Stochastic Frontiers Incorporating Exogenous Influences on Efficiency[1]

by

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Abstract

Until very recently, the incorporation of exogenous variables in the analysis of productive efficiency relative to a stochastic frontier was an arbitrary process. The practitioner had to assume that these variables were influencing either the structure of production possibilities or the degree of productive inefficiency. This paper proposes an alternative model which allows exogenous variables to influence both the structure of production possibilities and the degree of productive inefficiency. This new approach to the incorporation of exogenous influences on productive efficiency overcomes some of the economic and statistical problems that plague previous approaches.

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1. Introduction

The analysis of productive efficiency typically has two components. One is the estimation of a production (or cost or other) frontier, which serves as a benchmark against which the efficiency of each producer is measured. The other is the incorporation into the analysis of exogenous variables, which are neither inputs to the production process nor outputs of it, but which nonetheless influence the process. Examples range from input and output quality indicators, to service network characteristics, to the structure of property rights, and so on. These exogenous variables have been incorporated in a variety of ways, none of which is entirely satisfactory. The purpose of this paper is to propose a new way of integrating the two components into an analysis of productive efficiency. This new approach to the incorporation of exogenous influences on efficiency overcomes some of the economic and statistical problems that plague previous approaches.

In section 2 we describe the currently existing techniques for incorporating exogenous factors into frontier models, and we discuss the drawbacks of each approach. In section 3 we develop a model that incorporates exogenous influences on efficiency in a way that overcomes these problems. We also discuss least squares and maximum likelihood estimation of the model. In section 4 we conclude with a summary and some comments on the model.

2. Background

Let $y_i \in R_+$ denote output, $x_i = (x_{1i}, \ldots, x_{ni}) \in R_+^n$ denote a vector of inputs, and $z_i = (z_{1i}, \ldots, z_{mi}) \in R_+^m$ denote a vector of exogenous variables that influence the production process of producer $i, i = 1, \ldots, I$. The elements of z_i are neither outputs of the production process nor inputs to it. They capture features of the environment in which production takes place, and they are generally considered to be uncontrollable by those who manage the production process. The problem is to incorporate these exogenous variables into a model of productive efficiency measurement. The starting point is the concept of a production frontier.

Production frontiers can be either deterministic or stochastic. In the former case $\ln y_i = f(\ln x_i; \beta) - u_i$, where $f : R^n \to R$ is the deterministic production frontier and $u_i \geqq 0$ measures productive inefficiency[1].

In the latter case $\ln y_i = f(\ln x_i; \beta) + v_i - u_i$, where $v_i \sim N(0, \sigma_v^2)$ captures the effects of statistical noise, $u_i \geqq 0$ reflects productive inefficiency, and $[f(\ln x_i; \beta) + v_i]$ is the stochastic production frontier[2].

Both models are widely employed in empirical analysis, although the stochastic frontier model is preferred because it allows for measurement error and other sources of statistical noise.

Until very recently, exogenous variables were incorporated into the analysis of productive efficiency in three ways. Pitt and Lee (1981) assumed $f : R^{n+m} \to R$ and estimated the augmented stochastic production frontier model $\ln y_i = f(\ln x_i, \ln z_i; \beta) + v_i - u_i$, where $v_i \sim N(0, \sigma_v^2)$ and $u_i \geqq 0$. In this model, z_i is assumed to influence y_i directly, by influencing the structure of the production frontier relative to which the efficiency of producers is measured. Moreover, it is assumed that the elements of z_i, like the elements of x_i, are uncorrelated with each disturbance component v_i and u_i. The exogenous variables influence performance not by influencing efficiency, with which they are assumed to be uncorrelated, but by influencing the structure of production possibilities. Variation in efficiency is left unexplained in this formulation.

