A Non-formal Formula for the Rankin-Cohen Deformation Quantization

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Abstract. Don Zagier's superposition of Rankin-Cohen brackets on the Lie group $SL_2(\mathbb{R})$ defines an associative formal deformation of the algebra of modular forms on the hyperbolic plane [9]. This formal deformation has been used in [6] to establish strong connections between the theory of modular forms and that of regular foliations of co-dimension one. Alain Connes and Henri Moscovici also proved that Rankin-Cohen's deformation gives rise to a formal universal deformation formula (UDF) for actions of the group $ax + b$. In a joint earlier work [5], the first author, Xiang Tang and Yijun Yao proved that this UDF is realized as a truncated Moyal star-product. In the present work, we use a method to explicitly produce an equivariant intertwiner between the above mentioned truncated Moyal star-product (i.e. Rankin-Cohen deformation) and a non-formal star-product on $ax + b$ defined by the first author in an earlier work [2]. The specific form of the intertwiner then yields an oscillatory integral formula for Zagier's Rankin-Cohen UDF, answering a question raised by Alain Connes.

In this paper, we will study equivalences between two particular star-products on the symplectic manifold underlying the group $ax + b$. We start by recalling that for a symplectic manifold (M, ω), a *star-product* on M is a bilinear map $\star_\nu : C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[\nu]] : (f, g) \mapsto f \star_\nu g := \sum_{k=0}^{+\infty} \nu^k C_k(f, g)$, where ν is a formal parameter called *deformation parameter* and with $C^\infty(M)[[\nu]] := \left\{\sum_{k=0}^{+\infty} \nu^k f_k \mid f_k \in C^\infty(M)\ \forall k \in \mathbb{N}\right\}$, such that (i) for each $k \in \mathbb{N}\backslash\{0\}$, the map $C_k : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defines a bidifferential operator on M; (ii) the law $\star_\nu$, extended $\mathbb{C}[[\nu]]$-linearly to the space of formal power series $C^\infty(M)[[\nu]]$, gives an associative product on $C^\infty(M)[[\nu]]$; (iii) $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = c\{f, g\}$ for some constant $c \in \mathbb{C}$ depending on the chosen normalization and where $\{\cdot, \cdot\}$ means the Poisson structure associated with ω; (iv) $1 \star_\nu f = f \star_\nu 1 = f$. In what follows, we will assume that $\nu \in i\mathbb{R}_0$. Given a star-product $\star_\nu$ on (M, ω), a *derivation* D of $\star_\nu$ is a linear operator on $C^\infty(M)[[\nu]]$ of the form $D = \sum_{k=0}^{+\infty} \nu^k D_k$, where for every k, D_k is a $\mathbb{C}[[\nu]]$-linear differential operator, with D satisfying the following relation for every $f, g \in C^\infty(M)[[\nu]]$: $D(f \star_\nu g) = Df \star_\nu g + f \star_\nu Dg$. The space of derivations of $\star_\nu$ is denoted by $\mathrm{Der}(\star_\nu)$. Also, for any $\varphi, \psi \in C^\infty(M)[[\nu]]$, their $\star_\nu$-*commutator*

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is defined by $[\varphi, \psi]_{\star_\nu} := \varphi \star_\nu \psi - \psi \star_\nu \varphi$. A star-product $\star_\nu$ on a (M, ω) is said to be *invariant* under an action of $\mathfrak{g} = \mathrm{Lie}(G)$ on M (that is the data of a homomorphism $\mathfrak{g} \to \Gamma^\infty(TM) : X \mapsto X^\star$) if each $X^\star$ is a derivation of $\star_\nu$. In that case, we will say that $\star_\nu$ is $\mathfrak{g}$-invariant. Suppose now that $\star_\nu$ and $\star'_\nu$ are two star-products on the same symplectic manifold (M, ω). There are said to be *equivalent* if there exists a linear formal operator T on $C^\infty(M)[[\nu]]$ of the form $T = \sum_{k=0}^{+\infty} \nu^k T_k$, such that (i) $T_0 = \mathrm{Id}$; (ii) T_k is a $\mathbb{C}[[\nu]]$-linear differential operator for all $k \in \mathbb{N}$; (iii) T *intertwines* $\star_\nu$ and $\star'_\nu$, that is, for $f, g \in C^\infty(M)[[\nu]]$, $T(f \star_\nu g) = Tf \star'_\nu Tg$ (this property will be denoted shortly as: $T(\star_\nu) = \star'_\nu$). For this last reason, such an equivalence is also called an *intertwiner*. Let us now introduce the two star-products that we will link in the Sect. 2 by studying their intertwiners.

