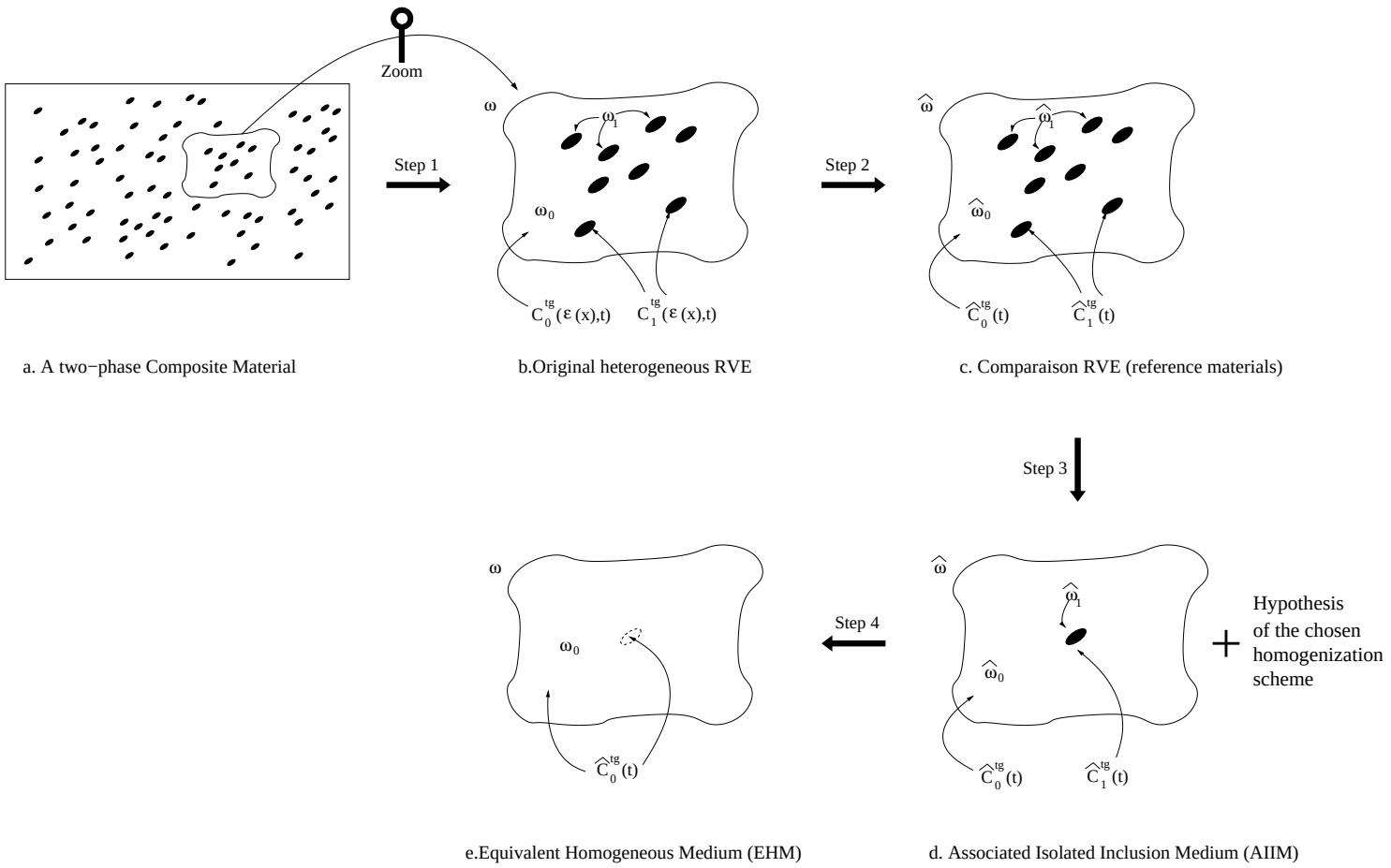


Figure 4.4: Schematic representation of homogenization procedure to obtain strain concentration tensor for inelastic two-phase composites.

Consequently we do not need to derive the expression of M-T model again since the final results of the model of the elastic problem can be directly used with the appropriate modifications. Thus, to obtain the strain concentration tensor $\tilde{A}^{MT}$ it suffices to replace the elastic operators with the tangent ones (those of the HLCM) in Eq.(2.86) of the elastic problem. The resulting expression of $\tilde{A}^{MT}$ in nonlinear regime is then written as:

$$\tilde{A}^{MT} = \left(I^4 - S_{I, \hat{C}_0^{tg}} : \left([\hat{C}_0^{tg}]^{-1} : \hat{C}_1^{tg} - I^4 \right)^{-1} + S_{I, \hat{C}_0^{tg}} \right)^{-1}. \quad (4.29)$$

Replacing $\tilde{A}^{MT}$ into Eq.(4.21) then the result into Eq.(4.28), one obtains the overall inelastic operator of a two-phase composite following Mori-Tanaka homogenization scheme. The complete derivation of the M-T scheme for the inelastic homogenization problem, is given in appendix F.

4.3.4 Other homogenization schemes

Similarly to the development of the Mori-Tanaka scheme in the context of nonlinear materials, extension of other homogenization schemes can be done. Final expressions of strain concentration tensors for the other homogenization models are given by:

1. Dilute inclusion model (DM):

$$A^{DM} = \left(I^4 + S_{I, \hat{C}_0^{tg}} : \left([\hat{C}_0^{tg}]^{-1} : \hat{C}_1^{tg} - I^4 \right)^{-1} \right)^{-1}. \quad (4.30)$$

2. Self-Consistent model (SC):

$$A^{SC} = \left(I^4 + S_{I, \bar{C}^{tg}} : \left([\bar{C}^{tg}]^{-1} : \hat{C}_1^{tg} - I^4 \right)^{-1} \right)^{-1}. \quad (4.31)$$

4.3.5 Eshelby's tensor computation issue

Analytical expressions of Eshelby's tensor can be obtained for various shapes of inclusions when the matrix phase of the composite is (i) isotropic or (ii) transversely isotropic with respect of the revolution axis of the spheroidal inclusions. Otherwise, and in particular in nonlinear regime where the tangent operator is anisotropic even for isotropic materials, the computation of Eshelby's tensor is not obvious and therefore still remains an open problem. One way to overcome this problem is to compute Eshelby's tensor numerically as proposed by Gavazzi and Lagoudas [27]. Starting from

Eshelby's well-known equation:

$$\varepsilon_{ij} = S_{ijkl} \varepsilon_{kl}^*, \quad (4.32)$$

and assuming the case of a matrix phase material with a tangent stiffness operator C^{tg} which is anisotropic, Eshelby's tensor S is given by the following surface integral over the unit sphere, as proposed by Mura [63]:

$$S_{ijkl} = \frac{1}{8\pi} C_{mnkl}^{tg} \int_{-1}^{+1} du \int_0^{2\pi} [G_{imjn}(\bar{\xi}) + G_{jmin}(\bar{\xi})] d\theta, \quad (4.33)$$

where the unit sphere is parametrized as:

$$\bar{\xi}_i = \frac{\xi_i}{a_i}; \quad \xi_1 = \sqrt{(1-u^2)} \cos \theta; \quad \xi_2 = \sqrt{(1-u^2)} \sin \theta; \quad \xi_3 = u; \quad (4.34)$$

with

$$G_{ijkl}(\bar{\xi}) = \bar{\xi}_k \bar{\xi}_l N_{ij}(\bar{\xi}) / D(\bar{\xi}), \quad (4.35)$$

and

$$D(\bar{\xi}) = \varepsilon_{mnl} K_{m1} K_{n2} K_{l3}, \quad (4.36)$$

$$N_{ij}(\bar{\xi}) = \frac{1}{2} \varepsilon_{ikl} \varepsilon_{jmn} K_{km} K_{ln}, \quad (4.37)$$

$$K_{ik}(\bar{\xi}) = C_{ijkl}^{tg} \bar{\xi}_j \bar{\xi}_l, \quad (4.38)$$

where ε_{ijk} is the permutation tensor and D can be regarded as the determinant of $K_{ik}(\bar{\xi})$.

Eq.(4.33) can be numerically evaluated using the following Gaussian quadrature formula as proposed by Gavazzi and Lagoudas [27]:

$$S_{ijkl} = \frac{1}{8\pi} \sum_{p=1}^{Mt} \sum_{q=1}^{Nt} C_{mnkl}^{tg} [G_{imjn}(\theta_q, u_p) + G_{jmin}(\theta_q, u_p)] W_{pq}, \quad (4.39)$$

where Mt and Nt designate the number of points used for the integration over θ_q and u_p , respectively and W_{pq} are the Gaussian weights. However, it was observed that the obtained macro predictions with the above mentioned numerical computation of S are generally too stiff.

Another way is to compute Eshelby's tensor with the isotropic part of the anisotropic nonlinear tangent operator of the matrix phase. There exists at least two general methods to perform the extraction of the isotropic part, see Bornert [8], and Doghri and Ouair [21]. In this work we use those which define the isotropic part of an anisotropic tangent tensor as a linear combination of the fourth rank tensors I^{vol} and I^{dev} , as follows:

$$C^{iso} \equiv (I^{vol} :: C^{tg}) I^{vol} + \frac{1}{5} (I^{dev} :: C^{tg}) I^{dev}, \quad (4.40)$$

with:

$$I^{vol} = \frac{1}{3} 1 \otimes 1, \quad I^{dev} = I - I^{vol}. \quad (4.41)$$

and 1 is the second-order identity tensor. The above equation can also be regarded as a projection of the tensor C^{tg} on the space spanned by I^{dev} and I^{vol} which represent the spherical and deviatoric parts of the fourth rank symmetric unit tensor I , respectively. The notation "::" designates the fourth-rank tensor contraction defined as follows:

$$I^{vol} :: C^{tg} = I_{ijkl}^{vol} C_{lkji}^{tg}, \quad (4.42)$$

Note that it is not necessary to compute the whole isotropic tangent operator since for, a spheroid (an ellipsoid with a revolution axis) and isotropic stiffness of the matrix phase, only the instantaneous Poisson's ratio " ν_t " is needed to be computed in order to evaluate Eshelby's tensor. One can proceed as follows:

$$\begin{aligned} I^{vol} :: C^{tg} &= 3\kappa_t = \frac{1}{3} C_{lljj}^{tg}, \\ I^{dev} :: C^{tg} &= 10\mu_t = C_{illi}^{tg} - \frac{1}{3} C_{lljj}^{tg}, \\ \nu_t &= \frac{3\kappa_t - 2\mu_t}{6\kappa_t + 2\mu_t}, \end{aligned} \quad (4.43)$$

where κ_t and μ_t are the instantaneous bulk and shear moduli, respectively. The use of the above method can be supported by the following arguments: In their paper on Eshelby's and Hill's tensors study [99], A. P. Suvorov and G. J. Dvorak [99] have derived the expression of S as follows:

$$S_{ijmn} = \frac{1}{16\pi} (M_{kijl} + M_{kjil} + M_{lijl} + M_{lilk}) C_{lkmn}^{tg}, \quad (4.44)$$

with

$$M_{kijl} = \int_{-1}^{+1} du \int_0^{2\pi} H_{kj}^{-1}(\bar{\xi}) \bar{\xi}_i \bar{\xi}_l d\theta, \quad (4.45)$$

and

$$H_{ij}(\bar{\xi}) = C_{ikjl}^{tg} \bar{\xi}_k \bar{\xi}_l. \quad (4.46)$$

It clearly appears from Eq.(4.45) that the analytical expression of H^{-1} is needed to compute M and then Eshelby's tensor S (Eq.4.44). The sole possible solution in this case is the use of a numerical procedure, since C^{tg} is an anisotropic tensor which would prevent us to have analytical expression of H^{-1} , the latter, however could be obtained if we use C^{iso} of C^{tg} . In other words, the use of C^{iso} allows the use of the analytical expressions of Eshelby's tensor. It was also showed in Doghri and Ouaar [21] that $C^{el} \leq C^{tg} \leq C^{iso}$ and more recently in Pierard and Doghri [74] that the overall homogenized tangent operator $\bar{C}^{tg}$ computed with Eshelby's tensor using C^{tg} is much more stiffer than those which is computed with Eshelby's tensor using C^{iso} . This

is the reason why, when the matrix is characterized by an anisotropic operator, the use of its isotropic part to compute Eshelby's tensor, leads to macro-estimates that display a better agreement with the target result provided by F.E. computation or by experiments.

4.4 Nonlinear Mori-Tanaka scheme using Hill's tensors

In this section we present Mori-Tanaka scheme expressed in terms of Hill's tensors for a two-phase inelastic composite. Hereafter are only presented the final results. First of all, expressions of Hill's tensor $P_{I, \hat{C}_0^{tg}}$ and Hill's constraint tensor C^* in terms of reference matrix material tangent operator $\hat{C}_0^{tg}$, are respectively written as:

$$P = S_{I, \hat{C}_0^{tg}} : (\hat{C}_0^{tg})^{-1}, \quad (4.47)$$

and as:

$$C^* = P_{I, \hat{C}_0^{tg}}^{-1} - \hat{C}_0^{tg}. \quad (4.48)$$

Then, the expression of the strain concentration tensor $\tilde{A}$ is written in terms of Hill's tensor $P_{I, \hat{C}_0^{tg}}$ is given by:

$$\tilde{A}^{MT} = \left(I^4 + P_{I, \hat{C}_0^{tg}} : [\hat{C}_1^{tg} - \hat{C}_0^{tg}] \right)^{-1}, \quad (4.49)$$

and in terms of Hill's constrain tensor C^* (using Eq.(4.48)) as:

$$\tilde{A}^{MT} = (C^* + \hat{C}_1^{tg})^{-1} : (C^* + \hat{C}_0^{tg}). \quad (4.50)$$

Eq.2.165 of strain concentration tensor A^{MT} is then rewritten as:

$$A^{MT} = \left[(1 - v_1) \left\{ (C^* + \hat{C}_1^{tg})^{-1} : (C^* + \hat{C}_0^{tg}) \right\} + v_1 I^4 \right]^{-1}. \quad (4.51)$$

Finally, the homogenized tangent operator $\bar{C}$ expression written terms of Hill's constrain tensor for a two-phase composite using Mori-Tanaka scheme is obtained by replacing the last equation into Eq.(4.28).

It should be worth to note that in the inelastic regime, Hill's tensor $P_{I, \hat{C}_0^{tg}}$ computation plays a major role for the accuracy of mechanical properties predictions. Examination of Eq.(4.47) leads to several possibilities to compute $P_{I, \hat{C}_0^{tg}}$, one can use for instance:

- (1) the anisotropic tensor $\hat{C}_0^{tg}$ with Eshelby's tensor $S_{I, \hat{C}_0^{tg}}$ computed with the isotropic part of $\hat{C}_0^{tg}$.

- (2) the isotropic part of $\hat{C}_0^{tg}$ with Eshelby's tensor $S_{I, \hat{C}_0^{tg}}$ computed with the isotropic part of $\hat{C}_0^{tg}$.

Note also that Pierard and Doghri [74] have proved for elastoplastic behavior that the homogenized stiffness operator written in terms of Hill's tensors is always symmetric since C^* and $P_{I, \hat{C}_0^{tg}}$ are symmetric.

4.5 Nonlinear multi-phase composites modeling

Homogenization of multi-phase composites in nonlinear regime follows the same steps as that for the linear elasticity one. However, to make this possible we first shall transform the nonlinear problem into a problem which is similar to the elasticity one. This is done making use of Linear Heterogeneous Comparison Material concept as viewed in above sections (section 4.2 in particular). To obtain homogenized mechanical properties of nonlinear multi-phase composites, many routes can be followed. A first one is, as viewed in chapter 2, the use of the Aggregate Decomposition Approach which was extended to inelastic regime by many researchers in the field, see for instance Doghri and Tinel [22], Friebel et al. [26] and Ouaar et al. [68]. A second possible way to tackle multi-phase composites problem is the use of what we will call the Direct Computation Approach. Note that the numerical implementation of the latter can be cumbersome and may exhibit some convergence problems. Direct Computation method is developed for some particular cases of nonlinear multi-phase composites in appendix A.

The aggregate decomposition approach was discussed in detail in section 2.13 of chapter 2 and is shortly recalled hereafter. The main steps of this method extended to nonlinear regime are presented. The method can be applied to any multi-phase composite (domain Ω) made of a matrix material (Ω_0) and N families of spheroidal inclusions or cavities. Consider that each family (i) is characterized by an aspect ratio, an orientation distribution function (ODF) $\psi_i(p)$ and a material for solid inclusions. The orientation of an individual spheroid is determined by a unit vector p along the revolution axis. The volume fractions of the matrix phase and the families of inclusions are designated by v_0 and v_i , respectively, $v_0 + \sum_{i=1}^N v_i = 1$.

In nonlinear regime, we start by replacing the nonlinear multi-phase RVE representing the nonlinear multi-phase composite with a Heterogeneous Linear Comparison Material with uniform linearized mechanical properties. In same manner as for a two-phase composite discussed above, the linearization is performed using the incremental formulation and the per-phase uniformization of tangent operators is achieved using the average strain of the considered phase. Then, the method starts with a virtual decomposition step by defining so-called pseudo-grains $\Omega_{i,p}$, each containing matrix material

ω_0 (in concentration v_0) and all inclusions ω_i , $i = 1, \dots, N$ of family (i) whose orientation is between p and $p + dp$. Next each pseudo-grain is homogenized using a model suitable for two-phase composites. Finally, the set of homogenized pseudo-grains is itself homogenized.

It can be shown that the volume average of the stress rate over the RVE can be expressed as follows:

$$\langle \dot{\sigma} \rangle_{\Omega} = \sum_{i=1}^N \frac{v_i}{1-v_0} \oint \langle \dot{\sigma} \rangle_{\Omega_{i,p}} \psi_i(p) dp \equiv \left\langle \langle \dot{\sigma} \rangle_{\Omega_{i,p}} \right\rangle_{i, \psi_i}, \quad (4.52)$$

where $\langle \dot{\sigma} \rangle_{\Omega_{i,p}}$ are the volume average of the stress rates over the pseudo-grains $\Omega_{i,p}$. The above equation can be rewritten as follows using the homogenized tangent operators of the RVE and of each pseudo-grain:

$$\bar{C}^{tg} : \langle \dot{\epsilon} \rangle_{\Omega} = \left\langle \bar{C}_{\Omega_{i,p}}^{tg} : \langle \dot{\epsilon} \rangle_{\Omega_{i,p}} \right\rangle_{i, \psi_i}. \quad (4.53)$$

We recall that the homogenized tangent operators $\bar{C}_{\Omega_{i,p}}^{tg}$ of each pseudo-grain $\Omega_{i,p}$ can be obtained using a suitable homogenization scheme for a two-phase composite.

As discussed in chapter 2, several possibilities can be considered to obtain the expression of the overall nonlinear operator $\bar{C}^{tg}$ of the multi-phase RVE Ω :

- A first possibility studied hereafter is to consider Voigt's model as a second homogenization scheme. The hypothesis of this model is written in terms of rate as:

$$\langle \dot{\epsilon} \rangle_{\Omega} = \langle \dot{\epsilon} \rangle_{\Omega_{1,p}} = \langle \dot{\epsilon} \rangle_{\Omega_{2,p}} = \dots = \langle \dot{\epsilon} \rangle_{\Omega_{N,p}}, \quad (4.54)$$

which leads to the following expression of the homogenized nonlinear operator $\bar{C}^{tg}$ for a multi-phase composite:

$$\bar{C}^{tg} = \left\langle \bar{C}_{\Omega_{i,p}}^{tg} \right\rangle_{i, \psi_i}. \quad (4.55)$$

- A second possibility is the use of Reuss's model for the second homogenization step. Reuss's hypothesis assumes that stress rates are the same everywhere in the RVE, it follows that:

$$\langle \dot{\sigma} \rangle_{\Omega} = \langle \dot{\sigma} \rangle_{\Omega_{1,p}} = \langle \dot{\sigma} \rangle_{\Omega_{2,p}} = \dots = \langle \dot{\sigma} \rangle_{\Omega_{N,p}}. \quad (4.56)$$

After some algebraic manipulations (see chapter 2 section 2.13), the final expression of the homogenized nonlinear operator $\bar{C}^{tg}$ for a multi-phase composite is then obtained. The final result is:

$$\bar{C}^{tg} = \left\langle \left(\bar{C}_p^{tg} \right)^{-1} \right\rangle_{i, \psi_i}^{-1}. \quad (4.57)$$

- Another possibility is to assume that the average strain rate of the matrix phase are the same for each pseudo-grain, so we have:

$$\langle \dot{\epsilon} \rangle_{\Omega_{1,p}^{\omega_0}} = \langle \dot{\epsilon} \rangle_{\Omega_{2,p}^{\omega_0}} = \dots = \langle \dot{\epsilon} \rangle_{\Omega_{N,p}^{\omega_0}} = \langle \dot{\epsilon} \rangle_{\Omega^{\omega_0}}. \quad (4.58)$$

Under this assumption, the strain rate averaged over Ω is written as:

$$\langle \dot{\epsilon} \rangle_{\Omega} = \left((1 - v_0) \left\langle \tilde{A}_{i,p} \right\rangle_{i,\psi_i} + v_0 I^4 \right) : \langle \dot{\epsilon} \rangle_{\Omega^{\omega_0}}, \quad (4.59)$$

and the stress rate averaged over Ω as

$$\langle \dot{\sigma} \rangle_{\Omega} = \left((1 - v_0) \langle C_i^{tg} : \tilde{A}_{i,p} \rangle_{i,\psi_i} + v_0 C_m^{tg} \right) : \langle \dot{\epsilon} \rangle_{\Omega^{\omega_0}}, \quad (4.60)$$

Combining the two last equations and identifying the result with $\langle \dot{\sigma} \rangle_{\Omega} = \bar{C}^{tg} : \langle \dot{\epsilon} \rangle_{\Omega}$ gives:

$$\bar{C}^{tg} = \left((1 - v_0) \langle C_i^{tg} : \tilde{A}_{i,p} \rangle_{i,\psi_i} + v_0 C_m^{tg} \right) : \left((1 - v_0) \left\langle \tilde{A}_{i,p} \right\rangle_{i,\psi} + v_0 I^4 \right)^{-1}. \quad (4.61)$$

Note that homogenization scheme derived below, can lead in some cases to inaccurate estimates. For instance, in case of multi-phase composite consisting in a matrix phase with two inclusion phases (with one inclusion phase having the same mechanical properties than the matrix), Doghri and Tinel [22] shown that the use of M-T scheme for the first homogenization step and Voigt's scheme for the second one, leads to a loss of the required symmetries for the macro stiffness operator, see also Pierard *et al.* [72].

4.6 Numerics

In this research work, two kinds of composites materials are considered: (i) brittle matrix composites (Concrete-like Matrix Composites) and (ii) elastoplastic matrix composites (Metal and Polymer Matrix composites). For each composite material, we propose a corresponding algorithm based on the two-step Mori-Tanaka/Voigt homogenization scheme. Those algorithms and them related topics are detailed and discussed in chapters 6 and 7 respectively.

4.7 Conclusion

In this chapter we developed an incremental method which allows to model the overall response of a class of nonlinear composites consisting in a hosting matrix with inclusions embedded therein. The method was derived within the framework of the micromechanics based on the mean-field approach and using the principle of Eshelby's Equivalent Inclusion. In a first part, we shown how the nonlinear homogenization problem is resolved by transforming it into a problem similar to a linear elastic one, making use of *Linear Heterogeneous Comparison Material* concept. The constitutive laws were linearized using an elastic-like relationship which relates the stress and strain rates respectively, through a tangent operator. The intra-phase homogeneity were obtained through uniform tangent operators which were computed using the average strain rate of the considered phase.

The various steps needed for the homogenization procedure were detailed. We started by deriving the overall expression of the homogenized tangent operator for a nonlinear two-phase composites. Afterwards the expression of strain concentration tensor for Mori-Tanaka scheme was derived and final results were presented for some homogenization schemes. Schematic representations were proposed in order to allow a better understanding of the used procedures. The crucial issue of Eshelby's tensor was examined. Finally, the proposed incremental formulation was then extended to handle the modeling of nonlinear multi-phase composites making use of both aggregate decomposition approach and direct computation method (see appendix A for the later one).

The numerical aspects (time discretization, algorithms, numerical simulations) of the proposed incremental formulation will be presented and discussed in chapters 6 and 7 respectively.

Chapter 5

Experimental program

5.1 Introduction

Concrete material lends itself to a variety of innovative designs but two characteristics have limited its use: it is brittle and weak in tension. One way to overcome this problem is the addition of metallic fibers to the concrete mix. As microcracks begin to appear due to shrinkage as water evaporates from the concrete (plastic shrinkage) or to loadings, the microcracks intersect with the fibers which stretch in order to block their growth. Fibers hinder microcracking, which improves the composite' ductility. Steel Fiber-Reinforced Concrete (SFRC) has captivated our interest because it has a large range of applications in civil engineering. It is used in bridges, pipes, airport runways, tunnel linings, etc. Steel fibers are much stiffer than concrete. However, as illustrated in the present study, the addition of fibers may induce porosity that reduces significantly the overall stiffness, see also B.R. Walkus [108].

SFRC has been the subject of previous investigations. Among researchers who focused on experimental aspects, B. R. Walkus [108] analyzed the response of concrete reinforced with various volume fractions of fibers under tensile loading. P. Soroushian [92] performed bending tests, and A. Van Hauwaert [106] investigated fiber adhesion. The experimental study reported hereafter deals with four-point bending tests campaign performed on three series of beam specimens made of:

- Plain concrete.
- Steel fiber-reinforced concrete with various volume fractions (1%, 3% and 4%) of fibers.

- Steel bar-reinforced concrete.

This study has mainly two objectives:

- The first one consists in the identification of the most important elements that govern the mechanical behavior of SFRC and to understand their role in order to achieve, afterwards, an appropriate modeling of the composite.
- The second followed goal is to evaluate the mechanical properties of both plain concrete and of steel fiber-reinforced concrete specimen, in the hardened state.

The collected results are treated and presented under curves format and then analyzed in order to clarify the guidelines for the achievement of an appropriate composite modeling. In a second part, we developed an incremental method which allowed us, in the nonlinear regime, to transform load-deflection curves onto stress-strain curves, the latter being more useful for both material parameters identification and numerical simulations validation.

This chapter is outlined as follows: in sections 2 and 3 a description of samples and test equipment is given. In section 4, the collected results from fourth-point bending tests are then treated and presented under curve format and then analyzed. The nondestructive campaign tests is presented in section 5 while porosity estimate is examined in section 6. Finally, in section 7, we developed a original and simple method, in nonlinear regime, enabling the transformation of load-deflection curves onto stress-strain curves which are more useful for both material parameters identification and numerical simulations validation.

5.2 Material

The various beam samples used in this study are made of plain concrete, steel fiber-reinforced concrete, and bar reinforced-concrete. Hereafter, we will give a description of the various materials (concrete mix and fibers) used to cast the samples.

5.2.1 Fibers

The America Concrete institute (ACI), Committee 544, defines four categories of Fiber-Reinforced Concrete (FRC) based on fiber material type. The most popular is Steel Fiber-Reinforced Concrete SFRC, for fibers made of steel. We also have GFRC for glass fibers, CFRC carbon fibers, and NFRC for natural fibers. In addition to this,

many more types of fibers are used today than just the straight-round ones that are simply cut from drawn wires. Steel fibers are available smoothed, corrugated, or with wide or hooked end for better anchorage, Figure 5.1. The fibers can be placed single or glued in form of mats. In SFRC applications, steel fibers usually range in volume fraction from 0.3 to 3

The type of fibers used in this investigation is steel hooked-end. The fibers, called ZAP30/0.5 were manufactured by Beckaert in Belgium. They have a length of 30mm with a diameter of 0.5mm, an aspect ratio $Ar = 60$, and have the following mechanical properties: Young's modulus $E = 210$ GPA, Poisson's ratio $\nu = 0.3$.

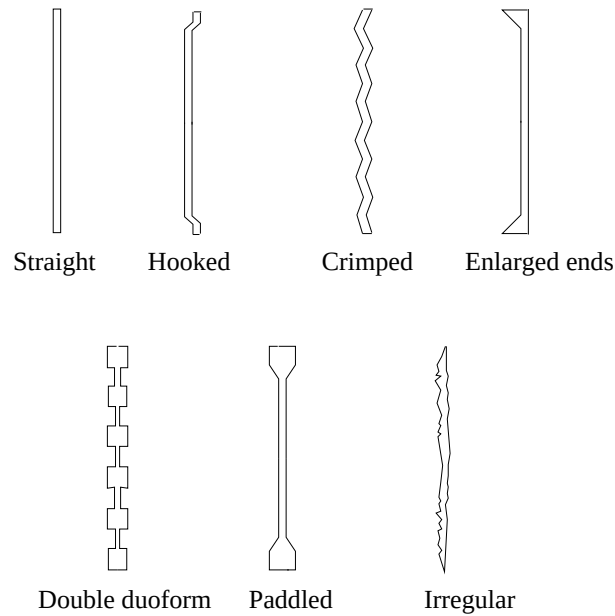


Figure 5.1: Steel fibers type

5.2.2 Concrete mix

The physical and mechanical properties of concrete are determined, in part, by the proportions of the three key ingredients, water, cement, and aggregate. The mixture should not be too stiff or too sloppy. It is difficult to form good test specimens if it is too stiff. If it is too sloppy, water may separate (bleed) from the mixture. Remember that water is the key ingredient. Too much water results in weak concrete. Too little

water results in a unworkable concrete.

In this program test, plain concrete mixture was prepared with a special cement 32.5 (LEAFAGE MORTIE'S), river sand 0/5, coarse aggregates 4/14 and water. Table 5.1 summarizes the ingredient proportions of the mixture.

Constituent	Quantity
Cement 32.5	18.5 kg
River sand 0/5	37.2 kg
Coarse aggregates 4/14	55.8 kg
Water	8.37 l

Table 5.1: Ingredient proportions of the concrete mixture.

5.2.3 Specimens manufacturing

All specimens were casted in 600x150x150mm steel molds, in accordance to the recommendation of JASE-SF4, NB B15-238 and AS TM C1018. Enrich SK1 mixer with a maximum capacity of 50 l was used in order to manufacture plain concrete specimens as displayed in Figure 5.2-(a). The mixer was first charged with aggregates, sand, and cement which were mixed during one minute, then water was added and the concrete was mixed for two more minutes. For SFRC specimens, fibers were introduced gradually as a last ingredient to concrete mix at time interval of two minutes. After two minutes of rest, the composite is mixed again for three minutes. Each mold was filled with material to be tested in two layers, and each layer was consolidated using a vibrating table, Figure 5.2-(b). Vibrating time depended on the workability of the mix and was limited to 5 to 10 seconds to avoid segregation of the constituents and fiber alignment. After 24 hours, the samples were removed from the molds and moist in air-conditioned room at 20°C in 90 % moisture. After 28 days of aging, plain concrete and SFRC samples are then submitted to four-point loading tests.

5.3 Test equipment

Four-point bending tests were conducted on MTS testing machine equipped with a load cell of 100 kN. The load is applied to the specimen throughout two rollers, 150 mm apart one of the other, Figure 5.3. Three kinds of recorders were used. Two linear voltage differential transformers (LVDT) were fixed on the lateral faces of the specimens. They record the vertical displacement of the upper side of the beam. Under



Figure 5.2: Plain concrete and SFRC specimen manufacturing.

the beam, an extensometer (or clip on gage) MTS 632.02F-20 was installed in order to measure the stretching of the lower fibers of the sample. Near the extensometer a strain gage of type CE-06-250UN-350 was glued to record the strain of the lower fibers.



Figure 5.3: Test setup of four-point bending.

All those recorders were related to the acquisition card which collects all measures and transforms them into workable quantities. The treated data are then sent to the computer which gives, throughout a code developed with LAB VIEW software, the results of each recorder in terms of a table. Figure 5.4 shows a schematic representation of the various components of the acquisition system.

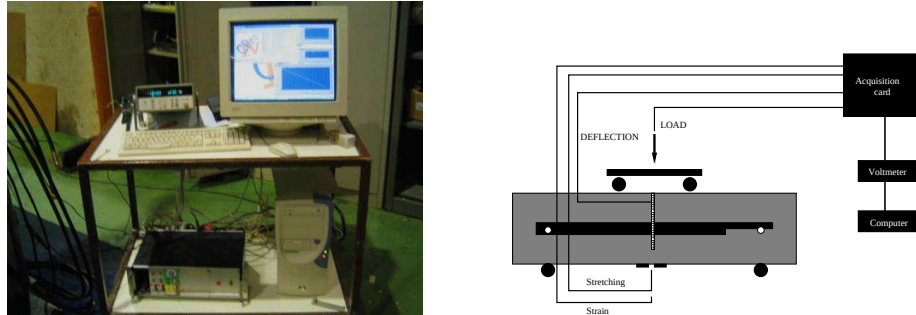


Figure 5.4: Acquisition chain used in four-point bending tests.

5.4 Four-point bending results and discussions

All results of the fourth-point bending tests campaign are given, under plots format, in appendix F. In particular in this section, attention is given to measurements captured by linear voltage displacement transformers (LVDT). Figure 5.6 displays load-deflection curves for plain concrete, concrete reinforced with 1%, 3% and 4% of steel fibers and concrete reinforced with long bars (4%) samples. We recall that long bars are the conventional reinforcement used in concrete in order to support tensile loadings.

Measurements collected by strain gages and clip on gages seems to be less accurate than those of LVDT recorders. It appears that the used strain gages were not appropriate. Indeed, the aggregates dimensions are in the same order than those of the strain gages and this leads to a local measurement of the strain field by the strain gage as viewed in Figure 5.5. It follows that strains measured by those recorders are overestimated. However, strain gages and extensometers measurements were used as qualitative indicators in order to confirm LVDT ones.

The analysis of results detailed in the appendix F has allowed us to make the following interesting observations.

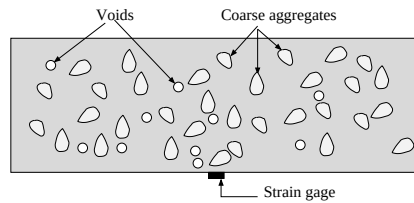


Figure 5.5: Overview on the size of strain gage and concrete mix constituents.

5.4.1 Stiffness

From tests results, it appears that fibers do not affect the overall stiffness of the composite. From Figure 5.6, slope analysis of load-deflection curves indicates that the overall stiffness of reinforced beams is not increased in a significant manner. At fiber volumes in excess of 3%, contrary to all expectations, one can remark a decrease of the stiffness of SFRC samples. This observation can be explained by:

- First, the low content of fibers.
- Second, the fact that fibers increase the porosity which counterbalances the expected effects of the reinforcement.

Fibers increase stiffness but also increase the initial porosity, as it can be confirmed by visual examination of specimen with 4% fibers, figure 5.7. This conclusion will have a crucial importance in the strategy that we will use for micromechanical modeling of steel fiber-reinforced concrete composites, see chapter 5.

5.4.2 Flexural strength

Two flexural strength values are commonly reported in four-point bending tests. The first, called the first-crack flexural strength, corresponds to the load at which the load-deflection departs from linearity (point A on Figure 5.8). The second, termed the ultimate flexural strength, corresponds to the maximum load achieved in the test (point C on Figure 5.8). Load-deflection curves, from Figures F.1 to F.12 of Appendix D, emphasize the confusion that can occur in reporting those two flexural strength values, especially in the case of small fiber volume fractions where the first-crack load may be the maximum load. For large amounts of fibers the two values are, however, quite distinct. Comparing plain concrete and reinforced concrete load-deflection curves, Figure 5.6, one can conclude that both fibers and long bars reinforcement improve the flexural strength of the concrete.

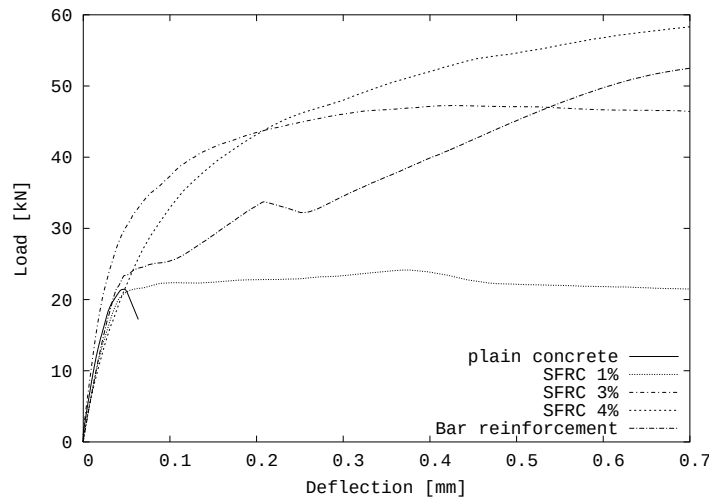


Figure 5.6: Plain concrete and SFRC (1%, 3% and 4%) under four-point bending: load-deflection curves.

5.4.3 Flexural toughness

Flexural toughness also called ductility, is an important characteristic required for steel fiber-reinforced concrete. Under static loading, ductility can be defined as the area located under load-deflection curve in bending test. This area represents the total energy absorbed by the specimen prior to its total failure. Examination of Figure 5.6 clearly indicates that SFRC beams have a higher ductility than plain concrete ones. It is observed that beyond first crack, SFRC specimens (and also bar-reinforced concrete specimens) do not lose their load-carrying capacity. Addition of fibers increases in a significant manner the load-deflection curve area of the composite. Fibers presence in concrete increases its ability to absorb energy during cracking. This is explained by the fact that when new microcracks open along the fiber, the fiber stretches in order to put to arrest the propagation of the new microcrack, as shown in Figure 5.9.

5.4.4 SFRC under loading/unloading

Concrete beams reinforced with 3% of fibers were also submitted to a cyclic four-point bending test. The LVDT records results are displayed in Figure 5.10 under the form of load-deflection curve. This test emphasizes the role of elastoplastic behavior fibers



Figure 5.7: Porosity increase of sample with high content of fibers.

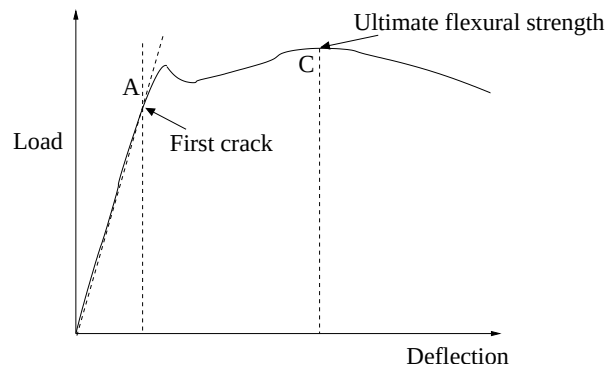


Figure 5.8: A typical load-deflection curve.

on the overall response of the composite. From Figure 5.10 it can be seen that beyond a certain level of loading, when unloading a specimen an irreversible or permanent deflection can be observed. A zero load corresponds to a non-zero deflection resulting from the fact that fibers become plastic. However, it should be pointed out that at this level of loading, the concrete matrix behavior can not be described by the means of the classical continuum damage mechanics theory. Fracture mechanics is, in this case, an more appropriate tool.

5.5 Non-destructive tests

In parallel with four-point bending tests, we achieved a non-destructive campaign tests. Grindo-Sonic test was performed according to NBN B15-230 standard. The

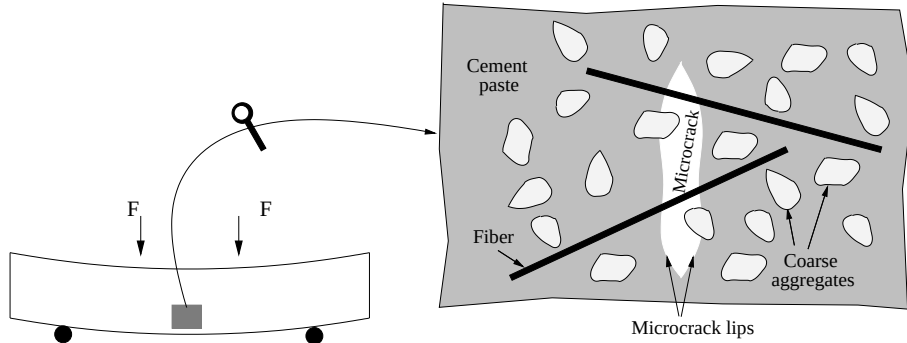


Figure 5.9: Role of fiber in the restriction of microcrack propagation.

aim of this method is to determine the resonance frequency of the body which is related to the dynamic elastic modulus E_{dyn} . For similar shapes and dimensions, the frequency of the body is proportional to its stiffness. When exciting a body with a frequency close to its natural frequency rise, the vibration amplitude increases considerably. This is what is called resonance.