Ali and Flinn (1989), Kalirajan (1990), and many others, formulated a two-stage model. The first stage consists of the basic stochastic production frontier model $\ln y_i = f(\ln x_i; \beta) + v_i - u_i$, from which measured inefficiency is determined from the conditional expectation $E(u_i \mid v_i - u_i)$. In the second stage measured inefficiency is associated with the exogenous variables by means of a regression of general form $E(u_i \mid v_i - u_i) = g(\ln z_i; \gamma) + e_i$, where $e_i \sim N(0, \sigma_e^2)$. Since $E(u_i \mid v_i - u_i)$ is bounded below by zero, some sort of limited dependent variable estimation technique is appropriate, although OLS is frequently used. In this for-

[1]See, for example, Aigner and Chu (1968), Afriat (1972), Richmond (1974), Førsund and Jansen (1977) and Greene (1980).
[2]See Aigner, Lovell and Schmidt (ALS) (1977), Battese and Corra (1977) and Meeusen and van den Broeck (1977). Both the stochastic and the deterministic production frontier models can be, and have been, converted to cost and other frontier models.

mulation z_i is assumed to influence y_i indirectly, through its effect on productive efficiency. Exogenous variables do not influence the structure of production possibilities, but they do influence the efficiency with which producers approach their production possibilities. In contrast to the first formulation, it is hoped that z_i and u_i are correlated, since no explanation is achieved if they are not. However in this formulation it is assumed that the elements of z_i are uncorrelated with the elements of x_i. If they are correlated, then estimates of β are biased and inconsistent due to the omission of the relevant variables z_i in the first-stage stochastic production frontier model.

The first formulation achieves no explanation of efficiency variation, while the second formulation provides an explanation at the cost of a potential omitted variables problem. Consequently the second formulation may explain biased efficiency scores because they have been measured relative to a biased representation of the structure of production possibilities. In an effort to overcome the drawbacks of both formulations, Deprins and Simar (1989a, 1989b) proposed a third formulation. Their model is a blend of the first two formulations. They begin with a deterministic production frontier model $\ln y_i = f(\ln x_i; \beta) - u_i$, where $u_i \geqq 0$ represents productive inefficiency. Inefficiency is assumed to depend on z_i according to the relationship $u_i = \exp\{\gamma^T z_i\} + e_i$, where $e_i \sim N(0, \sigma_e^2)$. The exponentiation operator ensures that the nonstochastic component of $u_i \geqq 0$, as required. Substituting yields the one-stage production frontier model $\ln y_i = f(\ln x_i; \beta) - \exp\{\gamma^T z_i\} - e_i$, where $E(-e_i) = 0$. This model is nonlinear in the parameters, and can be estimated by nonlinear least squares, or by MLE if a suitable one-sided distribution for the nonstochastic component of u_i is specified[3].

The third formulation is an improvement on the first formulation, since an explanation of inefficiency is obtained. It is also an improvement on the second formulation, since the omitted variables problem is avoided. The major difficulty with the third formulation is that it is based on a deterministic production frontier model. It would obviously be desirable to reformulate the third model in a stochastic production frontier context. Several recent studies have made progress in this direction.

[3]Deprins and Simar specified four one-sided distributions for the nonstochastic component of u_i, gamma, Weibull, log-normal and log-logistic

Kumbhakar, Ghosh and Mc Guckin (KGM) (1991) specified a stochastic production frontier model of general form $\ln y_i = f(\ln x_i; \beta) + v_i - u_i$, where $v_i \sim N(0, \sigma_v^2)$ and $u_i = \gamma^T z_i + e_i$. Inefficiency has a deterministic component associated with the exogenous variables z_i, and an unobserved random component. To ensure that the deterministic component is nonnegative, it is assumed that $u_i \sim N^+(\gamma^T z_i, \sigma_u^2)$. Note that if $\gamma^T z_i$ is a constant, independent of z_i, this model collapses to Stevenson's (1980) truncated normal stochastic frontier model, which in turn collapses to the ALS (1977) half-normal stochastic frontier model if the constant is zero. Substituting for u_i yields the augmented stochastic frontier model $\ln y_i = f(\ln x_i; \beta) - \gamma^T z_i + (v_i - e_i)$.[4] Although integration of exogenous influences on efficiency is achieved, the additive structure of the inefficiency relationship creates statistical problems. We return to this issue in section 3.