1 Rankin-Cohen and Bieliavsky-Gayral Star-Products

1.1 Rankin-Cohen Brackets and Formal Deformation Quantization

The first star-product that we consider is due to Zagier, Connes and Moscovici, but based on the researches of Rankin and Cohen concerning the modular forms. Let us consider the Hopf algebra $\mathcal{H}_1$ described by Connes and Moscovici in [6] in the particular case where all the δ_n are trivial. In this case, $\mathcal{H}_1$ reduces to the universal enveloping algebra $\mathcal{U}(\mathfrak{s})$ of the Lie algebra $\mathfrak{s}$ with basis $\{X, Y\}$ and bracket $[Y, X] = X$. Note that $\mathfrak{s} := \mathrm{Lie}(\mathbb{S})$, i.e. the Lie algebra associated with the $ax + b$ Lie group that we denote $\mathbb{S}$. Authors in [6] then show that the superposition of (a reduced form of) Rankin-Cohen brackets (denoted as RC_n^{red} for n a positive integer) gives rise to a formal universal deformation formula (UDF) $\star_\nu^{RC} := \sum_{n=0}^{+\infty} \nu^n RC_n^{red}$ for actions of the group $\mathbb{S}$. It is worth mentioning that the Lie group $\mathbb{S}$ corresponds to the Iwasawa component of $SL_2(\mathbb{R})$. Setting $\mathfrak{sl}_2(\mathbb{R}) := \mathrm{Lie}(SL_2(\mathbb{R})) = \mathrm{span}_{\mathbb{R}}\{H, E, F\}$ with brackets $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$, one has $\mathfrak{s} = \mathrm{span}_{\mathbb{R}}\{H, E\}$ and thus $\mathbb{S} = \exp(\mathrm{span}_{\mathbb{R}}\{H, E\})$ (it means that $X = E$ and $Y = \frac{H}{2}$ within notations in [6]). This leads to the following global coordinate system on $\mathbb{S}$:

$$\mathfrak{s} \simeq \mathbb{R}^2 \to \mathbb{S} : (a, \ell) \mapsto \exp(aH)\exp(\ell E). \tag{1}$$

We endow this space $\mathbb{S}$ with the symplectic form $\omega^{\mathbb{S}} = da \wedge d\ell$. As noticed in [5], the deformation $\star_\nu^{RC}$ defines a *left invariant* star-product on the symplectic manifold $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, meaning that for all $x \in \mathbb{S}$, the left action L_x of $\mathbb{S}$ is a *symmetry* of $\star_\nu^{RC}$, i.e. it satisfies the identity $L_x^\star \varphi \star_\nu^{RC} L_x^\star \psi = L_x^\star(\varphi \star_\nu^{RC} \psi)$, for all $\varphi, \psi \in C^\infty(\mathbb{S})[[\nu]]$. Hereafter, the star-product $\star_\nu^{RC}$ on the symplectic manifold $(\mathbb{S}, \omega^{\mathbb{S}})$ will be called the *Rankin-Cohen star-product* (or *RC star-product* in short). Let us now consider the symplectic vector space $\left(\mathbb{R}^2 = \{(q, p)\}, \omega := dq \wedge dp\right)$ and $\mathfrak{h} := \mathrm{span}_{\mathbb{R}}\{Q, P\} \oplus \mathbb{R}Z$ the corresponding Heisenberg algebra with brackets given by $[Q, P] = \omega(Q, P)Z = Z$, $[Y, Z] = 0\ \forall\ Y \in \mathrm{span}_{\mathbb{R}}\{Q, P\}$. We form the semi-direct product $\mathfrak{g}_{RC} := \mathfrak{sl}_2(\mathbb{R}) \ltimes \mathfrak{h}$. In [5], authors show that $\star_\nu^{RC}$ is $\mathfrak{g}_{RC}$-invariant, through the action of $\mathfrak{g}_{RC}$ on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ associated with the fundamental vector fields. Note that considering the natural action through matrix multiplication of $\mathbb{S} \subset SL_2(\mathbb{R})$ on $\left(\mathbb{R}^2, \omega\right)$, we can form the orbit $\mathcal{O} = \mathbb{S}.(0,1) = \{(\ell e^a, e^{-a}) \mid a, \ell \in \mathbb{R}\}$ (within coordinates (1)), corresponding to the upper half-plane of $\mathbb{R}^2$. As shown in [5], Proposition 3.4, the star-product $\star_\nu^{RC}$ realized on $\mathcal{O} \subset \mathbb{R}^2$ coincides with the restriction to $\mathcal{O}$ of the Moyal star-product on $(\mathbb{R}^2, \omega)$. A last property of $\star_\nu^{RC}$ is given below.

Proposition 1 ([4], Lemma 5.3). *The star-product $\star_\nu^{RC}$ is (up to a redefinition of the deformation parameter) the only $\mathfrak{s} \ltimes \mathfrak{h}$-invariant formal star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$.*

1.2 Bieliavsky-Gayral Star-Product and Non-formal Deformation Quantization

The second star-product that we consider constitutes a particular example of a construction presented in [3]. In what follows, the Poincaré algebra $\mathbb{R}H \ltimes (\mathbb{R}E \oplus \mathbb{R}F)$, with table given by $[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = 0$, will be denoted as $\mathfrak{g}_{BG}$. The method used in [3] consists in twisting the two-dimensional Moyal star-product $\star_\nu^M$ (in coordinates (1)) by means of an intertwiner T, in such a way that $T\left(\star_\nu^M\right)$ yields a left invariant star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. As pointed out by the authors, there exists a family (indexed by functions ϑ) of operators T leading to a *convergent form* of the corresponding star-product (as in the case of Moyal-Weyl). Indeed, by considering for instance the operator T_{ν,ϑ_0} associated with $\vartheta := \vartheta_0$ given by $T_{\nu,\vartheta_0} := \mathcal{F}^{-1} \circ \mathcal{M}_{\exp(\vartheta_0)} \circ \left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{F}$, where $\mathcal{F}$ is the partial Fourier transform in the second variable, $\mathcal{M}_{\exp(\vartheta_0)}$ is the multiplication by $\exp(\vartheta_0)$ whith $\vartheta_0(\xi) := -\frac{1}{4}\log\left(1 - 4\nu^2\xi^2\right)$ and where $\Psi_\nu(a,\xi) = \left(a, \frac{\sinh(2i\nu\xi)}{2i\nu}\right)$, the star-product $\star_{\nu,\vartheta_0}^{BG} := T_{\nu,\vartheta_0}\left(\star_\nu^M\right)$ on the space $\mathcal{E}_{\nu,\vartheta_0}^{BG} := T_{\nu,\vartheta_0}\left(\mathcal{S}(\mathbb{S})\right)$ (where $\mathcal{S}(\mathbb{S})$ is the Euclidean Schwartz space on $\mathbb{S} \simeq \mathbb{R}^2$ with respect to (1)) enjoys the following properties.