Many ways to vibrate a body are possible. In this study beam is vibrated in bending by exciting its upper face as displayed in Figure 5.11. Five attempts were carried out. Knowing the resonance frequency, dynamic elastic modulus E_{dyn} is then computed according to NBN B15-230 standard equation:

$$E_{dyn} = 4.10^{-6} \frac{\pi^2 L^4}{(4.73)^4 i^2} f_F^2 \rho C \quad [N/mm^2] \quad (5.1)$$

where:

- E_{dyn} is the dynamic elastic modulus
- L is specimen length
- i is gyration radius of the considered cross section
- f_F is the fundamental resonance frequency in bending
- ρ is sample density
- C is a corrective factor depending on i and on ν Poisson's ratio of the material.

In this study, according to tables given by Belgium Standard (NBN B15-230), for material with $\nu = 0.19$ and $i = 0.0722$, we have taken $C = 1.422$. For plain concrete specimens, the average dynamic elastic modulus E_{dyn} over the five tests was found equal to 38.7 GPa which is in a good agreement with the static elastic modulus ($E=36.9$ GPa) obtained in section 5.7.3.

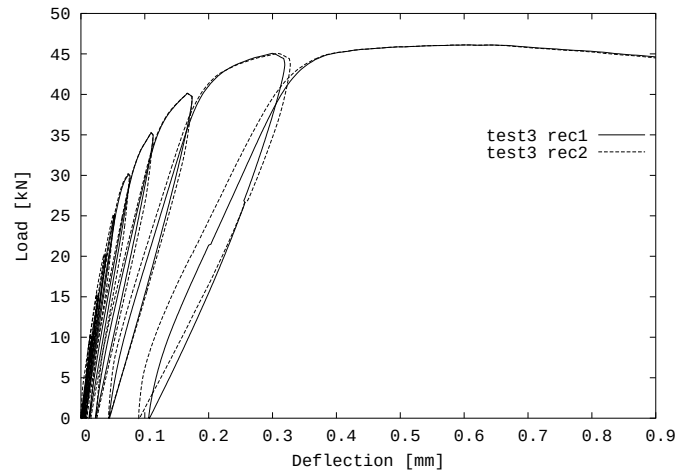


Figure 5.10: Load-deflection of SFRC (3% of fibers) under loading and unloading.

However, for steel fiber-reinforced concrete specimens, the above-mentioned test is out of application range. This can be explained by the fact that resonance frequency of concrete matrix material is quite different than those of fibers material one.

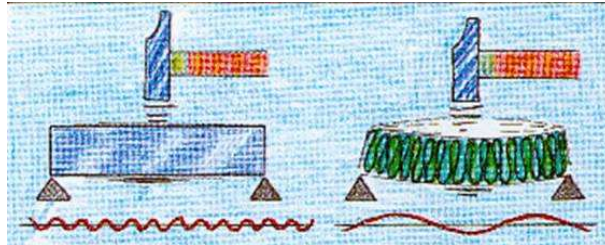


Figure 5.11: Schematic representation of non-destructive test.

5.6 Porosity estimate

Many techniques dealing with material porosity estimate can be found in the literature, see [71]. For example, Porosity Measurement with Mercury and Porosity Measurement with Triple Weighing Techniques can be cited. For a more detailed discussion

about those methods, the reader is referred to [71]. In our study, we proposed a simple method, accurate enough, in order to estimate the samples porosity. In this method we consider that plain concrete samples have a reference porosity which we assume equal to zero. This assumption has no negative consequence since we are only interested with porosity increase due to fibers addition. We proceed as follows: first, plain concrete, SFRC specimens, and the quantity of fibers presents in SFRC samples are weighed. Next, we determine ρ_{pc} , ρ_f plain concrete samples and fibers densities respectively. The total volume of each SFRC sample V_T , which is known, is written as:

$$V_T = V_{pc} + V_f + V_v, \quad (5.2)$$

where V_f is the volume of fibers present in the composite, V_{pc} is plain concrete material volume and V_v voids volume (porosity). V_f can be evaluated as:

$$V_f = \frac{m_f}{\rho_f}, \quad (5.3)$$

where m_f is fibers mass present in the composite. Since voids mass is assumed null ($m_v = 0$), plain concrete mass m_{pc} is then equal to:

$$m_{pc} = m_T - m_f, \quad (5.4)$$

consequently, the volume V_{pc} corresponding to plain concrete material is then given by:

$$V_{pc} = \frac{m_{pc}}{\rho_{pc}}, \quad (5.5)$$

it remains to compute voids volume V_v with:

$$V_v = V_T - V_f - V_{pc}. \quad (5.6)$$

5.7 Stress-strain curve construction from bending test

Due to its poor tensile strength, concrete is generally characterized in bending rather than in tension. Uniaxial stress-strain curves are, nevertheless, necessary in order to fit parameters of constitutive models in numerical simulations. Therefore, we developed an incremental method which enables us to construct a stress-strain curve from a load-deflection one.

Consider a concrete beam subjected to four-point bending test, as displayed in Figure 5.12. Assume a first load increment such as the specimen remains in the elastic domain. The diagram of normal stress over a cross section of the beam has a linear triangular form, Figure 5.12-(a). Note that as long as the material is elastic, the neutral axis will be located at the mid-height of the beam. With increasing load, microcracks

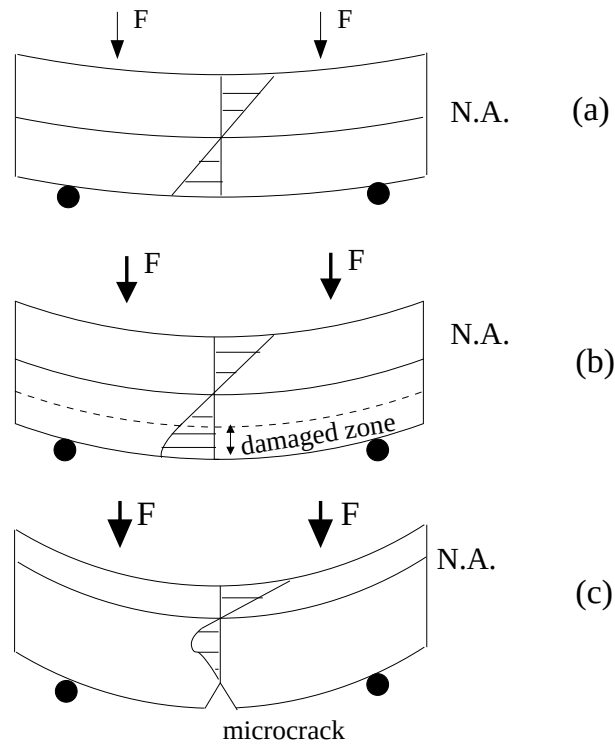


Figure 5.12: Cross section axial stress evolution during loading.

are mainly initiated in the stretched part of specimen. Damage takes place and the normal stress over the cross section diagram becomes nonlinear, Figure 5.12-(b). Microcracks cause a stress state relaxation. One can see, in Figure 5.12-(b), that stresses increase less in stretched part than in compressed one. Consequently, the neutral axis moves toward the compression zone in order to equilibrate normal strengths over the cross section. Under higher load, microcracks grow and nucleate giving rise to macrocrack in the compressed part. Microcracks zone is characterized by a loss of carrying capacity, stress decreases while strain increases as shown in Figure 5.12-(c).

5.7.1 Stress-strain-bending moment relationships

The aim of this section is to derive the normal stress-strain relationship using the experimental measurements of the deflection D_i , the maximum tensile strain ε_{it} , and

bending moment M_i at each “ i ” increment of loading. In the incremental method developed hereafter, we suppose that cross sections remain plane and that strain varies linearly with the height-coordinate of the cross section of the beam. The method is valid only for plain concrete and SFRC specimens. In the following, we only consider the normal stress σ_{xx} and strain ϵ_{xx} . For simplicity of notation, the subscript (xx) will be nevertheless omitted. For the apparatus sketched in Figure 5.13-(b), one obtains, $M_i = F_i l/2$, where F_i is the i^{th} increment of load F exerted on one roller. Note that all variables with subscript “ i ” are measured or computed at the “ i^{th} ” increment of loading.

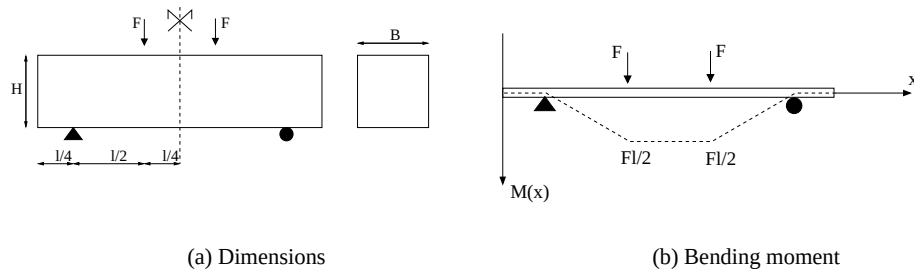


Figure 5.13: Beam model.

Consider a first loading step small enough to keep the material in the elastic domain and analyze the different fields acting over a cross section located where the maximum of the bending moment is reached, Figure 5.14. The bending moment M_1 can be related to the normal stress acting on the cross section throughout the following equation:

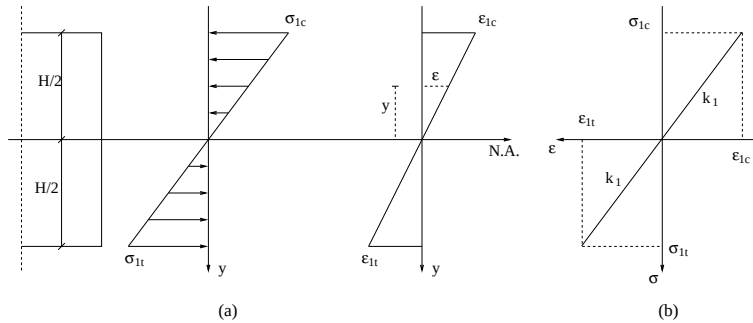


Figure 5.14: State stress and strain over a cross of a bent beam and stress-strain curve in elastic regime.

$$M_1 = F_1 \frac{l}{2} = B \left[\int_0^{H-C_1} \sigma_t(y) y dy + \int_{-C_1}^0 \sigma_c(y) y dy \right], \quad (5.7)$$

where F_1 is the first loading increment applied on each roller, $l/2$ the distance separating the roller from support, $\sigma_t(y)$ and $\sigma_c(y)$ are stresses in tensile and compression zone respectively, and B the thickness in the z direction.

Referring to Figure 5.14-(b), stresses are written as:

$$|\sigma(y)_t| = |\sigma(y)_c| = k_1 |\varepsilon(y)|. \quad (5.8)$$

assuming that strain varies linearly over the height of the beam, one can write:

$$\varepsilon(y) = \frac{y}{H-C_1} \varepsilon_{1t}, \quad (5.9)$$

with ε_{1t} the maximum tensile strain. However, since the first increment is assumed to be elastic, $C_1 = H/2$.

Now, consider a second loading step F_2 such as damage threshold of the material is reached. As represented in Figure 5.15, one can observe slope variations in stress diagram over the cross section. Material portion of cross section located between $y = -C_2$ and $y = H_1$ remains elastic. Are included here the compressive zone for $y \in [-C_2, 0]$ and a part of stretched one ($y \in [0, H_1]$). However, from $y = H_1$ to the $y = H - C_2$ the slope of the stress diagram has changed, it involves a loss of elasticity. In our case it means that material undergoes damage. Bending moment M_2 at step 2 is

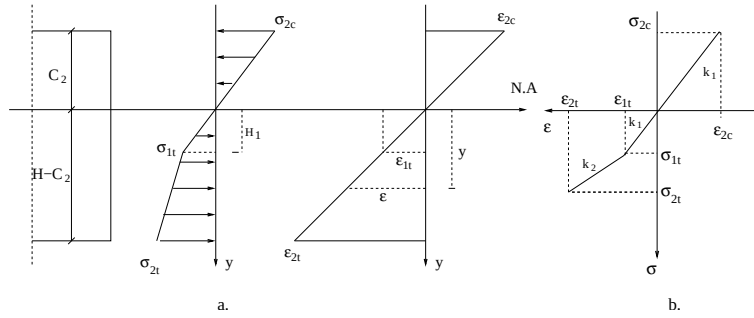


Figure 5.15: State stress and strain over a cross of a bent beam and stress-strain curve in nonlinear regime.

then written in term of stress as:

$$M_2 = F_2 \frac{l}{2} = B \left[\int_0^{H_1} \sigma_t(y) y dy + \int_{H_1}^{H-C} \sigma_t(y) y dy + \int_{-C}^0 \sigma_c(y) y dy \right]. \quad (5.10)$$

Expressions of stresses versus strains at each portion of the cross section can be derived from Figure 5.15-(b). One obtains:

$$\sigma_t(y) = k_1 \varepsilon(y), \quad \text{if } y \in [0, H_1] \quad (5.11)$$

$$\sigma_t(y) = k_1 \varepsilon_{1t} + k_2 (\varepsilon(y) - \varepsilon_{1t}), \quad \text{if } y \in [H_1, H - C_2] \quad (5.12)$$

$$\sigma_c(y) = k_1 \varepsilon(y) \quad \text{if } y \in [-C_2, 0]. \quad (5.13)$$

Strain versus height-coordinate of the cross section is written as:

$$\varepsilon(y) = \frac{y}{H - C_2} \varepsilon_{2t}. \quad (5.14)$$

The height-coordinate $y = H_1$ corresponding to σ_{1t} is then given by:

$$H_1 = \frac{\varepsilon_{1t}}{\varepsilon_{2t}} (H - C_2). \quad (5.15)$$

A basic condition to be satisfied in beam theory is that the external axial force F_E acting on a cross section, must balance the internal one, F_I . Since there is no external axial force and by neglecting friction, the internal force resultant must be zero. Thus, for the second step loading we have:

$$F_{2I} = B \left[\int_{-C_2}^0 \sigma(y) dy + \int_0^{H_1} \sigma(y) dy + \int_{H_1}^{H-C_2} \sigma(y) dy \right] = 0. \quad (5.16)$$

At each loading increment, this condition must be checked. Thus, at step n we have:

$$\underbrace{\int_{-C_n}^0 \sigma(y) dy}_{F_c} + \underbrace{\int_0^{H_1} \sigma(y) dy + \int_{H_1}^{H_2} \sigma(y) dy + \dots + \int_{H_{n-1}}^{H-C_n} \sigma(y) dy}_{F_t} = 0, \quad (5.17)$$

where the first term of the above equation represents compressive strength F_c acting over the part of section located between neutral axis and the upper side of the beam, and the others terms of the equation represent tensile strength F_t acting over the remaining section.

In summary, for each loading step we have two equations: the first equation relates the applied increment load (the bending moment) to the normal stress acting on the cross section, while the other one writes the equilibrium of the internal and external axial forces.

$$M_n = f(k_n, C_n), \quad (5.18)$$

$$F_{nI}(k_n, C_n) = 0. \quad (5.19)$$

Eqs 5.18 and 5.19 form a set of coupled equations with two unknowns : the slope " k_n " and position " C_n " of the neutral axis. This set of equations is resolved iteratively at each load increment. For convenience, the algorithm allowing the resolution of the above system of equations can be found in Figure 5.16.

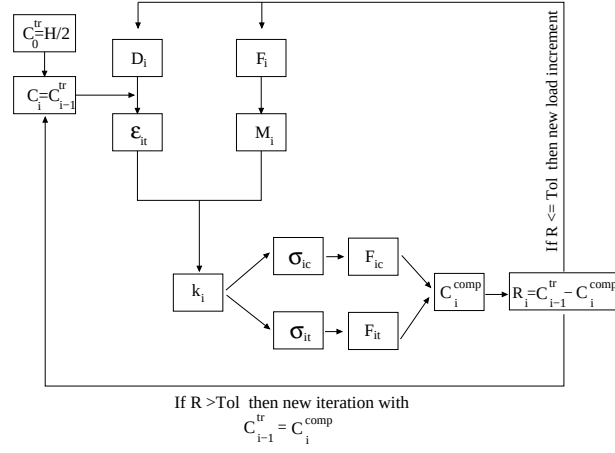


Figure 5.16: Algorithm for load-deflection curve onto stress-strain one.

5.7.2 Strain-deflection relationship

As mentioned above, measurement of maximum tensile strain captured by strain gages and clip on gages are not accurate. Therefore, ε_{it} is evaluated with help of the measurements of vertical deflection D_i .

Four-point bending creates a constant bending moment $M_i = F_i l / 2$ in the zone located between the two central rollers. Thus in this zone, according to classical beam theory, longitudinal material lines deform into concentric arcs

Let R_i denote the curvature radius of the neutral line located between the two rollers at the i^{th} increment of loading. As shown in Figure 5.17, R_i is linked to D_i with:

$$R_i = \frac{1}{2D_i} \left[\left(\frac{l}{2} \right)^2 + D_i^2 \right] + C_i. \quad (5.20)$$

The strain at a given height-coordinate of the cross section is related to curvature radius with [58]:

$$\varepsilon(y) = \frac{y}{R_i} \quad (5.21)$$

ε_{it} is obtained for $y = H - C_i$:

$$\varepsilon_{it} = \frac{H - C_i}{R_i}. \quad (5.22)$$

Now, it remains to relate ε_{it} to D_i . Isolating R_i from Eq.5.22 then substitution of this

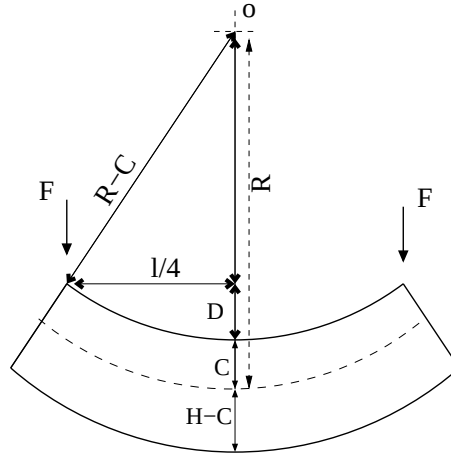


Figure 5.17: Radius curvature of beam submitted to pure bending.

value into Eq.5.20 and rearranging results in

$$\varepsilon_{it} = \frac{2D(H - C_i)}{\left(\frac{l}{2}\right)^2 + D_i^2 + 2D_iC_i}. \quad (5.23)$$

5.7.3 Resulting stress-strain curves

After treatment (sampling) of experimental load-deflection curves and applying the method detailed below, we obtain the strain-stress curves of plain concrete and steel fiber-reinforced concrete specimens as shown in figure 5.18. It can be noted that our results are in a good agreement with the experimentally-obtained stress-strain curve given in Pascal's thesis [71] for plain concrete specimen. Elastic modulus of each specimen is obtained by computing the slope of the elastic part of the considered curve. So we have:

- 36.9 GPa for Plain concrete specimens.
- 37.6 GPa for SFRC specimens with 1 % of steel fibers.
- 38.8 GPa for SFRC specimens with 3 % of steel fibers.

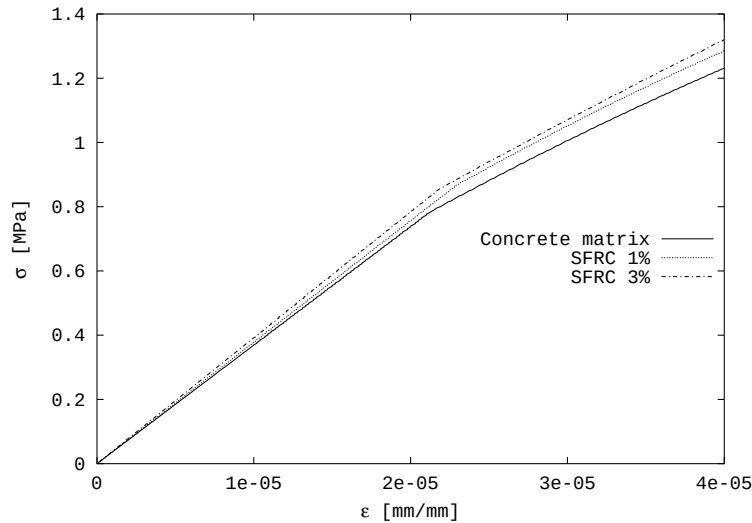


Figure 5.18: Tensile stress-strain curves of bent plain concrete and SFRC samples.

5.8 General conclusions and recommendations

In this chapter we presented an experimental study consisting in four-point bending tests performed on both Plain Concrete and Steel Fiber-Reinforced Concrete beams. The same beams were also submitted to a non-destructive test called Grindo-Sonic. From bending tests, results were collected under load-deflection curves which we converted onto stress-strain curves using an original method that we developed in the non-linear regime. This method was formulated using an incremental approach to handle material exhibiting inelastic behavior. It provides us a useful tool enabling the construction of the tension-compression behavior of a bent specimen where the only data needed, at each loading increment, are: deflection and bending moment or maximal strain and bending moment.

From material design point of view, it is concluded that fibers are the most promising reinforcement of concrete. In this study it was seen that they can improve the ductility, flexural strength, and at a less degree the stiffness. Other mechanical properties as impact resistance are improved as mentioned in Virginia transportation research council report [69]. Nevertheless, even if steel fibers have a positive effects on the mechanical properties of the composite, they can be used only as secondary reinforcement or in non-structural concrete. They do have a synergistic effect with steel bars (Rebars), but should not be used as a replacement for this primary reinforcement. On the other

hand, our experimental study showed that fiber addition may increase the porosity which will affect the mechanical performances of the concrete composite. Indeed, the porosity increase counterbalances the positive effect of fibers. This the reason why, when dealing with steel fiber-reinforced concrete with high volume fraction of fibers, we strongly recommend to achieve the addition process of fibers with a particular care. Fiber-reinforcement means more than just adding fibers in ordinary concrete.

From a micromechanical point of view, the conducted experimental study has allowed us to make a crucial statement: the increase of the porosity due to fibers addition in concrete material must be viewed as a full phase of the composite in order to account for its effect on the overall response of the composite, see chapter 6.

Finally, we would like to close this chapter by providing some recommendations which may be useful at the time of the achievement of bending test experiments. The most important points are summarized hereafter:

- (1) The use of appropriate strain or stress gages: Those must be compatible with the size of the biggest aggregates contained in the concrete mixture.
- (2) To capture the vertical displacement of both the upper and lower side of the bent specimen. That will have as consequence a more accurate construction of the normal stress-strain curve for materials which do not display the same behavior in tension that in compression as it is the case for concrete.
- (3) LDVT recorders must compute the vertical displacement between a point located at mid-span of the upper side of the beam and one of the two rollers, and between a point located at mid-span of the lower side of the beam and one of the two rollers, in order to be able to use beam theory tools. Indeed, it is in this part that beam deforms following concentric circles.

Chapter 6

Steel Fiber-Reinforced Concrete: Micromechanical modeling and numerical simulations

6.1 General Introduction

In this chapter, a particular attention will be paid to the modeling and simulation of Steel-Fiber Reinforced Concrete (SFRC) in particular and Concrete-like Matrix Composites (CMC) more generally.

Many researchers have investigated the behavior modeling of a Concrete-like material and numerous works have been done. To cite but a few, in the field of micromechanics, the modeling of tensile stiffness and strength of Fiber-Reinforced Concrete was investigated by A. Grimaldi and R. Luciano [32]. In their works, they modeled the concrete matrix as a no-tension material and considered a rigid plastic bonding between matrix and fibers. The homogenized mechanical properties were then obtained making use of homogenization procedure and the dual extreme principles expressed in terms of displacements and stresses. Christensen and Los' [14] three-phase sphere model for two-phase composites was improved by G. Li and Y. Zaho [53] to a four-phase sphere model to estimate the overall mechanical properties of concrete. The Concrete mate-

rial was modeled as a three-phase composite consisting in a continuous phase (cement paste), a dispersed phase (aggregates), and an interphase representing the Interface Transition Zone between cement paste and aggregates.

Other approaches such as finite element analysis were also used to predict the behavior of concrete composites, see [37], [104], [60], and [5]. For example, H.-T. Hu et al. [37] focused their research on concrete strengthened by fiber-reinforced plastics. The numerical computations were achieved using Abaqus finite element software in order to estimate the ultimate loading capacity of bar-reinforced concrete strengthened by fiber-reinforced plastics. The nonlinear bar-reinforced concrete behavior was handled by a suitable model available in the Abaqus material library while the nonlinear behavior of fiber-reinforced plastics was implemented by the authors and integrated to Abaqus via its User Defined Material option.

Under the lights of the experimental study presented in the previous chapter, we propose hereafter a micromechanical modeling of Steel Fiber-Reinforced Concrete and Concrete-like Matrix Composite, based on an incremental formulation of the two-step Mori-Tanaka/Voigt mean-field homogenization scheme.

At the micro-level, three processes which are responsible of stiffness decrease were identified:

- Porosity increase due to fibers addition into concrete mix,
- Material degradation due to voids and microcracks growth coalescence,
- Fiber debonding.

The influence of each one of these processes was taken into account through an appropriate modeling. Note that the last two cited processes are activated only when the material is submitted to a certain level of loading.

The experimental study led to the crucial finding that initial voids and microcracks coalescence due to damage process under loading, and porosity due to fiber addition must not be confused even if both cause a decrease of the composite stiffness. Assuming that concrete material including microcracks and initial voids represents one phase (the matrix) and fibers the other one (reinforcement phase), when those two phases are mixed in order to obtain SFRC, the initial porosity of the matrix phase increases in a significant manner. This porosity increase is viewed as a full phase and SFRC is modeled therefore as a three-phase composite. The first phase consists in concrete material, microcracks and initial voids, the second phase represents fibers while the third one represents the increase in porosity.

Activation of microcracks and coalescence of voids is handled by Ju's energy-based brittle damage model [40] which accounts for damage evolution only when the ma-

material is submitted to tensile strains. The latter are extracted using spectral decomposition and positive projection tensors [44]. The fibers are assumed elastoplastic and therefore modeled according to *textrmJ*₂ plasticity theory developed in chapter 3. All those constitutive laws and micromechanical models were then gathered into an algorithmic procedure enabling numerical simulations of SFRC and CMC materials. Our model predictions were validated against experimental results and F.E. computations performed on two types of unit cells representing composites with periodic microstructure. We also integrated our homogenization code into Abaqus software through its UMAT option in order to perform simulations on realistic structures with complex microstructures.

6.2 Porosity modeling

As suggested by the experimental study that we led, fiber addition to concrete mix induces a porosity increase of the composite, as displayed in Figure 6.1. This phe-

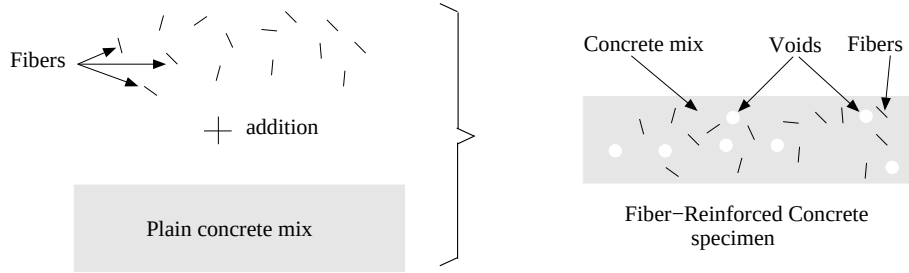


Figure 6.1: Schematic representation of porosity increase in Fiber-Reinforced Concrete composites due to fiber addition.

nomenon will affect the mechanical properties of the composite and therefore must be accounted for. In our work, the porosity increase is modeled as a full phase of the composite and its effect on the mechanical response is taken into account making use of micromechanical tools presented in chapters 2 and 4 respectively. To adapt homogenization methods of composites consisting in a matrix with embedded solid inclusions to composites which may contain void inclusions, it is enough to set the stiffness operator of the solid inclusion phase equal to zero ($\hat{C}_1^{tg} = 0$) in the expression of the strain concentration tensor. In particular for Mori-Tanaka homogenization scheme applied to a nonlinear two-phase composite where inclusions are cavities, the strain concentration tensor A^{MT} takes the following form:

$$A^{MT} = [I^4 - v_0 S_{I, \hat{C}_0^{tg}}]^{-1}$$

and therefore, the expression of the overall tangent operator becomes:

$$\bar{\mathcal{C}}^{tg} = \hat{\mathcal{C}}_0^{tg} - v_1 \hat{\mathcal{C}}_0^{tg} : A^{MT} \quad (6.1)$$

where v_1 is the cavities volume fraction.

6.3 Matrix damage

It is well-known that when brittle materials are submitted to a specific level of loadings, microcracks activate and voids grow leading to a degradation of the mechanical properties of the material. For SFRC as well as for CMC, it was observed that damage takes place only under tensile loading. To account for the influence of such phenomena in the mechanical response of SFRC, we used Ju's damage energy-based model developed in chapter 3.

6.4 Fiber-Matrix debonding

Another process which leads to the damage of Fiber (or Particle)-Reinforced composites is the fiber-matrix debonding. Extensive research has been done on the topics and many approaches were proposed to take into account the influence of the debonding on the composite response. In this work, fiber-matrix debonding is modeled using Abaqus capabilities. We use the traction-separation model provided by Abaqus, which

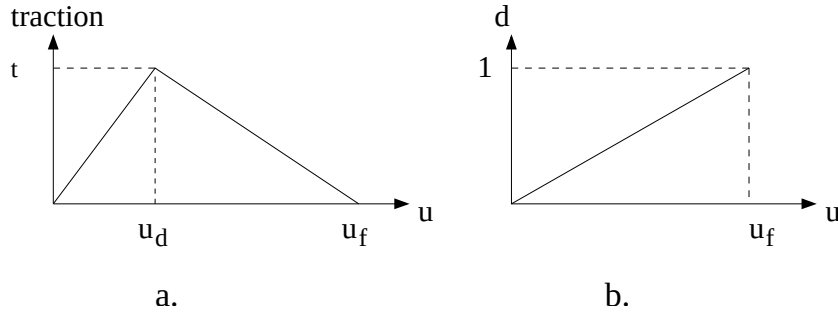


Figure 6.2: Traction separation model and damage evolution at the interface zone.

is based on the separation between the two faces of the interface. Cohesive forces evolution is displayed in Figure 6.2-a. The debonding is assumed to initiate when the

maximum nominal stress ratio reaches a value of one [1]:

$$\text{Max} \left\{ \frac{\langle t_n \rangle}{t_n^0}, \frac{t_s}{t_s^0}, \frac{t_t}{t_t^0} \right\} = 1 \quad (6.2)$$

where t_n^0 , t_s^0 and t_t^0 are the stresses in the normal, the first and second shear directions respectively. $\langle t_n \rangle$ indicates that only positive normal stress is considered (tension) otherwise it is set to zero. We also assume a linear evolution of the damage variable versus the total displacement at the interface zone as displayed in Figure 6.2-b and the total failure of the material is reached for $d = 1$. The damage stress components of the traction-separation model are then written according to:

$$t_n = \begin{cases} (1-d)\bar{t}_n, & \bar{t}_n > 0 \\ \bar{t}_n, & \text{otherwise} \end{cases}$$

$$t_s = (1-d)\bar{t}_s, \quad (6.3)$$

and:

$$t_t = (1-d)\bar{t}_t, \quad (6.4)$$

where $\bar{t}_n$, $\bar{t}_s$, and $\bar{t}_t$ are the current stresses predicted by the traction-separation behavior, see [1] for more elaborate discussion. Another method based on a micromechanical approach and using Weibull's statistical function is presented in appendix E.

6.5 Time integration and overall micro-macro algorithm

In this section, we show how homogenization schemes and constitutive models presented in chapters 3 and 4 are combined together in an algorithmic procedure in order to homogenize a multi-phase composite where the matrix obeys Ju's brittle damage model, the steel fibers (or particles) follow J_2 elasto-plasticity model and there is also a cavity phase. For a time interval $[t_n, t_{n+1}]$, what we need to compute are the macro stress $\bar{\sigma}$ and the overall tangent stiffness $\bar{C}^{tg}$. Operators considered for the (comparison material) matrix phase are the secant and tangent Ju's damage operators denoted $\hat{C}_0^d$ and $\hat{C}_0^{dtg}$ respectively. For the solid elastoplastic (comparison material) inclusion phase consider the algorithmic tangent operator $\hat{C}_1^{alg}$. In the following, in order to simplify notation, the domain of a pseudo-grain $\Omega_{i,p}$ is designated by ω and its constituents are denoted by ω_0 for the matrix and ω_1 for the inclusions. Time integration of constitutive equations and average (micro-macro) relationships are performed as follows. Consider finite time steps, strain and stress rates expressed as:

$$\Delta \epsilon = \dot{\epsilon}|_{t_n+\alpha} \Delta t \quad (6.5)$$

and;

$$\Delta\sigma = \dot{\sigma}|_{t_{n+\alpha}}\Delta t \quad (6.6)$$

note that the above quantities are computed at time $t_{n+\alpha}$, using the following generalized mid-point rule:

$$t_{n+\alpha} = (1 - \alpha)t_n + \alpha t_{n+1}; \quad (6.7)$$

with $\alpha \in [0, 1]$. For $\alpha = 0$ the integration scheme is explicit while for $\alpha = 1$ it is implicit (Backward Euler). Other values of α can be considered such as $\alpha = 1/2$ (mid-point rule) or $\alpha = 2/3$. Concerning the constitutive laws, time integration leads to:

(1) for the matrix phase:

$$\langle \Delta\sigma \rangle_{\omega_0} = \hat{C}_0^{dtg}|_{t_{n+\alpha}} : \langle \Delta\epsilon \rangle_{\omega_0} \quad (6.8)$$

(2) and for the inclusion phase:

$$\langle \Delta\sigma \rangle_{\omega_1} = \hat{C}_1^{alg}|_{t_{n+\alpha}} : \langle \Delta\epsilon \rangle_{\omega_1} \quad (6.9)$$

while for homogenization relationships, for instance, the average stress in the pseudo-grain ω is written as:

$$\Delta\bar{\sigma} \equiv \langle \Delta\sigma \rangle_{\omega} = (1 - v_0) \langle \Delta\sigma \rangle_{\omega_1} + v_0 \langle \Delta\sigma \rangle_{\omega_0} \quad (6.10)$$

Note that Eq.(6.9) is mainly used in the homogenization procedure to construct the expression of the overall tangent operator of the composite. Stresses in the inclusion phase are computed according to the return mapping algorithm detailed in section 3.3.6 of chapter 3.

The data are $\bar{\epsilon}_n$ (macro strain at t_n), $\Delta\bar{\epsilon}$ (macro strain increment), and history variables for each solid phase at t_n .

