

Stroboscopic exclusion process: a first-moment-driven dynamics

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Abstract – We define a new variant of exclusion processes in discrete time that has jump probabilities that depend on the last jump performed. In a particular limit for the jump probabilities and in suitable scaling limits for space and time, we compute the time evolution of the particle density starting from an arbitrary initial configuration, with closed boundary conditions. The core of the argument is the analysis of the time evolution of the moments. Numerical results are compared with the prediction and give excellent agreement.

Introduction. – Exclusion processes are one of the most studied interacting particle systems in statistical mechanics, among which are the now classic Asymmetric Exclusion Process (ASEP) [1, 2] and its numerous variants [3–22]. The vast majority of the literature focuses on non-equilibrium steady states (NESS) and studies the stationary current of particles as a function of the parameters of the model. However, a current of particles can also be observed in the relaxation of a system towards a NESS [23–26] or an equilibrium state [32]. The object of this paper is to study the latter case and derive the time evolution of the particle density relaxing towards equilibrium of yet another variant of the ASEP model. The main feature of this model is the dependence of the jump probabilities on the nature of the last jump performed, making this stochastic process somewhat reminiscent of the ones studied in [31]. We study in particular the regime where these probabilities vary with the system size N in such a way that the jumps tend to strictly alternate between left and right, in the scaling limit $N \rightarrow \infty$. The heuristics of this dynamics is that the nearly strict alternation between left and right jumps makes the evolution of the density’s first moment extremely slow, so that the system is constantly on a state with maximal entropy at fixed first moment. This permanent projection is a specific feature of the model, which was designed for that purpose. This unusual mechanism makes two time scales appear and permits to close the hierarchy of equations usually encountered in exclusion processes. A similar separation of

time scales was also observed in another model [27]. After an appropriate rescaling of the discrete time T , we show that the full dynamics of the model is tractable starting from any initial state. The fact that the time evolution of the system depends on concepts of equilibrium statistical mechanics makes it particularly unusual and interesting.

Definition. – We consider a system of αN particles ($0 < \alpha < 1$) located on a one-dimensional lattice of length N with *closed boundary conditions*. At each time step $T \rightarrow T + 1$ we choose uniformly at random one of the N sites. If the last move performed was to the right (resp. to the left), then with a probability p_r (resp. p_ℓ) we attempt a move to the right (resp. to the left) and with a probability $1 - p_r$ (resp. $1 - p_\ell$) we attempt a move to the left (resp. to the right). The attempted move is rejected:

- if the departure site is not occupied by a particle,
- if the arrival site is already occupied,
- if the arrival site does not exist (if we reach the extremities).

These update rules are sketched in Figure 1. In case of rejection, the configuration remains unchanged and a time step is still counted. By convention we fix that at $T = 0$ the last particle has been moved to the right.

A few comments on this dynamics are in order. Firstly, since the dynamics depends on the previous move, it is not Markovian. However, it can easily be made Markovian if one adds an extra degree of freedom to the state

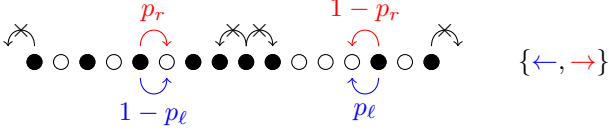


Fig. 1: The blue (resp. red) arrows represent the jump probabilities when the last move performed was to the left (resp. right).

of the model, taking the values in $\{\leftarrow, \rightarrow\}$ according to whether the last particle has been moved to the left or to the right. Secondly, our approach can be applied to variants of this dynamics: for example if we still count a time step for an attempted but rejected move, or if the time is made continuous by introducing random “clocks” at each site. Finally, let us mention that the closed boundary conditions force the system to reach an equilibrium state in the $T \rightarrow \infty$ limit, even when p_ℓ and p_r introduce a bias in one direction. This would not be the case for periodic boundary conditions or boundary conditions with particle reservoirs at the extremities.

On the finite lattice a configuration at a given time T is entirely described by $\{X_i(T)\}_{i=1, \dots, \alpha N}$, where $X_i(T)$ ($1 \leq X_i(T) \leq N$) is the position of the i -th particle. When we take the limit $N \rightarrow \infty$ and rescale distances by a factor N , the state of the system can be described (but a priori not fully) by a density $\rho(x, T)$, defined such that the number of particles at time T between Nx and $N(x + dx)$ is $\rho(x, T)Ndx$, for $0 \leq x \leq 1$. This density is equivalent to giving the counting function $z(x, T)$, defined to be such that there are $Nz(x, T)$ particles with position smaller than xN , at time T , namely

$$z(x, T) = \int_0^x \rho(u, T) du. \quad (1)$$

It will turn out to be fruitful to introduce the q -th moment

$$\langle x^q \rangle(T) = \frac{1}{\alpha N} \sum_{i=1}^{\alpha N} \left(\frac{X_i(T)}{N} \right)^q, \quad (2)$$

that can be expressed in the scaling limit $N \rightarrow \infty$ in terms of the density as

$$\langle x^q \rangle(T) = \frac{1}{\alpha} \int_0^1 x^q \rho(x, T) dx + o(N^0). \quad (3)$$

We note that $\alpha = \int \rho$ the average particle density of the model is conserved with time, and so is the zeroth moment $\langle x^0 \rangle$.