Reifschneider and Stevenson (RS) (1991) formulated a slightly different model, in which $\ln y_i = f(\ln x_i; \beta) + v_i - u_i$, with $v_i \sim N(0, \sigma_v^2)$ and $u_i = g(z_i; \gamma) + e_i$. Since u_i must be nonnegative, $g(z_i; \gamma) + e_i \geqq 0$ is ensured by specifying a functional form for the systematic component of inefficiency such that $g(z_i; \gamma) \geqq 0$, and also by assuming that the random component of inefficiency $e_i \sim N^+(0, \sigma_e^2)$. Thus their model becomes $\ln y_i = f(\ln x_i; \beta) - g(z_i; \gamma) + v_i - e_i$, and is structurally the same as the basic composed-error stochastic frontier model. Since $g(z_i; \gamma) \geqq 0$ is ensured by specification of the functional form for $g(\cdot)$, no modification of the standard MLE procedure is required. The assignment of a one-sided distribution to the random component of inefficiency simplifies estimation of the model by eliminating the statistical problems with the KGM additive formulation. Simplification comes at a cost, however: the conditions $g(z_i; \gamma) \geqq 0$ and $e_i \sim N^+(0, \sigma_e^2)$ are sufficient, but not necessary, for $u_i = g(z_i; \gamma) + e_i \geqq 0$.[5]

[4]The KGM production frontier has a homothetic Cobb-Douglas kernel, and it is imbedded in a system consisting of the production frontier and the first-order conditions for profit maximization. Huang and Liu (1994) developed a model very similar to that of KGM, but they required ui, rather than just its deterministic component, to be nonnegative. The Huang and Liu formulation has been extended to a panel data format by Battese and Coelli (1994). Both of these models are subject to the same statistical problems as the KGM model.

[5]The RS frontier is actually a cost frontier, and they compare three distributions for e_i, half-normal, exponential and gamma.

RS also proposed, but did not estimate, an additional specification of the model. In this specification $\ln y_i = f(\ln x_i; \beta) + v_i - u_i$, with $v_i \sim N(0, \sigma_v^2)$ as before, but with $u_i \sim N^+(0, \sigma_{u_i}^2)$. Thus inefficiency is allowed to vary with variation across producers in the variance of the normal distribution that is truncated below at zero. RS suggested the formulation $\sigma_{u_i} = \sigma_{u_0} + g(z_i; \gamma)$, with $g(z_i; \gamma) \geqq 0$. Mester (1993) estimated a restricted version of this formulation, in which z_i is a zero-one dummy variable assigning producers to one of two categories.

In section 3 we formulate a new stochastic frontier model that incorporates exogenous influences on efficiency. Our formulation differs from each of the above formulations in that it has a multiplicative, rather than an additive, structure. The multiplicative formulation overcomes the statistical problems that plague the additive formulations just discussed. It does, however, bear a resemblance to the previous formulations, particularly to the RS-Mester formulation.

3. The Model and its Estimation

3.1 The Model

We begin with the basic stochastic production frontier model

$$\ln y_i = f(\ln x_i; \beta) + v_i - u_i, \qquad i = 1, \ldots, I, \tag{1}$$

where $y_i \in I\!R_+$ is output, $x_i \in I\!R_+^n$ is an input vector, and β is a vector of parameters describing the structure of technology. We assume that $v_i \sim N(0, \sigma_v^2)$, and that $u_i \sim D^+(\cdot, \cdot)$, with $u_i \perp v_i$. We assume that inefficiency u_i is associated with a vector of exogenous variables $z_i \in I\!R^m$ by means of the relationship