Theorem 1 ([3], Theorem 4.5). *Within the previous notations,*

(i) $\left(\mathcal{E}_{\nu,\vartheta_0}^{BG}, \star_{\nu,\vartheta_0}^{BG}\right)$ *is an associative algebra, endowed with the Fréchet algebra structure transported from $\mathcal{S}(\mathbb{S}) \simeq \mathcal{S}\left(\mathbb{R}^2\right)$ under the operator T_{ν,ϑ_0}.*
(ii) For all compactly supported functions $\varphi, \psi \in \mathcal{D}(\mathbb{S}) \subset \mathcal{E}_{\nu,\vartheta_0}^{BG}$, the expression $\varphi \star_{\nu,\vartheta_0}^{BG} \psi$ admits an oscillatory integral representation.
(iii) The star-product $\star_{\nu,\vartheta_0}^{BG}$ is $\mathfrak{g}_{BG}$-invariant, through the action of $\mathfrak{g}_{BG}$ on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ associated with the fundamental vector fields. In particular, $\star_{\nu,\vartheta_0}^{BG}$ is also left invariant on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$.

The star-product given in Theorem 1 will be called the *strongly-closed Bieliavsky-Gayral star-product* and will be denoted as $\underline{\star}_\nu^{BG}$. It is worth mentioning that the choice $\vartheta \equiv 0$ leading to the associative algebra $\left(\mathcal{E}_{\nu,0}^{BG}, \star_{\nu,0}^{BG}\right)$ (where $\star_{\nu,0}^{BG} = T_{\nu,0}\left(\star_\nu^M\right)$) corresponds to that of [2], Theorem 6.13, where $\star_{\nu,0}^{BG}$ is also $\mathfrak{g}_{BG}$-invariant (and in particular left invariant). The star-product $\star_\nu^{\mathrm{BG}} := \star_{\nu,0}^{BG}$ will be called the *Bieliavsky-Gayral star-product* (or *BG star-product* in short). Note that we can easily show that $\underline{T}_\nu\left(\star_\nu^{BG}\right)=\underline{\star}_\nu^{BG}$, where $\underline{T}_\nu:=\mathcal{F}^{-1}\circ\mathcal{M}_{\exp(\vartheta_0)}\circ\mathcal{F}$. At this point, we can naturally ask whether the RC star-product also admits a convergent expression like the BG star-product (on a suitable subspace of $C^\infty(\mathbb{S})$). This is precisely the subject of the next section, where we start by describing a method to obtain the expressions of intertwiners between RC and BG star-products.

