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A geometrical approach to assess levels of structural robustness in statically indeterminate concrete elements

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Abstract

Graphic statics represents a powerful tool in the field of structural design, because the reciprocity between form and force diagrams allows constant and simultaneous control of both the geometry and magnitude of internal forces. The 21st century reinterpretation of graphic statics within interactive parametrical software has allowed its extension; in particular, admissible geometrical domains have been developed, which represent the set of permissible positions for the nodes of the force diagram. In the context of robustness assessment, admissible geometrical domains can indicate the range of alternative force distributions that can be activated in the structure in case of an unexpected event (such as loss of structural members, support displacement or increase of applied loads). This paper shows how computerized graphic statics can be employed to indicate the level of robustness of statically indeterminate reinforced concrete structures. The analysis of a simple case study highlights the set of potential force redistributions that can be activated in the structure, as well as their relative efficiency. The influence of the reinforcement layout on the range of possible force redistributions inside concrete is also outlined.

Keywords: structural robustness, structural design, graphic statics, strut-and-tie modelling, reinforced concrete, static indeterminacy, plastic theory.

1. Introduction

Structural robustness can be defined as a structure's aptitude to survive unforeseen or unexpected events, without being damaged to an extent that is disproportionate to the original cause (EN1990 [1]). This property is fundamental in ensuring human safety. Robustness became a compulsory Eurocode requirement after the terrorist attacks of September 11, 2001 (Gulvanessian and Vrouwenvelder [2]).

Many types of robustness assessment approaches are found nowadays in the literature (see Cavaco *et al.* [3] or Sorensen *et al.* [4] for more details). These approaches are all based on a numerical analysis of the structure's capacities for redistribution and resistance, calculated for given failure scenarios and for a design that has been determined beforehand. This leads to two main drawbacks. First, predefining potential failure scenarios does not allow one to cover the total range of events that could happen during the structure's lifetime. Secondly, applying numerical methods to previously designed structures makes these methods difficult to appropriate by project authors. Moreover these methods do not allow interaction with the design in seeking for optimal robustness performance.

To overcome these limitations, the authors of this paper have worked on a complementary approach to structural robustness, based on a geometrical rather than a numerical assessment (Zastavni *et al.* [5]). The approach enlarges the original method of graphic statics, in order to obtain a visual, interactive and damage independent robustness indicator: the admissible geometrical domain, which shows the set of potential force redistributions inside given structures. Section 2 first reviews graphic statics' main hypotheses. It then details how structural constraints can be added to the original method and, finally, shows how these constraints can be used to build the admissible geometrical domains that indicate the level of robustness. Section 3 illustrates the geometrical robustness analysis of a statically indeterminate reinforced concrete beam.

2. Geometrical robustness assessment

2.1. Graphic statics

Graphic statics was a method established in the 19th and 20th century for structural analysis (Maxwell [6]), further applied to trusses, arches and beams. The principle of graphic statics is to use a common framework to represent both the structure and its internal forces. For continuous structures, these forces are obtained from replacing the structure's principal internal stresses by their resultant normal force, forming what is called a strut-and-tie network (Muttoni *et al.* [7]). The strut-and-tie network gives graphic statics' form diagram *D*, from which the reciprocal force diagram D^* is drawn such as to respect (Zalewski and Allen [8]):

- parallelism: each bar i of the form diagram is related to a unique reciprocal bar i^* in the force diagram, such that the difference in angle between them is always zero;
- equilibrium: all the bars that connect in a same node in one of the diagrams have reciprocal bars that form a closed polygon in the other. Inversely, all the bars belonging to a same polygon in one of the diagrams must join in a unique node in the other;
- boundary: all bars and nodes of the strut-and-tie network have to remain entirely inside the defined structural envelope, and constraints and displacements have to be respected at nodes where external or support loads are applied;
- yield: material resistance must not be exceeded in any part of the structure. This consists in verifying that length L_{i^*} of each bar i^* of the force diagram conforms to:

  $$L_{i*} \leq F_i = \sigma_e A_i \tag{1}$$

  where F_i is the maximum axial force in bar i supposing that i is uniformly and fully stressed, σ_e is the material effective strength supposing a perfect rigid-plastic and thus ductile material behavior (Kostic [9]), and A_i is the areas chosen for the bars of the strut-and-tie network. Yield criterion makes strut-and-tie modeling applicable to any type of structure composed of any kind of material.