$$u_i = \exp\{\gamma^T z_i\}.\eta_i \tag{2}$$

with $\eta_i \geqq 0, E(\eta_i) = 1, V(\eta_i) = \sigma_\eta^2$, and the η_i are i.i.d. Note that $u_i \geqq 0$ as required. It follows from the assumptions on η_i that $E(u_i) = \exp\{\gamma^T z_i\}$, that $V(u_i) = \exp\{2\gamma^T z_i\}\sigma_\eta^2$, and that $C(u_i) = \sigma_\eta$. Thus the coefficient of variation of u_i is independent of i. However the variance of u_i depends on i, and so corresponds to the second specification proposed by Reifschneider and Stevenson (1991) and generalizes the specification estimated by Mester (1993).

Inserting (2) into (1) yields the stochastic frontier model incorporating exogenous influences on efficiency

$$\ln y_i = f(\ln x_i; \beta) - \exp\{\gamma^T z_i\}.\eta_i + v_i, \tag{3}$$

which can be written as

$$\ln y_i = f(\ln x_i; \beta) - \exp\{\gamma^T z_i\} + \varepsilon_i, \tag{4}$$

where

$$\begin{aligned} \varepsilon_i &= \ln y_i - E(\ln y_i) \\ &= v_i - \exp\{\gamma^T z_i\}.(\eta_i - 1) \end{aligned}$$

The ε_i are independently, but not identically, distributed, with

$$\begin{aligned} &E(\varepsilon_i) = 0 \\ &E(\varepsilon_i^2) = \sigma_\varepsilon^2 = \sigma_v^2 + \exp\{2\gamma^T z_i\}.\sigma_\eta^2 \\ &E(\varepsilon_i^3) = -\exp\{3\gamma^T z_i\}.E\big[(\eta - 1)^3\big] \end{aligned} \tag{5}$$

Equation(4) is the stochastic frontier model to be estimated. Its distinguishing feature is the way in which it incorporates exogenous factors influencing efficiency, as described in equation (2). In this specification u_i is a "scale" transformation of some underlying process η_i. This is a more convenient transformation than the "location" transformation $\big[u_i = \exp\{\gamma^T z_i\} + \eta_i\big]$ used by Deprins and Simar (1989a, 1989b) in a deterministic frontier setting, and later by KGM (1991), Huang and Liu (1994) and Battese and Coelli (1994) in a stochastic frontier setting. This is because with the location transformation the requirement that $u_i \geqq 0$ requires that $\eta_i \geqq -\exp\{\gamma T z_i\}$, which implies that the η_i are not i.i.d. A similar problem arises in the deterministic frontier model with a location transformation and with η_i distributed as half normal, exponential or Gamma ($P < 2$), as has been noted previously by Greene (1980,1993).

3.2 Estimation by Least Squares

The stochastic frontier model incorporating exogenous influences on efficiency as specified in equation (4) may be estimated by nonlinear least squares (NLLS) by means of

$$(\hat{\beta}, \hat{\gamma}) = \arg\min \sum_{i=1}^{I} \big[\ln y_i - f(\ln x_i; \beta) + \exp\{\gamma^T z_i\}\big]^2 \tag{6}$$

These estimators are consistent, although not efficient, since $E(\varepsilon_i^2) = \sigma_\epsilon^2$ is not independent of i as shown in equation (5). The quality of the estimators can be improved by the use of nonlinear weighted least squares (NLWLS), in which case

$$(\hat{\beta}, \hat{\gamma}) = \arg\min \sum_{i=1}^{I} \left[\frac{\ln y_i - f(\ln x_i; \beta) + \exp\{\gamma^T z_i\}}{\sigma_\varepsilon} \right]^2 \tag{7}$$

But since σ_ε is unknown, the problem is to construct a feasible NLWLS. This requires that we specify a distribution for $\eta \sim D^+(1, \sigma_\eta^2)$.[6]