- Loops over all pseudo-grains.
 - Voigt in final homogenization step: $\langle \Delta\epsilon \rangle_{\omega} = \Delta\bar{\epsilon}$
 - Initialization in inclusions: $\langle \Delta\epsilon \rangle_{\omega_1} = \langle \Delta\epsilon \rangle_{\omega}$
 - Loop within each pseudo-grain (iteration index omitted for simplicity):
 1. Check if solid inclusions or cavities:
 - (a) if solid inclusions:
 - Call J_2 elasto-plasticity constitutive box with $\langle \Delta\epsilon \rangle_{\omega_1}$ and $\langle \epsilon_n \rangle_{\omega_1}$ as arguments. The box returns $\langle \sigma_{n+1} \rangle_{\omega_1}$ and algorithmic tangent $\hat{C}_1^{alg}|_{n+1}$;

- (b) if cavities go directly to step 2
2. Compute average strain increment in the matrix phase:

$$\langle \Delta \epsilon \rangle_{\omega_0} = \frac{\langle \Delta \epsilon \rangle - (1 - v_0) \langle \Delta \epsilon \rangle_{\omega_1}}{v_0}$$

3. Call constitutive box of Ju's damage model with $\langle \Delta \epsilon \rangle_{\omega_0}$ and $\epsilon_n > \omega_0$ as arguments. The box returns $\langle \sigma_{n+1} \rangle_{\omega_0}$ and secant operator $\hat{C}_0^d|_{n+1}$.
4. Compute tangent operator for Ju's model, $\hat{C}_0^{dtg}|_{n+1}$.
5. Compute the instantaneous Poisson's ratio v_t from the isotropic part of $\hat{C}_0^{dtg}|_{n+1}$.
6. Compute Eshelby's tensor $S(v_t, A_r)$.
7. Compute strain concentration tensor A^{MT} using Mori-Tanaka. If solid inclusions:

$$A^{MT} = [v_0 \{I^4 + S : [(\hat{C}_0^{dtg}|_{n+1})^{-1} : \hat{C}_1^{alg}|_{n+1} - I^4]\} + (1 - v_0)I^4]^{-1}$$

If cavities:

$$A^{MT} = [I^4 - v_0 S]^{-1}$$

8. Check strain compatibility in the inclusions:

$$\langle \Delta \epsilon \rangle_{\omega_1} = ? \quad A^{MT} : \langle \Delta \epsilon \rangle_{\omega}$$

9. if yes: no more iterations for this pseudo-grain,
10. else, perform a new iteration with:

$$\langle \Delta \epsilon \rangle_{\omega_1} = A^{MT} : \langle \Delta \epsilon \rangle_{\omega}$$

- After convergence, compute effective tangent operator of the pseudo-grain. If it contains solid inclusions then:

$$\bar{C}_{\Omega_{i,p}}^{tg} = \hat{C}_0^{dtg}|_{n+1} + (1 - v_0)(\hat{C}_1^{alg}|_{n+1} - \hat{C}_0^{dtg}|_{n+1}) : A^{MT}$$

If the pseudo grain contains cavities, then:

$$\bar{C}_{\Omega_{i,p}}^{tg} = \hat{C}_0^{dtg}|_{n+1} - (1 - v_0)\hat{C}_0^{dtg}|_{n+1} : A^{MT}$$

- Save pseudo-grain's effective tangent and average stresses and strains.
- Compute RVE's effective tangent moduli and macro stresses

$$\bar{C}^{tg} = \left\langle \bar{C}_{\Omega_{i,p}}^{tg} \right\rangle_{i, \psi_i}, \quad \Delta \bar{\sigma} = \bar{C}^{tg} : \Delta \bar{\epsilon}$$

6.6 Uniaxial macro tensile test

By following the tensors storage given in appendix B, at each time increment, the overall stress-strain rate relationship given by our algorithm ($\Delta\bar{\sigma} = \bar{C}^{tg} : \Delta\bar{\varepsilon}$), can be written under the following matrix form [19]:

$$\begin{Bmatrix} \Delta\bar{\sigma}_1 \\ \Delta\bar{\sigma}_2 \\ \Delta\bar{\sigma}_3 \\ \Delta\bar{\sigma}_4\sqrt{2} \\ \Delta\bar{\sigma}_5\sqrt{2} \\ \Delta\bar{\sigma}_6\sqrt{2} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11}^{tg} & \bar{C}_{12}^{tg} & \bar{C}_{13}^{tg} & 0 & 0 & 0 \\ \bar{C}_{21}^{tg} & \bar{C}_{22}^{tg} & \bar{C}_{23}^{tg} & 0 & 0 & 0 \\ \bar{C}_{31}^{tg} & \bar{C}_{32}^{tg} & \bar{C}_{33}^{tg} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\bar{C}_{44}^{tg} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\bar{C}_{55}^{tg} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\bar{C}_{66}^{tg} \end{bmatrix} \begin{Bmatrix} \Delta\bar{\varepsilon}_1 \\ \Delta\bar{\varepsilon}_2 \\ \Delta\bar{\varepsilon}_3 \\ \Delta\bar{\varepsilon}_4\sqrt{2} \\ \Delta\bar{\varepsilon}_5\sqrt{2} \\ \Delta\bar{\varepsilon}_6\sqrt{2} \end{Bmatrix} \quad (6.11)$$

Our algorithm is strain driven. So, for a given strain increment, we compute the stress increment. Under the conditions of a uniaxial tension test performed along axis 1, the non-zero components of strain rate vector are $\Delta\bar{\varepsilon}_1$, $\Delta\bar{\varepsilon}_2$ and $\Delta\bar{\varepsilon}_3$, where only $\Delta\bar{\varepsilon}_1$ is known. So according to Eq.6.11 we have:

$$\begin{aligned} \Delta\bar{\sigma}_2 &= \bar{C}_{21}^{tg} \Delta\bar{\varepsilon}_1 + \bar{C}_{22}^{tg} \Delta\bar{\varepsilon}_2 + \bar{C}_{23}^{tg} \Delta\bar{\varepsilon}_3 \\ \Delta\bar{\sigma}_3 &= \bar{C}_{31}^{tg} \Delta\bar{\varepsilon}_1 + \bar{C}_{32}^{tg} \Delta\bar{\varepsilon}_2 + \bar{C}_{33}^{tg} \Delta\bar{\varepsilon}_3 \end{aligned} \quad (6.12)$$

with the condition that $\Delta\bar{\sigma}_2 = \Delta\bar{\sigma}_3 = 0$. To determine $\Delta\bar{\varepsilon}_2$ and $\Delta\bar{\varepsilon}_3$ which satisfy the above condition, one can use an iteration scheme [19]. Making use of Newton-Raphson scheme, equations system 6.12 are rewritten as:

$$\Delta\bar{\sigma}_2^{it} + (\bar{C}_{22}^{tg})^{it} (\Delta\bar{\varepsilon}_2^{it+1} - \Delta\bar{\varepsilon}_2^{it}) + (\bar{C}_{23}^{tg})^{it} (\Delta\bar{\varepsilon}_3^{it+1} - \Delta\bar{\varepsilon}_3^{it}) = 0 \quad (6.13)$$

$$\Delta\bar{\sigma}_3^{it} + (\bar{C}_{32}^{tg})^{it} (\Delta\bar{\varepsilon}_2^{it+1} - \Delta\bar{\varepsilon}_2^{it}) + (\bar{C}_{33}^{tg})^{it} (\Delta\bar{\varepsilon}_3^{it+1} - \Delta\bar{\varepsilon}_3^{it}) = 0 \quad (6.14)$$

Resolution of the above system where the unknowns are $\Delta\bar{\varepsilon}_2$ and $\Delta\bar{\varepsilon}_3$, leads to:

$$\Delta\bar{\varepsilon}_2^{it+1} = \Delta\bar{\varepsilon}_2^{it} + \left(\frac{(\bar{C}_{33}^{tg})^{it}}{(\bar{C}_{23}^{tg})^{it}} \Delta\bar{\sigma}_2^{it} - \Delta\bar{\sigma}_3^{it} \right) \left((\bar{C}_{32}^{tg})^{it} - \frac{(\bar{C}_{33}^{tg})^{it} (\bar{C}_{22}^{tg})^{it}}{(\bar{C}_{23}^{tg})^{it}} \right)^{-1} \quad (6.15)$$

$$\Delta\bar{\varepsilon}_3^{it+1} = \Delta\bar{\varepsilon}_3^{it} - \frac{1}{(\bar{C}_{32}^{tg})^{it}} \bar{\sigma}_2^{it} - \frac{(\bar{C}_{22}^{tg})^{it}}{(\bar{C}_{23}^{tg})^{it}} (\Delta\bar{\varepsilon}_2^{it+1} - \Delta\bar{\varepsilon}_2^{it}) \quad (6.16)$$

At each initial iteration it of a loading increment, our experience (see also [19]) showed that a fast convergence is obtained with: $\Delta\bar{\varepsilon}_2^{it} = \Delta\bar{\varepsilon}_2^{it} = -\frac{1}{2} \underbrace{\Delta\bar{\varepsilon}_1^{it}}_{\text{known}}$

6.7 Unit cell representations

For some finite element computations we assumed composites which have periodic microstructure with a square arrangement, Figures 6.3-a and 6.4-a. The different steps of the transition from a 3D representation to a 2D axisymmetric unit cell model are illustrated in Figures 6.3 and 6.4. Using the correct boundary conditions, the composite can be represented by a cylinder with hexagonal section, Figures 6.3-b and 6.3-b. This hexagonal section cylinder is approximated with a circular section cylinder (Figures 6.3-c and 6.4-c). Then this cylinder with circular section is itself represented by a cell with a plane geometry in the (r, z) plane, where r is the radial coordinate and z the coordinate along the symmetry axis as displayed in Figures 6.3-d and 6.3-d, see also [8]. The prescribed boundary conditions are as follows: Zero displacements are imposed in the radial direction to the left vertical side and in the vertical direction to the bottom horizontal side. The right vertical side is constrained to have uniform radial displacement. Uniform vertical displacement is imposed on the top horizontal side. Note that the use of unit cell representation is limited to composites with low to moderate volume fractions of inclusions and when the loading is axisymmetric. In

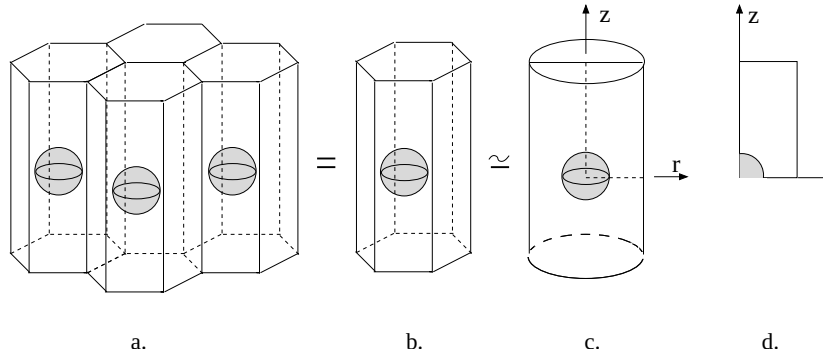


Figure 6.3: A 2D unit cell representation of composite with periodic microstructure: Single inclusion UC1.

cases of three-phase composites, a simple F.E. modeling can be achieved using the unit cell UC2 displayed in Figure 6.4. For this purpose, it suffices to affect to each entities of UC2 the correct mechanical properties. Nevertheless, it is interesting to note that if R and r denote the radius of the lower and upper inclusions respectively and H the height the UC2, the condition $R + 2r < H$ must be always satisfied which strongly limits, from a geometrical point of view, the use of UC2 to composites with low volume fractions of inclusions.

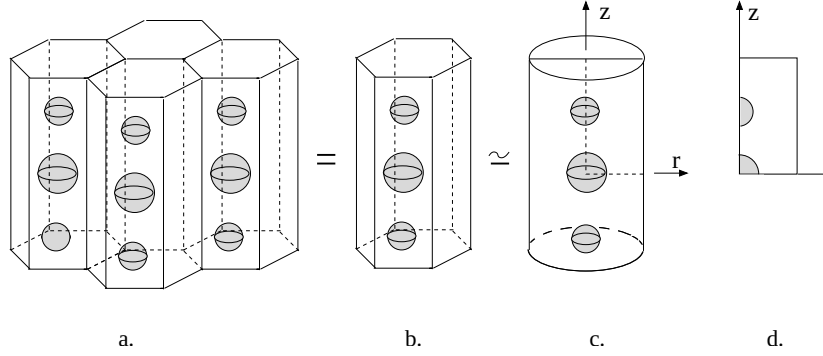


Figure 6.4: A 2D unit cell representation of composite with periodic micro-structure: Two inclusions UC2

6.8 A two-scale Approach

We developed a two-scale approach which enables the numerical simulation within reasonable CPU time, memory usage, and user time of realistic structures made of composite materials and subjected to complex loadings. The method consists in integrating our homogenization code into ABAQUS throughout its UMAT interface. At each iteration of the global equilibrium equations of the F.E. analysis and at each integration point of the macro F.E. mesh, the homogenization module is called. ABAQUS passes the macro-strain at time t_n , the macro strain increment, material parameters and history variables at time t_n . The homogenization module computes the macro-stress $\bar{\sigma}$ and the homogenized macro-tangent modulus $\bar{C}^{tg}$ and returns them to ABAQUS. Our homogenization code coupled with Abaqus supports two types of finite elements:

- (i) 3D solid stress-displacement elements C3D4 and C3D8, and
- (ii) Axisymmetric solid stress-displacement elements CAX3, CAX4, CAX8.

6.9 Numerical simulations

In numerical simulations presented in this section, the steel fibers or particles obey J_2 elasto-plasticity with Young's modulus $E_I = 210$ GPa, Poisson's ratio $\nu_I = 0.3$, initial yield stress $\sigma_Y = 210$ MPa, and power-law isotropic hardening ($R(p) = Kp^m$) with $K = 416$ MPa and $m = 0.3895$. For concrete material, the following expression for the damage evolution function $H(\xi)$ is used:

$$H(\xi) = (k + \frac{1}{2})AB \exp[B(\xi_0 - \xi)] \quad (6.17)$$

where k represents the microcrack density, ξ_0 the initial characteristic damage measure; A and B are material-dependent parameters. Note also that in the following, we will consider that: $\langle \cdot \rangle_\omega \equiv \langle \cdot \rangle$, $\langle \cdot \rangle_{\omega_0} \equiv \langle \cdot \rangle_0$, and $\langle \cdot \rangle_{\omega_1} \equiv \langle \cdot \rangle_1$.

6.9.1 Unreinforced concrete-like material

In the following simulations, we consider a fictitious plain concrete-like brittle material whose parameters are: $E = 25$ GPa, $\nu = 0.2$, $k = 0.7$, $A = 14.3$, $B = 14.8$ and $\xi_0 = 2.4 \times 10^{-5}$ MPa.

The first example simulates a uniaxial tension test with 3 different strain increments: $\Delta\bar{\epsilon} = \bar{\epsilon}/10$, $\bar{\epsilon}/100$, and $\bar{\epsilon}/1000$. The results are plotted in Figures 6.5 and do not show a significant influence of the strain increment values on the converged response. The second example considers the same material data except $\xi_0 = 2.4 \times 10^{-2}$ MPa

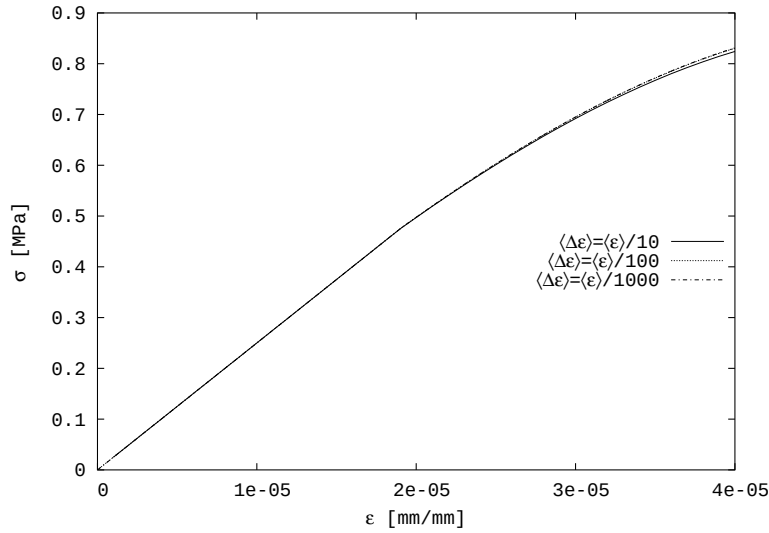


Figure 6.5: Convergence of Ju's model under various strain steps.

and studies the uniaxial response under cyclic strain history. After a first cycle, the peak strains are increased (in absolute values) in the second cycle. The stress-strain response and the strain history are displayed in Figures 6.6 and 6.7(a), respectively. It can be seen in Figure 6.6 that under tensile loading, the material first displays an elastic behavior then undergoes damage. Both in unloading and compressive stages,

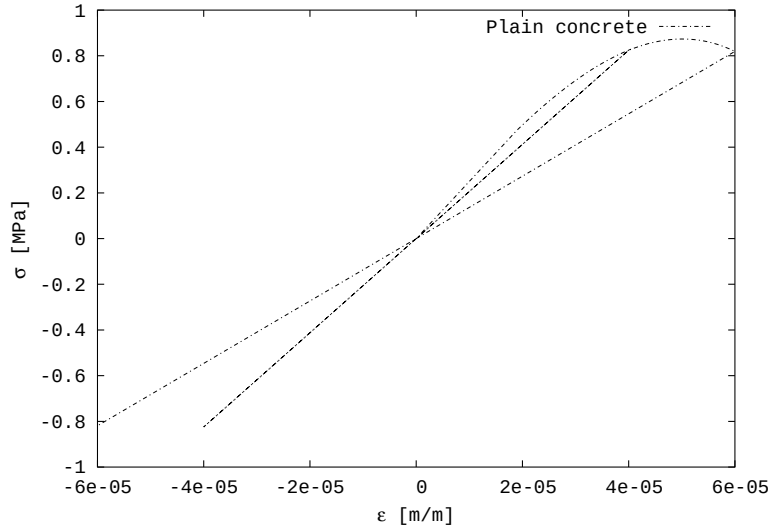


Figure 6.6: Concrete-like matrix under cyclic loading (traction/compression): Strain-stress curve

the material response is linear with a slope corresponding to the elastic damaged modulus. The material remains in the elastic domain until the previously reached damage threshold is attained again in tension. Those features are also reproduced by the scalar damage variable defined in Eq.(6.22) whose time-evolution is depicted in Fig.6.7(b).

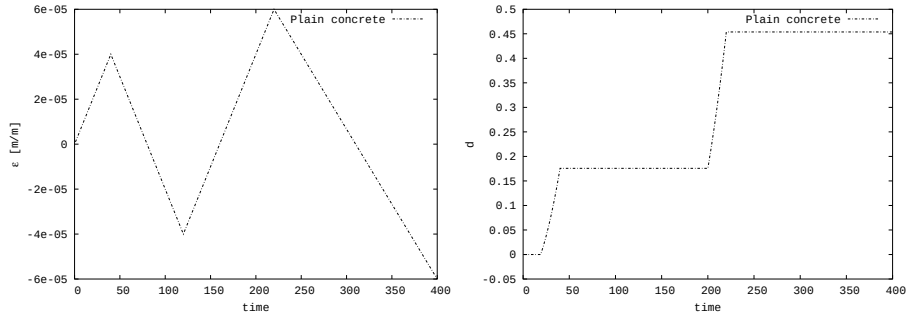


Figure 6.7: Loading history and damage evolution of Concrete-like matrix sample under uniaxial cyclic loading (traction/compression)

6.9.2 Validation against unit cell finite element results

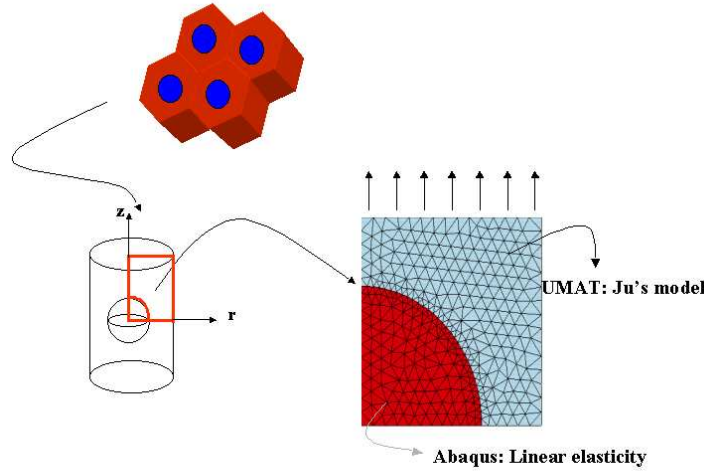


Figure 6.8: A 2D unit cell representation of composite with periodic micro-structure.

The mechanical response of a sphere-reinforced concrete-like matrix composite with a periodic micro-structure is studied. Figure 6.8 illustrates the different steps of the transition from a 3D representation to a 2D axisymmetric unit cell model. Zero displacements are imposed in the radial direction to the left vertical side and in the vertical direction to the left vertical side and in the vertical direction to the bottom horizontal side. The right vertical side is constrained to have uniform radial displacement. Uniform vertical displacement is imposed on the top horizontal side. The unreinforced matrix material obeys Ju's brittle damage model which was integrated into the Abaqus finite element program via a user-defined UMAT subroutine. The composite has the following mechanical properties: the concrete matrix has $E_0 = 36.9$ GPa, $\nu_0 = 0.19$, $k = 0.05$, $A = 14.3$, $B = 15.7$ and $\xi_0 = 2.4 \times 10^{-5}$ MPa, $k = 0.05$, while the aluminum spheres have $E_1 = 75$ GPa, $\nu_1 = 0.30$. with a volume fraction of 10%. The aim of this example is to evaluate the composite and 'per-phase' responses under tensile loading test. The simulations are achieved using both Abaqus finite element software and our homogenization code. For the F.E. calculations we used the unit cell UC1 meshed with 646 axisymmetric CAX3R elements with a total of 357 nodes. The F.E. average stress are computed as follows: For each phase, at each element integration point we compute the sum of the stresses weighted by the volume (see also [47]). The result is then divided by the total volume of the composite. So,

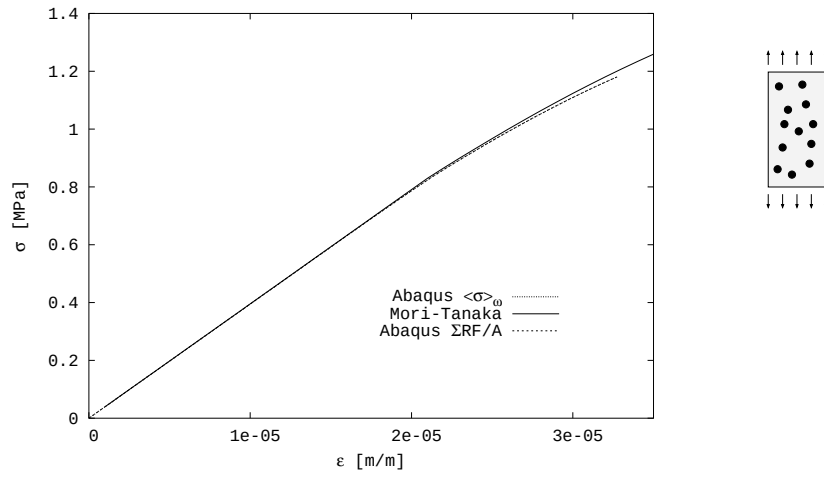


Figure 6.9: Concrete composite with 10% of aluminum spheres under tensile loading test: Mori-Tanaka and unit cell finite element predictions of $\langle \sigma \rangle_{\omega}$ the average stress in the whole composite.

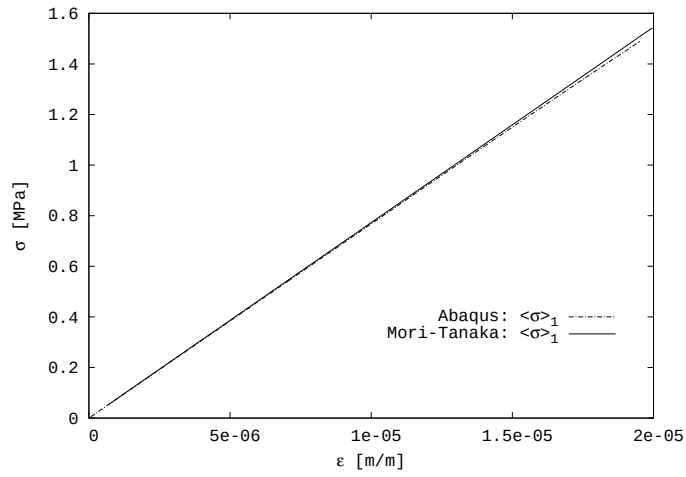


Figure 6.10: SSCC with 10% of aluminum spheres under tensile loading test: Mori-Tanaka and unit cell finite element predictions of $\langle \sigma \rangle_1$ the average stress in the inclusion phase of the composite.

we obtain:

$$\langle \sigma \rangle_r = \frac{\sum_{i=1}^{N_{el}} \sum_{j=1}^{N_{IP}} Ivol_{i,j} \sigma_{i,j}}{\sum_{i=1}^{N_{el}} \sum_{j=1}^{N_{IP}} Ivol_{i,j}}, \quad (6.18)$$

where N_{el} designates the number of elements in phase ' r ', N_{IP} the number of integration points per element, and $Ivol_{i,j}$ and $\sigma_{i,j}$ the volume and the stress at the integration point " j " of the element " i ". In the same manner, the per-phase average strain is computed. The result is:

$$\langle \varepsilon \rangle_r = \frac{\sum_{i=1}^{N_{el}} \sum_{j=1}^{N_{IP}} Ivol_{i,j} \varepsilon_{i,j}}{\sum_{i=1}^{N_{el}} \sum_{j=1}^{N_{IP}} Ivol_{i,j}}. \quad (6.19)$$

The average stress over the composite can be computed with:

$$\langle \sigma \rangle = \sum_{i=1}^{N_r} v_i \langle \sigma \rangle_i, \quad (6.20)$$

where N_r designates the phase number. or with:

$$\langle \sigma \rangle = \frac{\sum_{i=1}^{N_p} RF_i}{A}, \quad \langle \varepsilon \rangle = \frac{U}{l}, \quad (6.21)$$

where RF_i are the reaction forces at mesh points on which the displacement is applied, A the section area, U the imposed displacement, and l the height of the unit cell.

Figures 6.9, 6.10, and 6.11 show the average stress-strain curves of the composite, and inclusion and matrix phases respectively. A good agreement is observed between results predicted by our code and those obtained by Abaqus. It is also shown in Figures 6.9 that the two ways to compute the average stress in the composite given by Eqs. 6.20 and 6.21 respectively, give exactly the same results. It should be also interesting to point out that the average strains in the matrix and in the inclusion phases respectively are in the same range. This is explained by the fact that the aluminum spheres stiffness is only two times higher than the concrete matrix one.

Now let us have a look at the per-phase fluctuations of strain and stress fields respectively within the composite. For this purpose use a heterogeneity indicator. In this work, this indicator is represented by the cumulative probability for a given volume fraction of the phase " r " to be below a certain value of ε_{22} or σ_{22} , see also [88] and [75]. First, analyze strain fluctuation within the matrix phase at the final step loading

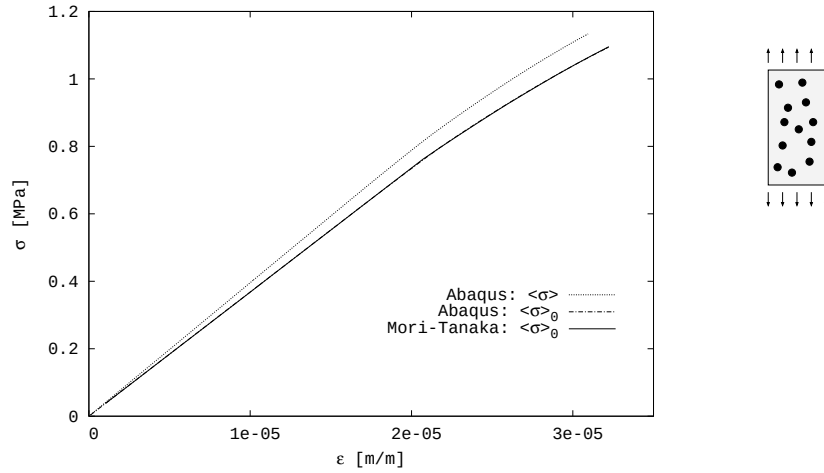


Figure 6.11: Concrete with 10% of aluminium spheres under tensile loading test: Mori-Tanaka and unit cell finite element predictions of $\langle \sigma \rangle_0$ the average stress in the matrix phase of the composite.

which corresponds to $\bar{\epsilon}_{22} = 3.2\%$). In Figure 6.12-a is plotted the cumulative probability of ϵ_{22} in the matrix. There are at least two ways to read this figure, one can say that 20% of the matrix undergoes an axial strain ϵ_{22} less than 3×10^{-5} for instance, or, more or less 90% of the matrix phase undergoes a strain ranging between 3×10^{-5} and 3.4×10^{-5} . Those results indicate that the axial strain field ϵ_{22} is rather homogeneous and that fluctuation of this field over the matrix phase is low. The less the cumulative probability curve is spread out the smaller the fluctuation is. Are also reported on this figure, as qualitative indicators, the matrix average strain values obtained using Abaqus and our code respectively. For more than 90% of the matrix phase, ϵ_{22} fluctuation around the predicted F.E. matrix average strain ($\langle \epsilon_{22} \rangle_0 = 3.22 \times 10^{-5}$) does not exceed 6%. By proceeding in a similar way, the axial stress field σ_{22} fluctuation is investigated. Figure 6.12-b reports the cumulative probability versus σ_{22} . It can be showed that more than 85% of the matrix phase undergoes a stress varying between 1 MPa and 1.15 MPa which indicates that σ_{22} fluctuation is not quiet important. In other words this means that the axial stress field σ_{22} in the concrete matrix material is rather homogeneous, since here again, more than 85% of the matrix phase material, σ_{22} fluctuation around the predicted F.E. matrix average stress ($\langle \sigma_{22} \rangle_0 = 1.094$ MPa) does not exceed 6%. The matrix average stress values predicted by Abaqus and our code respectively are reported on the same figure as qualitative indicators. For the inclusion phase, is plotted in Figure 6.13-a the curve representing the cumulative probability of ϵ_{22} in the inclusion phase. Here again, ϵ_{22} does not fluctuate in a significant manner. Variations of ϵ_{22} are between 1.9×10^{-5} and 2×10^{-5} leading to a maximum fluctuation around the F.E. predicted inclusion average strain

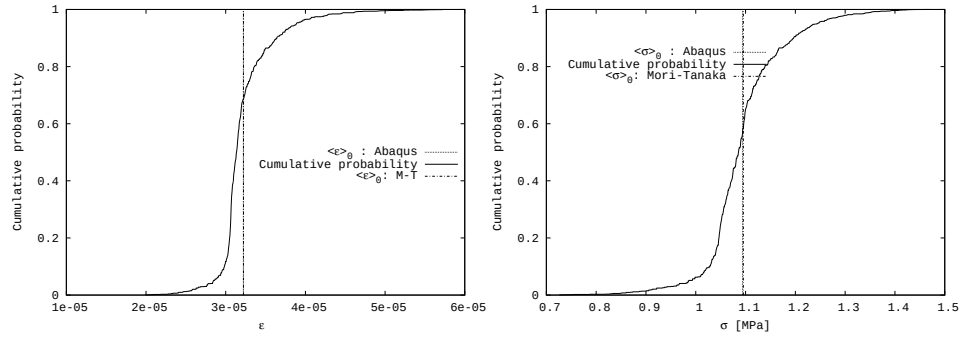


Figure 6.12: Concrete with 10% of aluminum spheres under tensile loading test: Axial strain ϵ_{22} and stress σ_{22} fluctuations within matrix phase.

($\langle \epsilon_{22} \rangle_1 = 1.955 \times 10^{-05}$) of more or less 3%. Concerning the axial stress σ_{22} , Figure 6.13-b indicates that the maximum fluctuation around the F.E. predicted inclusion average stress ($\langle \sigma_{22} \rangle_1 = 1.49$) is close to 2.7%. σ_{22} variations are ranged between 1.45 MPa and 1.53 MPa.

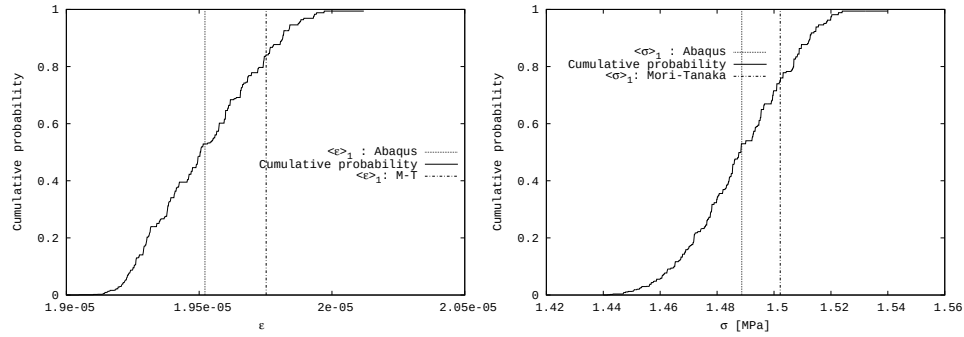


Figure 6.13: Concrete with 10% of aluminum spheres under tensile loading test: Axial strain ϵ_{22} and stress σ_{22} fluctuations within inclusion phase.

In some industrial applications, more than one type of inclusion are be added into the concrete mix. For instance, one needs to improve the mechanical and the acoustic properties of the concrete at the same time. In the following example, we simulated a tensile loading test performed on a three-phase concrete composite consisting of a concrete phase in which are embedded 5% of steel spherical inclusions to improve mechanical properties and 5% of polymer spherical inclusions for acoustic properties. The mechanical properties are: (i) for the concrete matrix see section 6.9.3 of this

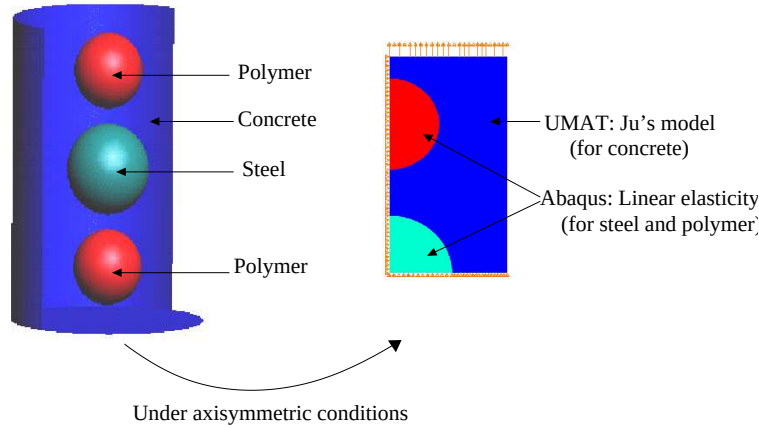


Figure 6.14: A 2D unit cell representation of composite with periodic micro-structure.

chapter, (ii) idem. for steel spheres, and (iii) $E_2 = 7GPa$, $\nu_2 = 0.3$ for polymer spheres. The F.E. computation is performed with the UC2 unit cell, Figure 6.14, which was meshed with 1317 CAX4 axisymmetric elements and 1436 nodes. Figure 6.15 shows the comparison between the estimate predicted using our M-T homogenization code and those using Abaqus F.E. computation. It is observed that Mori-Tanaka estimate of the damage part of the response is stiffer than F.E. one. However, it can be seen a good agreement between the two sets of results. In Figures 6.16, 6.17, and 6.18 per-phase fluctuations of strain and stress fields respectively, are reported. In Figure 6.16, one can observe that for more than 90% of the concrete matrix phase (ω_0) strain fluctuates between 2.8×10^{-05} and 3.2×10^{-05} with a maximum fluctuation of 10% around the F.E. predicted matrix average strain $\langle \epsilon_{22} \rangle_0 = 3.11 \times 10^{-05}$. The axial stress ranges between 0.94 and 1.1 MPa with a maximum fluctuation of 9.6% around $\langle \sigma_{22} \rangle_0$ which is equal 1.04 MPa. In the aluminium inclusions, the strain fluctuates between 1.6×10^{-05} and 1.8×10^{-05} . The maximum fluctuation of the strain field around $\langle \epsilon_{22} \rangle_1 = 1.73 \times 10^{-05}$ reaches 5.9%, while those of the stress field around $\langle \sigma_{22} \rangle_1$ which is equal 1.31 MPa, do not exceed 4.8%. In the polymer phase, Figure 6.18 shows that more than 90% of the polymer material undergoes an axial strain ranged between 4.8×10^{-05} and 5.55×10^{-05} leading to a maximum strain fluctuation of 8% around $\langle \epsilon_{22} \rangle_2$ which is equal to 5.22×10^{-05} . The axial stress mainly varies from 0.387 to 0.427 MPa with a maximum fluctuation of 5.6% around $\langle \sigma_{22} \rangle_2$ which is equal to 0.41 MPa. From the above, it clearly appears that both strain and stress fields fluctuations are not important within the various components of the concrete composite. In Figure 6.19 are displayed the maps of stress σ_{22} and strain ϵ_{22} fields variations over the unit cell in the loading direction. As expected, one can see that strains are higher in polymer inclusion while stresses are higher in steel inclusion. More validations tests of unit

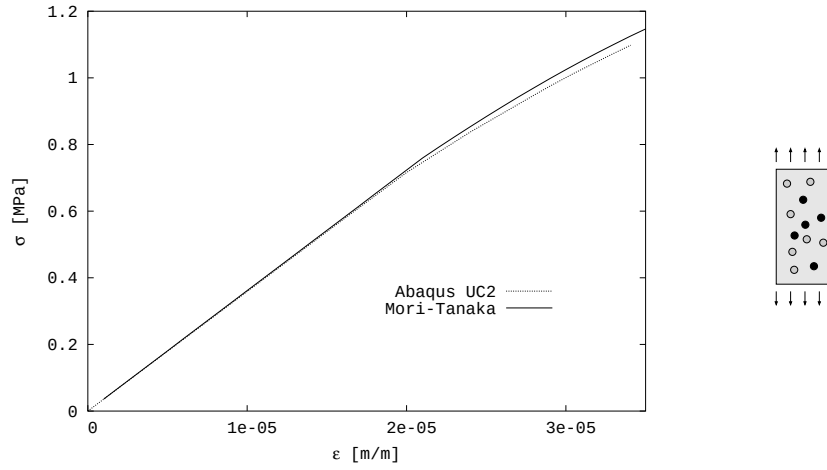


Figure 6.15: Three-phase concrete matrix composite under tensile loading test: Mori-Tanaka prediction versus F.E. computation.

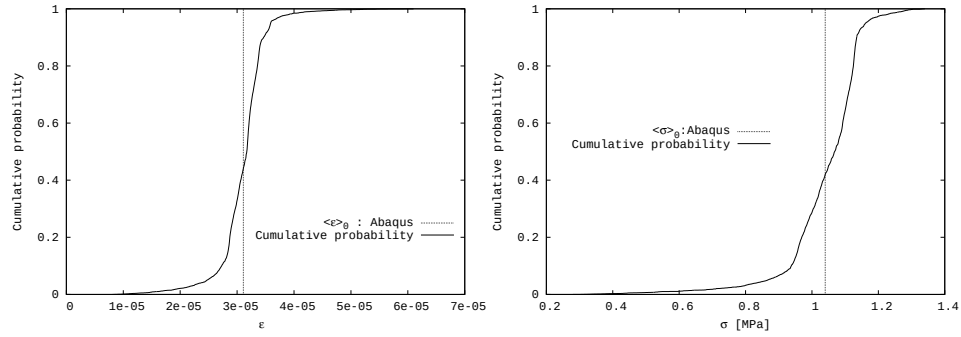


Figure 6.16: Concrete with 5% of aluminum spheres and 5% of polymer spheres under tensile loading test: Axial strain ϵ_{22} and stress σ_{22} fluctuations within matrix phase.

cell UC2 are presented in chapter 7.

6.9.3 Validation against experimental results

Before starting simulations of this section, we first would like to focus the attention of the reader on the fact that when dealing with brittle concrete-like materials, the non-linear response exhibited by the tested sample, is not pronounced as is the case for

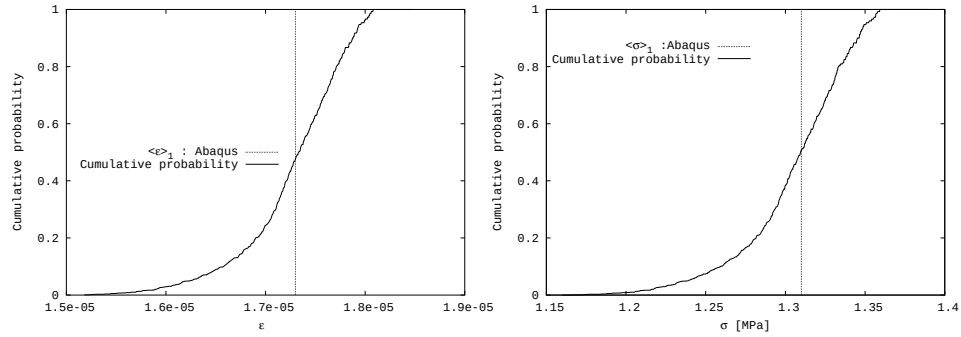


Figure 6.17: Concrete with 5% of aluminum spheres and 5% of polymer spheres under tensile loading test: Axial strain ϵ_{22} and stress σ_{22} fluctuations within steel inclusion phase.

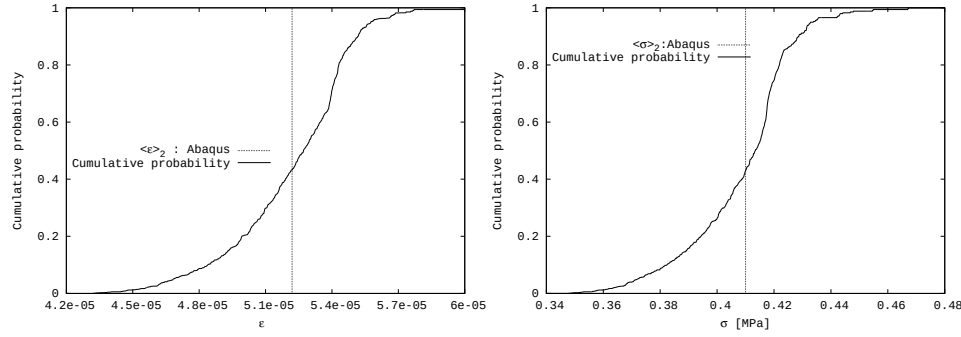


Figure 6.18: Concrete with 5% of aluminum spheres and 5% of polymer spheres under tensile loading test: Axial strain ϵ_{22} and stress σ_{22} fluctuations within polymer inclusion phase.

elasto-plastic behavior (see simulations of chapter 7). This observation is confirmed (supported) by other researchers in the field. In Figure 6.20 is plotted the stress-strain curve obtained by Maso, J.-C. [56] of concrete specimen under tensile loading test. It is observed that from point A, the nonlinearity begins which means that damage is initiated. Damage continues to evolve until point B and the observed nonlinearity along the path AB is not spectacular. Beyond point B, the specimen enters in the macro-cracking phase which corresponds to downward part of the stress-strain curve. From the experimentally-obtained load-deflection curves of bent plain concrete beams, an uniaxial tensile stress-strain curve was obtained with the incremental method developed in chapter 4. This curve (plotted in Figure 5.18) enabled the fitting of the parameters of Ju's brittle damage model. The following values were found: $E = 36.9$

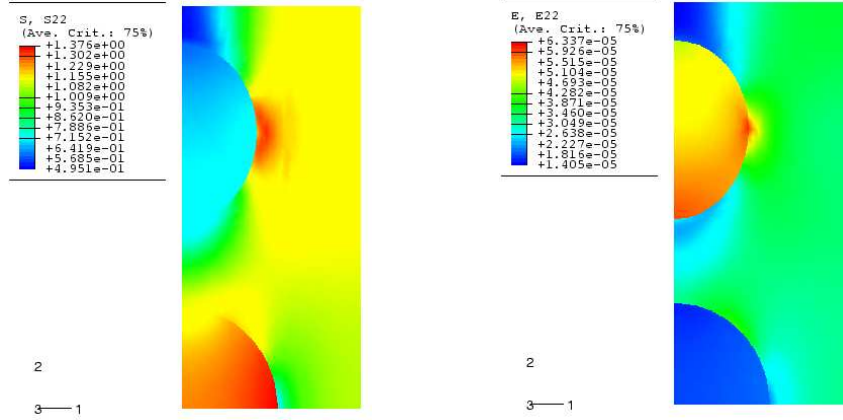


Figure 6.19: Stress and strain fields repartition over a three-phase Concrete containing Polymer and Steel Spheres: Uniaxial tensile loading test using F.E. analysis performed on UC2.

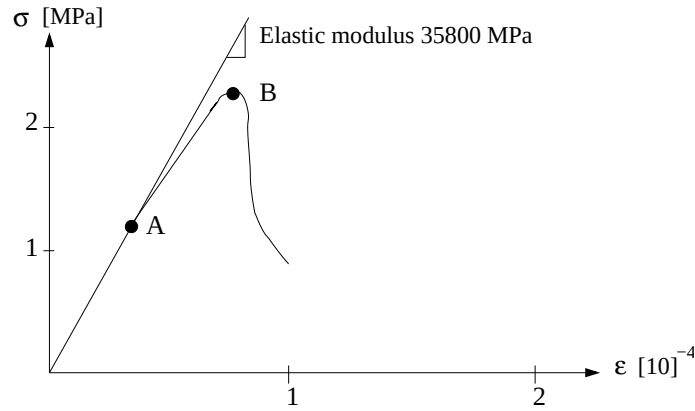


Figure 6.20: Tensile stress-strain curve of concrete specimen following Maso, J.-C. [56].