2.2. Constraint-based graphic statics

Graphic statics can be coupled to a set of geometrical constraints, which allow the conditional manipulation of the nodes in both reciprocal diagrams (Fivet [10]). These constraints are used to define affine relationships between the diagrams' nodes, such as relative position, proximity or equidistance. For example, considering the structure of Figure 1, the dark grey surface in force diagram D^* gives the set of positions that can be taken by the highlighted red node to ensure that:

- force magnitudes in the strut-and-tie network are below 10kN, defining the circular boundaries of the grey area;
- the strut-and-tie network remains inside the dashed rectangle in Figure 1's form diagram *D*, defining the linear boundaries of the grey area.

If the red node were to be given another position inside D^*'s dark grey area, this would result in drawing an alternative reciprocal strut-and-tie network in form diagram *D* that would still respect the abovementioned constraints. Hence, *D**'s dark grey area can be seen as a representation of the set of alternative force distributions that can happen in the structure, given some initial design constraints.

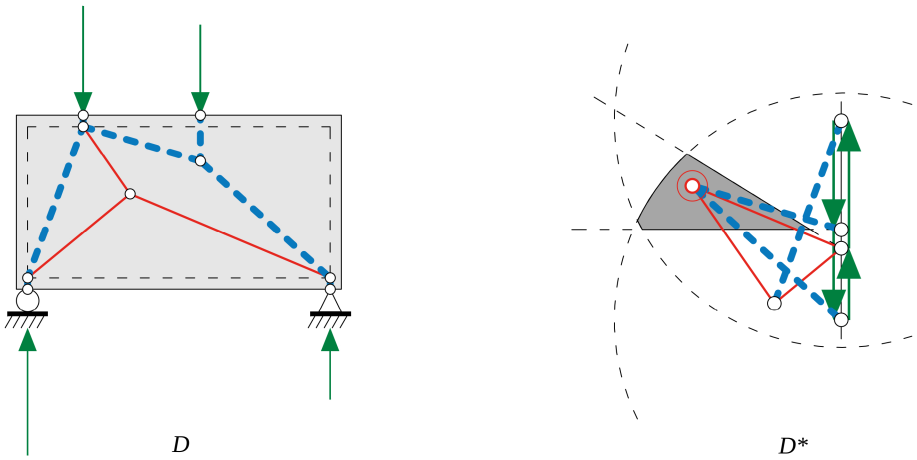

Figure 1: Constraining nodal positions in graphic statics' reciprocal diagrams (with struts in dashed blue and ties in plain red)

2.3. Admissible geometrical domains

In the context of robustness assessment, the set of possible force distributions inside a structure, given by force diagram's dark grey area, can be used to depict the structure's aptitude to find alternative structural workings in case of hazard. These force distributions are safe solutions to the structural problem: since they verify the conditions mentioned in Section 2.1, they satisfy the Lower Bound Theorem of Plastic Theory (Heyman [11]).

Sets of alternative structural workings will be called admissible geometrical domains. These domains can be used as early indicators of robust design: the larger their area, the more potential force distributions there are and the more robust the structure will be. The geometrical approach to robustness can thus be used by practitioners to gain insight into the redistribution properties of the design they are currently working on. As it is visual and potentially interactive, the method can help comparing design options and help orienting early design. Moreover, admissible geometrical domains are damage independent robustness indicators: thus damage (loss of a structural member, support displacement, increase of applied loads, etc) that could potentially happen during the structure's lifetime does not have to be predicted.

It has been shown in Section 2.2 that alternative force distributions can be reached in a structure by changing nodal positions in the strut-and-tie network, thus changing the network's shape. Another manner of redistributing forces inside a structure is to change nodal connections in the strut-and-tie network, thus changing the network's topology. The combination of both shape and topology problems is known as the layout problem, meaning a change in nodal position and connectivity at the same time (Rothwell [12]). In this case, the structural problem becomes highly complex: as an unbounded number of strut-and-tie networks are solutions, admissible geometrical domains are difficult to obtain. However, in practice, the process of modelling the states of internal forces as strut-and-tie networks can often result in a shape or a topology problem only, depending on the structure to be analyzed. For example, designing unreinforced masonry structures for robustness is reduced to a shape problem, as only funicular paths of internal forces can be activated in the structure (Deschuyteneer *et al.* [13]). It will here be shown how the topology problem can be implemented for the geometrical robustness assessment of reinforced concrete deep structures.