We begin by supposing that $\eta \sim \Gamma(P, \theta)$. Then

$$\begin{aligned} f(\eta) &= \frac{\theta^P \eta^{P-1} e^{-\theta\eta}}{\Gamma(P)} \\ E(\eta) &= \frac{P}{\theta} \Rightarrow P = \theta \qquad &&(\text{ since } \; E(\eta) = 1) \\ V(\eta) &= \frac{P}{\theta^2} = \frac{1}{\theta} &&(\text{ since } \; P = \theta) \\ E\Big[\big(\eta - E(\eta)\big)^3\Big] &= \frac{2P}{\theta^3} = \frac{2}{\theta^2} &&(\text{ since } \; P = \theta) \end{aligned} \tag{8}$$

These properties imply that

$$\begin{aligned} E(\varepsilon_i) &= 0 \\ E(\varepsilon_i^2) &= \sigma_v^2 + \exp\{2\gamma^T z_i\}.(\frac{1}{\theta}) \\ E(\varepsilon_i^3) &= -\exp\{3\gamma^T z_i\}.(\frac{2}{\theta^2}) \end{aligned} \tag{9}$$

A consistent estimate of σ_ε may now be obtained from the NLLS residuals by means of

$$\begin{aligned} &\frac{1}{I} \sum_{i=1}^{I} \frac{\hat{\varepsilon}_i^3}{\exp\{3\hat{\gamma}^T z_i\}} = \frac{-2}{\hat{\theta}^2} \;\Rightarrow \hat{\theta} \\ &\frac{1}{I} \sum_{i=1}^{I} \left[\hat{\varepsilon}_i^2 - \frac{\exp\{2\hat{\gamma}^T z_i\}}{\hat{\theta}} \right] = \hat{\sigma}_v^2 \end{aligned} \tag{10}$$

which, together with $\hat{\gamma}$, provide the estimate of

$$\hat{\sigma}_\varepsilon = \left[\hat{\sigma}_v^2 + \exp\{2\hat{\gamma}^T z_i\}. \left(\frac{1}{\hat{\theta}} \right) \right]^{\frac{1}{2}} \tag{11}$$

[6] Alternatively, it is possible to use an iterative technique. In the first step, NLLS generates an estimate of σ_ε, which is used in the second and succeeding steps in NLWLS.

to be used in NLWLS. A potential difficulty with the Gamma specification is that there is no guarantee that the left side of the first equation in (10) is negative. However this possibility provides a useful diagnostic test of the suitability of the Gamma specification.

The procedure is greatly simplified if we assume that η is exponentially distributed, since then $E(\eta) = 1 \ \wedge \ P = \theta \ \Rightarrow \theta = 1$, and there is no exponential parameter to be estimated. Proceeding as in the Gamma case, the first equation in (10) is dropped, and the second equation in (10) and the estimate of σ_ε in (11) are simplified by the deletion of $\hat{\theta}$.

When $\eta \sim \Gamma(\theta, \theta)$, it follows that $u_i \sim \Gamma\big(\theta, \theta \exp\{-\gamma^T z_i\}\big)$, which demonstrates clearly the sense in which our model generalizes the Gamma model of Greene (1990), in which (P, θ) are now $\big(\theta, \theta \exp\{-\gamma^T z_i\}\big)$. Similarly, when $\eta \sim$ exponential (1), $u_i \sim$ exponential $\big(\exp\{\gamma^T z_i\}\big)$, which generalizes the exponential model of Meeusen and van den Broeck (1977).