GPa, $\nu = 0.19$, $k = 0.3$, $A = 14.3$, $B = 15.8$ and $\xi_0 = 2.4 \times 10^{-5}$ MPa. The latter 4 values pertain to the damage evolution function $H(\xi)$, Eq.6.17. Figure 6.21 shows that the experimental curve is correctly simulated with the fitted values. The agreement might be improved by considering another expression of $H(\xi)$ in a future work. Next, the uniaxial macro response of concrete matrix reinforced with 1% and 3% of randomly oriented short fibers ($A_r = 60$) was numerically simulated using homogenization. Porosity increase due to fibers addition was included as a third phase made

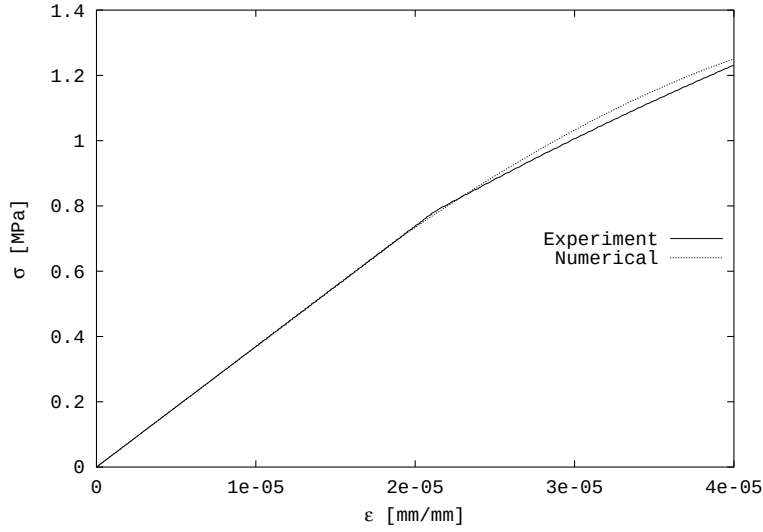


Figure 6.21: Ju's damage model identification from experimental results.

of spherical cavities of volume fraction equal to 0.5% and 0.85%, respectively. The numerical predictions are validated against experimental results in Figures 6.22 and 6.23, and a good agreement can be observed. In the version of Ju's damage model implemented in this work, the nonlinearity in the tensile stress-strain response of plain concrete is only due to damage evolution. Therefore, for a uniaxial stress state, a damage measure d can be defined simply by the decrease in secant stiffness, similarly to the well-known Lemaitre and Chaboche [50] model:

$$d = 1 - \frac{E_d}{E}, \quad d \in [0, 1] \quad (6.22)$$

The same definition can be applied to the composite material under uniaxial macro stress. The evolution of d as a function of tensile strain is plotted in Figure 6.24 for plain and reinforced concrete. One can see that damage is first initiated in concrete sample for a strain around 1.9×10^{-5} then for SFRC with 1% and 3% at a strain close to 2×10^{-5} . It is also observed on the same Figure, at the final loading step ($\bar{\epsilon} = 4 \times 10^{-5}$), that damage is more important for plain concrete sample than for SFRC ones which emphasizes the benefit effect of the fiber-reinforcement. Note also that for all samples damage evolves nonlinearly following a same trend.

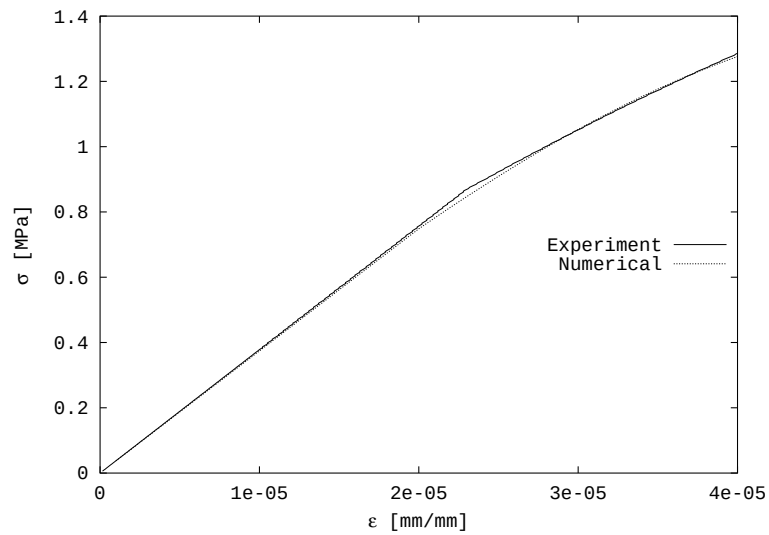


Figure 6.22: Concrete specimen reinforced with 1% of steel fibers under uniaxial tension:

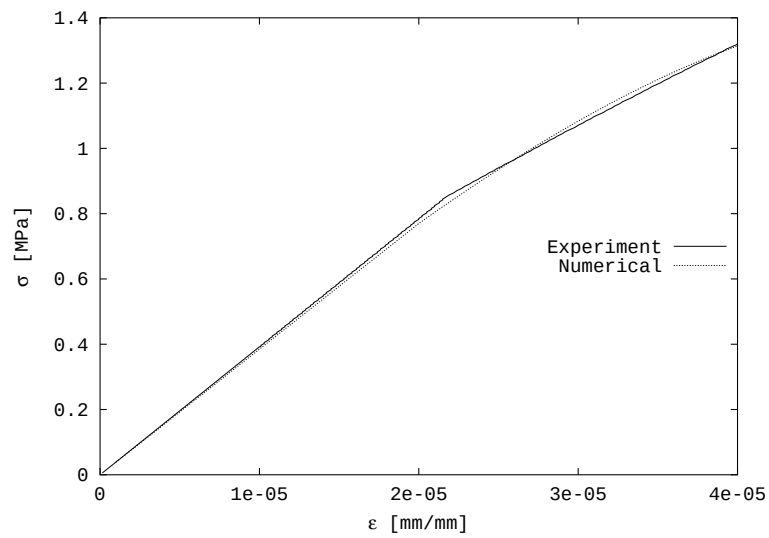


Figure 6.23: Concrete specimen reinforced with 3% of steel fibers under uniaxial tension:

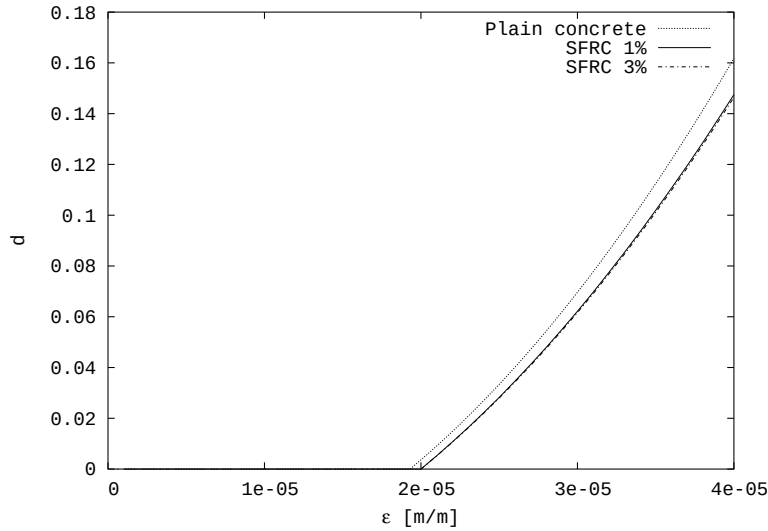


Figure 6.24: damage evolution of Plain Concrete and SFRC under uniaxial tensile loading.

6.9.4 Steel Fibers shape influence

The aim of the following simulations is to evaluate the influence of steel fiber shapes available in the industry on composite and per-phase responses. Both concrete and steel materials are assumed elastic with: $E_0=36900$ MPa, $\nu_0=0.19$ for concrete and $E_1=210000$ MPa, $\nu_1=0.3$ for steel. Two fiber shapes are considered: straight line and hooked-end, with $A_f=60$. F.E. computations are performed on two 3D unit cells displayed in Figure 6.25. The first unit cell represents SFRC with a hooked-end fiber of 3% of volume fraction while the second one represents a SFRC with a straight line fiber of 3% of volume fraction.

Unit cells are meshed using GMSH software [31] (developed by Prof. J.-Fr. Remacle of Civil Engineering department of Université catholique de Louvain) then subjected to uniaxial loading test simulated using Abaqus [1]. In figures 6.26, 6.27, and 6.28 are plotted the stress-strain curves of the composite, fiber and concrete matrix phases respectively predicted by F.E. analysis and using our homogenization code for two fiber shapes. It clearly appears that at low volume fractions of fibers (3%) the shape does not influence the overall response of the composite in a significant manner. However, the analysis of inclusion phases curves indicates an important difference for the two used fiber types. Axial stress is more important in the straight line fiber since all fiber material is aligned with the applied load. One can also observe, in the case

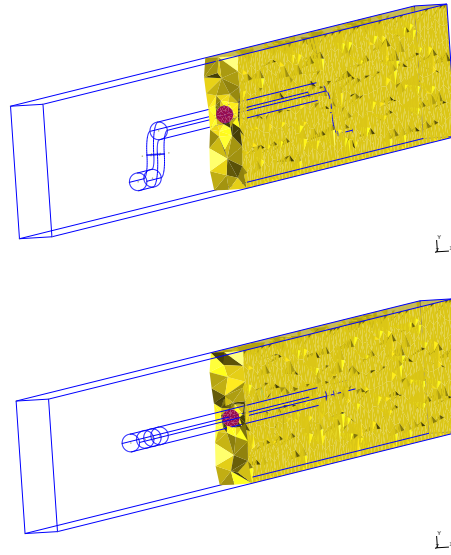


Figure 6.25: Various fiber geometries for SFRC composites.

of simulation with straight line fiber, a very good agreement between our estimates and F.E. Indeed, in our micromechanical model the inclusion shape is taken into account through Eshelby's tensor which cannot handle complex shapes such as those of hooked-end fibers.

Strains and stresses fluctuations over each phase of the two composites are reported in Figures 6.29 and 6.30. In the inclusion phase, it clearly appears that both strains and stresses are the same in more than 60% of the two fiber types. This can be explained by geometrical meanings. Indeed 64 % of the hooked-end fiber shape is similar to those of cylindrical fiber. In the matrix phase of the two composites, strains and stresses fluctuations are similar excepted for 15 % of the matrix material where a slight variation is observed. In Figure 6.31 is mapped the axial stress distribution over the two fiber types.

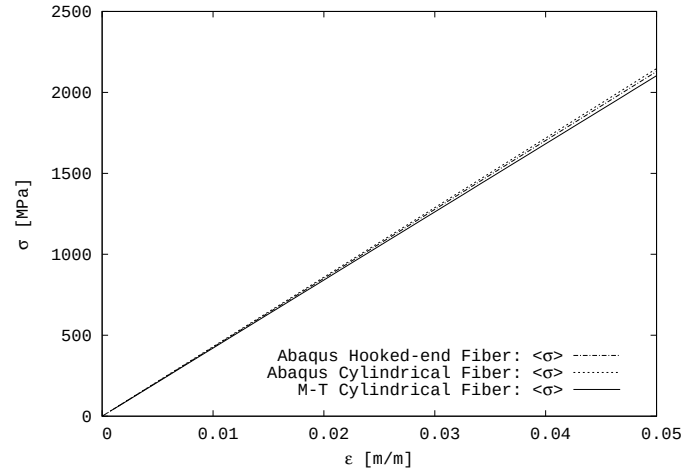


Figure 6.26: SFRC with 3% of fibers under tensile loading test: Mori-Tanaka and unit cell finite element predictions of $\langle \sigma \rangle$ the average axial stress of the composite.

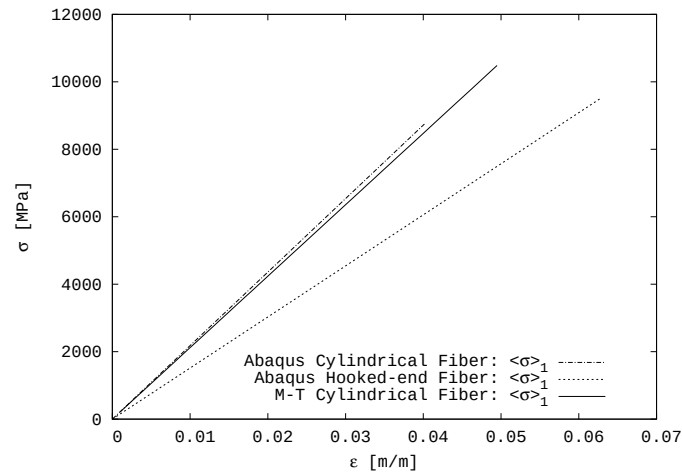


Figure 6.27: SFRC with 3% of fibers under tensile loading test: Mori-Tanaka and unit cell finite element predictions of $\langle \sigma \rangle_1$ the average axial stress in the inclusion phase of the composite.

6.9.5 Validation of a two-scale approach against the experimental bending test

The aim of the following example is to numerically simulate the four-point bending test which was performed experimentally (chapter 5). Simulation was performed us-

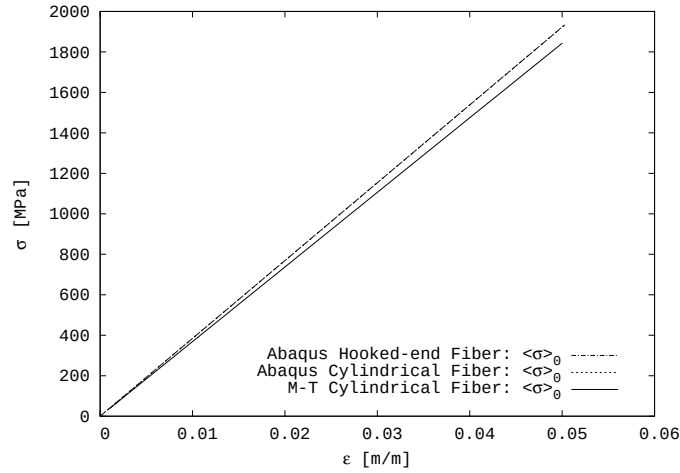


Figure 6.28: SFRC with 3% of fibers under tensile loading test: Mori-Tanaka and unit cell finite element predictions of $\langle \sigma \rangle_0$ the average axial stress in the matrix phase of the composite.

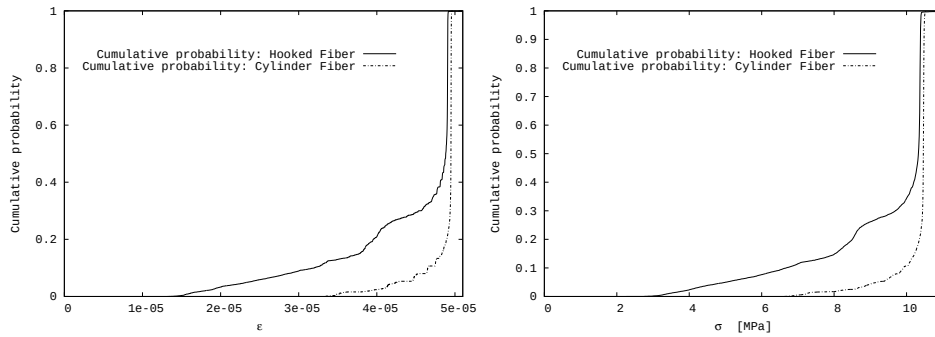


Figure 6.29: Concrete with 3% of steel fibers under tensile loading test: Axial strain and stress fluctuations within inclusion phase.

ing our homogenization code which was integrated into Abaqus via its User MATERIAL option, as discussed in section 6.8. The SFRC beam contains 3% of randomly oriented steel fibers and 0.85% of spherical cavities. The concrete matrix follows Ju's damage model with the experimentally-identified parameter values in section 6.9.3 of this chapter. The beam was meshed with 400 brick elements of type C3D8R with 612 nodes, Figure 6.32. The displacements (not the rotations) of the nodes located on the right support are zero while only the vertical displacement of the nodes located on the

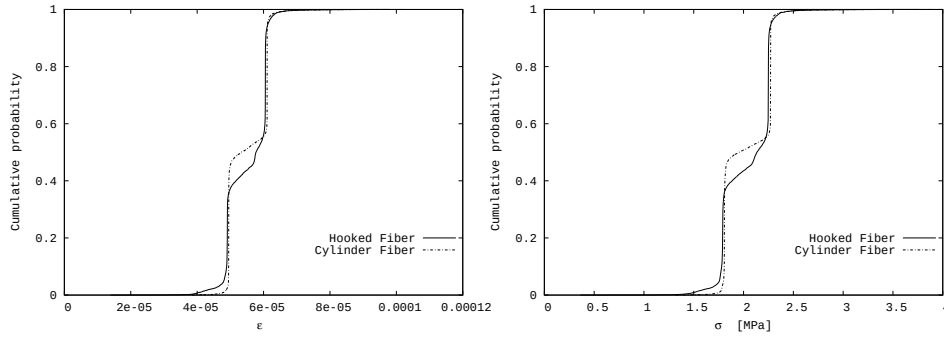


Figure 6.30: Concrete with 3% of steel fibers under tensile loading test: Axial strain and stress fluctuations within matrix phase.

left support is imposed to be zero. At the upper face of the beam a vertical displacement (0.02 mm) is applied throughout two rigid bodies. The numerically-obtained load-deflection curve is plotted in Figure 6.33 and superposed to the experimental curve. A fair agreement is observed although the linear elastic slopes are somewhat different for the two curves and damage occurs much later in the numerical simulation. A better agreement might be obtained with a better identification of Ju's model based on a damage evolution function different from the one considered here -Eq.6.17.

Focus now our interest on what happens in the mesh node number 33 (N33) which represents the more critical point of the beam (Figure 6.34). N33 with coordinates $x = 300$, $y = 0$, and $z = 75$ in mm, is located on the extremal stretched fiber where the bending momentum reaches its maximum value and consequently where the maximum normal tensile stress is expected to also reach its maximum value. In figure 6.35 are displayed the axial macro stress-strain curve and damage evolution versus the axial strain curve at N33. Damage “ d ” starts at a strain $\epsilon_{11} = 1.7 \times 10^{-5}$ and evolves until it reaches the value “ $d=0.3$ ” for $\epsilon_{11} = 5 \times 10^{-5}$. During this stage, the predicted $\sigma_{11} - \epsilon_{11}$ curve is nonlinear. Principal stresses are plotted versus their corresponding principal strains in Figure 6.36.

The predicted variation of the axial stress σ_{11} across the mid-span beam section is plotted in Figure 6.37. This curve represents σ_{11} variation versus y the height coordinate of the section. The 6.37 curve is linear over the compressed region of the section and in a part of the stretched one, while it becomes nonlinear in the remaining section in which damage has taken place, σ_{11} and y are then no longer proportional. At the micro level, per-phase axial stress-strain curves at integration point 3 of mesh element 6 are displayed in Figure 6.38. Location and details of element 6 and its integration points are given in Figure 6.34. At that same point, are also plotted in

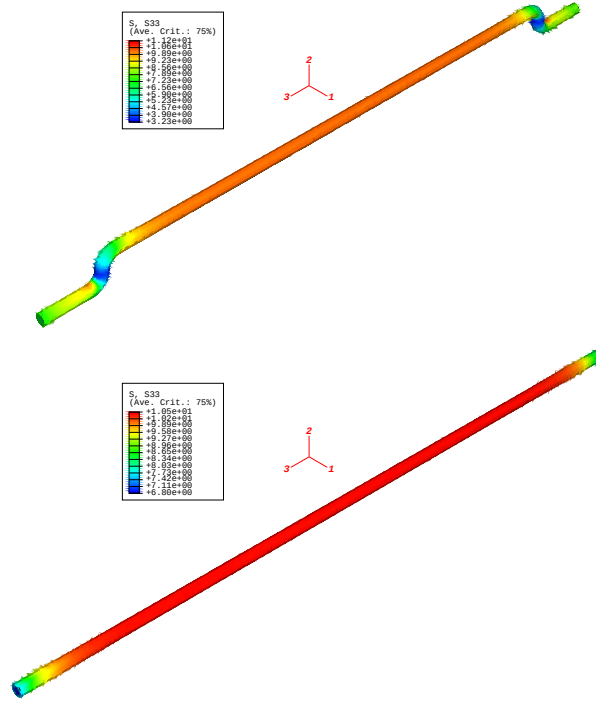


Figure 6.31: Axial stress distribution over hooked-end and straight line fibers contained in a tensile unit cell.

Figure 6.39, the curves $\sigma_{11} - \varepsilon_{11}$, $\sigma_{22} - \varepsilon_{22}$, $\sigma_{33} - \varepsilon_{33}$, and $\sigma_{12} - \varepsilon_{12}$. Examination of the Figure clearly indicates that the considered material point is under a multi-axial stress/strain state. Note also that the presence of the shear stress σ_{12} , even if it is lower than the other nonzero stresses, indicates that Bernoulli's hypotheses are not satisfied. This is certainly due to beam dimensions. Indeed the ratio, representing beam length divided by its height, is equal to 5, while to be under Bernoulli's hypotheses it must be ≥ 10 . Hereafter are given the maximum principal strains ε_{Max} at half-time of loading ($t = 0.5$):

$$[\varepsilon_P]_{t=0.5} = 10^{-05} \begin{pmatrix} 1.99 & 0 & 0 \\ 0 & -0.30 & 0 \\ 0 & 0 & -0.37 \end{pmatrix}; \quad (6.23)$$

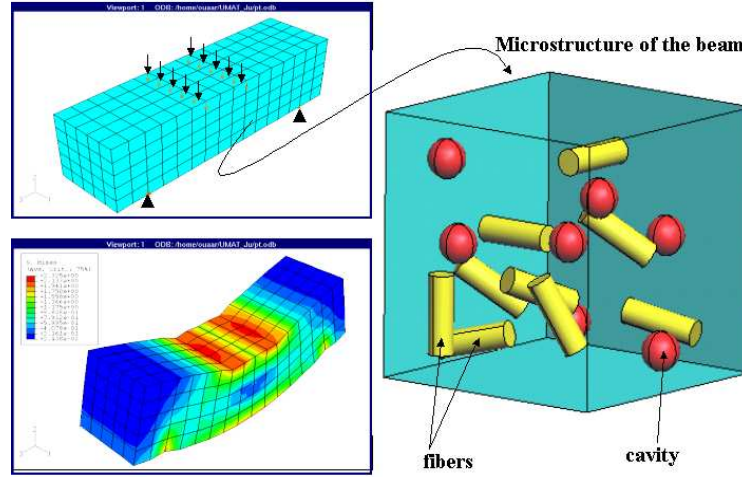


Figure 6.32: 3D modeling of a beam submitted to a four-point bending test: A two-scale numerical approach.

and the final time $t = 1$:

$$[\varepsilon_P]_{t=1} = 10^{-05} \begin{pmatrix} 4.38 & 0 & 0 \\ 0 & -0.68 & 0 \\ 0 & 0 & -0.82 \end{pmatrix} \quad (6.24)$$

Eqs.(6.23) and (6.24) indicates that in the principal strain system two principal strains are compressive strains while the last one is a tensile strain, which emphasizes Poisson's effect. The principal strain directions denoted **I**, **II**, and **III** are displayed in Figure 6.40 where 1, 2, 3 represent the beam axis. It is shown that, from time loading $t = 0.5$ to $t = 1$, the principal strain directions do not evolve in significant manner.

6.9.6 Fiber-matrix debonding simulation

The simulation of composites' mechanical response using codes based on homogenization methods may eclipse some local phenomena such as debonding which may occur at the micro-scale. To emphasize this phenomenon, consider two 2D axisymmetric unit cells:

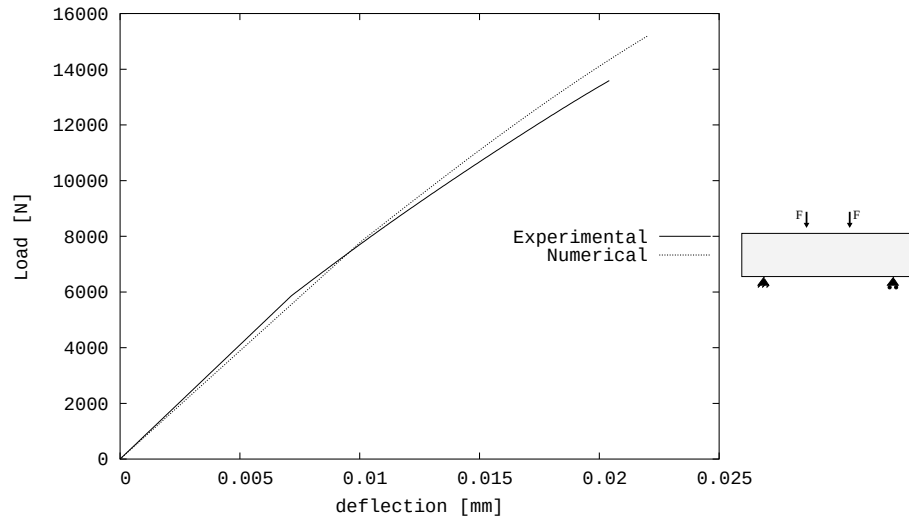


Figure 6.33: SFRC (3 percent of fibers) under bending: Load-deflection curve, numerical versus experimental results.

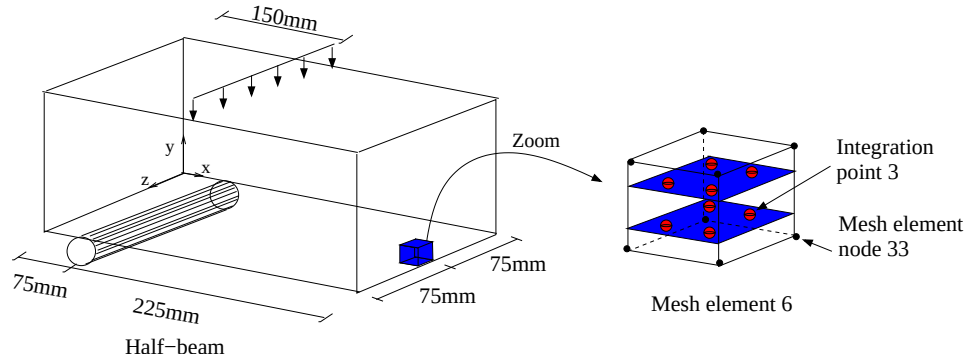


Figure 6.34: Localization of mesh element number 6.

- (1) one homogeneous modeled using our homogenization code interfaced with Abaqus, and
- (2) one heterogeneous, consisting of a concrete matrix in which a steel fiber with $A_r = 60$ is embedded. The elastic-damage behavior of the concrete phase is handled by Ju's model which we have implemented and integrated to Abaqus.

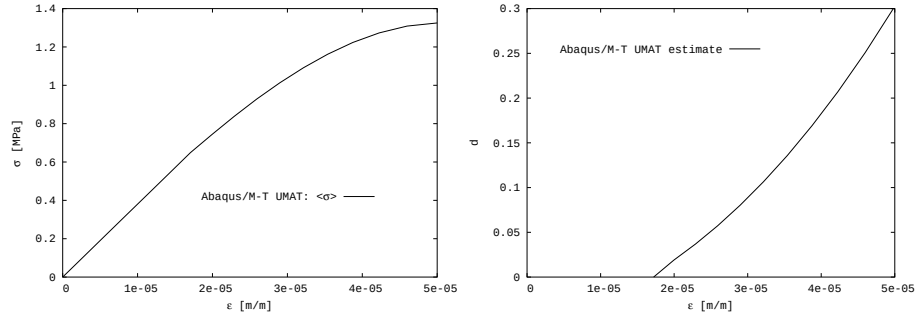


Figure 6.35: Axial stress-strain curve $\sigma_{11} - \epsilon_{11}$ and damage evolution “ d ” of a bent beam: Results predicted using Abaqus coupled with our homogenization code at mesh node 33.

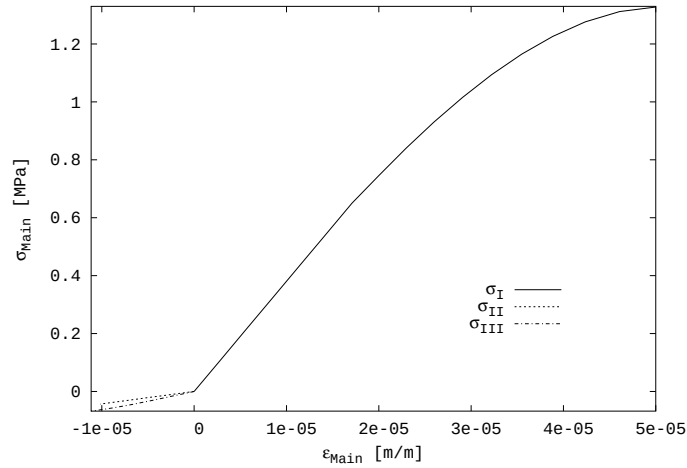


Figure 6.36: Principal stress-strain curves at mesh node 33 .

In both cases, unit cells have an aspect ratio similar to those of the fiber ($A_r=60$). Note that making some assumptions, those unit cells may represent a material point located on the maximum stretched fiber of the bent SFRC specimen studied in the previous section (section 6.9.5). Subject, for instance, those two unit cells to the maximum tensile strain reached during the bending test of section 6.9.5, i.e. $\epsilon_{11} = 5 \times 10^{-5}$. As displayed in Figure 6.41, the simulation in the heterogeneous unit cell is stopped due to convergence problems, while the one performed on the homogeneous unit cell is suc-

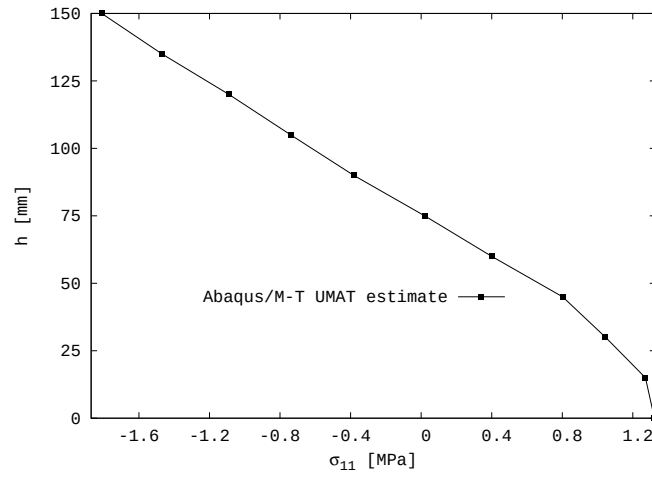


Figure 6.37: Axial stress σ_{11} evolution across the mid-span section.

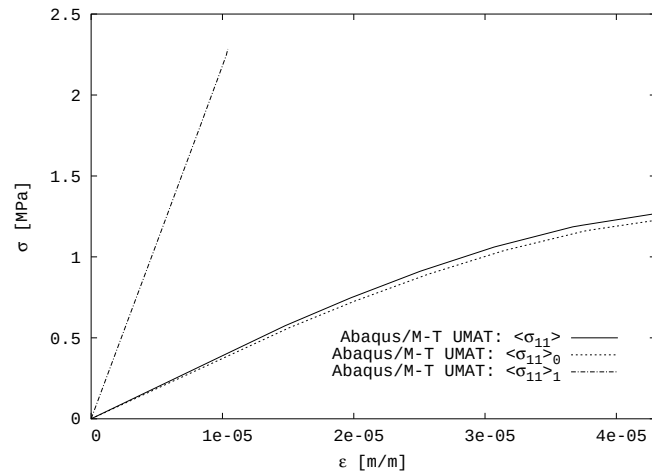


Figure 6.38: Composite and per-phase axial stress-strain curves at integration point 3 of mesh element 6.

cessfully completed. Let us have a look at what happens on the fiber-matrix interface zone and consider the mesh node A_1 shared by the two materials displayed in Figure 6.43. In Figure 6.42 is reported the axial stress-strain curve at the considered point for the two unit cells. It can be shown for heterogeneous unit cell computation that the

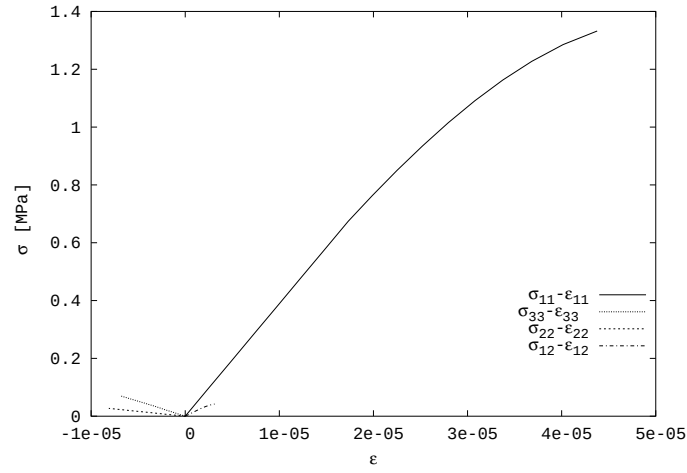


Figure 6.39: Stress-strain state of the composite at integration point 3 of mesh element 6.

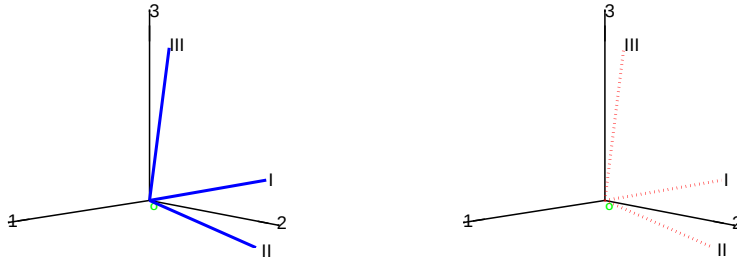


Figure 6.40: Principal strain directions at half-time and final-time loading computed at integration point 3 of mesh element 6.

stress-strain curve becomes horizontal which corresponds to a zero tangent modulus. Indeed, the matrix phase stiffness decreases dramatically due to the presence of the stiffer fiber. Damage tangent operator becomes singular leading to convergence problems both in our module (which is integrated to Abaqus) and in the F.E.computation.

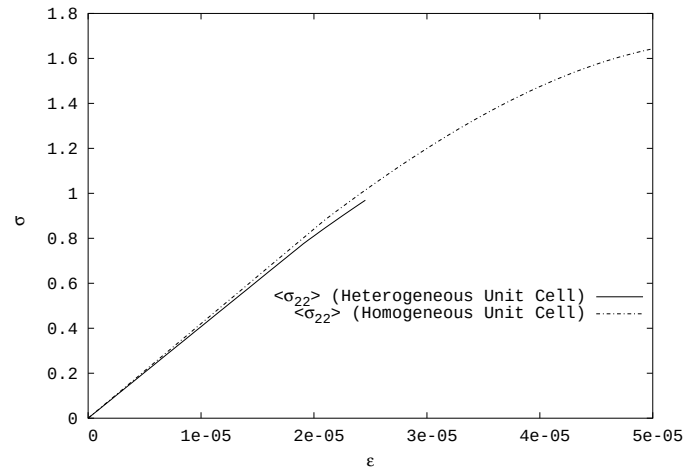


Figure 6.41: Axial average stress-strain curves for SFRC (3%) sample modeled with Homogeneous and Heterogeneous unit cells.

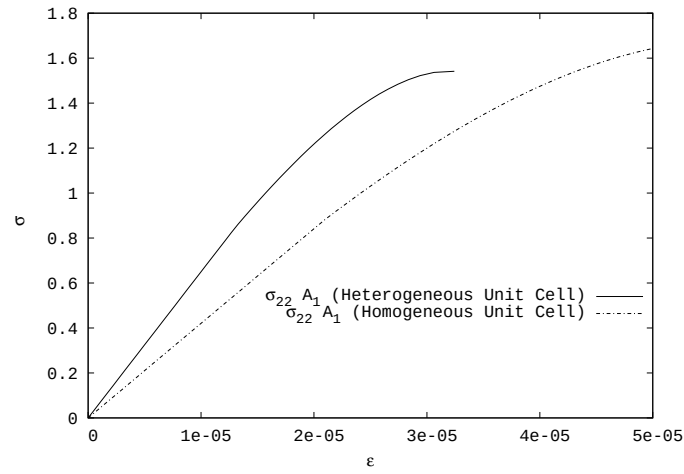


Figure 6.42: Axial stress-strain curves at mesh node A_1 : SFRC (3%) modeled with Heterogeneous and Homogeneous unit cells.

The decrease in the matrix phase's stiffness observed in the previous example suggests that fiber-matrix debonding may take place in this zone. An important question is to

determine which of the matrix damage or fiber-matrix debonding occurs first. The answer is not obvious without experimental investigation. In this study we assume that matrix material damage first takes place then under increased loading the fiber-matrix decohesion occurs. So, investigate the problem and consider the 2D axisymmetric unit

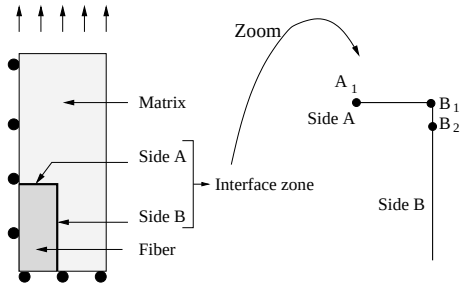


Figure 6.43: Interface zones on 2D unit cell.

cell displayed in Figure 6.43. The interface zone (Sides A and B in Fig 6.43) is meshed using axisymmetric cohesive elements COHAX4 with the following characteristics: the stiffness is equal to 75×10^7 MPa, damage takes place under a normal stress of 1.3 MPa for the normal opening mode (mode I) and under a shear stress of 0.9 MPa for the shear mode opening (mode II), and the complete failure of the element is assumed to occur for an opening of 0.00024 mm. The unit cell is subjected to an axial displacement of 0.0024 mm corresponding to an axial strain of 5×10^{-5} (the value of the tensile strain reached during the bending test of the previous example). We consider the cases of purely elastic and elastic-damage behaviors respectively of the concrete matrix. Details of interface zone (sides A and B) before and after debonding are provided in Figures 6.44 and 6.45 respectively.

As expected by physical reasoning, examination of Figures 6.44, 6.45, and 6.50 indicates that two modes of debonding are activated: one in mode I under normal stress (on side A) and one in mode II under shear stress (on side B). In figure 6.46 is displayed the axial stress at concrete matrix element mesh node A_1 of side A. In results plotted in Figure 6.46), three cases are considered: in the first one we consider an elastic behavior of the concrete matrix without fiber-matrix debonding (denoted Elastic in Fig 6.46), in the second case we consider an elastic behavior of the concrete matrix with debonding (denoted Elastic+Debonding in Fig 6.46), while in the third case damage of concrete matrix as well as the fiber-matrix debonding is taken into account (denoted Damage+Debonding in Fig 6.46). Examination of Figure 6.46 shows for the case of the matrix obeying Ju's damage model that the nonlinearity starts from a stress around 1 MPa which corresponds to the damage of the matrix then the debonding in the normal direction starts for a stress around 1.3 MPa. In Figure 6.47 is reported the

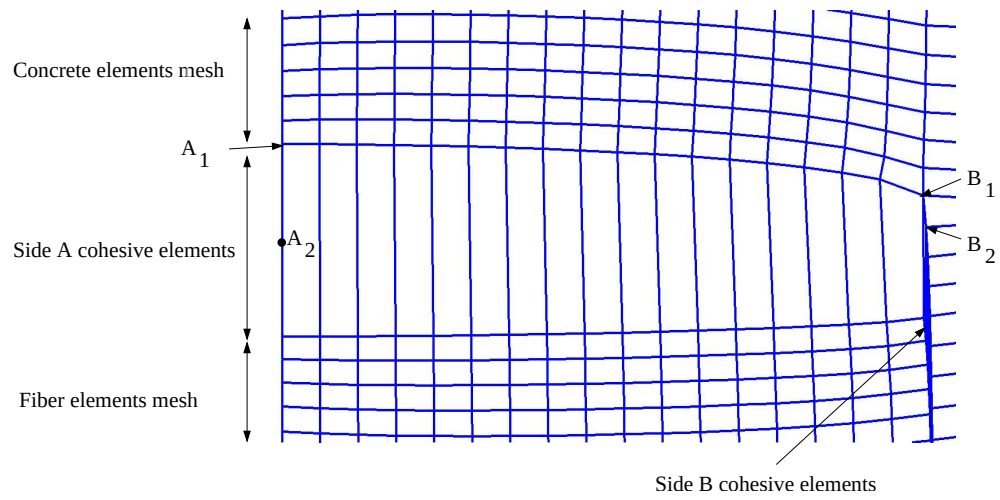


Figure 6.44: Mesh detail at Side A of the interface zone.