3. Case study example

3.1. Description of the structure

The example developed here consists in a reinforced concrete deep beam, which is 2.1m long, 1.1m high and 20cm thick (Figure 2). It is simply supported, and loaded at the top by a vertical point load Q = 10kN. For the sake of the exercise, steel rebars are placed inside the beam as depicted by the layout of dashed lines $F_{i=0...10}$, the outer ones being placed at 5cm from the edge of the beam.

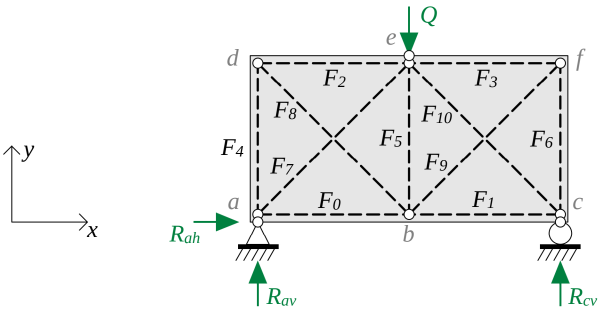

Figure 2: The structure under consideration with its nodal grid and candidate bars

Following the graphic statics approach, internal stresses inside such a continuous structure can be represented by a resultant strut-and-tie network. The set of valid strut-and-tie networks defines the structure's admissible geometrical domain, depicting its level of robustness in terms of force redistributions. In the present case, possible paths for the networks' ties are limited to locations where rebars are present, according to the layout of dashed lines $F_{i=0...10}$. Supposing the same hypothesis for struts, this means that the set of possible strut-and-tie networks can be described by a unique topology, composed of 6 nodes and 11 bars. Assessing the structure's level of robustness thus consists in finding the set of internal force distributions that follow the given topology, and for which material resistance is not exceeded in any bar. To speed up the analysis of the structural problem, a numerical pre-processing will first be applied in order to obtain the limit combinations of internal forces, as these will be the only pertinent ones for obtaining the structure's admissible geometrical domain.

3.2. Structural analysis through equilibrium matrix and Gauss-Jordan elimination

The structure's equilibrium equations are first written in a matrix format such that (A | b) x = 0. This equilibrium matrix possesses 12 lines, one for each node's –x and –y directions. It has 14+1 columns, one per bar F_i, one per reaction force R_i and one for the external load Q = 10kN:

$$
\begin{array}{c}
\left[\begin{array}{ccccccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0.71 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.71 & 0.71 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0.71 & 0.71 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.71 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0.71 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -0.71 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & -0.71 & 0 & 0 & 0.71 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -0.71 & 0 & 0 & -0.71 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -0.71 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -0.71 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\\
\underbrace{\hspace{12em}}_{F_i} \quad \underbrace{\hspace{4em}}_{R_i} \quad Q
\end{array}
\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ R_{ah} \\ R_{av} \\ R_{cv} \\ 10 \end{bmatrix} = 0 \qquad (2)
$$

Matrix (2) is composed of 12 equations (12 lines) and has 14 unknowns (11 F_i's and 3 R_i's). Thus it is $14 - 12 = 2$ times indeterminate. To solve the associated system, two indeterminate unknowns can be chosen, let them be F_0 and F_1 (as highlighted in green in both matrices (2) and (3)). This means that all the other internal forces $F_{i=2...10}$ and reaction forces R_i can be expressed as a function of F_0, F_1 and Q only. To obtain the equilibrium equations under this simplified form, a Gauss-Jordan elimination can be performed on matrix (2), for which the two columns corresponding to the indeterminate unknowns have been placed just before the column of the external loads. This gives:

$$\begin{array}{c}\left[\begin{array}{cccccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0.5 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0.5 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0.5 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1.41 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1.41 & 0 & -0.71 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1.41 & -0.71 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1.41 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -0.5
\end{array}\right] \\ \hspace{18em}\mathbf{a}\quad\;\mathbf{b}\quad\;\mathbf{c}\end{array}
\left[\begin{array}{c} F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ R_{ah} \\ R_{av} \\ R_{cv} \\ F_0 \\ F_1 \\ 10 \end{array}\right] = 0 \qquad (3)$$