We next consider a second nested pair of distributions for η, truncated normal and half normal. If $\eta \sim N^+(\mu, \sigma^2)$, then

$$
\begin{aligned}
&f(\eta) = \Big[\Phi(\frac{\mu}{\sigma})\Big]^{-1} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}(\frac{\eta-\mu}{\sigma})^2} \\
&E(\eta) = 1 = \mu + c\sigma \Rightarrow c = \frac{\phi(\mu/\sigma)}{\Phi(\mu/\sigma)} \\
&\sigma_\eta^2 = V(\eta) = \sigma^2 + \mu - 1 = \sigma^2 - c\sigma \\
&E\Big[\big(\eta - E(\eta)\big)^3\Big] = -c\sigma^3 + c^2\sigma^2 + c\sigma
\end{aligned}
\tag{12}
$$

This specification is unattractive for practical purposes because the constraint $E(\eta) = 1$ is not solved analytically as in the Gamma and exponential cases, but instead must be solved numerically, with c being defined in terms of the p.d.f. and the c.d.f. of the standard normal evaluated at (μ/σ). Given this difficulty, however, it is possible to proceed as above to obtain from equations (12)

$$
\begin{aligned}
&E(\varepsilon_i) = 0 \\
&E(\varepsilon_i^2) = \sigma_v^2 + \exp\{2\gamma^T z_i\}.(\sigma^2 - c\sigma) \\
&E(\varepsilon_i^3) = -\exp\{3\gamma^T z_i\}.(-c\sigma^3 + c^2\sigma^2 + c\sigma)
\end{aligned}
\tag{13}
$$

Using the NLLS results, the two equations

$$\begin{aligned} &\frac{1}{I}\sum_{i=1}^{I}\left(\frac{\hat{\varepsilon}_i}{\exp\{\gamma^T z_i\}}\right)^3 = (-c\sigma^3 + c^2\sigma^2 + c\sigma) \\ &\mu + c\sigma = 1 \end{aligned} \tag{14}$$

provide estimates $\hat{\mu}$ and $\hat{\sigma}$. These in turn yield an estimate $\hat{c}$. Together these yield an estimate of σ_v^2 via

$$\hat{\sigma}_v^2 = \frac{1}{I}\sum_{i=1}^{I}\left[\hat{\varepsilon}_i^2 - \exp\{2\hat{\gamma}^T z_i\}.(\hat{\sigma}_\eta^2 - \hat{c}\hat{\sigma}_\eta)\right] \tag{15}$$

This provides all the information required to estimate σ_ε, which is used in NLWLS.

Again the procedure is simplified if we restrict the truncated normal distribution for η to be half normal, in part because there is no half normal parameter to be estimated, but more importantly because it is possible to solve analytically for $E(\eta) = 1$. If $\eta \sim N^+(0, \sigma^2)$ then $\mu = 0 \Rightarrow c = \sqrt{2/\pi}$ and $\sigma = \sqrt{\pi/2}$, so that $\sigma_\eta^2 = (\pi/2) - 1$. Hence it follows immediately that

$$E(\varepsilon_i^2) = \sigma_v^2 + \exp\{2\gamma^T z_i\}.\left(\left(\frac{\pi}{2}\right) - 1\right)$$

and

$$\hat{\sigma}_v^2 = \frac{1}{I}\sum_{i=1}^{I}\left(\hat{\varepsilon}_i^2 - \exp\{2\hat{\gamma}^T z_i\}.\left(\left(\frac{\pi}{2}\right) - 1\right)\right)$$

which provides all the necessary information required to estimate σ_ε and perform NLWLS.

When $\eta \sim N^+(\mu, \sigma^2)$ with the restriction $\mu + c\sigma = 1$, it follows that $u_i \sim N^+(\mu_i, \sigma_i^2)$ with $\mu_i = \mu\exp\{\gamma^T z_i\}$ and $\sigma_i = \sigma\exp\{\gamma^T z_i\}$, subject to the same restriction ($\mu + c\sigma = 1$). This generalizes Stevenson's (1980) truncated normal model. When $\eta \sim N^+(0, \sigma^2)$ we have $u_i \sim N^+\left(0, \sqrt{\pi/2}\, \exp\{\gamma^T z_i\}\right)$, which generalizes the ALS (1977) half normal model.