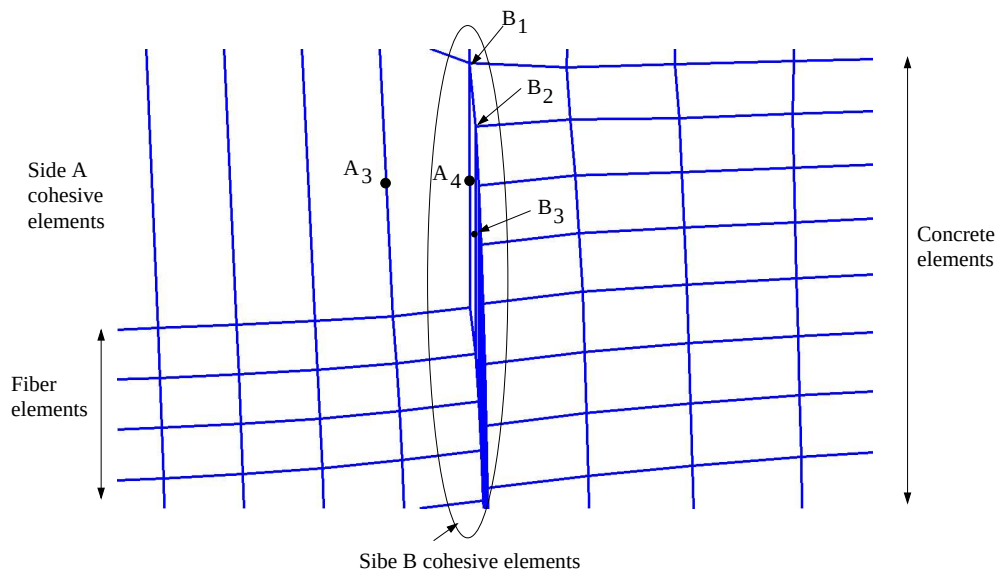


Figure 6.45: Mesh detail at Side B of the interface zone.

axial stress evolution along the side A of the interface zone. Stresses are computed at integration points A_2 , A_3 , and A_4 of the cohesive elements in the case of damage

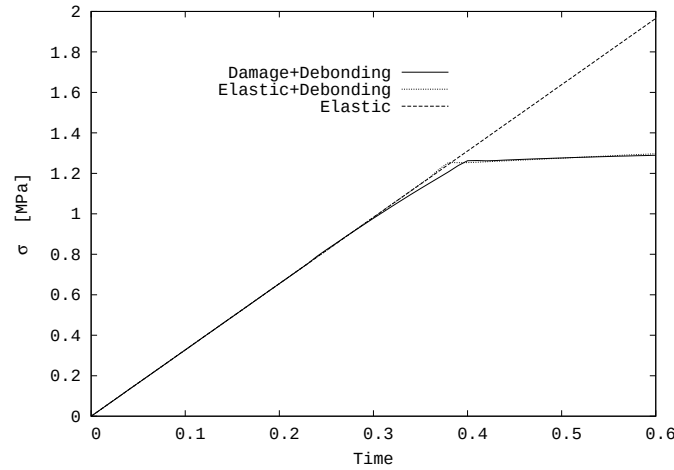


Figure 6.46: Axial stresses at node A_1 of concrete material element mesh: Elasticity plus debonding and Damage plus debonding cases.

matrix behavior. One can see that debonding takes place much earlier for points located close to corner of the fiber (point B in Fig. 6.43). This is certainly due to the fact that, in this zone, the debonding in normal and shear directions are combined. The shear stresses evolution versus time at point B_1 and B_2 of concrete matrix mesh element on side B, are reported in Figure 6.48. Here again are compared the cases of elastic and damage behavior of the concrete matrix. In Figure 6.49 are plotted axial and shear stresses respectively at point A_4 shared by both sides A and B. The axial stress is computed at integration point A_4 of the cohesive element of side A, while the shear stress is computed at integration point A_4 of the cohesive element located on the side B. It is interesting to note an increase of the slope of the shear stress curve at a time t around 0.25 which coincides with the debonding time in the normal direction of the cohesive element of side A. The debonding under the shear stress occurs much later at a time around 0.4. In Figure 6.50, the equivalent von Mises stress is mapped over the critical zone of fiber-matrix interface and the surrounding fiber and matrix materials respectively.

6.9.7 Particle- and fiber-reinforced concrete-like matrix materials

This section is devoted to a parametric study of the proposed model. We consider a fictitious concrete-like material which is reinforced with either steel particles or fibers.

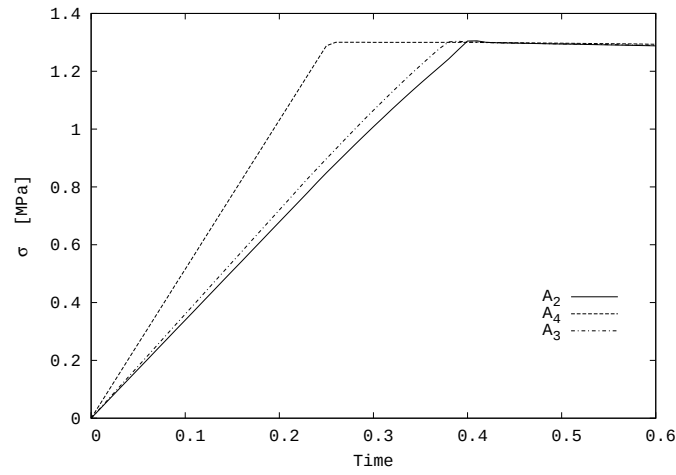


Figure 6.47: Axial stresses evaluated at the integration points A_2 , A_3 , and A_4 of cohesive elements of side A: Damage plus debonding case.

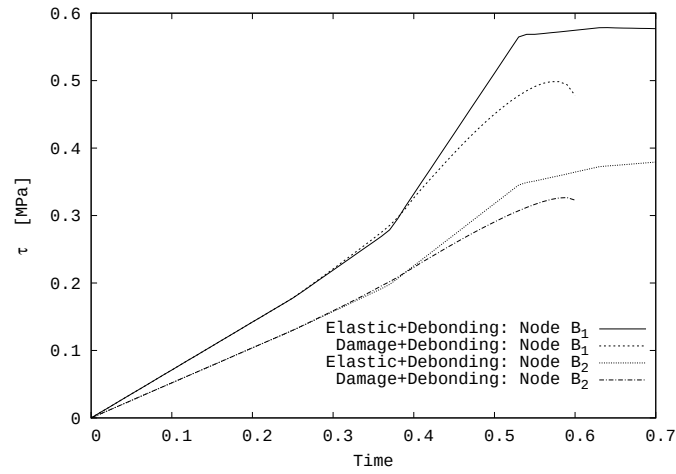


Figure 6.48: Shear stresses evaluated at nodes B_1 and B_2 of concrete material elements mesh of side B: Elasticity plus debonding and Damage plus debonding cases.

These reinforcements lead to composite materials designated as SPRC and SFRC, respectively. The unreinforced matrix material has the same properties as in section

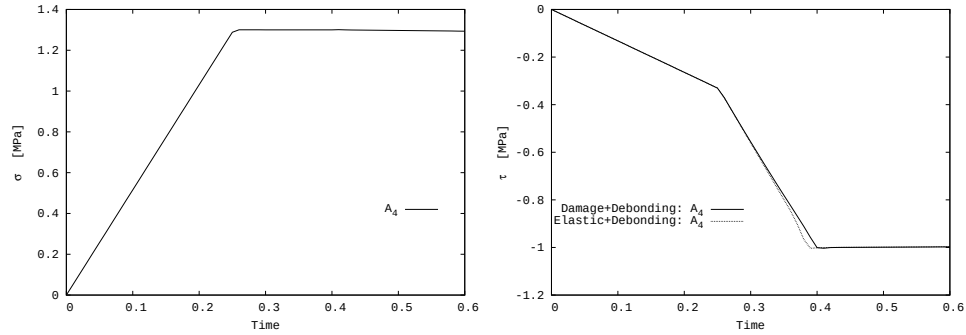


Figure 6.49: Axial stresses evaluated at the integration points A_2 , A_3 , A_4 of cohesive elements of side A: Damage plus debonding

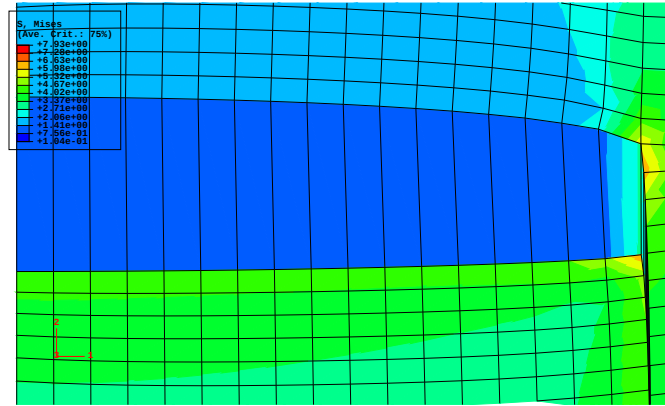


Figure 6.50: von Mises equivalent stress at critical zone of the fiber-matrix interface.

6.9.1 except when indicated otherwise. In the first example, we study the influence of the matrix' microcrack density parameter k -Eq.6.17- which takes 3 values: $k = 0.3$, 0.5 and 0.7 . Spherical particles of volume fraction $v_1 = 10\%$ are embedded in the matrix. The influence of k on the composite response is displayed in Figure 6.51 which shows that the composite softens with increasing values of k .

In the second example, we illustrate the influence of the reinforcements shape. The matrix parameter k is set to $k = 0.7$ and elastic steel inclusions ($v_1 = 10\%$) are considered in three shapes: spherical ($A_r = 1$), long ($A_r > 100$) and short ($A_r = 50$) fibers. The fibers are aligned in the macro tension direction. As expected, Fig.6.52 shows that the composite stiffens with increasing values of the aspect ratio A_r .

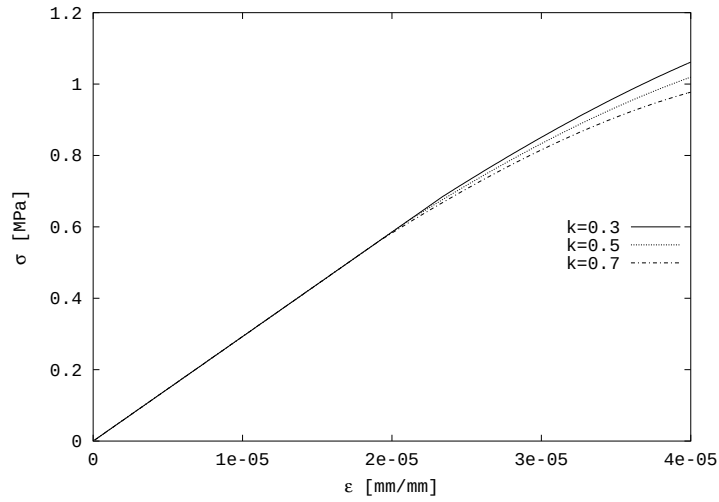


Figure 6.51: Influence of crack density factor “k” on the macro-response of SPRC under tensile loading.

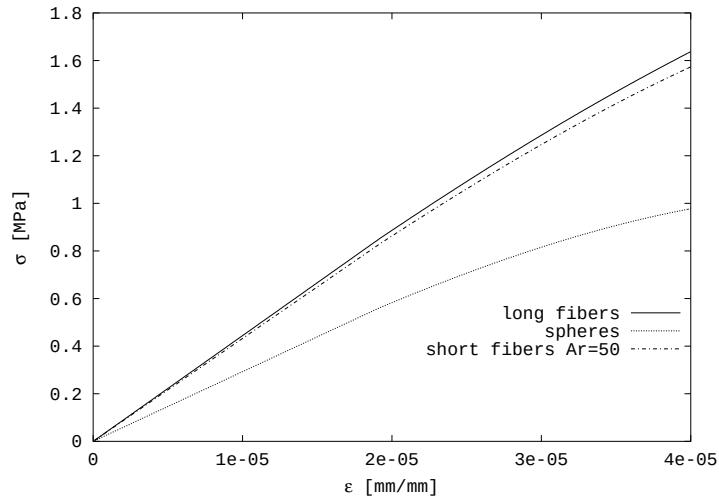


Figure 6.52: Influence of the shape of the reinforcement on the response of reinforced concrete matrix submitted to tension

In a third example, we investigate the influence of the matrix’ parameter A - Eq.6.17- which takes 3 values: $A = 10.$, 14.3 and 17 . The matrix is reinforced with spherical

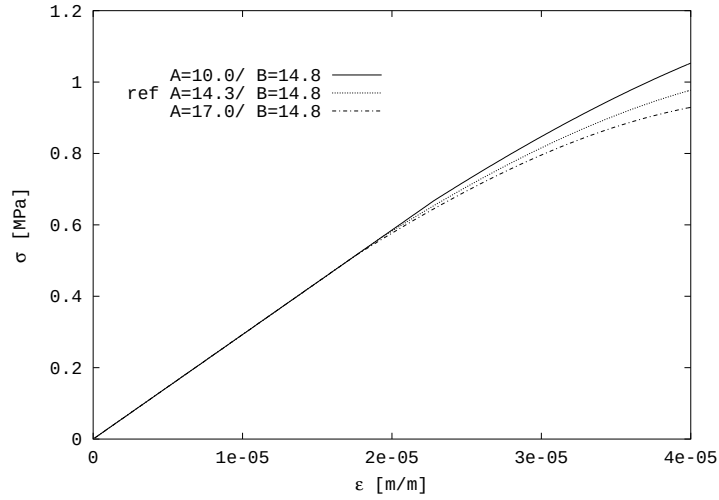


Figure 6.53: Influence of material parameter A on the response of 10% sphere reinforced concrete submitted to tension.

steel particles ($v_1 = 10\%$). Simulation of uniaxial tension tests show that the composite softens considerably with increasing values of A , as illustrated in Figure 6.53.

The same composite is considered in a fourth example which illustrates the influence of the matrix' parameter B which now takes 3 values $B = 10.$, 14.8 and 17 . Figure 6.54 shows that the composite softens dramatically with increasing values of B .

In a fifth example, the response of SFRC under cyclic loading is simulated. The parameters of the matrix material are: $E_m = 20$ GPa, $\nu_m = 0.2$, $A = 0.8$, $B = 0.16$, $k = 0.7$ and $\xi_0 = 2.4 \times 10^{-2}$ MPa. The matrix is reinforced with short steel fibers ($A_r = 50$ and $v_1 = 20\%$) which are elasto-plastic with linear isotropic hardening ($m = 1$). The composite is subjected to cyclic tension/compression in the fiber direction under imposed macro strain. The resulting stress-strain response is plotted in Fig.6.55 together with those of the fibers and matrix taken separately. Both elasto-plastic and damage phenomena are illustrated. First, the composite behaves elastically (step 1), then when the yield stress hardening of the fibers is reached, the response of the composite becomes elasto-plastic, step 2. Under higher tensile strain, the composite undergoes damage, step 3. During unloading and compressive loading phases damage does not evolve, the composite exhibits elastic then elasto-plastic behavior: steps 4 and 5, respectively. During step 6, the composite goes elastic then elasto-plastic under tension (step 7). No more damage develops in this step since the damage threshold is not reached.

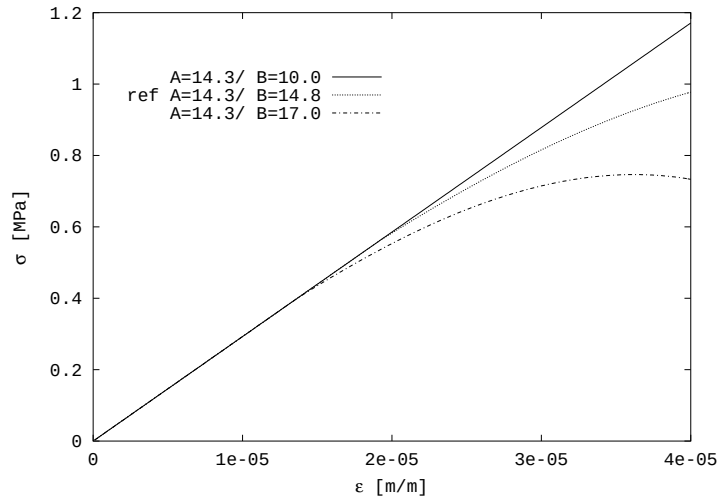


Figure 6.54: Influence of material parameter B on the response of (10%) spheres reinforced concrete submitted to tension.

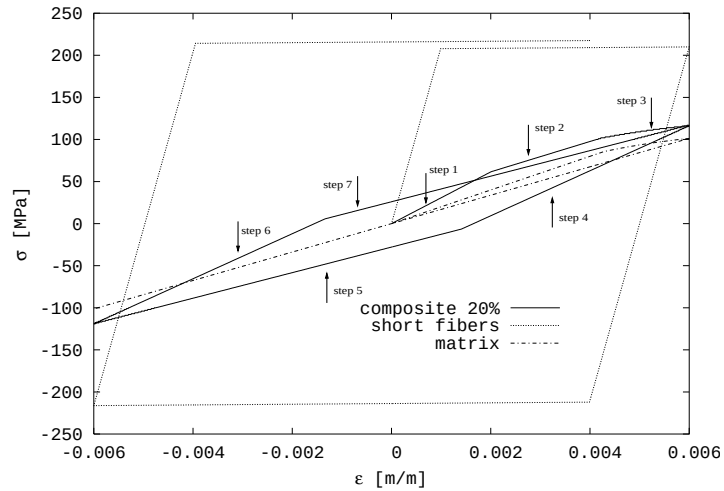


Figure 6.55: SFRC under cyclic loading: Effect of damage and of plasticity on the macroscopic response of the composite

6.10 Conclusion

In this chapter, we developed a micromechanical-based model and the corresponding algorithms which try to reproduce the mechanical behavior of a class of concrete-like

matrix composites. Various damage processes such as porosity increase due to fibers addition, microcracks activation and growth and voids coalescence, and fiber-matrix debonding were taken into account in the modeling. A two-step Mori-Tanaka/Voigt homogenization scheme was proposed to handle the modeling of multi-phase concrete-matrix composites where inclusion phases can either be solids or cavities. As an application we simulated the mechanical behavior of both Plain concrete and Steel Fiber-Reinforced Concrete composite under several discriminating tests: monotonic tension, compression, unloading and cyclic loadings. Our model estimates were validated against both experimental results obtained in chapter 5 and direct finite element computations performed using ABAQUS software. For F.E. simulations we considered a periodic microstructure represented with two axisymmetric unit cells. Our homogenization code was also interfaced to the finite element package ABAQUS through its user-defined material interface option in order to perform two-scale simulations of 3D realistic structures. In general, our estimates showed a fair agreement with both F.E. simulations and experimental results. However, in some cases, our predictions seem to be overestimated in comparison with F.E. ones. Thinking that this is due to intra-phase heterogeneity which is not taken into account in our model, we investigate strain and stress fluctuations within the composite. The analysis of results did not show a clear tendency which allows to make a direct link between our results overestimation and the intra-phase heterogeneity.

Chapter 7

Numerical simulations of elasto-plastic matrix composites

7.1 General Introduction

This chapter is devoted to the simulation of elasto-plastic-matrix composites (EMC) using both homogenization methods and finite element analysis. In this study, two classes of EMC are considered: Metal Matrix Composites (MMC) and Polymer Matrix Composites (PMC). Those composites are widely used in industrial applications, for instance in boat hulls and aircraft wings for PMC, and in engine components for MMC. Many researchers were interested by the modeling and simulation of such composites behavior. Among them, Tandon and Weng [102](with a secant formulation) and Doghri and Ouaar [21] (with an incremental formulation) studied the case of a polymer matrix (epoxy) reinforced with spherical inclusion made of silica. Suquet et al. [97](with a modified secant formulation), Segurado et al. [86] (3D and 2D F.E. analysis), Doghri and Ouaar [21], and Doghri and Friebel (with an incremental formulation) [20] investigated the behavior of a MMC consisting in aluminum alloy matrix in which are embedded ceramic inclusions.

This chapter is organized as follows: in section 2 we present an algorithm which handles the modeling of elasto-plastic matrix composites behavior within the framework of the homogenization mean-field approach. The crucial issue of the choice of the tangent operator and the Eshelby's tensor computation is discussed in section 2. Finally, in sections 3 and 4 we simulate the behavior of MMC and PMC under several

discriminating loading tests, where the accuracy of the proposed incremental formulation is validated against both finite element computations and available results in the literature.

7.2 Overall micro-macro algorithm

In this section we propose an algorithm which, making use of constitutive law and homogenization schemes developed in chapters 4 and 5 respectively, enables us to simulate the deformation behavior of elasto-plastic matrix composites (EMC). For this purpose, the nonlinear operator C^{tg} used in homogenization procedure of chapter 5 is replaced with C^{alg} since we deal with materials modeled with J_2p theory. The choice of the consistent operator C^{alg} in place of the continuum operator C^{ep} will be justified hereafter, in section 7.3.

For a time interval $[t_n, t_{n+1}]$, what we need to compute are the macro stress $\bar{\sigma}$ and the overall tangent stiffness $\bar{C}^{alg}$. The data are $\bar{\varepsilon}_n$ (macro strain at t_n), $\Delta\bar{\varepsilon}$ (macro strain increment), and history variables for each solid phase at t_n . In the following, in order to simplify notation, the domain of a pseudo-grain $\Omega_{i,p}$ is designated by ω and its constituents are denoted by ω_0 for the matrix and ω_1 for the inclusions.

- Loops over all pseudo-grains.
 - Voigt in final homogenization step: $\langle \Delta\boldsymbol{\varepsilon} \rangle_\omega = \Delta\bar{\boldsymbol{\varepsilon}}$.
 - Initialization in inclusions: $\langle \Delta\boldsymbol{\varepsilon} \rangle_{\omega_1} = \langle \Delta\boldsymbol{\varepsilon} \rangle_\omega$.
 - Loop within each pseudo-grain (iteration index omitted for simplicity):
 1. Check if solid inclusions or cavities:
 - (a) if solid inclusions:
 - * Call J_2 elasto-plasticity constitutive box with $\langle \Delta\boldsymbol{\varepsilon} \rangle_{\omega_1}$ and $\langle \boldsymbol{\varepsilon}_n \rangle_{\omega_1}$ as arguments. The box returns $\langle \boldsymbol{\sigma}_{n+1} \rangle_{\omega_1}$ and algorithmic tangent $\hat{C}_1^{alg}|_{n+1}$;
 - * Compute Algorithmic operator at $t_{n+\alpha}$ with:

$$\hat{C}_1^{alg}|_{n+\alpha} = (1 - \alpha)\hat{C}_1^{alg}|_n + \alpha\hat{C}_1^{alg}|_{n+1}.$$
 - (b) if cavities go directly to step 2
 2. Compute average strain increment in the matrix phase:

$$\langle \Delta\boldsymbol{\varepsilon} \rangle_{\omega_0} = \frac{\langle \Delta\boldsymbol{\varepsilon} \rangle - (1 - v_0) \langle \Delta\boldsymbol{\varepsilon} \rangle_{\omega_1}}{v_0}.$$

3. Call J_2 elasto-plasticity constitutive with $\langle \Delta \varepsilon \rangle_{\omega_0}$ and $\langle \varepsilon_n \rangle_{\omega_0}$ as arguments. The box returns $\langle \sigma_{n+1} \rangle_{\omega_0}$ and algorithmic tangent $\hat{C}_0^{alg}|_{n+1}$.
4. Compute Algorithmic operator at $t_{n+\alpha}$ with:

$$\hat{C}_0^{alg}|_{n+\alpha} = (1 - \alpha)\hat{C}_0^{alg}|_n + \alpha\hat{C}_0^{alg}|_{n+1}.$$

5. Compute the instantaneous Poisson's ratio ν_t from the isotropic part of $\hat{C}_0^{alg}|_{n+\alpha}$
6. Compute Eshelby's tensor $S_{(\nu_t, A_r)}$
7. Compute strain concentration tensor A using Mori-Tanaka scheme. If solid inclusions:

$$A^{MT} = [\nu_0\{I + S_{(\nu_t, A_r)} : [(\hat{C}_0^{alg}|_{n+\alpha})^{-1} : \hat{C}_1^{alg}|_{n+\alpha} - I]\} + (1 - \nu_0)I]^{-1}.$$

If cavities:

$$A^{MT} = [I - \nu_0 S_{(\nu_t, A_r)}]^{-1}.$$

8. Check strain compatibility in the inclusions:

$$\langle \Delta \varepsilon \rangle_{\omega_1} = ? \quad A^{MT} : \langle \Delta \varepsilon \rangle_{\omega}.$$

9. if yes: no more iterations for this pseudo-grain,
10. else, perform a new iteration with:

$$\langle \Delta \varepsilon \rangle_{\omega_1} = A^{MT} : \langle \Delta \varepsilon \rangle_{\omega}.$$

- After convergence, compute effective tangent operator of the pseudo-grain. If it contains solid inclusions then:

$$\bar{C}_{\Omega_{i,p}}^{alg} = \hat{C}_0^{alg}|_{n+\alpha} + (1 - \nu_0)(\hat{C}_1^{alg}|_{n+\alpha} - \hat{C}_0^{alg}|_{n+\alpha}) : A^{MT}.$$

If the pseudo grain contains cavities, then:

$$\bar{C}_{\Omega_{i,p}}^{alg} = \hat{C}_0^{alg}|_{n+\alpha} - (1 - \nu_0)\hat{C}_0^{alg}|_{n+\alpha} : A^{MT}.$$

- Save pseudo-grain's effective tangent and average stresses and strains.

- Compute RVE's effective tangent moduli and macro stresses

$$\bar{C}^{alg} = \left\langle \bar{C}_{\Omega_{i,p}}^{alg} \right\rangle_{i, \psi_i}, \quad \Delta \bar{\sigma} = \bar{C}^{alg} : \Delta \bar{\varepsilon}.$$

Note that in the following, for the simplicity of the notation, we will consider that: $\langle \cdot \rangle_{\omega} \equiv \langle \cdot \rangle$, $\langle \cdot \rangle_{\omega_0} \equiv \langle \cdot \rangle_0$, and $\langle \cdot \rangle_{\omega_1} \equiv \langle \cdot \rangle_1$.

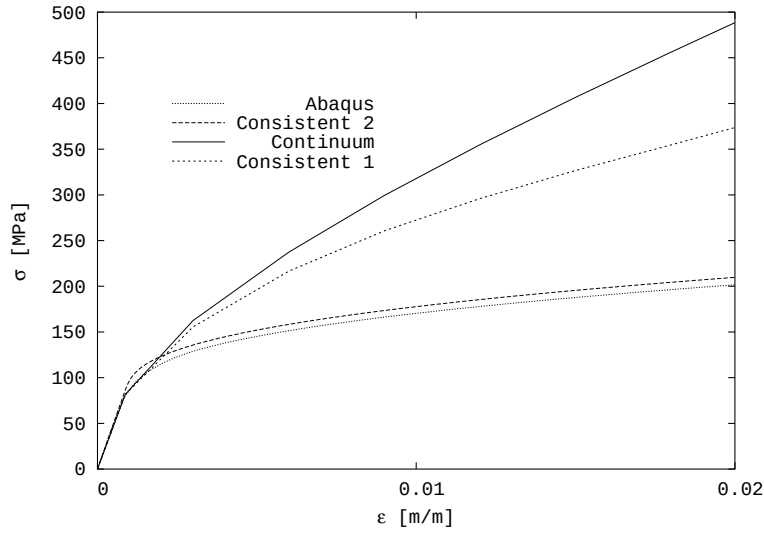


Figure 7.1: Tangent operators

7.3 Choice of tangent operator and Eshelby's tensor computation

For materials that obey to an elasto-plastic behavior modeled with J_2p , we showed in chapter 4 that two tangent operators can be defined: the continuum tangent operator and the consistent one, also called algorithmic operator. In all coming simulations we will use the “algorithmic” operator C^{alg} . This choice is motivated by results of the study on tangent operators achieved by Doghri and Ouair [21]. They showed that the continuum tangent operator C^{tg} is stiffer than the “algorithmic” (C^{alg}) one, see [21] for details. This analytical result is also observed by the means of numerical simulations. Consider metal matrix composite with 20% of spherical inclusions submitted to tensile loading test. The matrix obeys to J_2p and the inclusions are elastic. Figure 7.1 illustrates simulations obtained with various tangent operators. As expected by the analytical result (see [21], pp1689-1690), the response computed with the continuum operator, is too stiff compared to those computed with the algorithmic one, denoted “Consistent 1” in the Figure 7.1. Note that for those two cases, represented by “Continuum and Consistent 1” in the Figure 7.1, Eshelby’s tensor is computed numerically using a function provided by Lagoudas [27]. It also appears that the estimate obtained when we use the algorithmic operator (Consistent 1) remains too stiff compared to the F.E one which is considered as the target result. However, this estimate is significantly

improved if the Eshelby's tensor is computed using the isotropic part of the algorithmic operator [21] as displayed in Figure 7.1 with the curve denoted "Consistent 2". We concluded from this example that the better results are those obtained by using the algorithmic operator as phase moduli and when Eshelby's tensor is computed using the isotropic part of the algorithmic operator of the considered phase.

7.4 Metal Matrix Composites

In this section we will study the mechanical response of a class of metal matrix composites (MMC). The response of MMC samples under several discriminating tests will be simulated using both homogenization method and F.E. analysis. In all following examples of this section, we consider two constitutive materials:

- (1) Aluminum alloy material assumed elasto-plastic and modeled with $J_2 p$, with the following mechanical properties: $E = 75$ GPa, $\nu = 0.3$, the yield stress $\sigma_Y = 75$ MPa, with the power-law isotropic hardening $R(p) = kp^m$ where $k = 416$ MPa and $m = 0.3895$.
- (2) Ceramic material assumed to behave elastically following Hooke's law with $E = 400$ GPa and $\nu = 0.2$.

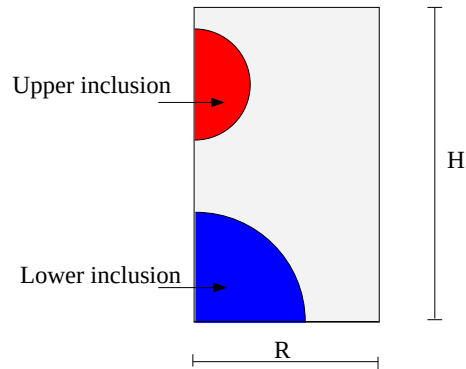


Figure 7.2: UC2 Unit cell dimensions and inclusions location

7.4.1 Solid Particles Reinforced Metal Matrix

Consider a two-phase composite consisting of a metallic matrix made of aluminum alloy in which are embedded ceramic spherical inclusions.

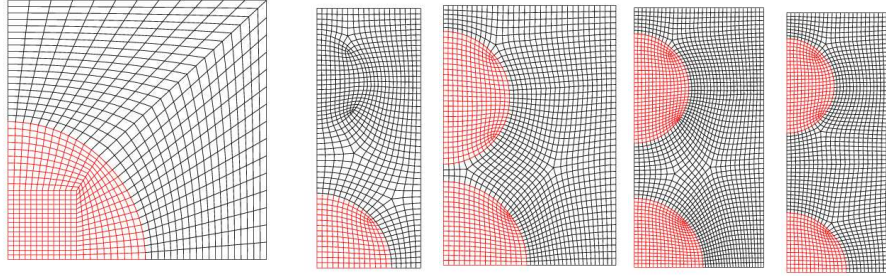


Figure 7.3: Unit cells UC1 and UC2 representation for various values of the height-width ratio

Unit cells geometry influence

First of all, through the following example we wish to study the influence of the ratio H/R of Unit Cells dimensions on the tensile response of the composite with 10% of ceramic spheres. H represents the height of the considered unit cell while R is its width, as displayed in Figure 7.2. Simulations are carried out on UC1 and UC2 (see section 6.7 of chapter 6), both meshed using axisymmetric CAX4R elements to accommodate the plastic strains of the matrix phase and to prevent the shear locking phenomena.. For UC1 (Unit cell with one inclusion) we considered two values of H/R : $H/R = 1$ and $H/R = 2.5$, while for UC2 (Unit cell with two inclusions) we considered three values: $H/R = 1.5$, $H/R = 2$, and $H/R = 2.5$, as displayed in Figure 7.3. UC2 is designed such as the total volume fraction of inclusions is 10%: upper inclusion represents 2x2.5% of the total volume of the composite while the central inclusion represents 5%. It can be shown in Figure 7.4 that UC1 with $H/R = 1$ and UC2 with $H/R = 2$ and 2.5 give quiet similar predictions while those of UC1 with $H/R = 2.5$ and of UC2 with $H/R = 1.5$ seem to be too stiff. If UC1 with $H/R = 1$ estimate is considered as the target one, the use of UC2 with $H/R = 2.5$ for F.E. computations leads to satisfying predictions.

Tensile loading test

In the following, the mechanical response of the MMC with four values of volume fractions ($v_1 = 5\%, 10\%, 20\%, 30\%$) under macro-tension tests is simulated with two

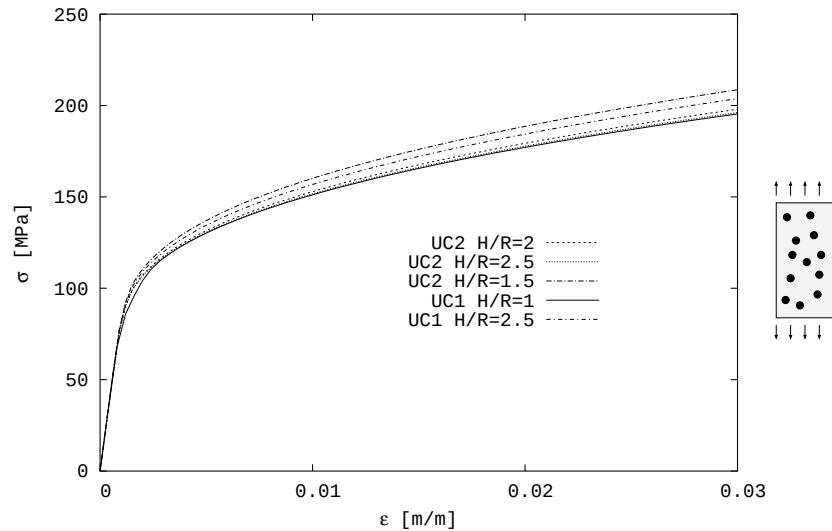


Figure 7.4: MMC with 10% of ceramic spheres under tensile loading test: Unit cells size effects influence

different methods: (i) Unit cells (UC1 and UC2) finite elements analysis and (ii) Mori-Tanaka homogenization scheme. It should be interesting to note that even if M-T scheme is derived for composites with random microstructures, the estimate obtained with this homogenization scheme can be compared with those obtained with unit cell F.E. which model composites with composites with periodic microstructures, see section 6.7 of chapter 6. Indeed, it was shown in Sagurado [55] that F.E. computations on 2D axisymmetric unit cell representing composites on periodic microstructures and computations on 3D RVE representing composites with random microstructures, lead to quite similar estimates in the average sense. In Figures 7.5, 7.6, 7.7, and 7.8 are plotted the stress-strain curves simulated with the two methods. It clearly appears that the elastic part of the response is captured well by our model while it tends to slightly overestimate the response in the plastic part. In general a good agreement is seen between the two sets of results.

The tendency to overestimate the plastic part of the response exhibited by our homogenization code can be explained by the fact that, in our incremental formulation, tangent operators describing the mechanical properties of our Linear Heterogeneous Comparison Material (LHCM) are computed with the average strain of the considered phase. The main inconvenience of this method is that the intra-phase strain and stress fluctuations, which can be important, are not accounted for. This argument is supported by many researchers in the field. Focus our attention on the composite with 10% of inclusions for instance, and first analyze fluctuations of various fields within

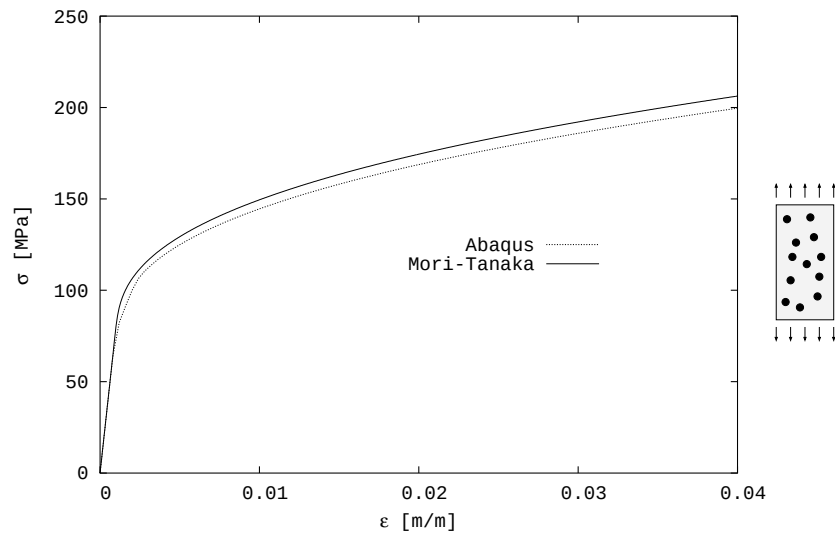


Figure 7.5: MMC with 5% of ceramic spheres under tensile loading test: Mori-Tanaka predictions versus the unit cell UC1 finite element computations.

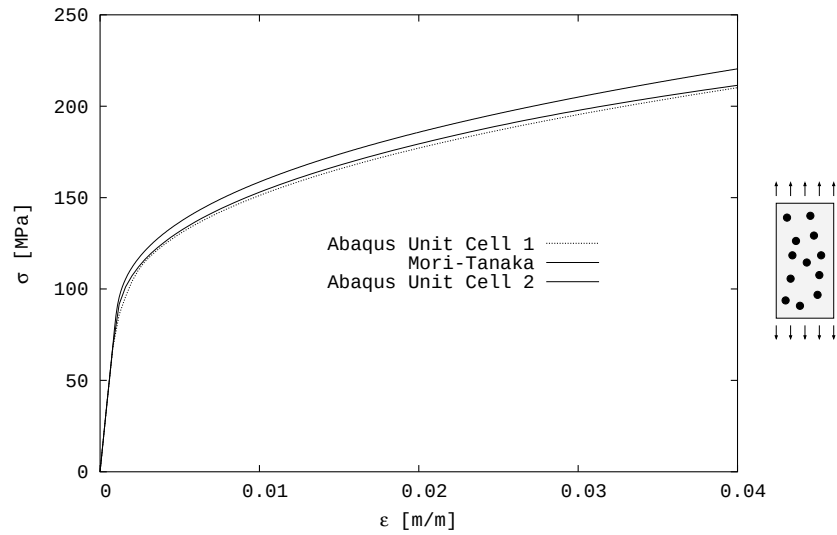


Figure 7.6: MMC with 10% of ceramic spheres under tensile loading test: Mori-Tanaka predictions versus the unit cell UC1 finite element computations.