2 Non-formal Rankin-Cohen Star-Product

2.1 Guessing the Intertwiners Between RC and BG Star-Products

Consider the symplectic manifold $\left(\mathbb{S},\omega^{\mathbb{S}}\right)$. Thanks to the particular invariances of $\star_\nu^{BG}$ and $\star_\nu^{RC}$ mentioned previously, we get the following injections: $\mathfrak{g}_{BG}\hookrightarrow\mathrm{Der}\left(\star_\nu^{BG}\right):Y\mapsto Y^\star$ and $\mathfrak{g}_{RC}\hookrightarrow\mathrm{Der}\left(\star_\nu^{RC}\right):Y\mapsto Y^\star$, where $Y^\star$ denotes the fundamental vector field associated with Y. Note that $\mathfrak{s}$ is a subalgebra of both $\mathfrak{g}_{BG}$ and $\mathfrak{g}_{RC}$. By virtue of [1], Theorem 4.1, one knows that the set of $\mathbb{S}$-equivalence classes of left invariant star-products on $\mathbb{S}$ is canonically parametrized by the set of sequences of elements belonging to the second de Rham cohomology space $H^2_{dR}(\mathbb{S})^{\mathbb{S}}$ of the $\mathbb{S}$-invariant de Rham complex on $\mathbb{S}$. Thanks to the fact that every Chevalley-Eilenberg 2-cocycle on $\mathfrak{s}$ is a coboundary, it turns out that there exists at least one intertwiner T on $C^\infty(\mathbb{S})[[\nu]]$ between $\star_\nu^{BG}$ and $\star_\nu^{RC}$ that is $\mathbb{S}$-*equivariant*, i.e. such that for all $x\in\mathbb{S}$, $L_x^\star T=TL_x^\star$. At the level of the space $\mathcal{D}(\mathbb{S})$ of smooth compactly supported functions, we can then express T as a (formal) convolution operator by means of the Schwartz kernel theorem. Consequently, there exists a distribution u such that for any $\varphi\in\mathcal{D}(\mathbb{S})$ and $x\in\mathbb{S}$, $T\varphi(x)=\int_{\mathbb{S}}u(x,y)\varphi(y)d_{\mathbb{S}}^L(y)$, where $d_{\mathbb{S}}^L$ denotes a left invariant Haar measure on $\mathbb{S}$. Using the $\mathbb{S}$-equivariance of T and setting $u_T(x):=u\left(e,x^{-1}\right)$ (for $e\in\mathbb{S}$ the neutral element), we get the following particular form: $T\varphi(x)=\int_{\mathbb{S}}u_T\left(y^{-1}x\right)\varphi(y)d_{\mathbb{S}}^L(y)$. This yields in turn the following algebra morphism: $D:\mathfrak{g}_{RC}\to\mathrm{Der}\left(\star_\nu^{BG}\right):Y\mapsto D_Y:=T^{-1}Y^\star T$, extending to $\mathfrak{g}_{RC}$ the injection $\mathfrak{s}\hookrightarrow\mathrm{Der}\left(\star_\nu^{BG}\right)$. Since $H^1_{dR}(\mathbb{S})=\{0\}$ thanks to the Poincaré lemma, we deduce from [8], Theorem 8.2 that for any $Y\in\mathfrak{g}_{RC}$, the associated derivation D_Y is inner, i.e. $D_Y=\frac{1}{2\nu}\left[\varPhi_Y,\cdot\right]_{\star_\nu^{\mathrm{BG}}}$ for some $\varPhi_Y\in C^\infty(\mathbb{S})[[\nu]]$. For instance, in the basis $\{H,E\}$ of $\mathfrak{s}$, $\varPhi_H(a,\ell)=\ell$ and $\varPhi_E(a,\ell)=\frac{1}{2}e^{-2a}$ (within (1)). Now, remark that the algebra $\mathfrak{g}_{RC}$ decomposes (as vector space) as $\mathfrak{s}\oplus V_{RC}$, with $V_{RC}:=\mathrm{span}_{\mathbb{R}}\{F,Q,P,Z\}$. For any element X_k in the basis of $\mathfrak{s}$ and any element Y_l in the basis of V_{RC}, one has: $[X_k,Y_l]=\sum_a C^a_{kl}X_a+\sum_b\underline{C}^b_{kl}Y_b$, with $a\in\{1,2\}$, $b\in\{1,\dots,4\}$ and where $C^a_{kl},\underline{C}^b_{kl}$ are some real constants. The fact that D is a Lie algebra morphism then leads to the following ODE: $X_k^\star\left(\varPhi_{Y_l}\right)=\sum_a C^a_{kl}\varPhi_{X_a}+\sum_b\underline{C}^b_{kl}\varPhi_{Y_b}$, up to additive formal constants. Once the solution $\varPhi_{Y_l}$ is obtained for any Y_l in the basis of V_{RC}, we can deduce the expression of D_Y for any $Y\in\mathfrak{g}_{RC}$ by means of linear extension. Regarding the convolution kernel, recall first that $d_{\mathbb{S}}^L(y)$ is preserved under $Y^\star$ for all $Y\in\mathfrak{g}_{RC}$. Since the inverse T^{-1} of T is also a convolution operator with associated kernel

v, for $x, x_0 \in \mathbb{S}$ and $Y \in \mathfrak{g}_{RC}$, one has: $D_Y|_{x_0} v\left(x^{-1}x_0\right) = -\left.Y^\star\right|_x v\left(x^{-1}x_0\right)$, where $|_{x_0}$ (respectively $|_x$) means that the operator is applied by considering the function on its right as depending on the x_0-variable (resp. x-variable) only. A straightforward computation then leads to the observation that any intertwiner T such that $T\left(\star_\nu^{BG}\right)$ is a $\mathfrak{g}_{RC}$-invariant star-product admits as an inverse a convolution operator $T^{-1} = v \times \cdot$, with v a solution of the following two equations:

$$D_Q|_{x_0} v\left(x^{-1}x_0\right) = -\left.Q^\star\right|_x v\left(x^{-1}x_0\right) \;,\;\; D_P|_{x_0} v\left(x^{-1}x_0\right) = -\left.P^\star\right|_x v\left(x^{-1}x_0\right), \tag{2}$$

for $D_Q|_{(a,\ell)} = \frac{1}{2\nu}\left[K_1 e^{-a}, \cdot\right]_{\star_\nu^{BG}}$, $D_P|_{(a,\ell)} = \frac{1}{2\nu}\left[\left(K_2 - K_1\ell\right)e^a, \cdot\right]_{\star_\nu^{BG}}$, $Q^\star|_{(a,\ell)} = -e^{-a}\partial_\ell$ and $P^\star|_{(a,\ell)} = e^a\left(\partial_a - \ell\partial_\ell\right)$ (within the chart (1)), for some parameters $K_1, K_2 \in \mathbb{C}[[\nu]]$. We set $\boldsymbol{K} := (K_1, K_2)$. Solving the previous equations then shows that to the corresponding intertwiner $T_{\nu,\boldsymbol{K}} = v \times \cdot$ is formally given by:

$$T_{\nu,\boldsymbol{K}}\varphi(a_0,\ell_0) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left[\xi\left(\ell - K_1^2\eta_\nu^2(r)\ell_0\right) - \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu\xi)\right]}\underline{\eta_\nu}(\xi)\varphi\left(a_0 - \log(K_1\eta_\nu(\xi)), \ell\right)d\ell d\xi, \tag{3}$$

where $\eta_\nu : \mathbb{R} \to \mathbb{R}_0^+ : \xi \mapsto \sqrt{\frac{2}{1+\sqrt{1-4\nu^2\xi^2}}}$ and $\underline{\eta_\nu} := \eta_\nu \exp(2\vartheta_0)$. From now on, we will always assume that K_1 is a strictly positive real number and K_2 a real number, satisfying the limit condition $\lim_{\nu\to 0}(K_1, K_2) = (1, 0)$. The next section is devoted to an analytical study of the intertwiner $T_{\nu,\boldsymbol{K}}$ as well as the formula $\sharp_{\nu,\boldsymbol{K}} := T_{\nu,\boldsymbol{K}}\left(\star_\nu^{BG}\right)$.