3.3 Estimation by Maximum Likelihood

More efficient estimators of β and γ can be obtained by using maximum likelihood techniques. In the Gamma case, the loglikelihood for one observation i can be obtained from Greene (1990) where (P, θ) are replaced by $\big(\theta, \theta \exp\{-\gamma^T z_i\}\big)$. However the general Gamma case requires difficult numerical optimization[7] and so we consider only its special case: $\theta = 1$ (exponential). Letting[8] $e_i = v_i - u_i = y_i - f(\ln x_i; \beta)$, we have

$$\begin{aligned}\ln f(e_i) = &- \gamma^T z_i + \exp\{-\gamma^T z_i\}\big(y_i - f(\ln x_i; \beta)\big) + \frac{\sigma_v^2}{2} \exp\{-2\gamma^T z_i\} \\ &+ \ln \Phi\left(-\frac{y_i - f(\ln x_i; \beta) + \sigma_v^2 \exp\{-\gamma^T z_i\}}{\sigma_v}\right)\end{aligned} \tag{16}$$

and so

$$(\hat{\beta}, \hat{\gamma}, \hat{\sigma}_v) = \arg\max \sum_{i=1}^{I} \ln f(e_i) \tag{17}$$

In the truncated normal case, $\ln f(e_i)$ can be obtained from Stevenson (1980), where (μ, σ) is replaced by $\big(\mu \ \exp\{\gamma^T z_i\}, \sigma \ \exp\{\gamma^T z_i\}\big)$ with the additional constraint $\mu + c\sigma = 1$.

Then we have

$$\begin{aligned}\ln f(e_i) = &- \ln \Phi\left(\frac{\mu}{\sigma}\right) - \frac{1}{2} \ln\left(\sigma_v^2 + \sigma^2 \exp\{2\gamma^T z_i\}\right) \\ &+ \ln \phi\left(\frac{y_i - f(\ln x_i; \beta) + \mu \exp\{\gamma^T z_i\}^{\frac{1}{2}}}{\left(\sigma_v^2 + \sigma^2 \exp\{2\gamma^T z_i\}\right)}\right) \\ &+ \ln \Phi\Bigg(\left(\sigma_v^2 + \sigma^2 \exp\{2\gamma^T z_i\}\right)^{-\frac{1}{2}} \times \\ &\quad \times \left(\frac{\sigma_v}{\sigma \exp\{\gamma^T z_i\}\mu} - \frac{\sigma \exp\{\gamma^T z_i\}}{\sigma_v}\Big(y_i - f(\ln x_i; \beta)\Big)\right)\Bigg)\end{aligned} \tag{18}$$

and so

$$\begin{aligned}(\hat{\beta}, \hat{\gamma}, \hat{\mu}, \hat{\sigma}, \hat{\sigma}_v) &= \arg\max \sum_{i=1}^{I} \ln f(e_i) \\ s.t. \qquad \mu + &\frac{\phi(\mu/\sigma)}{\Phi(\mu/\sigma)}.\sigma = 1\end{aligned} \tag{19}$$

In the half normal case the problem is easy to solve since $\mu = 0$ and $\sigma = \sqrt{\pi/2}$, but in the general case the optimization problem (19) is not to easy to solve due to

[7]Numerical optimization problems associated with the Gamma distribution have been noted by Greene (1990) and Ritter and Simar (1994).

[8]In the case of a cost frontier, $e_i = v_i + u_i$. So in this case, the formulae of the loglikelihood are still valid replacing $(y - f(\ln x_i; \beta))$ by its reverse $(f(\ln x_i; \beta) - y_i)$ since v_i is symmetric.

the nonlinear contraint. This difficulty can be overcome when there is a constant term in z_i by reparameterizing the loglikelihood (18) as follows: let $\gamma^T = (\gamma_0\ \gamma_1^T)$ and $z_i^T = (1\ z_{i,1}^T)$; then $u_i = \exp\{\gamma^T z_i\}\eta = \exp\{\gamma_1^T z_{i,1}\}\tilde{\eta}$ where now $E(\tilde{\eta}) = e^{\gamma_0}$ and $V(\tilde{\eta}) = e^{2\gamma_0}\sigma_\eta^2$.