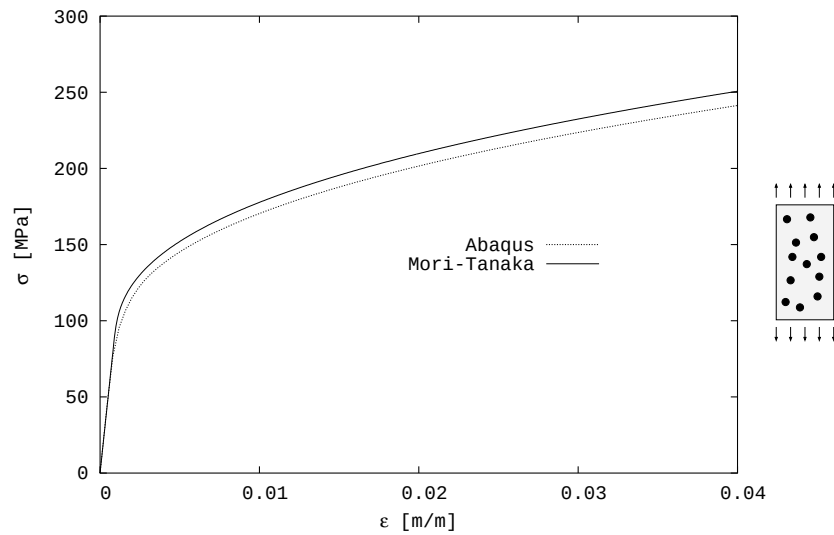


Figure 7.7: MMC with 20% of ceramic spheres under tensile loading test: Mori-Tanaka predictions versus the unit cell UC1 finite element computations.

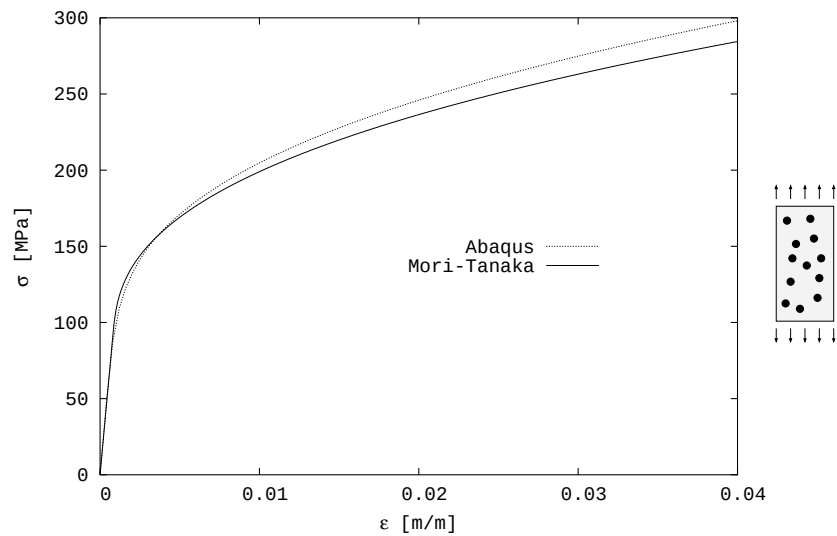


Figure 7.8: MMC with 30% of ceramic spheres under tensile loading test: Mori-Tanaka predictions versus the unit cell UC1 finite element computations.

the matrix phase at the final step loading which corresponds to $\bar{\epsilon}_{22} = 3\%$. In Figure 7.9 is plotted the cumulative probability for that a given volume fraction of the matrix material undergoes a certain value of ϵ_{22} versus this value of ϵ_{22} (see section 6.9.7 of chapter 6 and also Pierard et al. [75]). There are many manners to read this figure, one can say that 20% of the matrix undergoes an axial strain ϵ_{22} less than ± 0.03 ($\pm 3\%$) for instance, or 60% of the matrix phase undergoes a strain ranged from 0.03 to 0.042. On Figure 7.9 are also reported, as qualitative indicators, the matrix average strain values obtained using Abaqus and our code. This graphics also provides an information on the heterogeneity state of ϵ_{22} , the less the cumulative probability curve is spread out the less the dispersion (the fluctuation) is important. From Figure 7.9, it clearly appears that a great part of the matrix undergoes an ϵ_{22} which fluctuates mainly from $\epsilon_{22} = 0.028$ to $\epsilon_{22} = 0.043$, while the F.E. axial average matrix strain $\langle \epsilon_{22} \rangle_0 = 0.033$. It can be concluded that ϵ_{22} fluctuations around the axial average matrix strain value $\langle \epsilon_{22} \rangle_0$ predicted by F.E. as well as by our code, are quite important. The other fields that it should be interesting to analyze are the axial matrix stress

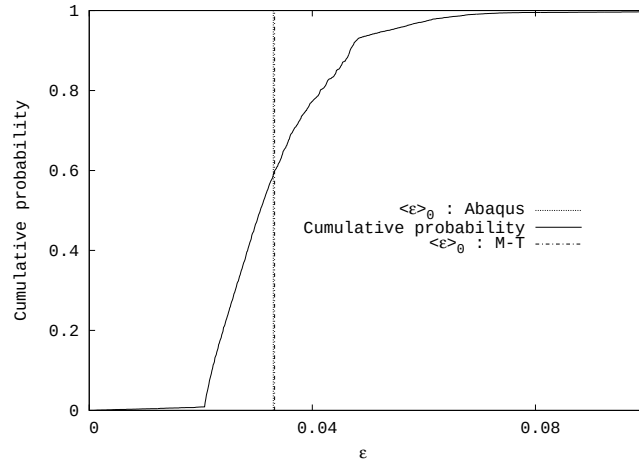


Figure 7.9: MMC with 10% of ceramic spheres under tensile: Axial strain ϵ_{22} fluctuations in the matrix phase.

σ_{22} and the von Mises equivalent stress σ_{eq} fluctuations, since they both intervene in the yield criterion (plastic surface) expression ($\sigma_{eq} - R(p) - \sigma_Y$, Eq3.60 of chapter 3) and consequently influence the plastic part of the composite response.

The average von Mises equivalent stress $\langle \sigma_{eq} \rangle$ obtained from F.E. analysis is com-

puted as follows:

$$\langle \sigma_{eq} \rangle = \frac{\sum_{i=1}^{N_{el}} \sum_{j=1}^{N_{IP}} Ivol_{i,j} \sigma_{i,j}^{eq}}{\sum_{i=1}^{N_r} \sum_{j=1}^{N_{IP}} Ivol_{i,j}}, \quad (7.1)$$

while the von Mises equivalent average stress, $\langle \sigma \rangle_{eq}$, provided by our homogenization code is computed using:

$$\begin{aligned} \langle \sigma \rangle_{eq} = & \frac{1}{2} \left[(\langle \sigma_{11} \rangle - \langle \sigma_{22} \rangle)^2 + (\langle \sigma_{22} \rangle - \langle \sigma_{33} \rangle)^2 + (\langle \sigma_{33} \rangle - \langle \sigma_{11} \rangle)^2 \right. \\ & \left. + 6 (\langle \sigma_{12} \rangle^2 + \langle \sigma_{13} \rangle^2 + \langle \sigma_{23} \rangle^2) \right]. \end{aligned} \quad (7.2)$$

In the above, N_{el} and N_{IP} represent elements and integration points numbers respec-

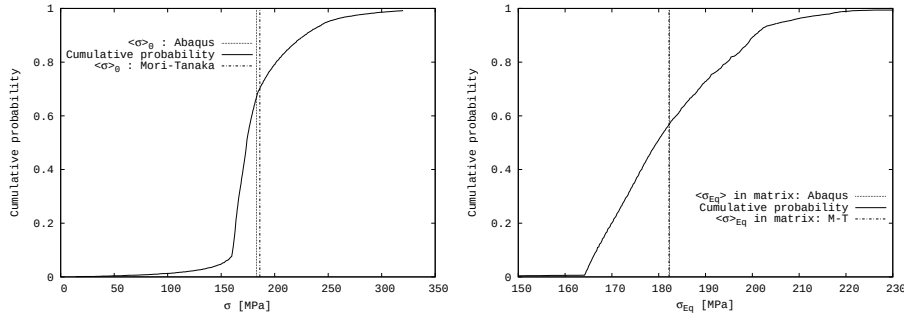


Figure 7.10: MMC with 10% of ceramic spheres under tensile: Axial stress σ_{22} and von Mises Equivalent stress σ_{Eq} fluctuations in the matrix phase.

tively and $Ivol$ the integration point volume. Note that $\langle \sigma \rangle_{eq} \neq \langle \sigma_{eq} \rangle$. Examination of Figure 7.10-a clearly indicates that stress fluctuations around $\langle \sigma_{22} \rangle > 0$ (predicted using both Abaqus and our homogenization code) are quite important. Figure 7.10-b leads to the same conclusion for σ_{eq} fluctuations.

Concerning the inclusion phase, since it behaves elastically it is enough to only analyze axial strain and stress fields respectively. In Figure 7.11-a is plotted the cumulative probability versus the axial strain in the inclusion phase. It is also reported on the figure the inclusion average strain obtained using both F.E. analysis and our homogenization code. The dispersion of the cumulative probability around the inclusion average strain provided by F.E. computation, clearly indicates that ϵ_{22} fluctuates in significant manner within the inclusion phase. Examination of Figure 7.11-b leads to the same conclusion concerning stress σ_{22} fluctuation.

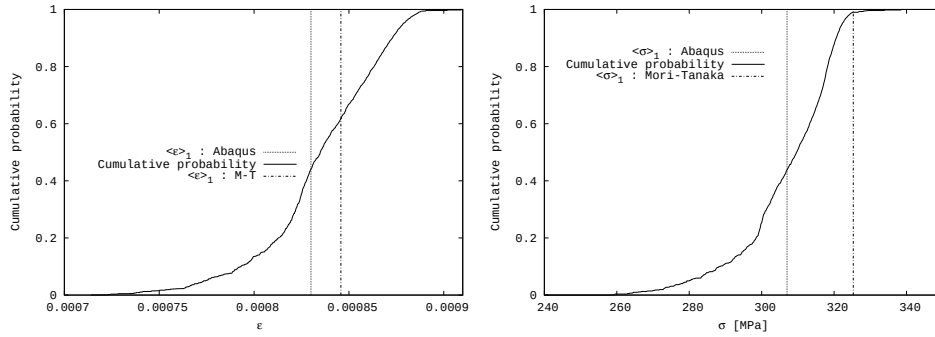


Figure 7.11: MMC with 10% of ceramic spheres under tensile: ϵ_{22} and σ_{22} fluctuations in the inclusion phase.

However the analysis of per-phase average stress-strain curves displayed in Figure 7.12 seems to indicate that the average stress-strain curve predicted by our code is much closer to F.E. for the matrix phase than those for the inclusion one. This observation is somewhere in contradiction with the proposed argument to explain the fact that the predicted response by our model is overestimated for the plastic part. Figure

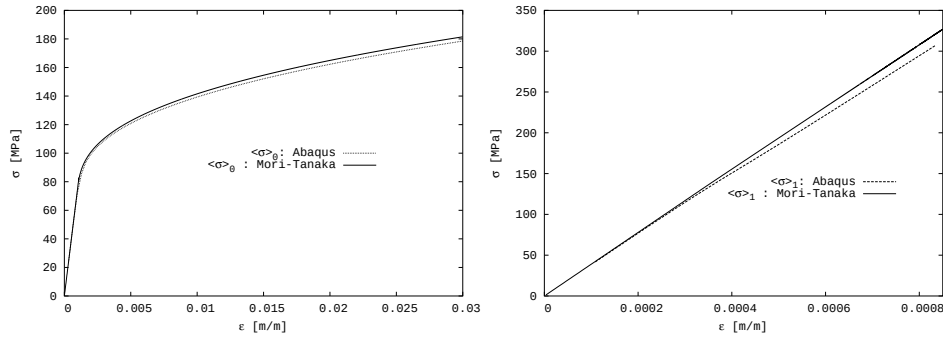


Figure 7.12: MMC with 10% of ceramic spheres under tensile: Per-phase average strain-stress curve.

7.13 renders the stress and strain fields fluctuations drawn under isocontour form on the unit cell UC2 ($H/R = 2.5$) for the MMC with 10% of ceramic spheres.

The previous MMC with 30% of inclusions was also studied by other researchers (Suquet [98] with a modified secant formulation) who used other formulations to simulate its response under tensile loading test. The F.E. simulation is obtained using the 2D unit cell UC1. The various results are plotted in Figure 7.14. It can be shown that the

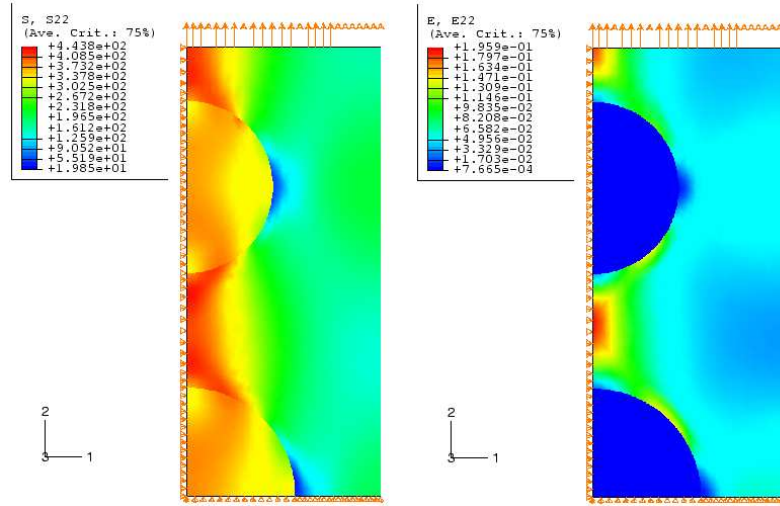


Figure 7.13: MMC with 10% of ceramic spheres under tensile: Stress and strain fields over UC2 unit cell ($H/R = 2.5$).

secant total formulation gives prediction that is very close to F.E. one for the plastic part of the response, while our incremental formulation seems to underestimate it. In general, a good agreement is shown between all simulations.

In a next example, we simulated a tension/compression cyclic loading test of the same MMC as that for the previous examples (with 20% of inclusions). Our prediction is compared with F.E. computations in Figure 7.15 and a good agreement is shown between the two sets of results.

Macro shear test

In order to illustrate the capability of our code and the robustness of its algorithms, we simulated the response of the previous MMC under shear test. The predicted shear-stress response under a prescribed macro shear strain $\bar{\epsilon}_{12} = 0.03$ for MMC with various volume fractions of reinforcement (10, 20, and 30%) are plotted in figure 7.16.

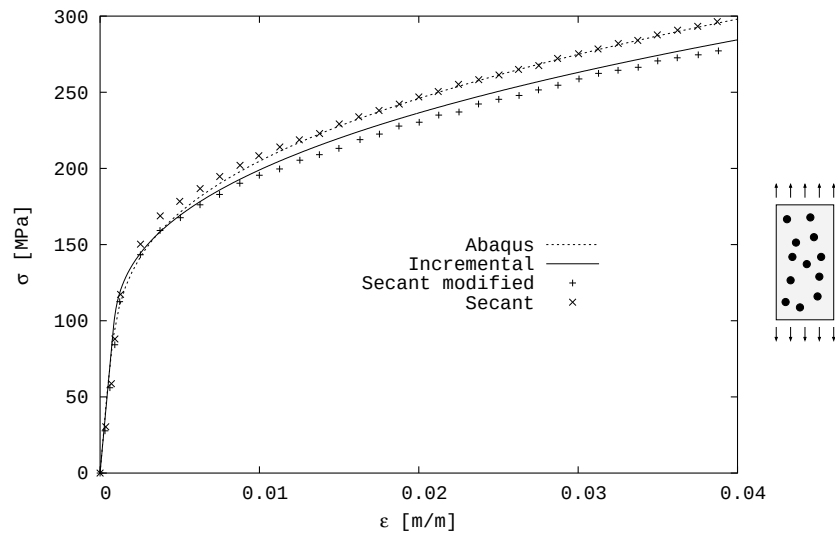


Figure 7.14: MMC with 30% of ceramic spheres under tensile loading test: Incremental, secant, secant modified formulations, and F.E. analysis

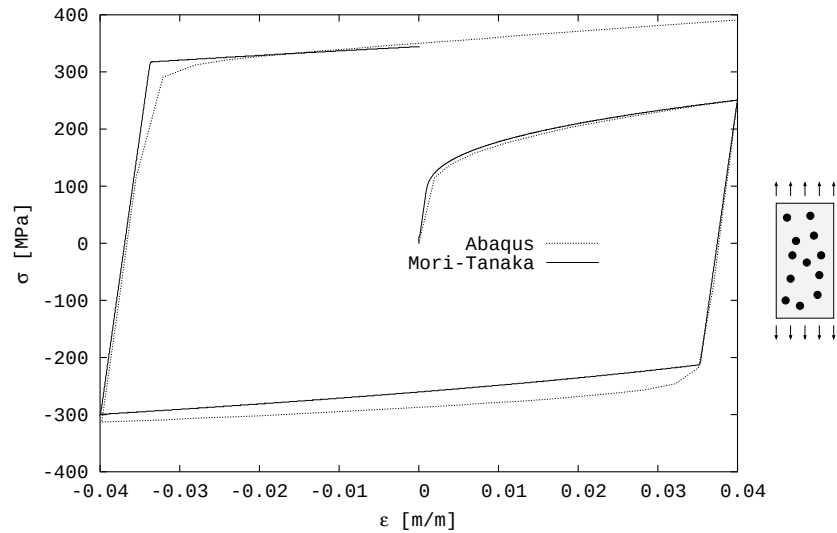


Figure 7.15: MMC with 20% of ceramic spheres under cyclic loading test: Mori-Tanaka against unit cell UC1 finite element predictions.

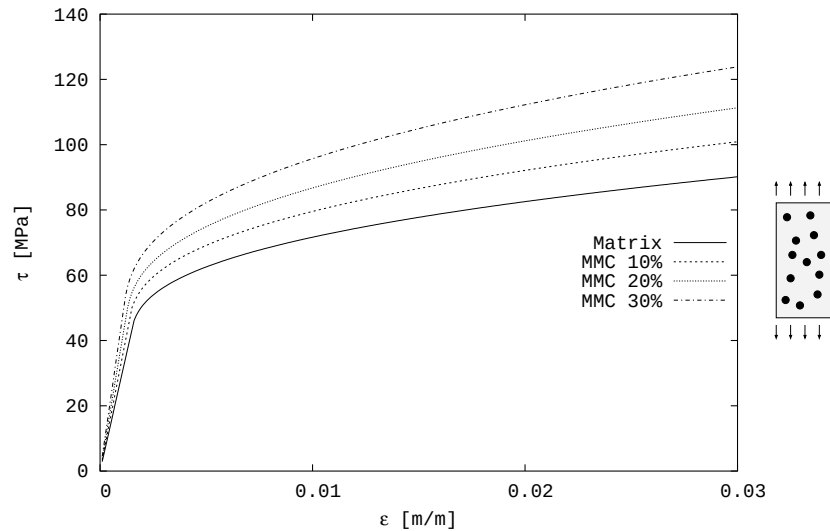


Figure 7.16: MMC with various volume fraction of ceramic spherical inclusions under shear test: Predicted Mori-Tanaka shear stress-strain curves

Inclusion aspect ratio influence

The aim of the following example is to study the influence of reinforcement shape on the response of MMC. The same MMC (with 30% of inclusions) as the previous examples is considered with various values of the aspect ratio of the reinforcement ($A_r = 1, 5, 10$ and 100) aligned with the loading axis. Figure 7.17 displays predicted stress strain curves under tension tests. It can be deduced that the higher the value of A_r is, the stiffer the response of the composite is.

Material hardening parameter influence

Now we wish to evaluate the impact of the parameter m of the hardening function on the stress-strain relationship of the composite. Two values of matrix hardening exponent m are considered: (i) $m = 1$ which corresponds to a linear hardening and (ii) $m = 0.3895$ which corresponds to a nonlinear hardening. The simulated tensile stress-strain responses are plotted in Figure 7.18. One can see a good agreement between our code estimates and F.E ones. It can be shown also that the composite with $m = 0.3895$ is stiffer than the composite with $m = 1$. This result can be explained as follows: at the microscale, (1) in the matrix phase the accumulative plastic strain p is smaller than 1,

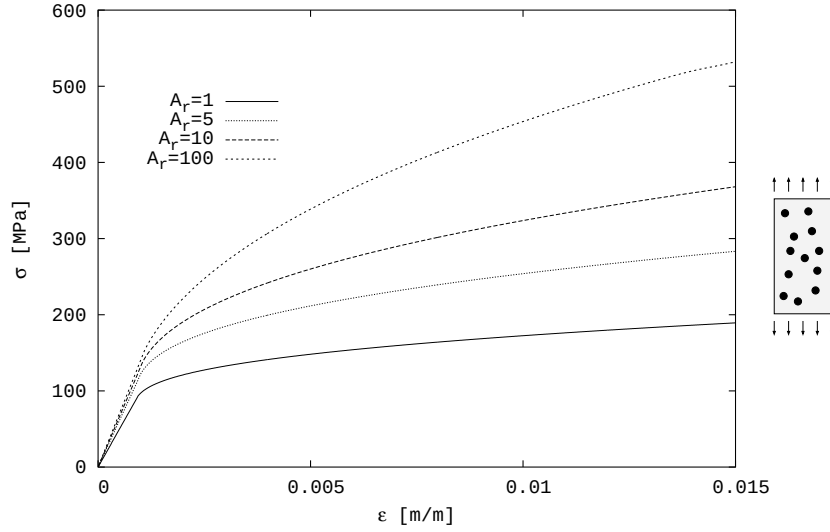


Figure 7.17: Influence of shape reinforcement on MMC response under tension.

the quantity p^m is larger for $m = 0.3895$ than for $m = 1$, consequently it follows that the stress is higher for $m = 0.3895$ than for $m = 1$ since $\sigma = \sigma_Y + kp^m$. (2) in the two considered composites, the inclusions behave elastically, one can then assume that the trend composite stress trend is the same that those of matrix phase.

A three-phase MMC under tension

As a last example in this section, we simulated tensile response of a fictitious three-phase composite consisting of an Aluminum alloy matrix reinforced with 10% of ceramic spheres and 5% of steel spheres. Steel material is assumed elastic with $E=210$ GPa and $\nu = 0.3$. F.E. computation is performed on UC2 with $H/R = 2.5$ in which ceramic and steel inclusions are located as displayed in Figure 7.19. Two homogenization schemes are used: (i) two-step Mori-Tanaka/Voigt homogenization scheme and (ii) Direct Mori-Tanaka scheme, see appendix A. As displayed in Figure 7.20 a very nice agreement can be seen between the three sets of the predicted macro-tensile responses.

At the micro-scale, the per-phase responses are plotted in Figure 7.21 for the matrix phase and in Figure 7.22 for ceramic and steel inclusions phases respectively. A very good agreement is shown between our prediction and F.E. computations for the matrix phase, while for ceramic and steel phases our code overestimates responses comparing

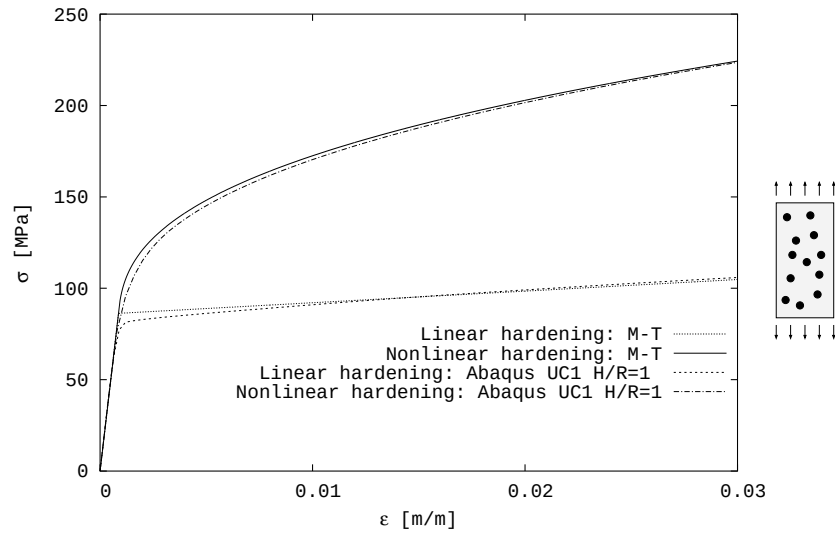


Figure 7.18: MMC with linear hardening $m = 1$ and nonlinear hardening $m = 0.385$: Mori-Tanaka predictions versus F.E. ones

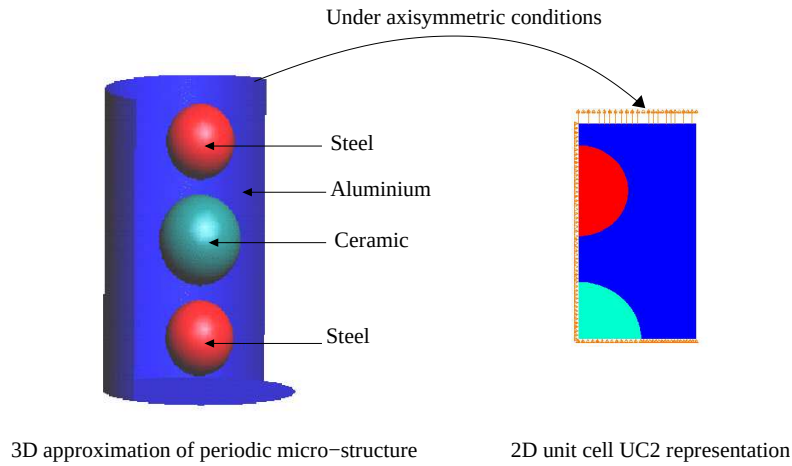


Figure 7.19: MMC three-phase composite with periodic micro-structure: 3D and 2D representation.

to F.E. ones. The analysis of axial strain ϵ_{22} fluctuations within each phase leads to surprising and interesting remarks. In Figures 7.23 and 7.24 is plotted for each

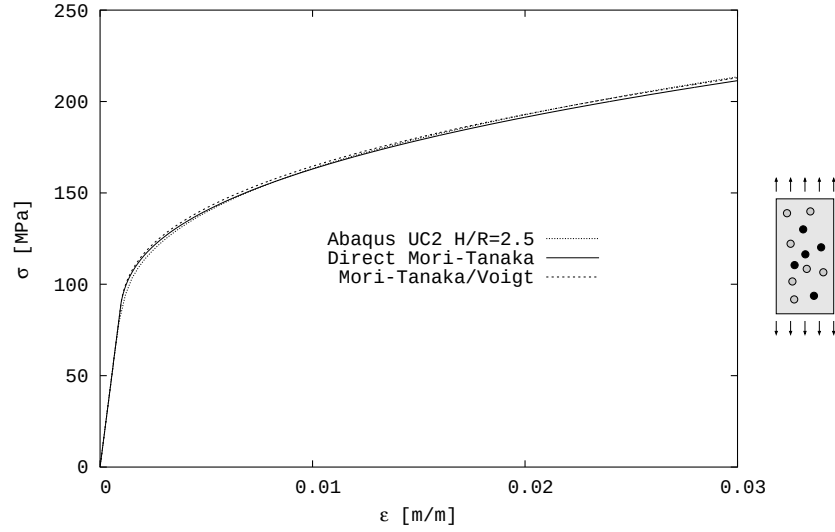


Figure 7.20: Tensile response of MMC reinforced with 10% of ceramic spheres and 5% of steel spheres: The two-step Mori-Tanaka/Voigt scheme versus Direct Mori-Tanaka scheme and F.E. analysis predictions respectively.

phase the cumulative probability versus ϵ_{22} in the considered phase. Examination of those figures clearly indicates that strain fluctuations for the three phases is important, however in the matrix this fluctuation varies in larger proportions. This results suggests that our code should overestimate the matrix phase response which is not the case, see Figure 7.21. In figure 7.25 one can find, mapped on UC2, the stress and strain fields fluctuations under isocontour form..

Eshelby and Hill's predictions

As viewed in section 4.4 of chapter 4, one can use Hill's tensors to formulate the homogenization problem based on Eshelby's Equivalent Inclusion. In the following example we consider the previously studied two-phase MMC with $v_1 = 30\%$ and $Ar = 10$ submitted to a tensile loading test. Simulations are performed using P tensor computed as follows:

(1) First way:

$$P_{I, \hat{C}_0^{alg}} = S_{(v_t, A_r)} : C_0^{alg}; \quad (7.3)$$

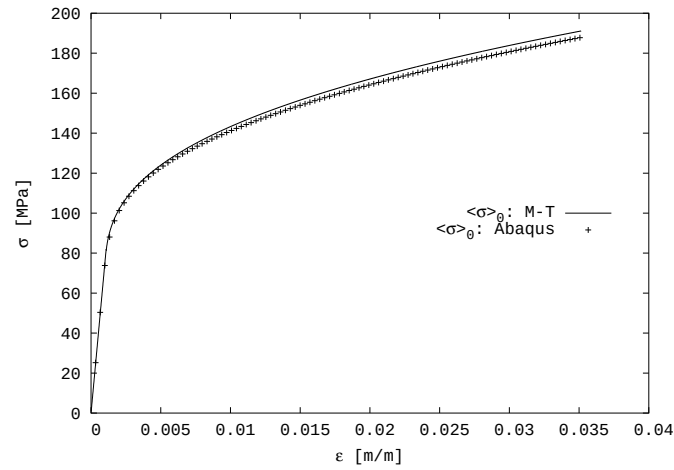


Figure 7.21: Matrix stress-strain curve for a three-phase MMC reinforced with 10% of ceramic spheres and 5% of steel spheres under tension.

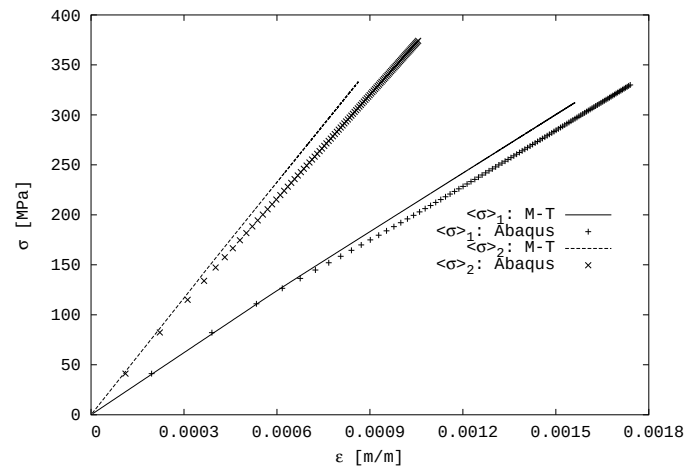


Figure 7.22: Ceramic and steel phases stress-strain curves for a three-phase MMC reinforced with 10% of ceramic spheres and 5% of steel spheres under tension.

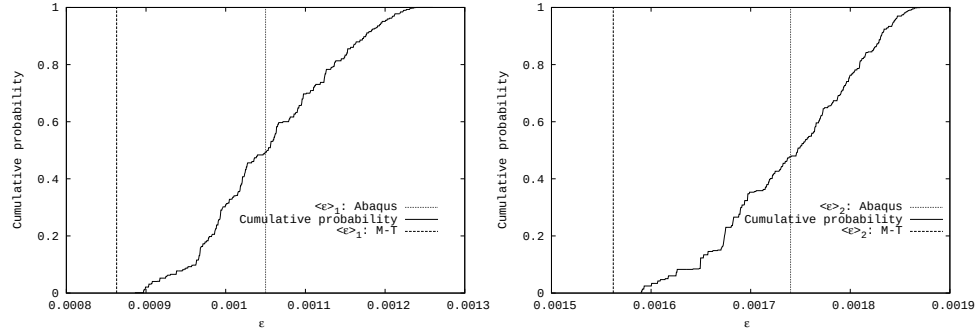


Figure 7.23: Axial strain ε_{22} fluctuation in ceramic and steel inclusions phases of a three-phase MMC reinforced with 10% of ceramic spheres and 5% of steel spheres under tension.

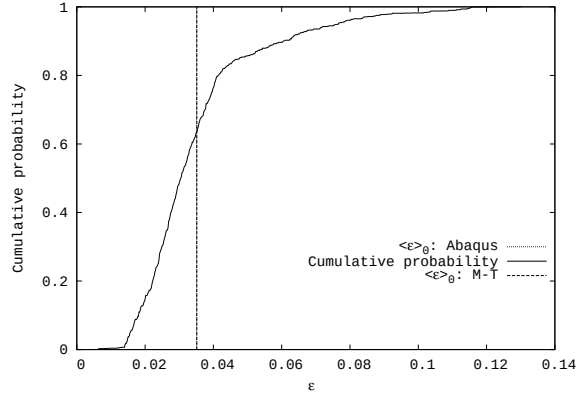


Figure 7.24: Axial strain ε_{22} fluctuation in the matrix phase of a three-phase MMC reinforced with 10% of ceramic spheres and 5% of steel spheres under tension.

(2) Second way:

$$P_{I, \hat{C}_0^{Iso}} = S_{(v_t, A_r)} : C_0^{Iso}. \quad (7.4)$$

where C_0^{Iso} is the isotropic part of C_0^{alg} computed using Eq.(4.40). In Figure 7.26 are reported the simulated stress-strain curves of the composite using Eshelby's and Hill's tensors. One can see that simulation carried out using $P_{I, \hat{C}_0^{Iso}}$ computed with Eq.(7.4) gives a too stiff estimate compared to those obtained using $P_{I, \hat{C}_0^{alg}}$ which is exactly similar to Eshelby's response. If the load is increased, the stress-strain curves computed

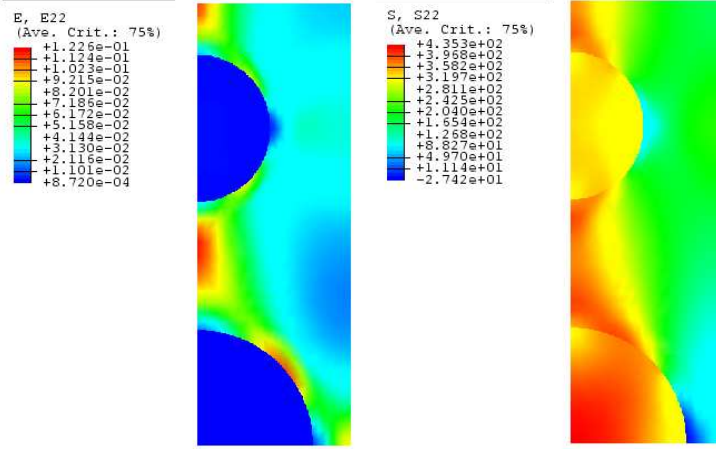


Figure 7.25: Stress and strain fields repartition over UC2: Three-phase MMC reinforced with 10% of ceramic spheres and 5% of steel spheres.

with both $S_{I,\hat{C}_0^{alg}}$ or $P_{I,\hat{C}_0^{alg}}$, diverge as seen in Figure 7.27. This is explained by the fact that $P_{I,\hat{C}_0^{alg}}$ computed with (1) is not necessary symmetric (idem for Eshelby's tensor) and leads to a non-symmetric overall nonlinear operator of the composite. While the use of $P_{I,\hat{C}_0^{Iso}}$ computed with C_0^{Iso} ensures the symmetry of $\bar{C}^{alg}$.

7.4.2 Porous Metal Matrix

In this section we first focus our interest on porous metal matrix which we model as a two-phase composite where the aluminum alloy matrix represents one phase and voids represents the other one. Simulations are achieved using both our homogenization code and Abaqus F.E. software. The F.E. computations are performed using unit cells UC1 and UC2 displayed in Figures 6.3 and 6.4 of chapter 6.

In figures 7.28, 7.29, and 7.33 are illustrated tensile stress-strain curves of porous aluminum alloy matrix with various volume fractions voids 5%, 10%, and 15% respectively. For F.E. computations, stresses are computed as follows;

$$\langle \sigma \rangle = \frac{\sum_{i=1}^{N_{mn}} RF_i}{A}, \quad (7.5)$$

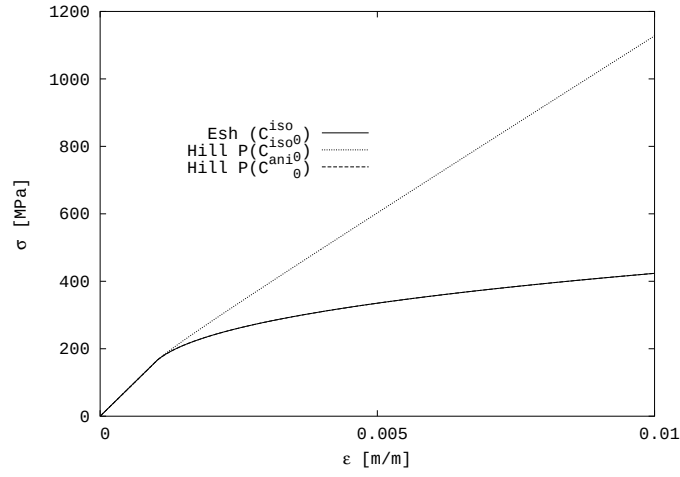


Figure 7.26: Eshelby's estimate against Hill's one: MMC with 30% of aligned ceramic fibers with $Ar=10$

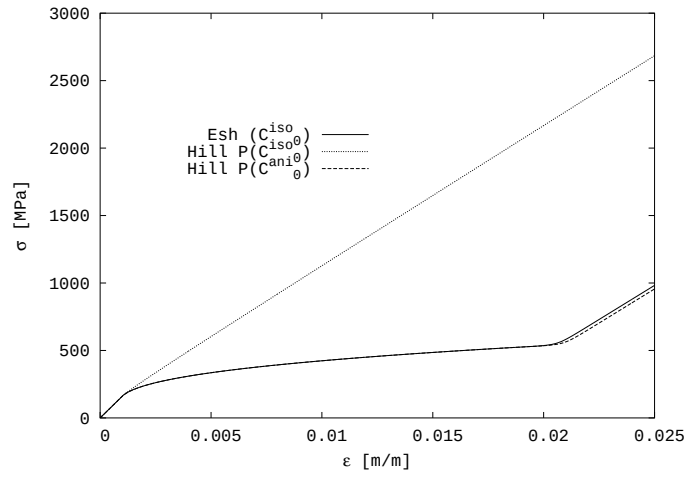


Figure 7.27: Eshelby's estimate against Hill's one: MMC with 30% of aligned ceramic fibers with $Ar=10$

where N_{mn} is mesh nodes number, RF the reaction force, and A the section area. In Figure 7.29, for instance, it can be shown that UC1 and UC2 F.E. results computing stress with $\sum RF$ are very close while our M-T homogenization code tends to give predictions that are stiffer, in particular for the plastic part of the response. In general, it seems

that the margin existing between M-T predictions and F.E. ones does not increase with porous volume fraction increase. Let us now study the fluctuation of the various fields

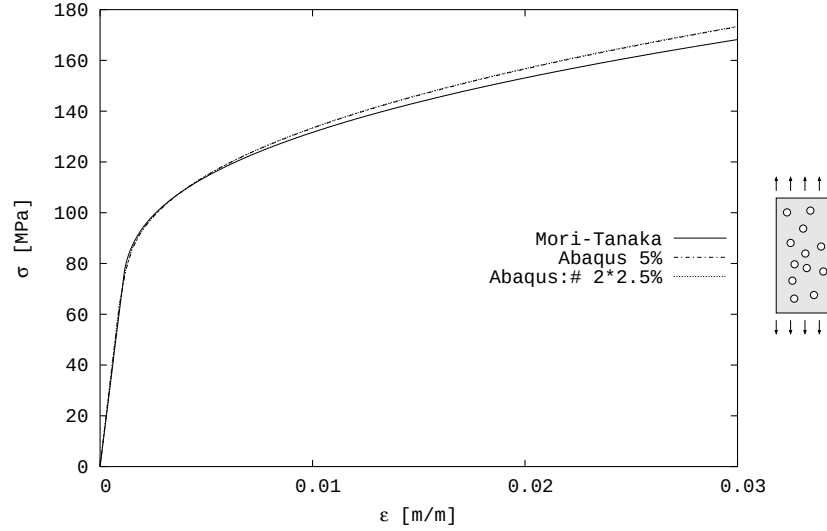


Figure 7.28: Stress-strain response of porous metal matrix with 5% of voids under tension: Mori-Tanaka predictions versus various unit cell F.E. computations

acting on the porous metal matrix. Examination of Figure 7.30 indicates that more than 70% of the matrix phase undergoes an axial strain ϵ_{22} ranged between 2 and 3% with a maximal fluctuation of ϵ_{22} around the average strain value $\langle \epsilon_{22} \rangle_0 = 2.7\%$ in matrix phase predicted by F.E. reaching 30%. Concerning the maximal fluctuation of σ_{22} around the F.E. predicted average stress value $\langle \sigma_{22} \rangle_0 = 2.7\%$, Figure 7.31-a shows that it is close to 30% for more than 80% of the matrix phase. Figure 7.31-b indicates that in a large part of the matrix phase, the fluctuation of the von Mises equivalent stress σ_{eq} around the average von Mises equivalent stress $\langle \sigma_{eq} \rangle$ computed using Abaqus does not exceed 15%. In general, the three investigated fields ϵ_{22} , σ_{22} and σ_{eq} displayed important fluctuations. Axial stress and displacement fields fluctuations along loading axis for aluminum matrix with 10% of voids are illustrated in Figure 7.32 under isocontour form.