To guarantee admissible sets of internal force distributions inside the topology, material resistance must not be exceeded in any part of the structure. This means that the following inequality has to be verified for each bar F_i of the network:

$$-F_{C,max} \leq F_i \leq F_{T,max} \qquad (4)$$

with $F_{C,max}$ as the maximum axial force in the struts, and $F_{T,max}$ these in the ties. These forces depend on the strength class and on the cover for concrete struts, and on the number and diameter of rebars for steel ties. For C25/30 concrete (f_{cd}^* = 14MPa) and for circular struts the radius of which equals the cover (e=5cm) we find $F_{c,max}$ = 110kN; for 2Ø12 BE500s steel rebars (f_y = 435MPa) we have $F_{T,max}$ = 98.4kN. Based on matrix (3), inequality (4) can be rewritten for bars $F_{i=2...10}$ as:

$$-F_{C,max} \leq F_i = -(aF_0 + bF_1 + cQ) \leq F_{T,max} \qquad (5)$$

with coefficients *a*, *b*, and *c* given in the last three columns of matrix (3). Note that the same reasoning could be applied for constraining reaction forces R_i between specific values. However, for the sake of the present example, these forces will remain unbounded. (4) and (5) thus give 11 pairs of inequalities, which have to be verified by the combinations of internal forces in each strut-and-tie in order to be admissible in terms of internal forces. From a geometrical point of view, these inequalities car be turned into half plane equations in a 2D graph (the dashed lines in Figure 3), with the –x axis standing for the force inside bar F_0 and the –y axis for those in F_1. The light grey surface of Figure 3, formed inside the dashed boundary lines, then depicts the set of valid combinations (F_0, F_1) which are solutions to the structural problem. In particular, the six corner points of this surface are associated with (F_0, F_1) combinations for which the corresponding strut-and-tie network has reached yielding in two bars.

The light grey surface of Figure 3 can further be analyzed, by assessing the efficiency of the strut-and-tie networks associated to each of its valid point coordinates (F_0, F_1). The efficiency is obtained from the formula given by Maxwell in 1890 [14], later reinterpreted by Baker *et al*. [15]:

$$\sum_i L_{t,i}F_{t,i} + \sum_j L_{c,j}F_{c,j} = \sum_i L_{t,i}A_{t,i}\frac{F_{t,i}}{A_{t,i}} + \sum_j L_{c,j}A_{c,j}\frac{F_{c,j}}{A_{c,j}} = V_t\sigma_{e,t} + V_c\sigma_{e,c} \quad (6)$$

This means that the sum of the products of each bar length in tension in the form diagram $L_{t,i}$ by its corresponding length (force) in the force diagram $F_{t,i}$, plus the same sum of products for compressive bars (with *c* indices), is directly proportional to the volume of material *V* that has to be implemented in the strut-and-tie network to carry the applied load. As in equation (1), $\sigma_{e,t}$ and $\sigma_{e,c}$ values give material effective strength for both steel and concrete; $A_{t,i}$ represents the rebars area and $A_{c,j}$ the maximum circular area that can be given to the concrete struts (which is a function of the cover). Equation (6) then helps ranking the set of possible internal force combinations in the strut-and-tie by order of efficiency: the less volume needed the more efficient the structure is. Efficiency is given by the polygonal curves in Figure 3, with the associated values in [kNm]. Equation (6) also helps assessing that combination ($F_0 = F_1 = 5kN$) gives the most efficient path for internal forces among the set of valid ones: this path is the one associated with a minimum volume of material, thus giving the smallest solution to equation (6).