2.2 Towards a Non-formal RC Star-Product

In order to give rise to an oscillatory integral formula for the RC star-product, we will focus on the intertwiner between $\underline{\star}_\nu^{BG}$ (the strongly-closed BG star-product) and $\star_\nu^{RC}$, rather than the one between $\star_\nu^{BG}$ and $\star_\nu^{RC}$. The reason is that Proposition 4.10 in [3] exhibits explicitly a space $\mathcal{S}^{BG}(\mathbb{S})$ of Schwartz-type functions such that, endowed with the multiplication $\underline{\star}_\nu^{BG}$, $\left(\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S}), \underline{\star}_\nu^{BG}\right)$ becomes an associative Fréchet algebra. It turns out that $\mathcal{S}^{BG}(\mathbb{S})$ corresponds to the usual Schwartz space in coordinates $(x, y) = (\sinh(2a), \ell)$. In what follows, the idea is then to transport that space under the operator $\underline{T}_{\nu,\boldsymbol{K}} := T_{\nu,\boldsymbol{K}} \circ \underline{T}_\nu^{-1}$.

Proposition 2. *Let $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$. Then, $\underline{T}_{\nu,\boldsymbol{K}}\varphi \in \mathcal{S}(\mathbb{S})$.*

Proof. Let $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$. Note first that $\mathcal{S}^{BG}(\mathbb{S}) \subset \mathcal{S}(\mathbb{S})$. Also, the element $\mathbf{E} := e^{i\left[r\left(z - K_1^2\eta_\nu^2(r)\ell_0\right) - \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu r)\right]}$ satisfies $\frac{1}{1+r^2}\left(\mathrm{Id} - \partial_z^2\right)\mathbf{E} = \mathbf{E}$. Thanks to the self-adjointness of the differential operator $D := \mathrm{Id} - \partial_z^2$ (w.r.t. the $L^2(\mathbb{R})$-inner product), we get the following expression for $\underline{T}_{\nu,\boldsymbol{K}}$:

$$\underline{T}_{\nu,\boldsymbol{K}}\varphi(a_0,\ell_0) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\mathbf{E}\frac{\tilde{\eta}_\nu(r)}{1+r^2}D\varphi\left(a_0 - \log(K_1\eta_\nu(r)), z\right)drdz, \tag{4}$$

where $\tilde{\eta}_\nu : \mathbb{R} \to \mathbb{R}_0^+ : r \mapsto \eta_\nu(r)\exp(\vartheta_0(r))$. The Lebesgue dominated convergence theorem and the mean-value theorem then imply that $\underline{T}_{\nu,\boldsymbol{K}}\varphi$, $\partial_1\underline{T}_{\nu,\boldsymbol{K}}\varphi$ and $\partial_2\underline{T}_{\nu,\boldsymbol{K}}\varphi$ (where ∂_j denotes the partial derivative w.r.t. the jth-variable) are continuous, which means that $\underline{T}_{\nu,\boldsymbol{K}}\varphi \in C^1(\mathbb{S})$. Denoting by $f_\varphi(a_0,\ell_0,r,z)$ the integrand in (4), remark now that for $m,n \in \mathbb{N}$, $\partial_1^m\partial_2^n f_\varphi(a_0,\ell_0,r,z) = \left(-iK_1^2 r\eta_\nu^2(r)\right)^n \mathbf{E}\frac{\tilde{\eta}_\nu(r)}{1+r^2}\partial_1^m(D\varphi)\left(a_0 - \log(K_1\eta_\nu(r)), z\right)$. An induction on the degree of the partial derivatives then shows that $\underline{T}_{\nu,\boldsymbol{K}}\varphi \in C^\infty(\mathbb{S})$. Now, thanks to the Fubini theorem, notice that performing the integral w.r.t. the z-variable within the expression of $\underline{T}_{\nu,\boldsymbol{K}}\varphi$ leads to

$$\underline{T}_{\nu,\boldsymbol{K}}\varphi(a,\ell) = \int_{\mathbb{R}} e^{i\left[-K_1^2 r\eta_\nu^2(r)\ell - \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu r)\right]}\tilde{\eta}_\nu(r)\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right)dr \tag{5}$$