In a second example we consider a three-phase porous metal matrix composite with 5% of voids, and reinforced with 5% of ceramic spheres. F.E. computations are performed using UC2 with two possible arrangements of ceramic spheres and voids as displayed in Figure 7.34. A uniaxial tensile test is simulated and results are plotted in Figure 7.35. The F.E. results suggest that the proposed arrangements of voids and solids inclusions inside the composite do not influence in a significant manner the average response of the composite. Note also that predictions made by our code is close enough to F.E. ones.

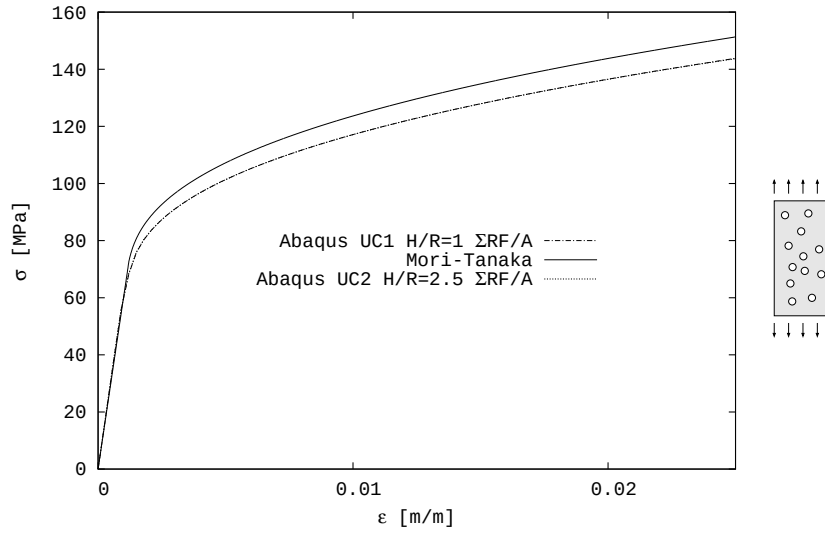


Figure 7.29: Stress-strain response of porous metal matrix containing 10% of voids under tension: Mori-Tanaka predictions versus UC1 and UC2 unit cells F.E. computations

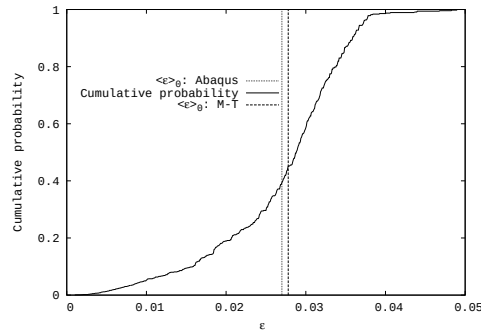


Figure 7.30: Metal Matrix with 10% of spherical voids under uniaxial tension: ε_{22} fluctuation over the matrix phase.

7.5 Polymer Matrix Composites

This section is devoted to numerical simulations of a class of Polymer Matrix Composites (PMC). In all following examples, we consider two constitutive materials:

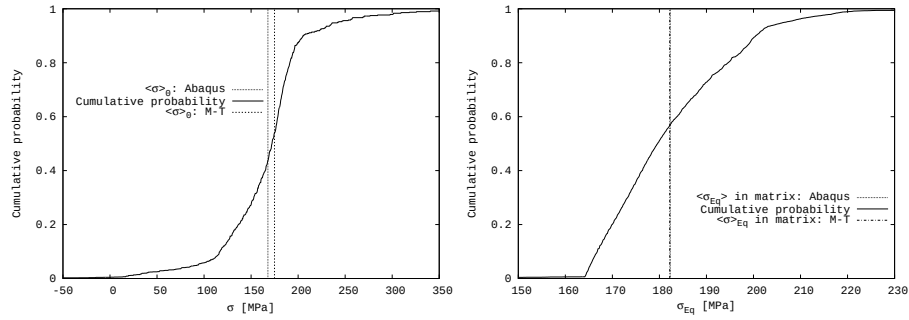


Figure 7.31: Metal Matrix with 10% of spherical voids under uniaxial tension: Axial stress σ_{22} and σ_{eq} von Mises equivalent stress fluctuations over the matrix phase.

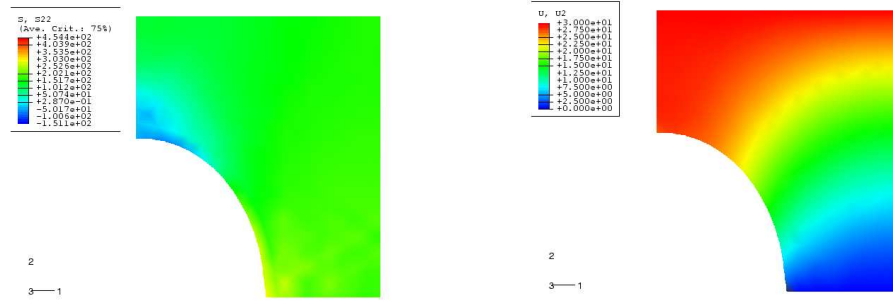


Figure 7.32: Aluminum matrix with 10% of voids: Stress σ_2 and displacement u_2 fields repartition over UC1

- (1) Epoxy material assumed elastoplastic and modeled with J_2p , with the following mechanical properties: $E = 3.16$ GPa, $\nu = 0.35$, the yield stress $\sigma_Y = 75.86$ MPa, with the power-law isotropic hardening $R(p) = kp^m$ where $k = 32.18$ MPa and $m = 0.26$.
- (2) Aluminum material assumed to behave elastically with the following elastic properties: $E = 73.1$ GPa and $\nu = 0.18$.

Consider a two-phase composite made of a polymer matrix (epoxy) containing spherical silica inclusions. In a first example, the uniaxial-tensile response of specimen with two values of volume fractions of silica spheres ($v_i = 15\%, 35\%$) is simulated with our Mori-Tanaka homogenization code and using Abaqus F.E. software. The F.E. computations were performed on a unit cell UC1 meshed using axisymmetric CAX8R elements. The results of the two sets of simulations are plotted in Figures 7.36-7.37. In general, a fair agreement between the two prediction methods is shown. Therefore, it should be interesting to notice that for Mori-Tanaka simulations, both elastic and plastic parts are not accurately predicted. This is mainly due to the very

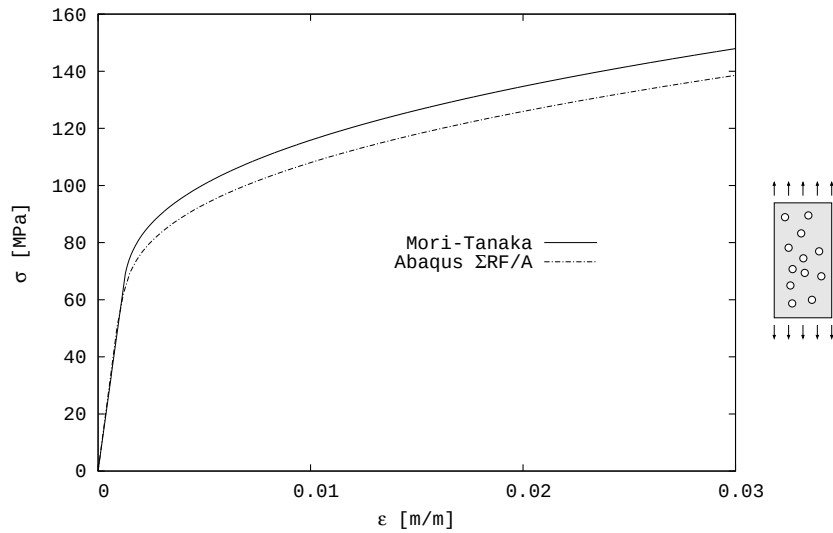


Figure 7.33: Stress-strain response of porous metal matrix containing 15% of voids under tension: Mori-Tanaka predictions versus various unit cell F.E. computations

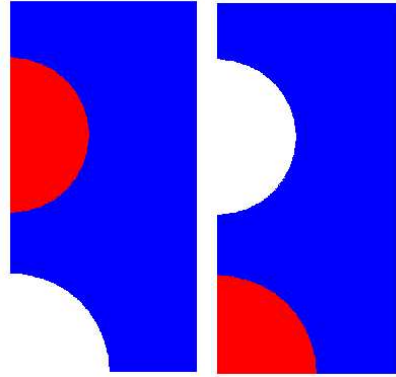


Figure 7.34: Three-phase porous metal matrix composites with two possible arrangements of the periodic microstructure

high stiffness contrast between the matrix and the reinforcement. Indeed, the stiffness of the silica is more or less 23.1 times higher than the epoxy polymer matrix one, and this could lead to important strain (or other field) fluctuations in each phase (or between phases) which are not accounted for in our incremental formulations. To verify this argument let us analyze fluctuations of ε_{22} , σ_{22} , and σ_{eq} over each phase.

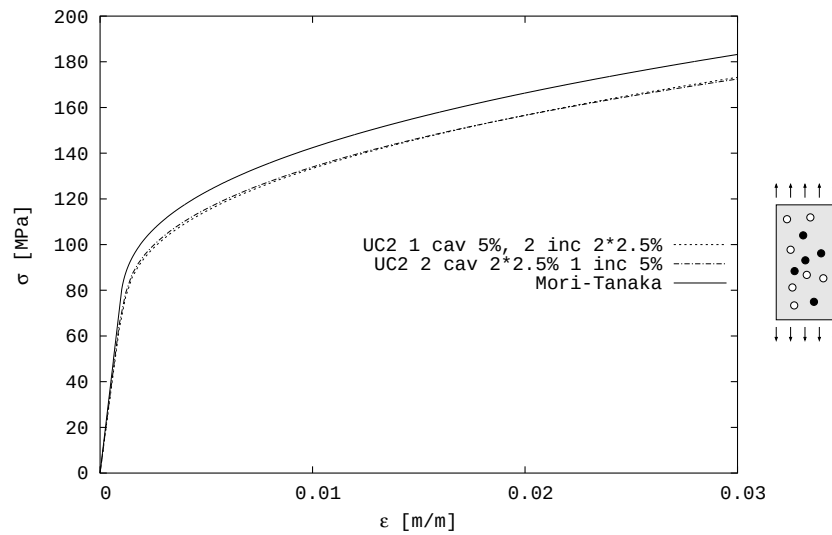


Figure 7.35: Stress-strain response of metal matrix composite reinforced with 5% of carbon sphere and containing 5% of voids under tension: Mori-Tanaka versus F.E. estimations

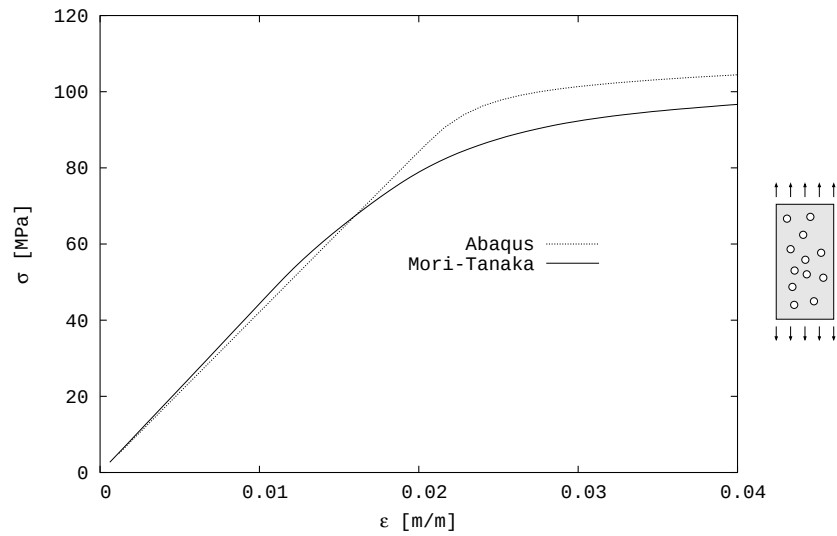


Figure 7.36: PMC with 15% of silica spheres under tensile loading test: Mori-Tanaka prediction versus unit cell finite element computations

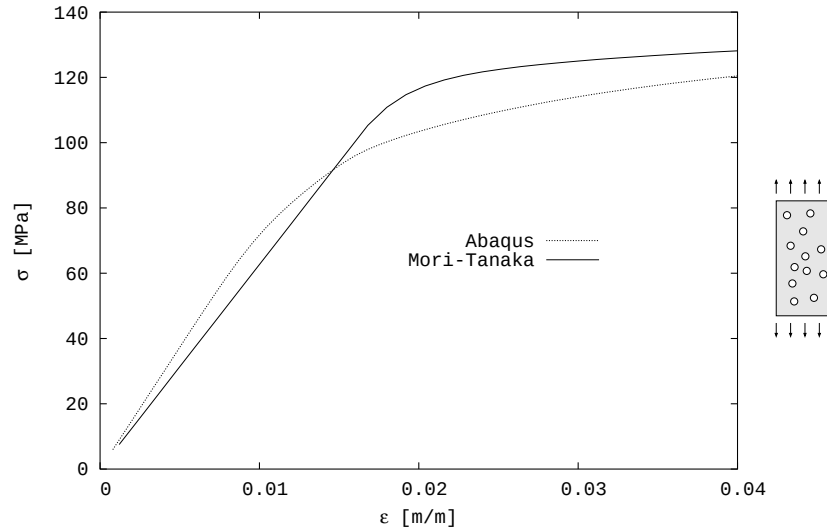


Figure 7.37: PMC with 35% of silica spheres under tensile loading test: Mori-Tanaka predictions versus unit cell finite element computations.

First, focus our interest on what happens on the matrix phase. Figure 7.38 indicates that more than 90% of the matrix phase undergoes an axial strain ϵ_{22} varying between 2 and 7% with a maximal fluctuation of ϵ_{22} around the average matrix strain value $\langle \epsilon_{22} \rangle_0 = 4.7\%$ predicted by F.E. reaching 49%. Note also that for more than 90% of the matrix phase the maximal fluctuation of σ_{22} around the F.E. predicted average stress value $\langle \sigma_{22} \rangle_0 = 86.6\text{ MPa}$ can reach 21%, as displayed in Figure 7.39-a. Concerning σ_{eq} , Figure 7.31-b shows that in a large part of the matrix phase (more than 90%), the fluctuation of the von Mises equivalent stress σ_{eq} around the F.E. predicted average von Mises equivalent stress $\langle \sigma_{eq} \rangle$ does not exceed 15%. For the inclusion phase, since it behaves elastically only axial strain and stress fields respectively are investigated. In this case, examination of the Figure 7.40 indicates that ϵ_{22} and σ_{22} fluctuations are less important in the inclusion phase than those in matrix phase. More than 90% of the inclusion material undergoes an ϵ_{22} varying within a range of 0.16% to 0.21% which leads to a maximum fluctuation around the F.E. predicted value of the average axial strain ($\langle \epsilon_{22} \rangle_1 = 0.192\%$) reaching 10%. In the inclusion phase, as reported by Figure 7.40-b the axial stress acting on more than 90% of the inclusion material, is varying between 120 MPa and 145 MPa which leads to a maximal fluctuation reaching 9% around the predicted F.E. average axial $\langle \sigma_{22} \rangle_1 = 133\text{ MPa}$.

In a last example we simulated a tension-compression loading-unloading cyclic test of the previous PMC with 42% of silica sphere inclusions. Figure 7.41 illustrates our M-T prediction compared with F.E. one and a fair agreement is shown between the

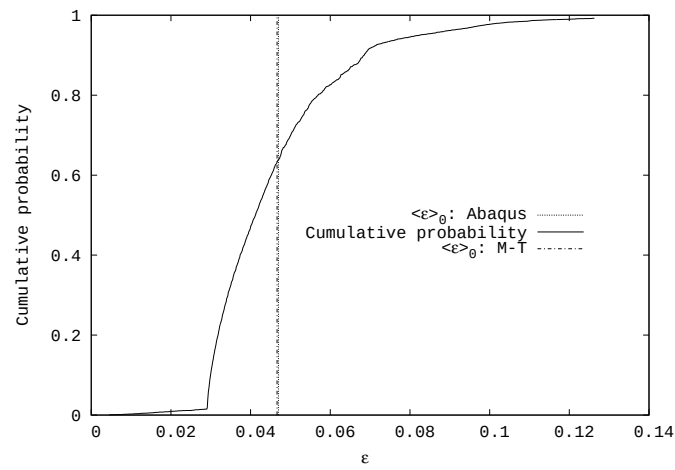


Figure 7.38: PMC with 15% of silica spheres under tensile: Axial strain ϵ_{22} fluctuations over the matrix phase.

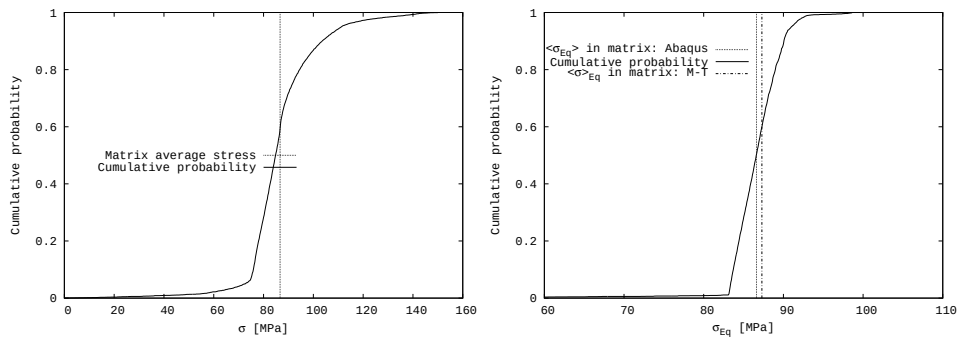


Figure 7.39: PMC with 15% of silica spheres under tensile: ϵ_{22} and σ_{22} and σ_{eq} fluctuations over the matrix phase.

two sets of results.

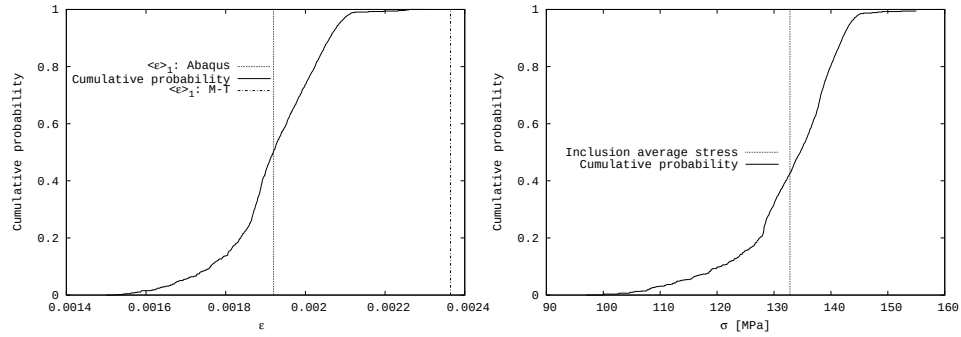


Figure 7.40: PMC with 10% of silica spheres under tensile: ϵ_{22} and σ_{22} fluctuations in the inclusion phase.

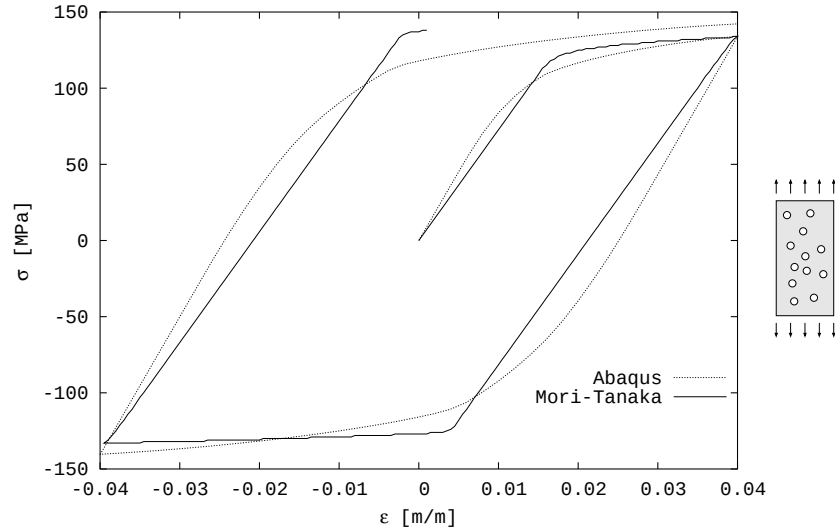


Figure 7.41: PMC with 42% under cyclic loading test: Mori-Tanaka and Finite element stress-strain curves

7.6 Conclusion

In this chapter, we proposed a micromechanical-based model and its corresponding algorithms which enables the simulation of the mechanical response of a class of elastoplastic matrix composites. The proposed model is based on a two-step Mori-Tanaka/Voigt homogenization scheme to handle the modeling of elastoplastic multi-

phase composites with matrix displaying an elastoplastic behavior and where phases can either be solid inclusions or cavities. Both Eshelby's and Hill's formulations were investigated and taken into account into our code. We simulated the mechanical behavior of both Metal Matrix Composites and Polymer Matrix Composites under several discriminating tests: monotonic tension, unloading and cyclic loadings. The accuracy of the proposed approach was validated against both direct finite elements computations performed using Abaqus software and available results in the literature. For F.E. simulations we considered a periodic microstructure represented by various types of 2D (and 3D) unit cells. In many cases, our predictions show a fair agreement with both F.E. simulations and results available in the literature. However, in general, it appears from the above simulations that our homogenization code lead to estimates that are stiff compared with F.E. ones. As many researchers interested with this topic, we thought that the main reason of this, is that tangent moduli, which in theory should vary in space, are assumed to be uniform and computed with the average strain of the considered, leading to overestimated mechanical properties in cases of high strain and stress fields fluctuations. However, per-phase strain and stress fluctuations analysis did not show clear tendencies to support this argument.

Chapter 8

Conclusion

In this thesis we proposed a micromechanical model for a class of rate-independent multi-phase composites. As applications, we focused our interest in particular on Steel Fiber-Reinforced Concrete (SFRC) and on a class of Elasto-Plastic Matrix Composites (EMC). Experimental (only for SFRC), modeling, and simulation aspects of the study of those composites were investigated. Under the lights of these results, several interesting remarks and conclusions can be made.

8.1 Micromechanical modeling

Based on the so-called Hill's incremental formulation, we proposed an extension of some homogenization schemes to the nonlinear regime. The problem of nonlinear homogenization was investigated within the framework of the mean-field approach. We derived the overall expression of the nonlinear tangent operator for a two-phase composite and pointed out the difficulties encountered during homogenization and averaging procedures. The homogenization of the nonlinear composites was tackled using the concept of Linear Heterogeneous Comparison Material which is based on two main ideas:

- (1) the linearization of the per-phase constitutive law and,
- (2) space uniformization of the intra-phase mechanical characteristic such as stiffness operator.

The linearization was done using tangent operators and the per-phase uniformization of the mechanical properties is achieved by computing tangent operators with the average strain of the considered phase. The crucial issue of Eshelby's tensor computation was discussed. The proposed incremental formulation was extended to handle homogenization of nonlinear multi-phase composites.

8.2 Constitutive models

In this work, two models were selected to describe the mechanical behavior at the micro-level. The first one is Ju's energy-based damage model for concrete-like material. In this model, it is assumed that damage takes place only under tensile strains which are extracted using spectral decomposition and positive projection tensors. The second one is J_2 plasticity model to account for history and rate-independent effects, and strain hardening material behavior typically exhibited by steels and solid polymers. Those two constitutive models have made possible the modeling of a large range of composite materials (concrete-matrix composites, metal-matrix composites and polymer-matrix composites) under the infinitesimal strain hypothesis.

8.3 Experimental study

Four-point bending tests were performed on several beams made of Plain concrete, Steel Fiber-Reinforced Concrete (with 1%, 2%, 3%, and 4 % of fibers volume fraction), and Steel Bar-Reinforced Concrete. The bending test results were collected under load-deflection curves and were transformed onto stress-strain curves using an incremental method which we developed for materials exhibiting nonlinear behavior. This test campaign has provided data for Ju's model parameters fitting (tests on plain concrete specimen) and data for our numerical simulations validation (tests on SFRC specimens). This experimental study has also allowed a better understanding of the physical and mechanical aspects which govern the behavior of SFRC in order to achieve the correct modeling of the composite.

The analysis of experimental results has emphasized the beneficial effects of fibers addition in concrete material. Indeed, it was observed that some mechanical properties of SFRC such as flexural strength, ductility... etc were improved. Note however that composite' stiffness has not been increased significantly. This observation led to the crucial finding that initial voids and microcracks coalescence due to damage process, and porosity increase due to fiber addition must not be confused even if both cause

a decrease of the composite's stiffness. Assuming that concrete material including initial voids and microcracks represents one phase (the matrix) and fibers the other one (reinforcement phase), when those two phases are mixed in order to obtain Steel Fiber-Reinforced Concrete, the initial porosity of the matrix phase increases in a significant manner. This observation has a major consequence for micromechanical modeling of SFRC composites. Indeed, this porosity increase is viewed as a full phase and therefore SFRC is modeled as a three-phase composite. The first phase consists in concrete material, initial voids and microcracks, the second phase represents steel fibers while the third one represents the increase in porosity. In parallel to the bending tests, non-destructive Grindo-Sonic tests were conducted in order to validate the first campaign tests results.

8.4 Numerical simulations

The efficiency and the accuracy of the proposed nonlinear homogenization model were evaluated through several numerical simulations. Algorithms, based on the two-step Mori-Tanaka/Voigt homogenization schemes, enabling the simulation of Concrete-Matrix Composites (CMC) and Elastoplastic-Matrix Composites (EMC) were presented and implemented. Several discriminating tests of CMC and EMC were simulated using two different ways:

- (1) Finite element computations performed on two different unit cells representing composites with periodic microstructure.
- (2) Our homogenization code.

For the simulation of realistic structures with complex microstructures we developed a two-scale approach. This method consists in connecting our homogenization module with Abaqus via its UMAT option so that, at each integration point, Abaqus sees the micro-structure through the information (stresses and tangent moduli) provided by our code.

Numerical simulations presented in chapters 6 and 7, show that in some cases our model leads to overestimated predictions. Whether for elasto-damaged matrix composites or elastoplastic matrix composites, our estimates are stiffer than the F.E. ones which are considered as the target results. As many researchers in the field, we thought that this overestimation is probably due to the fact that our model does not account for intra-phase heterogeneity. However, per-phase stress and strain fluctuation investigation showed that there is not a direct link between the observed overestimation of predictions made using homogenization model and the intra-phase heterogeneity.

In general numerical results showed that the proposed nonlinear homogenization model is able to predict accurately enough the behavior of both damage-matrix and elastoplastic-matrix composites. It was also shown that the method reduced considerably the modeling and calculation time.

8.5 Summary of the contributions

The main contributions of the thesis can be summarized in the following points:

1. We proposed an incremental formulation and the corresponding algorithms which enable, within reasonable CPU time and memory usage the simulation of multi-phase composites with matrices displaying elastoplastic or elastic-damaged behaviors.
2. We conducted an experimental study consisting in four-point bending tests performed on several samples of both Plain concrete and Steel Fiber-Reinforced Concrete (SFRC) with various volume fractions of fibers. The study has:
 - Provided a better understanding of the physics and the mechanics which govern the deformation behavior of Steel Fiber-Reinforced Concrete Composites in order to achieve an appropriate modeling.
 - Provided experimental data for Ju's model parameters fitting (for plain concrete material) and for our numerical predictions validation.
 - Provided Stress-strain curves of concrete and SFRC specimens in tension.
 - Emphasized the positive effect of fiber addition in concrete material.
3. We developed a simple and original incremental method enabling the transformation of load-deflection curves onto stress-strain ones when the bent sample displays a nonlinear behavior. This method was implemented giving rise to a useful tool.
4. As suggested by the experimental study, SFRC was modeled as a three-phase composite where the first phase represents concrete material, initial voids and microcracks, the second phase is fibers, while the third one represents porosity increase due to fiber addition.
5. We simulated the response of several classes of nonlinear composites under several discriminating tests (compression, tension and cyclic loadings) using our homogenization code.

6. We carried out several F.E. simulations on SFRC and elastoplastic matrix composites (EMC) represented with two types of unit cells.
7. We integrated Ju's damage model to the Abaqus F.E. software through its User MATerial option (UMAT) to enable the simulation of the anisotropic brittle behavior of concrete-like materials.
8. We also integrated our homogenization module into Abaqus via its UMAT option in order to simulate the behavior of realistic structures made of nonlinear composites.

8.6 Future prospects

As future developments, many things remain to be done. Some of the interesting ideas that might be investigated are summarized below.

- In the experimental study some points should be improved: (1) The use of appropriate strain gages in four-point bending tests; (2) a better measurement of porosity increase due to fiber addition in concrete mix; (3) perform other bending tests with other fiber types; and (4) a better measurement of fiber orientations in SFRC specimens.
- The improvement of the numerical implementation of Ju's damage model. For instance: (1) The numerical computation of the tangent operator must be improved using second order perturbation method; (2) To make a better choice of the damage evolution function H , (3) In Finite element analysis, convergence problems due to strain localization during the softening regime must be investigated. Regularization techniques such as relative viscosity or characteristic length must be introduced in Ju's model.
- To take into account the nondeterministic aspect in the micromechanical modeling, because the problem is not only to know the computation result for a given configuration, but also to determine the relevance of the result compared to uncertainties of the data input as it is the case when we deal with concrete-like materials modeling.
- The investigation of fiber-matrix debonding problem within the framework of homogenization methods.
- The development of a micromechanical approach for macro-crack propagation which will enable to study the ductile behavior of Steel Fiber-Reinforced Concrete and contrast it with the brittle behavior of Unreinforced Plain Concrete.

8.7 List of Publications

This research work has contributed to the publication of the following papers:

Refereed papers for international journals:

- Doghri, I., Ouaar, A., *Homogenization of two-phase elastoplastic composite materials and structures. Study of tangent operators, cyclic plasticity and numerical algorithms.* Internat. Journal of Solids and Structures, 40 (2003) 1681-1712.
- Ouaar, A., Doghri, I., Delannay, L., Thimus, J.Fr., *Micromechanics of the deformation and damage of steel fiber-reinforced concrete.* Inter. Journal of Damage Mechanics, in press.

Conference proceedings and workshops:

- Ouaar, A., Doghri, I., Thimus., J-Fr., *Modeling and simulation of fiber-reinforced concrete using homogenization schemes.* 5th National Congress on Theoretical and Applied mechanics. p 355-358. Louvain-la-Neuve, Belgium. 2000.
- Ouaar, A., Doghri, I., Thimus., J-Fr., *Micro-mechanical modeling and numerical simulations of a two-phase material .* Colloque Français de Mécanique MECAMAT. Janvier 2001.
- Doghri, I., Ouaar, A., *Nonlinear mechanics of rate-independent, inclusion-reinforced materials: Modeling and computation.* 6th International Workshop on Nonlinear Elasticity in Materials. June 18-22, 2001.
- Doghri, I., Ouaar, A., *Homogenization of rate-independent inclusion-reinforced materials: Some numerical aspects.* Abaqus Conference. Paris 2002.
- Ouaar, A., Doghri, I., Thimus, J.Fr., *Micro-mechanics of two-phase nonlinear composite materials. Application to reinforced mortar and concrete: Modeling and Simulation.* 6th National Congress on Theoretical and Applied mechanics. NCTAM-2003-048. Ghent, Belgium. May 2003.
- Ouaar, A., Doghri., I., Thimus, J-Fr., Huge., F., *Micromechanical modeling of the deformation and damage of inelastic brittle three-phase composites: Application to Fiber-Reinforced Concrete.* 21st International Congress of Theoretical and Applied Mechanics. SM4L 11959, p229. Warsaw, Poland, 2004.

- Ouaar, A., Thimus, J.Fr., Doghri, I., Remacle, J-Fr., *Concrete Matrix Composites: Some micromechanical modeling aspects and numerical simulations*. Congrès International Rhéhabilitation des Constructions et Développement Durable, Tome II, pp 182-190. Alger, Mai 2005
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- Ouaar, A., Remacle, J-Fr., Doghri, I. Chevaugeon, N., *A Micromechanical model for a class of brittle multi-phase composites: Application to Steel Fiber-Reinforced Concrete*., 7th World Congress on Computational Mechanics, Los Angeles, California, July 16 - 22, 2006.

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Chapter 9

Notations and units

Fourth order tensors

Symbol	Definition	Units
A	Strain concentration tensor such as $\langle \varepsilon \rangle_{\omega_1} = A : \langle \varepsilon \rangle_{\omega}$	-
$\tilde{A}$	Strain concentration tensor such as $\langle \varepsilon \rangle_{\omega_1} = \tilde{A} : \langle \varepsilon \rangle_{\omega_0}$	-
C^{el}	Elastic operator	MPa
C^*	Hill's constraint tensor	MPa
C^d	Ju's damage secant operator	MPa
C^{dtg}	Ju's damage tangent operator	MPa
C^{alg}	Algorithmic tangent operator	MPa
C^{ep}	Continuum tangent operator (el.-pl.)	MPa
C^{sec}	Secant stiffness operator	MPa
$\mathbb{A}$	Strain concentration tensor $\varepsilon(x) = \mathbb{A} : \bar{\varepsilon}$	-
B	Stress concentration tensor such as $\langle \sigma \rangle_{\omega_1} = B : \langle \sigma \rangle_{\omega}$	-
$\tilde{B}$	Stress concentration tensor such $\langle \sigma \rangle_{\omega_1} = \tilde{B} : \langle \sigma \rangle_{\omega_0}$	-
I^{vol}	Volumetric unit tensor	-
I^{dev}	Deviatoric unit tensor	-
I	Symmetric unit tensor	-
P	Hill's tensor	1/MPa
D	Compliance	1/MPa
S	Eshelby's tensor	-

Second order tensors

Symbol	Definition	Units
G	Displacement gradient	-
N	Normal to the yield surface in stress space	-
1	Unit tensor	-
ε	Total infinitesimal strain	-
ε^e	Elastic strain	-
ε^p	Plastic strain	-
ε^d	Damage strain	-
ε^*	Eigenstrain	-
$\varepsilon^\infty, \varepsilon^0$	Far field strain	-
σ	Cauchy stress	<i>MPa</i>
τ	Polarization tensor	<i>MPa</i>

Vectors

Symbol	Definition	Units
n	outer unit normal to a surface	-
t	Stress vector (traction)	<i>MPa</i>
u	Displacement	<i>mm</i>
x	Position	<i>mm</i>

Scalars

Symbol	Definition	Units
f	Yield function	<i>MPa</i>
p	Accumulated plastic strain	-
v_1	Volume fraction of inclusions	-
w	Stress potential	<i>MPa</i>
E	Young's modulus	<i>MPa</i>
G	Shear modulus	<i>MPa</i>
$R(p)$	Hardening function	<i>MPa</i>
S	Area	<i>mm</i> ²
V	Volume	<i>mm</i> ³
$\dot{\gamma}$	Plastic multiplier	1/s
κ	Bulk modulus	<i>MPa</i>
λ	Lamé coefficient	<i>MPa</i>
μ	Lamé coefficient	<i>MPa</i>
ν	Poisson's ratio	-
σ_{eq}	Equivalent stress	<i>MPa</i>
σ_{eq}^{tr}	Trial equivalent stress after an elastic time step	<i>MPa</i>
σ_Y	Initial yield stress	<i>MPa</i>

Symbols

Symbol	Definition
$\hat{x}_r$	Reference state in phase r
$\langle x \rangle_{\omega_r}$	Average value over phase r
$\langle x \rangle_{\omega}$	Average value over all the volume
$\bar{x}$	Macroscopic value
$C > 0$	$\varepsilon : C : \varepsilon > 0$ for all ε
Δx	Increment
x_0	Refers to the matrix
x^{dev}, x^{iso}	Deviatoric and Isotropic part of the tensor
x^{eff}	Effective value
$\dot{x}$	Time derivative
$\otimes$	Tensor product
$:$	Contraction over two indices
$::$	Contraction over four indices

Appendix A

Direct computation method for multi-phase composites

Consider the multi-phase RVE (Ω) consisting in a matrix material with inclusions made of several constitutive materials embedded therein. Focus our interest only on the case of composites with spherical inclusion or aligned inclusions if their aspect ratio $A_r \neq 1$. Dealing with nonlinear materials, the original RVE is replaced with an Heterogeneous Linear Comparison Material with uniform linearized mechanical properties. We use an incremental formulation for the linearization process and the per-phase uniformization of tangent operators is achieved using the average strain of the considered phase.

The aim of what follows is to derive the overall tangent operator of the Equivalent Homogeneous Medium of the RVE (Ω) in terms of micro-constituents (ω_r) mechanical and geometrical properties. In the RVE (Ω), the average stress rate is written as:

$$\begin{aligned}\langle \dot{\sigma} \rangle_{\Omega} &= v_0 \langle \dot{\sigma} \rangle_{\omega_0} + v_1 \langle \dot{\sigma} \rangle_{\omega_1} + v_2 \langle \dot{\sigma} \rangle_{\omega_2} + \cdots + v_N \langle \dot{\sigma} \rangle_{\omega_N} \\ &= v_0 C_0^{tg} : \langle \dot{\epsilon} \rangle_{\omega_0} + v_1 C_1^{tg} : \langle \dot{\epsilon} \rangle_{\omega_1} + v_2 C_2^{tg} : \langle \dot{\epsilon} \rangle_{\omega_2} + \cdots + v_N C_N^{tg} : \langle \dot{\epsilon} \rangle_{\omega_N}\end{aligned}\quad (\text{A.1})$$

Strain rate concentration tensor relationships read:

$$\langle \dot{\epsilon} \rangle_{\omega_r} = A_r : \langle \dot{\epsilon} \rangle_{\Omega} \quad r = 1, \cdots N \quad (\text{A.2})$$

The average strain rate over Ω is written as:

$$\begin{aligned}\langle \dot{\epsilon} \rangle_{\Omega} &= v_0 \langle \dot{\epsilon} \rangle_{\omega_0} + v_1 \langle \dot{\epsilon} \rangle_{\omega_1} + v_2 \langle \dot{\epsilon} \rangle_{\omega_2} + \cdots + v_N \langle \dot{\epsilon} \rangle_{\omega_N} \\ &= v_0 \langle \dot{\epsilon} \rangle_{\omega_0} + v_1 A_1 : \langle \dot{\epsilon} \rangle_{\Omega} + \cdots + v_N A_N : \langle \dot{\epsilon} \rangle_{\Omega}\end{aligned}\quad (\text{A.3})$$

with help of Eq.A.2. Then, isolating $\langle \dot{\epsilon} \rangle_{\omega_0}$ from the above equation leads to:

$$\begin{aligned} \langle \dot{\epsilon} \rangle_{\omega_0} &= \frac{1}{v_0} [\langle \dot{\epsilon} \rangle_{\Omega} - v_1 A_1 : \langle \dot{\epsilon} \rangle_{\Omega} - \cdots - v_N A_N : \langle \dot{\epsilon} \rangle_{\Omega}] \\ &= \frac{1}{v_0} \left[I - \sum_{i=1}^N v_i A_i \right] : \langle \dot{\epsilon} \rangle_{\Omega} \end{aligned} \quad (\text{A.4})$$

Finally, substitution of the last equation and Eq.A.2 into Eq.A.1 gives:

$$\begin{aligned} \langle \dot{\sigma} \rangle_{\Omega} &= v_0 C_0^{tg} : \frac{1}{v_0} \left[I - \sum_{i=1}^N v_i A_i \right] : \langle \dot{\epsilon} \rangle_{\Omega} + v_1 C_1^{tg} : A_1 : \langle \dot{\epsilon} \rangle_{\Omega} + \cdots + v_N C_N^{tg} : A_N : \langle \dot{\epsilon} \rangle_{\Omega} \\ &= \left[C_0^{tg} + \sum_{i=1}^N v_i (C_i^{tg} - C_0^{tg}) : A_i \right] : \langle \dot{\epsilon} \rangle_{\Omega} \end{aligned} \quad (\text{A.5})$$

In Equivalent Homogeneous Medium, the average stress rate is written as:

$$\langle \dot{\sigma} \rangle_{\Omega} = \bar{C}^{tg} : \langle \dot{\epsilon} \rangle_{\Omega} \quad (\text{A.6})$$

Identification of the last equation with Eq(A.5) gives the expression of the overall tangent operator of the multi-phase composite obtained using the so-called direct computation method. It remains now to replace A_i according to the suitable homogenization scheme.