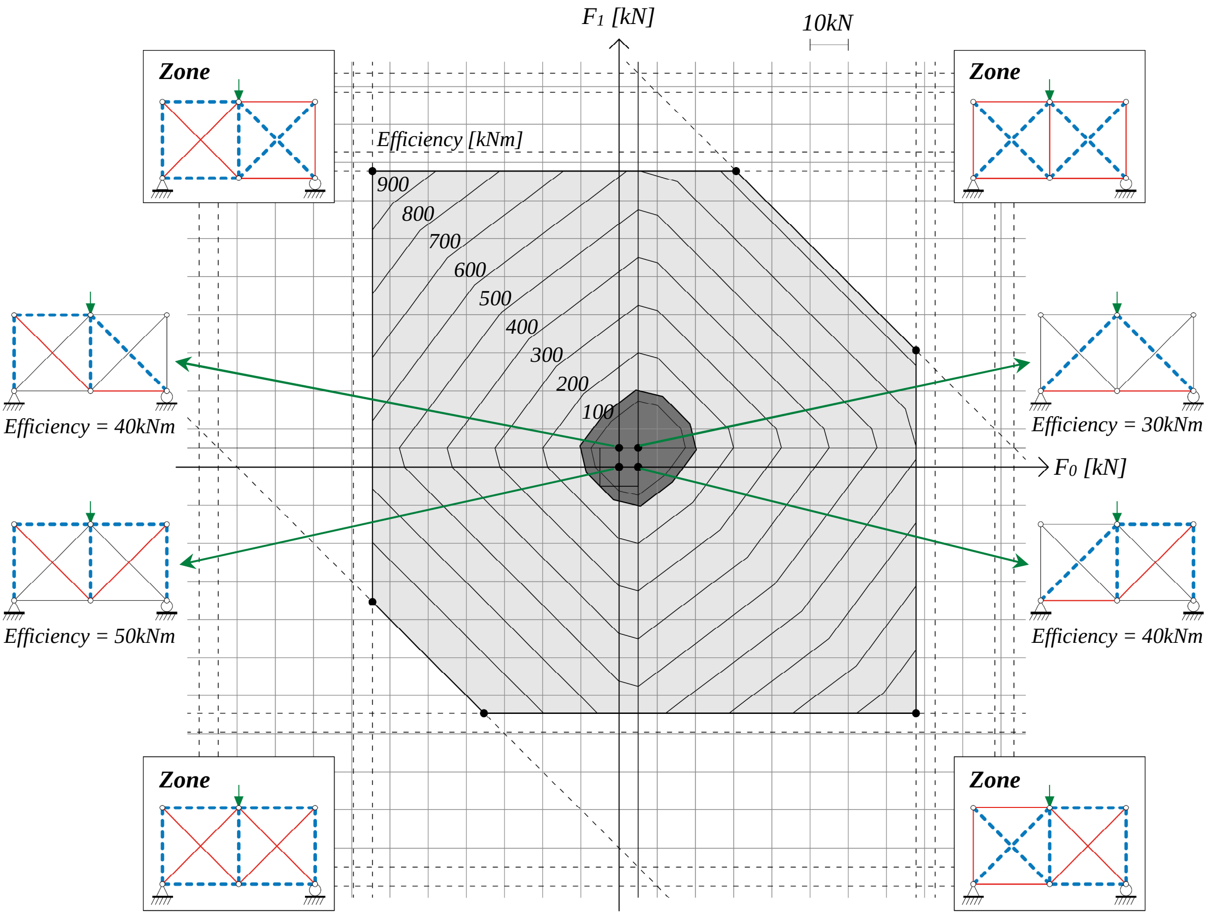

Figure 3: The solution surface associated with the structural problem, representative combinations of internal forces inside the strut-and-tie network topology and related curves of efficiency

The solution surface of Figure 3 can be used to relate the location of rebars in concrete to the level of robustness of the associated structure in terms of force redistributions. Here Figure 3 has been obtained for a structure in which tensile forces can be activated in any bar of the initial strut-and-tie topology (see Figure 2). If the rebar layout was to be limited to bars F_0, F_1, F_8, and F_9 only (corresponding to a classical rebar layout for this type of structure), the solution space would be limited to the square area comprised between the four central nodes of the solution surface. Hence, this reduced rebar layout would still provide some redistribution properties to the structure, even if at a lower level.

Figure 3 also helps knowing whether a given combination of internal forces is relevant or penalized by the self-stress state. Relevant combinations are defined as the ones at least as efficient as the one for which forces F_0 and F_1 are opposite to those in the most efficient strut-and-tie network. For the present structure, the combination of internal forces leading to maximal efficiency (30kNm) is the one for which $F_0 = F_1 = 5kN$. Thus pertinent combinations of internal forces are the ones at least as efficient as the solution for which $F_0 = F_1 = -5kN$. The associated solution sub-surface is given by the dark grey area in Figure 3.