with $\hat{\varphi} := \mathcal{F}^{-1}\varphi$. An obvious but important fact about η_ν, $\log\eta_\nu$, $\tilde{\eta}_\nu$ and arcsinh is that those functions belong to the space $\mathcal{O}_M(\mathbb{R})$ of Schwartz multipliers on $\mathbb{R}$. In particular, $\tilde{\eta}_\nu\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right)$ is a Schwartz function on $\mathbb{R}$ (for fixed a). That being said, we set $\mu(r) := r\eta_\nu^2(r)$, $\underline{\mathbf{E}} := e^{-i\mu(r)\ell}$, $\tilde{\mathbf{E}} := e^{-\frac{K_2}{2\nu K_1}\operatorname{arcsinh}(2i\nu r)}$ and $A := i\left(\frac{d}{dr}\mu(r)\right)^{-1}\partial_r$. Note that $\frac{1}{1+\ell^2}\left[\mathrm{Id} + A^2\right](\underline{\mathbf{E}}) = \underline{\mathbf{E}}$. Remark also that $\left(\frac{d}{dr}\mu\right)^{-1}$, $\frac{d^2}{dr^2}\mu \in \mathcal{O}_M(\mathbb{R})$. Denoting by $A^\dagger$ the adjoint operator of A w.r.t. the $L^2(\mathbb{R})$-inner product, notice that the coefficients of the differential operator $\mathrm{Id} + \left(A^\dagger\right)^2$ as well as the element $\tilde{\mathbf{E}}$ belong to $\mathcal{O}_M(\mathbb{R})$ (w.r.t. the r-variable). Using now an integration by parts argument (combined with the Lebesgue dominated convergence and the mean-value theorem) leads to the observation that $\sup_{(a,\ell)\in\mathbb{S}}\left|a^k\ell^p\partial_a^m\partial_\ell^n(\underline{T}_{\nu,\boldsymbol{K}}\varphi)(a,\ell)\right| < +\infty$, which concludes the proof.

For a given function f, let supp$\,(f)$ denotes its support. Note that using (1), a direct computation leads to the following interesting result.

Lemma 1. *For any* $\psi \in \underline{T}_{\nu,\boldsymbol{K}}\left(\mathcal{S}^{BG}(\mathbb{S})\right)$, *supp* $(\mathcal{F}\psi) \subset \mathbb{R} \times \left[-\frac{K_1^2}{|i\nu|}, \frac{K_1^2}{|i\nu|}\right]$.

Following notations due to Gel'fand and Shilov, for a given $\sigma \in \mathbb{R}^+$, we set $\mathcal{S}_\sigma := \left\{f \in \mathcal{S}^{BG}(\mathbb{S}) \mid \mathrm{supp}(f) \subset \mathbb{R} \times [-\sigma,\sigma]\right\}$, $\underline{\mathcal{S}}_\sigma := \cup_{0\leq\epsilon<\sigma}\mathcal{S}_\epsilon$ and $\mathcal{S}^\sigma$ (respectively $\underline{\mathcal{S}}^\sigma$) the space $\mathcal{F}^{-1}\left(\mathcal{S}_\sigma\right)$ (resp. $\mathcal{F}^{-1}\left(\underline{\mathcal{S}}_\sigma\right)$).

Proposition 3. *The operator* $\underline{\tau}_{\nu,\boldsymbol{K}}$ *defined by*

$$\underline{\tau}_{\nu,\boldsymbol{K}}\psi(a_0,\ell_0) = \frac{K_1^2}{2\pi}\int_{\mathbb{R}^2} e^{i\left[r\left(\ell_0 - K_1^2\eta_\nu^2(r)z\right) - \frac{K_2}{2i\nu K_1}\,arcsinh(2i\nu r)\right]}\tilde{\eta}_\nu(r)\psi\left(a_0 + \log(K_1\eta_\nu(r)), z\right)drdz \tag{6}$$

is an inverse for $\underline{T}_{\nu,\boldsymbol{K}}$. *More precisely, the following identities hold:*

$$\left(\underline{\tau}_{\nu,\boldsymbol{K}} \circ \underline{T}_{\nu,\boldsymbol{K}}\right)\big|_{\mathcal{S}^{BG}(\mathbb{S})} = Id_{\mathcal{S}^{BG}(\mathbb{S})} \quad \textit{and} \quad \left(\underline{T}_{\nu,\boldsymbol{K}} \circ \underline{\tau}_{\nu,\boldsymbol{K}}\right)\big|_{\underline{\mathcal{S}}^{K_1^2/|i\nu|}} = Id\big|_{\underline{\mathcal{S}}^{K_1^2/|i\nu|}}.$$