Appendix B

Second and fourth-rank tensors storage

Consider a symmetric fourth-rank tensor C with respect of the first and second pairs of its subscript such as:

$$C_{ijkl} = C_{klij} \quad (\text{B.1})$$

Consider also that C relates the stress and strain second-rank tensors, σ and ε respectively through:

$$\sigma_{ij} = C_{ijkl} : \varepsilon_{lk} \quad (\text{B.2})$$

To be numerically useful, it is convenient to express the above stress-strain equation under vectors and matrices form. For this purpose, the stress and strain tensors are stored in six by one column vectors, and C in a six by six matrix as follows:

1. for the second-order stress tensor, we define a 6x1 vector with the following correspondence of subscripts:

$$\begin{array}{lll} \sigma_1 = \sigma_{11} & \sigma_2 = \sigma_{22} & \sigma_3 = \sigma_{33} \\ \sigma_4 = \sigma_{12} = \sigma_{21} & \sigma_5 = \sigma_{23} = \sigma_{32} & \sigma_6 = \sigma_{13} = \sigma_{31} \end{array} \quad (\text{B.3})$$

Taking advantage of symmetry, the stress tensor is then stored as follows:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4\sqrt{2} \\ \sigma_5\sqrt{2} \\ \sigma_6\sqrt{2} \end{pmatrix} = \underbrace{\{\sigma\}}_{1 \times 6}$$

2. In the same manner, the second-order strain tensor is stored. The result is:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \equiv \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4\sqrt{2} \\ \varepsilon_5\sqrt{2} \\ \varepsilon_6\sqrt{2} \end{pmatrix} = \underbrace{\{\varepsilon\}}_{1 \times 6}$$

3. For the fourth-order tensor C_{ijkl} , the two first subscripts ij are denoted by one subscript, and the second two subscripts kl by one subscript. For example we have:

$$C_{1111} = C_{\underbrace{11}_1 \underbrace{11}_1} = C_{11}, \quad \text{and,} \quad C_{2313} = C_{\underbrace{23}_5 \underbrace{13}_6} = C_{56} \quad (\text{B.4})$$

Finally equation (B.2) is written under following matrix form:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4\sqrt{2} \\ \sigma_5\sqrt{2} \\ \sigma_6\sqrt{2} \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14}\sqrt{2} & C_{15}\sqrt{2} & C_{16}\sqrt{2} \\ C_{21} & C_{22} & C_{23} & C_{24}\sqrt{2} & C_{25}\sqrt{2} & C_{26}\sqrt{2} \\ C_{31} & C_{32} & C_{33} & C_{34}\sqrt{2} & C_{35}\sqrt{2} & C_{36}\sqrt{2} \\ C_{41}\sqrt{2} & C_{42}\sqrt{2} & C_{43}\sqrt{2} & 2C_{44} & 2C_{45} & 2C_{46} \\ C_{51}\sqrt{2} & C_{52}\sqrt{2} & C_{53}\sqrt{2} & 2C_{54} & 2C_{55} & 2C_{56} \\ C_{61}\sqrt{2} & C_{62}\sqrt{2} & C_{63}\sqrt{2} & 2C_{64} & 2C_{65} & 2C_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4\sqrt{2} \\ \varepsilon_5\sqrt{2} \\ \varepsilon_6\sqrt{2} \end{pmatrix}$$

Appendix C

Coordinate system change

Hereafter are briefly given the suitable formulas for coordinate system change. For more elaborate discussion see [19]. Consider two orthogonal bases e_i and e_i^* with $i=1,2,3$. The matrix $[P]$ which enables to pass from basis e to basis e^* is defined as follows:

$$P_{ij} = e_i^* \cdot e_j \quad (C.1)$$

If one deals with second-rank tensors, σ for instance, stored as a six by one vector as above-indicated, the following transformation relationship hold:

$$\{\sigma\} = [q]^T \{\sigma^*\} \quad (C.2)$$

where σ and σ^* are six by one stress vectors expressed in basis e_i and e_i^* respectively, and the matrix $[q]$ is given by:

$$q = \begin{bmatrix} P_{11}P_{11} & P_{12}P_{12} & P_{13}P_{13} & P_{11}P_{12}\sqrt{2} & P_{12}P_{12}\sqrt{2} & P_{11}P_{13}\sqrt{2} \\ P_{21}P_{21} & P_{22}P_{22} & P_{23}P_{23} & P_{21}P_{22}\sqrt{2} & P_{22}P_{23}\sqrt{2} & P_{21}P_{23}\sqrt{2} \\ P_{31}P_{31} & P_{32}P_{32} & P_{33}P_{33} & P_{31}P_{32}\sqrt{2} & P_{32}P_{33}\sqrt{2} & P_{31}P_{33}\sqrt{2} \\ P_{11}P_{21}\sqrt{2} & P_{12}P_{22}\sqrt{2} & P_{13}P_{23}\sqrt{2} & P_{11}P_{22} + P_{12}P_{21} & P_{12}P_{23} + P_{13}P_{22} & P_{11}P_{23} + P_{13}P_{21} \\ P_{21}P_{31}\sqrt{2} & P_{22}P_{32}\sqrt{2} & P_{23}P_{33}\sqrt{2} & P_{21}P_{32} + P_{22}P_{31} & P_{22}P_{33} + P_{23}P_{32} & P_{21}P_{33} + P_{23}P_{31} \\ P_{11}P_{31}\sqrt{2} & P_{12}P_{32}\sqrt{2} & P_{13}P_{33}\sqrt{2} & P_{11}P_{32} + P_{12}P_{31} & P_{12}P_{33} + P_{13}P_{32} & P_{11}P_{33} + P_{13}P_{31} \end{bmatrix}$$

In the case of fourth-rank tensor C stored as a six by six matrix as $[C]$, the transformation rule gives:

$$[C] = [q][C^*][q]^T \quad (C.3)$$

where $[C]$ and $[C^*]$ are the stiffness matrices expressed in basis e_i and e_i^* respectively

Appendix D

Analytical expressions of Eshelby's tensors

Hereafter are presented analytical expression of Eshelby's tensor for a spheroidal inclusion aligned along axis 1 and characterized by an aspect ratio A_r as shown in the Figure D.1. The ratio A_r is defined by:

$$A_r = \frac{l}{d} \quad (\text{D.1})$$

where l and d represent the characteristic dimension of the ellipsoid I , as displayed in Figure D.1. The general form of S is:

$$\begin{aligned} S_{1111} &= \frac{1}{2(1-\nu)} [(1-2\nu)(1-g) + (3A_r^2(1-g) - 1)/(A_r^2 - 1)] \\ S_{2222} &= S_{3333} = \frac{1}{8(1-\nu)} [2(1-2\nu)g + 3(A_r^2 - 3g/2)/(A_r^2 - 1)] \\ S_{1122} &= S_{1133} = \frac{1}{2(1-\nu)} [-(1-2\nu)(1-g) + (3g/2 - 1)/(A_r^2 - 1)] \\ S_{2211} &= S_{3311} = \frac{1}{4(1-\nu)} [-(1-2\nu)g + A_r^2(3g - 2)/(A_r^2 - 1)] \\ S_{P(22,33)} &= \frac{1}{8(1-\nu)} [-2(1-2\nu)g + (A_r^2 - 3g/2)/(A_r^2 - 1)] \\ S_{P(2,3)P(2,3)} &= \frac{1}{2}(S_{2222} - S_{2233}) \\ S_{P(1,2)P(1,2)} &= S_{P(1,3)P(1,3)} = \frac{1}{4(1-\nu)} [(1-2\nu)(1-g/2) + (3g/2 - 1)] \end{aligned} \quad (\text{D.2})$$

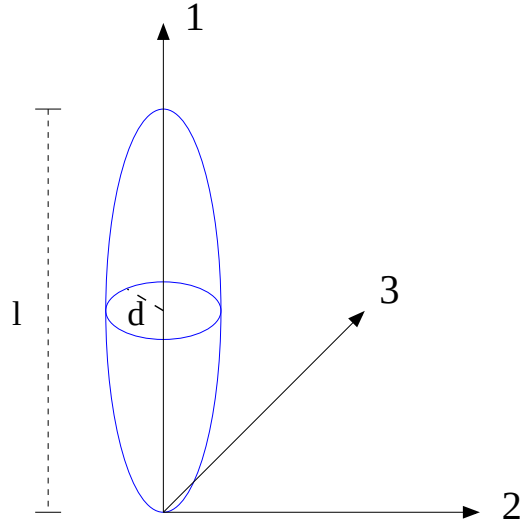


Figure D.1: Inclusion system axis for Eshelby's tensor computation

where g is a function of A_r and $P(.,.)$ are all possible permutation of tensor subscripts: for example $S_{P(1,2)P(1,2)}$ give S_{1212} , S_{1221} , S_{2112} and S_{2121} while $S_{P(11,22)}$ give S_{1122} and S_{2211} . For a prolate spheroid ($1 < A_r < \infty$), g is expressed as follows:

$$g = \frac{A_r}{2(A_r^2)} \left(A_r - \frac{\cosh^{-1} A_r}{\sqrt{A_R^2 - 1}} \right), \quad (\text{D.3})$$

and for an oblate spheroid ($1 < A_r < \infty$)

$$g = \frac{A_r}{2(A_r^2)} \left(A_r - \frac{\cosh^{-1} A_r}{\sqrt{1 - A_R^2}} \right) \quad (\text{D.4})$$

For particular inclusion shapes, one can obtain the corresponding expressions of Eshelby's tensor by affecting to A_r the value which corresponds to the desired inclusion shape. For example we have:

- for a sphere $A_r = 1$, the non-zero components of S_{ijkl} are:

$$\begin{aligned}
 S_{1111} &= S_{2222} = S_{3333} = \frac{7-5\nu}{15(1-\nu)} \\
 S_{P(11,22)} &= S_{P(11,33)} = S_{P(22,33)} = \frac{5\nu-1}{15(1-\nu)} \\
 S_{P(1,2)P(1,2)} &= S_{P(1,3)P(1,3)} = S_{P(2,3)P(2,3)} = \frac{4-5\nu}{15(1-\nu)}
 \end{aligned} \tag{D.5}$$

- for a flat disk $A_r \rightarrow 0$, the non-zero components of S_{ijkl} are:

$$\begin{aligned}
 S_{1111} &= 1 \\
 S_{1122} &= S_{1133} = \frac{\nu}{1-\nu} \\
 S_{P(1,2)P(1,2)} &= S_{P(1,3)P(1,3)} = \frac{1}{2}
 \end{aligned} \tag{D.6}$$

- for a long continuous fiber $1 < A_r \rightarrow \infty$, the non-zero components of S_{ijkl} are:

$$\begin{aligned}
 S_{2222} &= S_{3333} = \frac{5-4\nu}{8(1-\nu)} \\
 S_{2211} &= S_{3311} = \frac{\nu}{2(1-\nu)} \\
 S_{2233} &= S_{3322} = \frac{4\nu-1}{8(1-\nu)} \\
 S_{P(2,3)P(2,3)} &= \frac{3-4\nu}{8(1-\nu)} \\
 S_{P(1,2)P(1,2)} &= S_{P(1,3)P(1,3)} = \frac{1}{4}
 \end{aligned} \tag{D.7}$$

Appendix E

Micromechanical modeling of Fiber-matrix debonding

There are many ways to model the fiber-matrix debonding process, for instance one can cite F.E. one which was used in section of chapter 6. Another interesting approach which can be easily combined with homogenization methods is those proposed by J.W. Ju and H.K. Lee [41]. In this method, the debonding is governed by the internal average stress of a fiber and the Weibull parameters. Consider a composite consisting in a matrix phase in which are firmly embedded inclusions as represented in Figure E.1-a. Under increased loads, some fibers may undergo a total interface debonding (Figure E.1-b), and not playing any mechanical role can then be regarded as void particles (Figure E.1-c). Assuming that Weibull statistics control the decohesion process and that this process occur only under tensile loading conditions (for composites with brittle matrix phase material), the cumulative probability function of fiber debonding is written as:

$$P_d[\langle\sigma_{11}\rangle_1] = 1 - \exp\left[-\left(\frac{\langle\sigma_{11}\rangle_1}{S_0}\right)^M\right] \quad (\text{E.1})$$

where $\langle\sigma_{11}\rangle_1$ is the tensile average stress in the inclusion phase denoted by the subscript “1” in the direction “1”, and M and S_0 are the Weibull parameters. Accordingly, the volume fraction of voids created under a given level of $\langle\sigma_{11}\rangle_1$ can be computed by:

$$v_v = v_1 P_d[\langle\sigma_{11}\rangle_1] = v_1 \left\{ 1 - \exp\left[-\left(\frac{\langle\sigma_{11}\rangle_1}{S_0}\right)^M\right] \right\} \quad (\text{E.2})$$

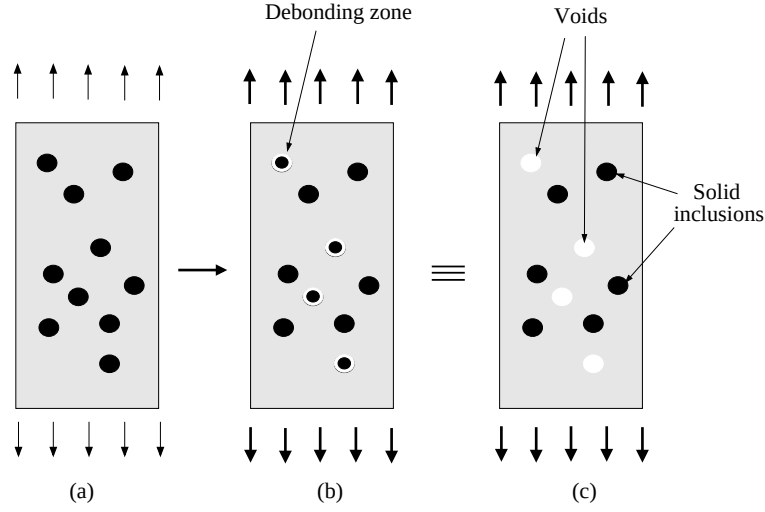


Figure E.1: Schematic representation of inclusion debonding process.

For the case of biaxial or triaxial tensile loadings, it is enough to replace $\langle \sigma_{11} \rangle_1$ in Eqs.(E.1) and (E.2) by: $(\langle \sigma_{11} \rangle_1 + \langle \sigma_{22} \rangle_1)/2$ and $(\langle \sigma_{11} \rangle_1 + \langle \sigma_{22} \rangle_1 + \langle \sigma_{33} \rangle_1)/3$ respectively. Note that choices other than the internal stress $\langle \sigma_{11} \rangle_1$ are possible. To accommodate the brittle nature of concrete, Lee H. K. and Liang Z. [49] used the experimentally adjusted value of $\langle \sigma_m \rangle_1$:

$$\langle \sigma_m \rangle_1 = \left(1 - \frac{2}{3}v_1\right) \left(\frac{\langle \sigma_{11} \rangle_1 + \langle \sigma_{22} \rangle_1 + \langle \sigma_{33} \rangle_1}{3}\right) \quad (\text{E.3})$$

where v_1 is the Poisson's ratio of fibers. From a physical point of view, the approach exposed above does not reflect the truly observed debonding phenomenon, in particular during uniaxial multi axial test. However this method provides a simple way to account for debonding which can be easily into a homogenization code. Note that for a hydrostatic tensile loading test the proposed model agrees with the physical observations.

Hereafter is summarized the iterative algorithm used to account for the evolutionary complete debonding of fibers.

- (1) Initialize: $it = 0$; $v_1|_{n+1}^{(0)} = v_1|_n$; $v_v|_{n+1}^{(0)} = v_v|_n$.
- (2) Compute average stress $\langle \sigma_{11} \rangle_1^{(it)}|_{n+1}$ in fibers

(3) Compute the Weibull probability distribution function:

$$P_d \left[\langle \sigma_{11} \rangle_1 |_{n+1}^{(it)} \right] = 1 - \exp \left[- \left(\frac{\langle \sigma_{11} \rangle_1 |_{n+1}^{(it)}}{S_0} \right)^M \right]$$

(4) Compute volume fractions of voids and inclusions respectively:

$$v_v |_{n+1}^{(it)} = v_1 \left\{ 1 - \exp \left[- \left(\frac{\langle \sigma_{11} \rangle_1}{S_0} \right)^M \right] \right\}$$

$$v_1 |_{n+1}^{(it)} = v_1^{ini} - v_v |_{n+1}^{(it)}$$

where v_1^{ini} is the initial volume fraction of fibers.

(5) Convergence test checking:

– if

$$\left| \frac{v_1 |_{n+1}^{(it)} - v_1 |_{n+1}^{(it-1)}}{v_1 |_{n+1}^{(it-1)}} \right| \leq TOL(e.g., 10^{-8}) \quad (E.4)$$

then update the following quantities:

$$v_1 |_{n+1} = v_1 |_{n+1}^{(it)}, v_v |_{n+1} = v_v |_{n+1}^{(it)}, \langle \sigma_{11} \rangle_1 |_{n+1} = \langle \sigma_{11} \rangle_1 |_{n+1}^{(it)};$$

– else: set $it = it + 1$ and go to step (2)

Appendix F

Mori-Tanaka model for nonlinear two-phase composites

In this section, we derive Mori-Tanaka scheme in nonlinear regime. Consider first a RVE representing a nonlinear two-phase composite made of a matrix and inclusion materials both displaying a nonlinear behavior. The original nonlinear RVE is replaced with a Heterogeneous Linear Comparison Material (LHCM) which has a linearized constitutive law and uniform mechanical properties per phase (see section 4.3.3 of chapter 4). Then we associate to the LHCM an RVE representing the Associated Isolated Inclusion Medium (AIIM) according to M-T scheme hypothesis (Figure F.1). Accordingly this AIIM is subjected to a far-field strain rate $\dot{\epsilon}^0$ and to the nonuniform perturbation strain rate $\dot{\epsilon}''(x)$ resulting from the presence of the neighboring inclusions, as shown in the Figure F.1-b. The perturbation strain $\dot{\epsilon}^{pr}(x)$ due to the presence of the inclusion I in the matrix is defined as follows:

$$\dot{\epsilon}^{pr}(x) = \begin{cases} S_{I, \hat{C}_0^{tg}} : \dot{\epsilon}^* & \text{inside the isolated inclusion} \\ \dot{\epsilon}'(x) & \text{inside the matrix} \end{cases} \quad (\text{F.1})$$

define also:

$$\dot{\epsilon}'''(x) = \dot{\epsilon}'(x) + \dot{\epsilon}''(x) \quad (\text{F.2})$$

the resulting perturbation strain rate of all inclusions inside the RVE. of course $\dot{\epsilon}'(x) = 0$ inside the isolated inclusion.

In the matrix phase of the AIIM RVE (figure 2.10-b), the strain and stress rates are:

$$\dot{\epsilon}(x) = \dot{\epsilon}^0 + \dot{\epsilon}'(x) + \dot{\epsilon}''(x) \equiv \dot{\epsilon}^0 + \dot{\epsilon}'''(x) \quad (\text{F.3})$$

$$\dot{\sigma}(x) = \hat{C}_0^{tg} : (\dot{\epsilon}^0 + \dot{\epsilon}'''(x)) \quad (\text{F.4})$$

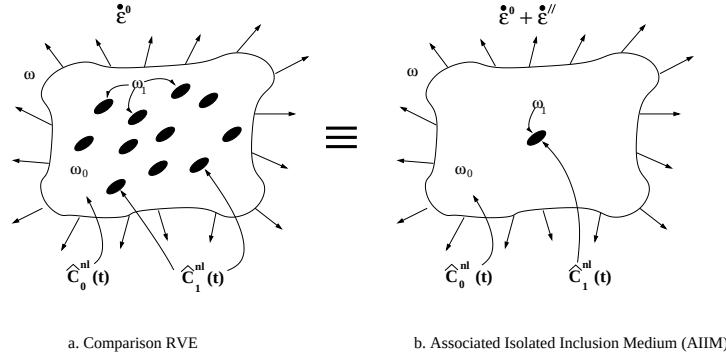


Figure F.1: Schematic representation of Mori-Tanaka model hypotheses in nonlinear regime.

In the inclusion, the strain and stress rates are:

$$\dot{\epsilon}(x) = \dot{\epsilon}^0 + S_{I, \hat{C}_0^{tg}} : \dot{\epsilon}^* + \dot{\epsilon}'''(x) \quad (F.5)$$

$$\dot{\sigma}(x) = \hat{C}_1^{tg} : (\dot{\epsilon}^0 + S_{I, \hat{C}_0^{tg}} : \dot{\epsilon}^* + \dot{\epsilon}'''(x)) \quad (F.6)$$

In the inclusion zone of the equivalent homogeneous medium, the strain and stress rates are written as:

$$\dot{\epsilon}(x) = \dot{\epsilon}^0 + S_{I, \hat{C}_0^{tg}} : \dot{\epsilon}^* + \dot{\epsilon}'''(x) - \dot{\epsilon}^* \quad (F.7)$$

$$\dot{\sigma}(x) = \hat{C}_0^{tg} : (\dot{\epsilon}^0 + S_{I, \hat{C}_0^{tg}} : \dot{\epsilon}^* + \dot{\epsilon}'''(x) - \dot{\epsilon}^*) \quad (F.8)$$

To fulfill the mechanical equivalent condition between the two medium displayed in the Figure F.2, the stress rate in the inclusion zone of the Homogeneous equivalent medium given by Eq.(F.8) and the stress rate in inclusion zone of the substitution RVE given by Eq.(F.6)) must be the equal. It follows that:

$$\hat{C}_1^{tg} : (\dot{\epsilon}^0 + S_{I, \hat{C}_0^{tg}} : \dot{\epsilon}^* + \dot{\epsilon}'''(x)) = \hat{C}_0^{tg} : (\dot{\epsilon}^0 + S : \dot{\epsilon}^* + \dot{\epsilon}'''(x) - \dot{\epsilon}^*) \quad (F.9)$$

The above mechanical equivalence condition can be fulfilled by considering the average over inclusion material ω_1 (all inclusions), consequently, Eq.(F.9) then becomes:

$$\hat{C}_1^{tg} : \dot{\epsilon}^0 + S_{I, \hat{C}_0^{tg}} : \dot{\epsilon}^* + \langle \dot{\epsilon}'''(x) \rangle_{\omega_1} = \hat{C}_0^{tg} : (\dot{\epsilon}^0 + S : \dot{\epsilon}^* + \langle \dot{\epsilon}'''(x) \rangle_{\omega_1} - \dot{\epsilon}^*) \quad (F.10)$$

rearranging we obtain the suitable strain rate $\dot{\epsilon}^*$ for homogenization as:

$$\dot{\epsilon}^* = -([(\hat{C}_0^{tg})^{-1} : \hat{C}_1^{tg} - I^4]^{-1} + S_{I, \hat{C}_0^{tg}})^{-1} : (\dot{\epsilon}^0 + \langle \dot{\epsilon}'''(x) \rangle_{\omega_1}) \quad (F.11)$$

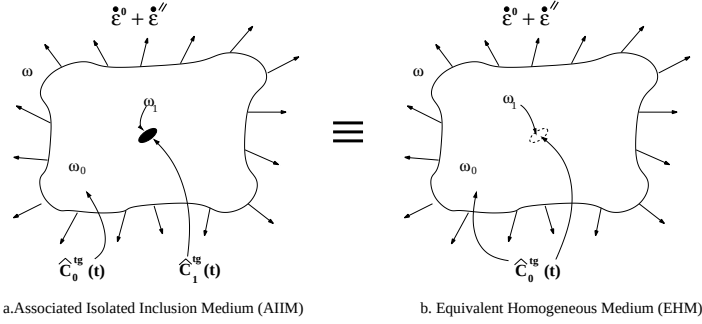


Figure F.2: Mechanical Equivalent between the homogeneous equivalent medium and the equivalent isolated inclusion medium.

replacing the last equation into Eq.(F.7) and averaging, the average strain in the inclusion is then found as:

$$\langle \dot{\epsilon} \rangle_{\omega_1} = \left(I^4 - S_{I, \hat{C}_0^{tg}} : \left([(\hat{C}_0^{tg})^{-1} : \hat{C}_1^{tg} - I^4]^{-1} + S_{I, \hat{C}_0^{tg}} \right)^{-1} \right) : (\dot{\epsilon}^0 + \langle \dot{\epsilon}'''(x) \rangle_{\omega_1}) \quad (F.12)$$

The assumption made by Mori and Tanaka is that the disturbance strain $\epsilon''(x)$ due to the surrounding inclusions are on average the same for the matrix phase and for any inclusion. So, we have

$$\langle \dot{\epsilon}'''(x) \rangle_{\omega_0} = \langle \dot{\epsilon}'''(x) \rangle_{\omega_1} \quad (F.13)$$

and that:

$$\dot{\epsilon}^0 + \langle \dot{\epsilon}'''(x) \rangle_{\omega_1} = \dot{\epsilon}^0 + \langle \dot{\epsilon}'''(x) \rangle_{\omega_0} = \langle \dot{\epsilon} \rangle_{\omega_0} \quad (F.14)$$

Eq.(F.12) is then rewritten as:

$$\langle \dot{\epsilon} \rangle_{\omega_1} = \left(I^4 - S_{I, \hat{C}_0^{tg}} : \left([(\hat{C}_0^{tg})^{-1} : \hat{C}_1^{tg} - I^4]^{-1} + S_{I, \hat{C}_0^{tg}} \right)^{-1} \right) : \langle \dot{\epsilon} \rangle_{\omega_0} \quad (F.15)$$

then, identification of the last equation with Eq.4.20 leads to:

$$\tilde{A}^{MT} = \left(I^4 - S_{I, \hat{C}_0^{tg}} : \left([(\hat{C}_0^{tg})^{-1} : \hat{C}_1^{tg} - I^4]^{-1} + S_{I, \hat{C}_0^{tg}} \right)^{-1} \right) \quad (F.16)$$

Replacing $\tilde{A}^{MT}$ into Eq.(4.21) then the result into Eq.(4.28), one obtains the overall inelastic operator of a two-phase composite following Mori-Tanaka homogenization scheme.

Appendix G

Experimental program results

In this appendix, we present the various results of four-point bending tests performed on both plain concrete and steel fiber-reinforced concrete specimens. In each test, measurements are captured by

- Two linear voltage displacement transformers (LVDT) for load-deflection measurements.
- One strain gage and one extensometer for load-strain measurements.

In each figure for example, the notations "rec 1", "rec 2" and "test 1" in the legend refer to the measure captured by recorder 1 and 2 of the test 1 performed on the concerned specimen. The notations "CP1" and "Gage 1" refer to extensometers and strain gage measurements corresponding to test 1.

G.1 Plain Concrete samples results

The results of plain concrete samples tests are gathered in the figures hereafter. Figure G.1 shows load-deflection curves of the various vertical displacement recorders. Figure G.2 displays the stretching of fibers located on the lower side of the specimen versus the applied load. The strain gage measurements versus load are plotted in Figure G.3.

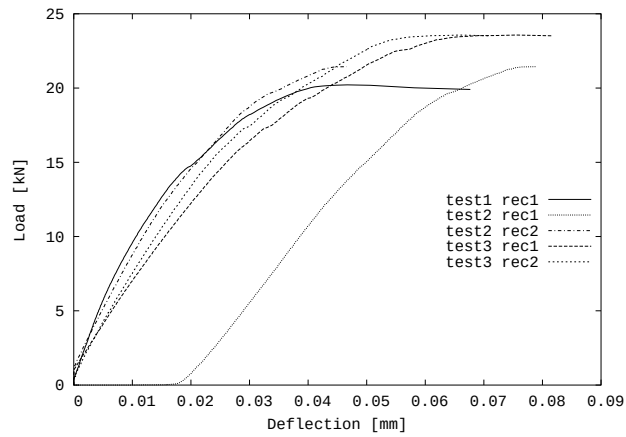


Figure G.1: Plain concrete under four point bending: Load-Deflection curves.

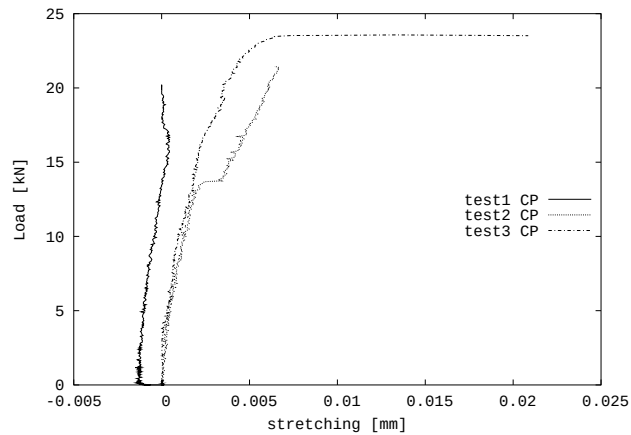


Figure G.2: Plain concrete under four point bending: Load versus clip on gage measurements curves.

G.2 Steel fiber-reinforced concrete (1 percent of fibers) results

Here, are reported results of four-point bending tests submitted on concrete matrix sample reinforced with one percent of fibers . The load-deflection curve is plotted in

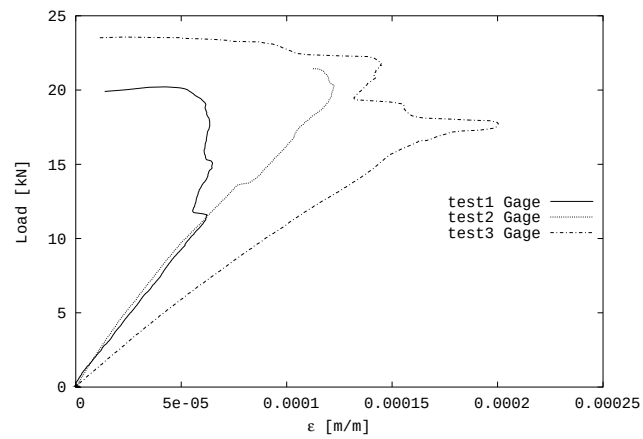


Figure G.3: Plain concrete under four point bending: Load versus strain gage measurements curves.

Figure G.4. The load-clip on gage measurement versus load curves are drawn in figure G.5. The strain gage measurements versus load are plotted in Figure G.6

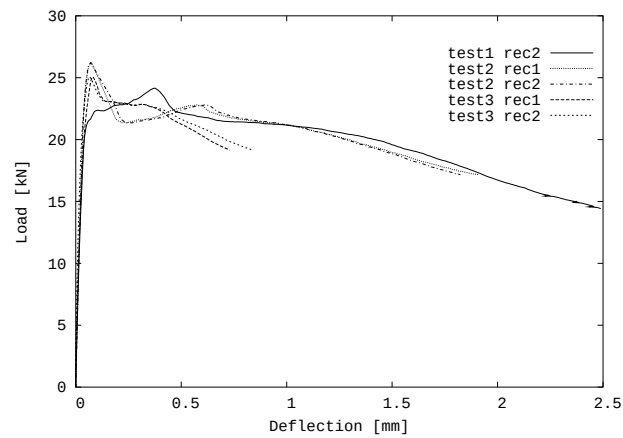


Figure G.4: SFRC (1 percent of fibers) under four point bending: Load-Deflection curves.

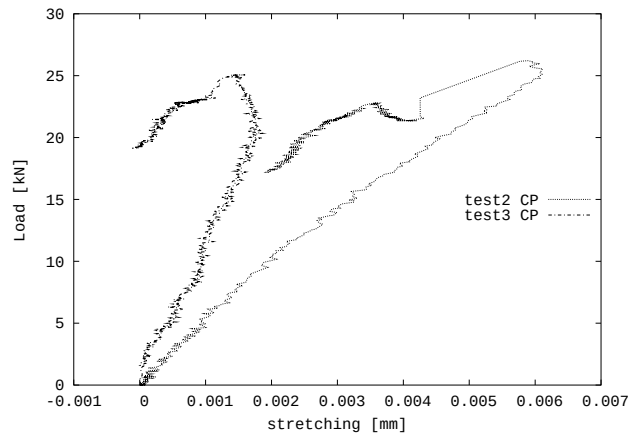


Figure G.5: SFRC (1 percent of fibers) under four point bending: Load versus clip on gage measurements curves.

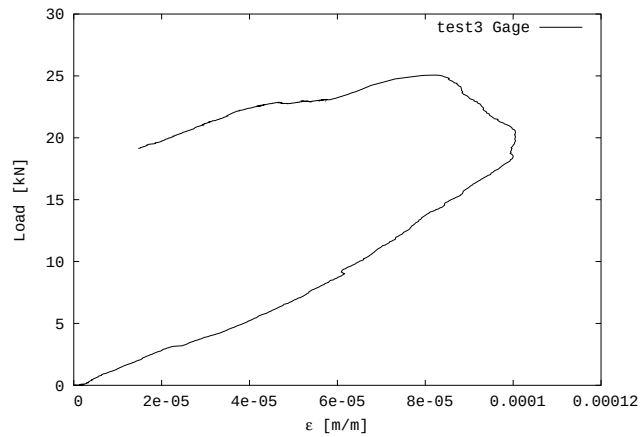


Figure G.6: SFRC (1 percent of fibers) under four point bending: Load versus strain gage measurements curves.

G.3 Steel fiber-reinforced concrete (3 percent of fibers) results

Figure G.7 displays the results of the vertical displacement recorders versus load, while Figure G.8 shows the clip on gage measurement versus load.

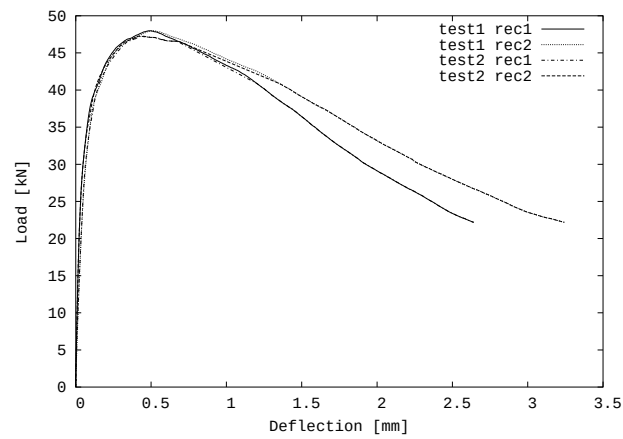


Figure G.7: SFRC (3 percent of fibers) under four point bending: Load-Deflection curves.

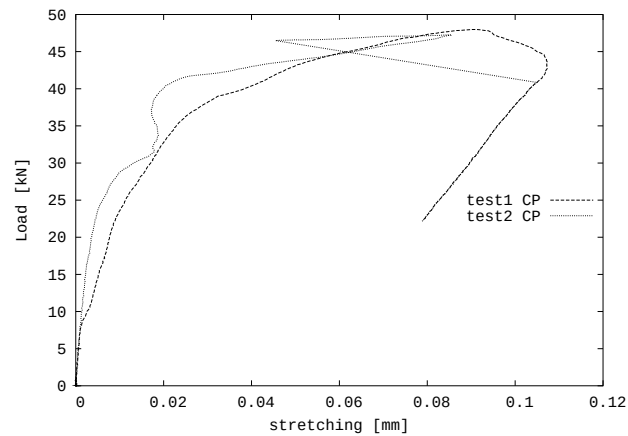


Figure G.8: SFRC (3 percent of fibers) under four point bending: Load versus clip on gage measurements curves.

G.4 SFRC with 4% of fibers results

Figures G.9 and G.10 illustrate the load versus LVDT measurements of the deflection and load versus extensometer measurements on the bottom face of sample.

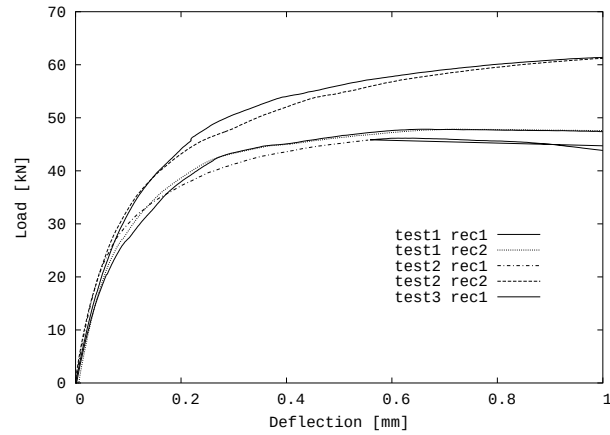


Figure G.9: SFRC (4 percent of fibers) under four-point bending: Load versus LVDT measurements curves.

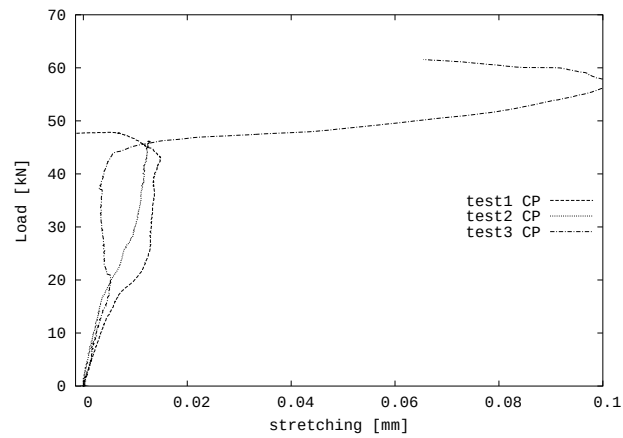


Figure G.10: SFRC (4 percent of fibers) under four point bending: Load versus extensometer measurements curves.

G.5 Bar-reinforced concrete results

We present here the results of bar reinforced concrete, with 4% of volume fraction of steel bars. The load-deflection curves and load-extensometer measurements on bottom face are displayed in Figures G.11 and G.12 respectively.

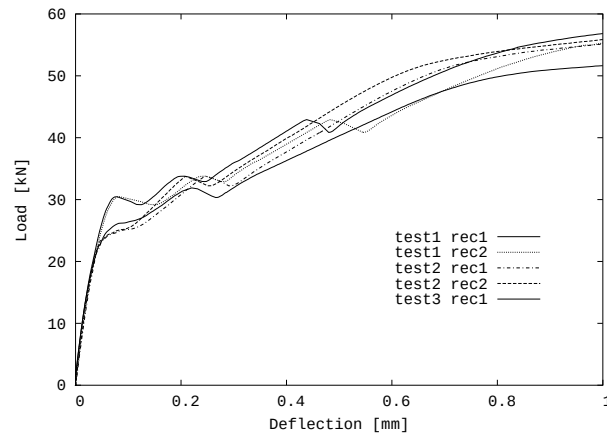


Figure G.11: Bar-reinforced concrete under four point bending: Load versus LVDT measurements curves.

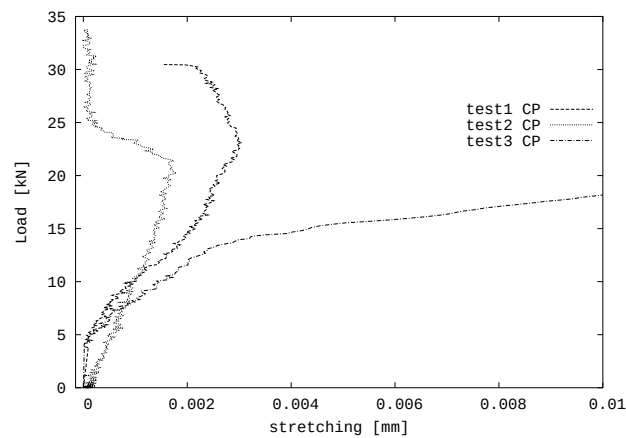


Figure G.12: Bar-reinforced concrete under four point bending: Load versus extensometer measurements curve.