3.3. Conversion into admissible geometrical domains

To assess geometrically the capacities for force redistributions of the given structure, the 6 corner points of the solution surface of Figure 3 give the 6 possible limit combinations of internal forces (F_0, F_1). These are associated with the 6 strut-and-tie networks that determine the boundaries of the admissible geometrical domain. In this case, the domain is composed of 8 segments – some are partly superimposed –, as depicted in light grey in Figure 4's force diagram *D**. This domains represents the capacity of the given structure to redistribute internal forces. The strut-and-tie network represented in Figure 4's form diagram *D* is associated to the bottom right point of Figure 3's light grey solution surface, reached when F_0 = 7.775kN and F_1 = -6.458kN. This solution yields bar F_7 in compression (in light blue) and bar F_9 in tension (in orange).

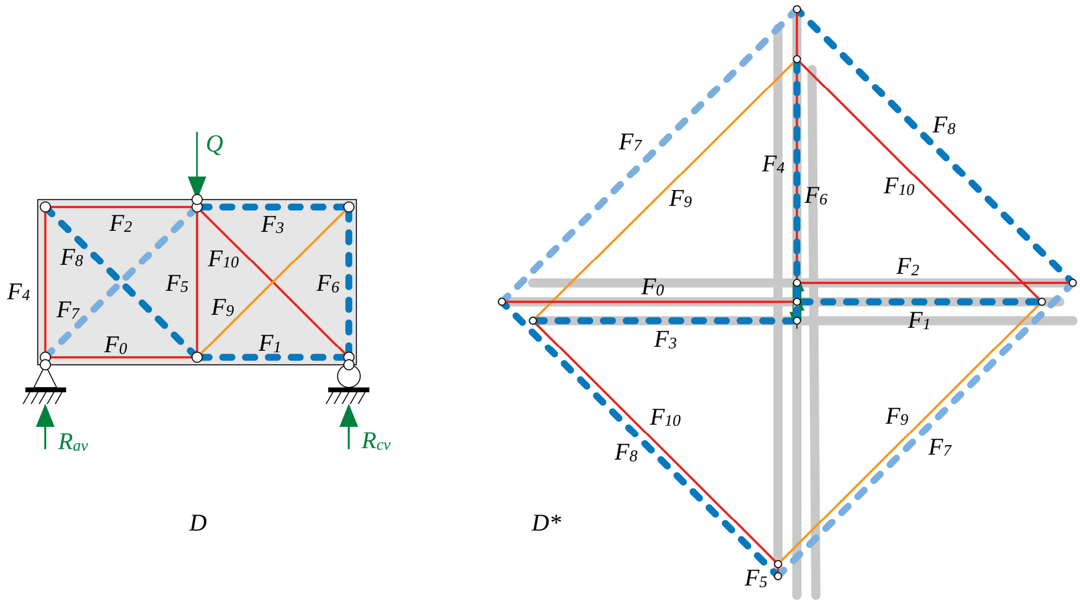

Figure 4: Admissible geometrical domain depicting the structure's level of robustness in terms of force redistributions

4. Conclusion and perspectives

This paper presents a geometrical method that helps characterizing levels of structural robustness. Based on graphic statics and strut-and-tie modeling, the method develops admissible geometrical domains as robustness indicators. Domains depict the range of possible nodal positions inside given structures, indicating their capacity for force redistributions. Hence, comparing domain areas for several rebar configurations helps highlighting the loss or gain in the aptitude to redistribute forces.

As the proposed geometrical method is visual, it can easily be appropriated by project authors at the design stage. When implemented into parametric software (such as Geogebra or Grasshopper under Rhino), the method can further be turned into an interactive tool, showing which design parameters will enhance robustness properties as well as efficiency. In particular, for concrete structures, the method can help choosing a pertinent rebar layout in order to reach required robustness performances.

Even if it helps understanding the basic functioning of the geometrical approach for robustness assessment, the case study presented in this paper remains quite simplistic. In order to be fully useful to study realistic structures, the method is now being extended to:

- larger structures, with more bars;
- structures with several external loads;
- structures with multiple supports and external static indeterminacy;
- structures with holes modeling openings (doors or windows);
- structures for which bars of the initial grid are not limited to orthogonal and 45° angles.

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