Proof. Let $\gamma_{\nu,\boldsymbol{K}}$ be the smooth map defined by $\gamma_{\nu,\boldsymbol{K}} : \mathbb{R}^2 \to \mathbb{R}^2 : (a,\xi) \mapsto (a + \log(K_1\eta_\nu(\xi)), K_1^2 r\eta_\nu^2(\xi))$ and let $\mathcal{M}_{\nu,\boldsymbol{K}}$ be the operator defined for any smooth function f on $\mathbb{R}^2$ by $\mathcal{M}_{\nu,\boldsymbol{K}}(f) := \left[(a,\xi) \mapsto K_1^2\tilde{\eta}_\nu(\xi)e^{-\frac{K_2}{2\nu K_1}\operatorname{arcsinh}(2i\nu\xi)} f(a,\xi)\right]$. A direct computation shows that $\underline{\tau}_{\nu,\boldsymbol{K}}$ given by (6) admits the following decompostion: $\underline{\tau}_{\nu,\boldsymbol{K}} = \mathcal{F}^{-1} \circ \mathcal{M}_{\nu,\boldsymbol{K}} \circ \gamma^\star_{\nu,\boldsymbol{K}} \circ \mathcal{F}$. The first identity is then a consequence of Proposition 2 combined with the decomposition above. For the second identity, let us choose a function $\psi \in \mathcal{S}^\sigma$, for $0 \leq \sigma < \frac{K_1^2}{|i\nu|}$ (i.e. $\psi \in \underline{\mathcal{S}}^{K_1^2/|i\nu|}$). It means that there exists $\psi_0 \in \mathcal{S}_\sigma$ such that $\psi = \mathcal{F}^{-1}\psi_0$. Thanks to the decomposition of $\underline{\tau}_{\nu,\boldsymbol{K}}$, we have $\underline{\tau}_{\nu,\boldsymbol{K}}\psi = \mathcal{F}^{-1}\left(\mathcal{M}_{\nu,\boldsymbol{K}}\left(\gamma^\star_{\nu,\boldsymbol{K}}\psi_0\right)\right)$. Note that the function $\mathcal{M}_{\nu,\boldsymbol{K}}\left(\gamma^\star_{\nu,\boldsymbol{K}}\psi_0\right)(a,\xi)$ vanishes identically for ξ outside an interval $[-\tilde{\sigma},\tilde{\sigma}]$, for a certain $0 \leq \tilde{\sigma} < +\infty$. Moreover $\partial_1\mathcal{M}_{\nu,\boldsymbol{K}}\left(\gamma^\star_{\nu,\boldsymbol{K}}\psi_0\right) = \mathcal{M}_{\nu,\boldsymbol{K}}\left(\gamma^\star_{\nu,\boldsymbol{K}}\partial_1\psi_0\right)$. It implies that $\mathcal{M}_{\nu,\boldsymbol{K}}\left(\gamma^\star_{\nu,\boldsymbol{K}}\psi_0\right) \in \mathcal{S}(\mathbb{R}^2)$, which in turn implies that $\underline{\tau}_{\nu,\boldsymbol{K}}\psi \in \mathcal{S}(\mathbb{S})$. Plugging the function $\varphi := \underline{\tau}_{\nu,\boldsymbol{K}}\psi$ into the expression (5) then concludes the proof.

We can now provide an oscillatory integral formula for the Rankin-Cohen deformation by transporting the space $\mathcal{S}^{BG}(\mathbb{S})$ under the operator $\underline{T}_{\nu,\boldsymbol{K}}$.

Theorem 2. *Let* $\mathcal{E}_{\nu,\boldsymbol{K}} := \underline{T}_{\nu,\boldsymbol{K}}\left(\mathcal{S}^{BG}(\mathbb{S})\right)$.

(i) The following inclusions hold: $\underline{T}_{\nu,\boldsymbol{K}}(\mathcal{D}(\mathbb{S})) \subset \underline{\mathcal{S}}^{K_1^2/|i\nu|} \subset \mathcal{E}_{\nu,\boldsymbol{K}} \subset \mathcal{S}(\mathbb{S})$.

(ii) For $\varphi,\psi \in \mathcal{E}_{\nu,\boldsymbol{K}}$, *the formula* $\varphi\sharp_{\nu,\boldsymbol{K}}\psi := \underline{T}_{\nu,\boldsymbol{K}}\left(\underline{\tau}_{\nu,\boldsymbol{K}}\varphi \star^{BG}_\nu \underline{\tau}_{\nu,\boldsymbol{K}}\psi\right)$ *defines an associative product on* $\mathcal{E}_{\nu,\boldsymbol{K}}$. *The algebra* $(\mathcal{E}_{\nu,\boldsymbol{K}}, \sharp_{\nu,\boldsymbol{K}})$ *is then endowed with the Fréchet algebra structure transported under* $\underline{T}_{\nu,\boldsymbol{K}}$ *from* $\mathcal{S}^{BG}(\mathbb{S})$.

(iii) The star-product $\sharp_{\nu,\boldsymbol{K}}$ *is* $\mathfrak{g}_{RC}$*-invariant (and, up to a redefinition of the deformation parameter, it coincides with* $\star^{RC}_\nu$*). In particular,* $\sharp_{\nu,\boldsymbol{K}}$ *is left invariant.*

(iv) Denoting by R *the right action of* $\mathbb{S}$, *the star-product* $\sharp_{\nu,\boldsymbol{K}}$ *admits the following integral representation at the level of functions belonging to* $\underline{\mathcal{S}}^{K_1^2/|i\nu|}$:

$$\varphi\sharp_{\nu,\boldsymbol{K}}\psi = \frac{K_1^4}{(2\pi i\nu)^2}\int_{\mathbb{R}^4}\frac{1}{\cosh(a_1)\cosh(a_2)\cosh(a_1-a_2)}e^{\frac{K_1^2}{\nu}[\tanh(a_2)\ell_1-\tanh(a_1)\ell_2]} \times R^\star_{\left(a_1+\log\frac{\cosh(a_1-a_2)}{\cosh(a_2)},\ell_1\right)}\varphi R^\star_{\left(a_2+\log\frac{\cosh(a_1-a_2)}{\cosh(a_1)},\ell_2\right)}\psi da_1d\ell_1da_2d\ell_2.$$

(v) Let $\boldsymbol{K}_0 := (1,0)$. *For every* $\varphi,\psi \in \underline{T}_{\frac{\theta}{i},\boldsymbol{K}_0}(\mathcal{D}(\mathbb{S}))$ *and for every* $x \in \mathbb{S}$, *the map* $\mathbb{R} \to \mathbb{C} : \theta \mapsto \left(\varphi\sharp_{\frac{\theta}{i},\boldsymbol{K}_0}\psi\right)(x)$ *is smooth. Its Taylor series at* 0 *defines an associative star-product on* $\left(\mathbb{S},\omega^{\mathbb{S}}\right)$ *which coincides with* $\star^{RC}_{\frac{\theta}{i}}$.