Therefore we have again $\mu_i \sim N^+(\mu_i, \sigma_i^2)$ but now, $\mu_i = \tilde{\mu}\ \exp\{\gamma_1^T z_{i,1}\}$ and $\sigma_i = \tilde{\sigma}\exp\{\gamma_1^T z_{i,1}\}$. In this parameterization $\tilde{\mu} = e^{\gamma_0}\mu$ and $\tilde{\sigma} = e^{\gamma_0}\sigma$ are unconstrained, since $E(\tilde{\eta}) = \tilde{\mu} + c\tilde{\sigma} = e^{\gamma_0}$ and γ_0 is free. Writing

$$
\begin{aligned}
\ln f(e_i) = &- \ln \Phi\left(\frac{\tilde{\mu}}{\tilde{\sigma}}\right) - \frac{1}{2}\ \ln\big(\sigma_v^2 + \tilde{\sigma}^2\ \exp\{2\gamma_1^T z_{i,1}\}\big) \\
&+ \ln\phi\left(\frac{y_i - f(\ln x_i;\beta) + \tilde{\mu}\ \exp\{\gamma_1^T z_{i,1}\}}{\big(\sigma_v^2 + \tilde{\sigma}^2 \exp\{2\gamma_1^T z_{i,1}\}\big)^{\frac{1}{2}}}\right) \\
&+ \ln\Phi\Bigg(\big(\sigma_v^2 + \tilde{\sigma}^2\ \exp\{2\gamma_1^T z_{i;1}\}\big)^{-\frac{1}{2}} \times \\
&\quad \times \left(\frac{\sigma_v}{\tilde{\sigma}\ \exp\{\gamma_1^T z_{i,1}\}}\tilde{\mu} - \frac{\tilde{\sigma}\ \exp\{\gamma_1^T z_{i,1}\}}{\sigma_v}\Big(y_i - f(\ln x_i;\beta)\Big)\right)\Bigg)
\end{aligned}
\tag{20}
$$

we have

$$
(\hat{\beta}, \hat{\gamma}_1, \hat{\tilde{\mu}}, \hat{\tilde{\sigma}}, \hat{\sigma}_v) = \arg\max \sum_{i=1}^{I} \ln f(e_i) \tag{21}
$$

and finally

$$
\gamma_0 = \ln\left[\tilde{\mu} + \frac{\phi(\frac{\tilde{\mu}}{\tilde{\sigma}})}{\Phi(\frac{\tilde{\mu}}{\tilde{\sigma}})}.\tilde{\sigma}\right] \tag{22}
$$

4. Summary and Conclusions

The problem of incorporating exogenous influences on efficiency has not previously been solved. As Battese and Coelli (1994) note, the two - stage procedure of estimating efficiency, and then regressing estimated efficiencies against a vector of exogenous variables, involves an inconsistency in the assumptions made on the inefficiency error component. The multiplicative approach of Deprins and Simar (1989a, 1989b) avoids this inconsistency, but at the cost of assuming a deterministic frontier. The general approach followed by KGM (1991), Reifschneider and Stevenson (1991), Huang and Liu (1994) and Battese and Coelli (1994) sensibly incorporates the exogenous influences directly into the inefficiency error component, but it does so in an additive manner which causes statistical problems. In this paper we have extended the Deprins and Simar multiplicative formulation from a deterministic frontier model to a stochastic frontier model. Such a formulation avoids the statistical problems encountered in the additive formulation. In Section 3 we showed how to estimate the model, under a variety of distributional assumptions, using either least squares or maximum likelihood techniques. In a forthcoming paper we provide an empirical illustration of both estimation techniques.

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