Proof. We start by item (i). For the first inclusion, it comes from the fact that the range of the map $\left[B \to \mathbb{R} : r \mapsto K_1^2 r\eta_\nu^2(r)\right]$ is included in an interval $[-\epsilon,\epsilon]$ for a certain $0 \leq \epsilon < \frac{K_1^2}{|i\nu|}$. Choosing $\psi = \underline{T}_{\nu,\boldsymbol{K}}\varphi$ with $\varphi \in \mathcal{D}(\mathbb{S}) \subset \mathcal{S}^{BG}(\mathbb{S})$, a direct computation of $\mathcal{F}\left(\underline{T}_{\nu,\boldsymbol{K}}\varphi\right)$ shows that $\psi \in \underline{\mathcal{S}}^{K_1^2/|i\nu|}$. The second inclusion is straightforward and the last one is a consequence of Proposition 2. The item (ii) comes from the associativity of $\underline{\star}_\nu^{BG}$ (see Theorem 1). The item (iii) comes from the construction of $\sharp_{\nu,\boldsymbol{K}}$ itself (cf. the method used in Sect. 2.1) and Proposition 1. Moreover, notice that for $\varphi =: \underline{T}_{\nu,\boldsymbol{K}}\varphi_0$ and $\psi =: \underline{T}_{\nu,\boldsymbol{K}}\psi_0$ belonging to $\underline{T}_{\nu,\boldsymbol{K}}\left(\mathcal{S}^{BG}(\mathbb{S})\right)$, we have $\varphi\sharp_{\nu,\boldsymbol{K}}\psi := \underline{T}_{\nu,\boldsymbol{K}}\left(\varphi_0\underline{\star}_\nu^{BG}\psi_0\right)$. The left invariance of $\sharp_{\nu,\boldsymbol{K}}$ is then a direct consequence of that of $\underline{\star}_\nu^{BG}$ (see Theorem 1, item (ii)) and $\underline{T}_{\nu,\boldsymbol{K}}$. For the item (iv), we start by choosing σ such that $\sigma < \frac{K_1^2}{|i\nu|}$ and two functions $\varphi, \psi \in \mathcal{S}^\sigma$. Using the expression of $\underline{\star}_\nu^{BG}$ given in [3] and performing the change of variables $\ell_1 \mapsto \ell_1 + \ell \exp\left[-2\left(a_1 + \log\frac{\cosh(a_1-a_2)}{\cosh(a_2)}\right)\right]$ and $\ell_2 \mapsto \ell_2 + \ell \exp\left[-2\left(a_2 + \log\frac{\cosh(a_1-a_2)}{\cosh(a_1)}\right)\right]$ into the integral expression corresponding to $\frac{(2\pi i\nu)^2}{K_1^4}\underline{T}_{\nu,\boldsymbol{K}}\left(\underline{\mathcal{T}}_{\nu,\boldsymbol{K}}\varphi\underline{\star}_\nu^{BG}\underline{\mathcal{T}}_{\nu,\boldsymbol{K}}\psi\right)(a,\ell)$, we get the announced integral formula in the item (iv), thanks to the fact the right action R on $\mathbb{S}$ reads $R_{(a,\ell)}(a',\ell') = \left(a'+a, \ell+\ell' e^{-2a}\right)$. Regarding the item (v), we choose $\boldsymbol{K}_0 := (1,0)$ without any loss of generality (indeed, formula in item (iv) is the same for any K_1 up to a redefinition of ν and independent from K_2). Note first that using similar arguments to that of the proof of Proposition 2, we can see that both functions $\left[(x,\theta) \mapsto \underline{T}_{\frac{\theta}{i},\boldsymbol{K}}\varphi(x)\right]$ and $\left[(x,\theta) \mapsto \left(\psi_1\sharp_{\frac{\theta}{i},\boldsymbol{K}_0}\psi_2\right)(x)\right]$ are elements of $C^\infty(\mathbb{S}\times\mathbb{R})$. Theorem 40.1 in [7] then implies that for every $\varphi \in \mathcal{D}(\mathbb{S})$, the function of $\theta \in \mathbb{R}$ given by $\underline{T}_{\frac{\theta}{i},\boldsymbol{K}_0}\varphi$ belongs to $C^\infty\left(\mathbb{R}, C^\infty(\mathbb{S})\right)$ and for every $\varphi, \psi \in \underline{T}_{\frac{\theta}{i},\boldsymbol{K}_0}(\mathcal{D}(\mathbb{S}))$, the function of $\theta \in \mathbb{R}$ given by $\varphi\sharp_{\frac{\theta}{i},\boldsymbol{K}_0}\psi$ belongs to $C^\infty\left(\mathbb{R}, C^\infty(\mathbb{S})\right)$. Moreover, by virtue of the associativity of $\underline{\star}_{\frac{\theta}{i}}^{BG}$, the Taylor series at 0 of the map $\left[\mathbb{R}\to\mathbb{C} : \theta \mapsto \left(\varphi\sharp_{\frac{\theta}{i},\boldsymbol{K}_0}\psi\right)(x)\right]$ yields an associative star-product which is $\mathfrak{g}_{RC}$-invariant by construction. We then deduce from Proposition 1 that the formal version of $\sharp_{\nu,\boldsymbol{K}_0}$ coincides with $\star_\nu^{RC}$.

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