Strong memory limit and separation of time scales. — The model becomes tractable in the case where the jump probabilities p_ℓ, p_r depend on N and get vanishingly small as $N \rightarrow \infty$. More precisely, we consider the *strong memory limit* $p_\ell, p_r \rightarrow 0$, as well as the *weakly asymmetric regime* $p_r - p_\ell = (p_r + p_\ell) \mathcal{O}(\frac{1}{N})$. In the strong memory limit, it becomes increasingly likely that the sequence of jumps strictly alternates between left and right,

hence the name *stroboscopic exclusion process*. We will now show that in the strong memory limit, once the distances are rescaled by N , there are two possible rescalings of T such that the rescaled system undergoes a spatial evolution of order $\mathcal{O}(1)$ during a $\mathcal{O}(1)$ time lapse.

Let us consider two consecutive time steps in which a particle is moved and study the possible modifications of the moments (2). During the two time steps the quantities $\sum_i X_i^q$ for $q \geq 2$ have a probability $\mathcal{O}(1)$ of being modified, since the probability that the second move cancels the modification of the first one is $\mathcal{O}(1/N)$. However, if the two jumps are performed in opposite direction, the quantity $\sum_i X_i$ will remain the same, irrespectively of the particles selected for each jump. Hence, $\sum_i X_i$ only has a probability $\mathcal{O}(p_\ell + p_r)$ of being modified during these two time steps, since the second move will be in the same direction as the first one with probability p_ℓ or p_r . It follows that the q -th moments for $q \geq 2$ should evolve at the same rescaled time denoted τ , whereas the first moment should evolve at a time scale $t = (p_\ell + p_r)\tau$. In the strong memory regime $p_\ell + p_r$ tends to zero with N , so that the time scale t is then infinitely slower than τ . Hence, on the τ scale, the system effectively evolves at fixed first moment. These order of magnitude considerations will be confirmed in a more quantitative way in the next sections.

The probability distribution of observing a certain sequence of particles and holes can be expressed in terms of the density only upon a mean-field approximation, and so the density does not fully characterize the system. In our case however, when $\tau \rightarrow \infty$ at fixed t the mean field approximation is quite natural due to the scales separation and we expect it to be exact, and is also numerically observed to hold quite well.

To support this claim, we first recall that in the scaling limit, the evolution under time τ at fixed t exactly alternates between left and right moves. Let us include the direction of the last move $\rightarrow, \leftarrow$ to the state of the system so that the evolution is Markovian. Notice that the strict alternation imposes that the reachable configurations have the same first moment, up to $\pm \frac{1}{\alpha N^2}$. With this strict alternating dynamics, the transition probability between two configurations is symmetric. It follows that the uniform probability distribution on the reachable configurations satisfies detailed balance, hence is the stationary probability distribution that is reached when $\tau \rightarrow \infty$. Entropic arguments ensure then that the configurations with a precise density distribution $\rho(x, t)$ are exponentially more numerous, so that with probability 1 in the scaling limit the system is observed in a fixed density $\rho(x, t)$. The question is now to understand the finer details of the configurations beyond the density, namely the probability distribution of observing a certain sequence of particles and holes. We will restrict for simplicity to clusters of particles, but similar arguments can be applied to more general sequences.

To that end, we divide the interval $[0, 1]$ into windows small enough so that ρ can be approximated to be a con-

stant K/N on them, with K the number of particles in a window. On each of these windows, since the microscopic configurations are equiprobable, the study of the sizes of the clusters of particles reduces to studying the combinatorial quantity for a fixed w

$$C_K^N = \sum_{\substack{I \subset \{1, \dots, N\} \\ |I|=K}} h_w(I), \quad (4)$$

where $h_w(I)$ is the sum of w^{m_i} if in the configuration I the particles are grouped into clusters of size $m_1, m_2, \dots$. Indeed, the coefficient in front of w^n in C_K^N is the number of clusters of n particles among all the configurations. Considering the first cluster of b particles appearing after $a-1$ holes and followed by one hole, one finds the following recurrence relation

$$C_K^N = \sum_{a=1}^N \sum_{b=1}^K \left[w^b \binom{N-a-b}{K-b} + C_{K-b}^{N-a-b} + \delta_{K-b,0} \delta_{N+1-a-b,0} w^b \right], \quad (5)$$

from which one deduces the generating function

$$\sum_{K,N} C_K^N x^K y^N = \frac{wxy(1-xy)^2}{(1-y(1+x))^2(1-wxy)}. \quad (6)$$

From this function one finds that for $K, N \rightarrow \infty$ at K/N fixed we have [28]

$$C_K^N = \frac{wK(1-K/N)^2}{1-wK/N} \binom{N}{K} + \mathcal{O}\left(\binom{N}{K}\right). \quad (7)$$

Upon expanding in w at order n , it yields that the probability of observing a cluster of n particles is $(1-\rho)^2 \rho^n$, which is indeed the mean field result. Hence, the stationary state reached for $\tau \rightarrow \infty$ at fixed t exactly satisfies the mean field relations.

Time evolution of the moments . – We now wish to determine the slowest time evolution of the system, i.e. the evolution of the system on the time scale t . We will see that it is entirely determined by the evolution of the first moment.

To that end, we define $\Delta_q(T) = \langle x^q \rangle(T+1) - \langle x^q \rangle(T)$ ($q \geq 1$). In what follows we will keep the T dependence implicit to write more concise expressions. One has

$$\Delta_q = P_r \int_0^1 P \left(\begin{smallmatrix} \bullet & \circ \\ x_N & x_{N+1} \end{smallmatrix} \right) \delta_r \langle x^q \rangle dx + P_\ell \int_0^1 P \left(\begin{smallmatrix} \circ & \bullet \\ x_N & x_{N+1} \end{smallmatrix} \right) \delta_\ell \langle x^q \rangle dx, \quad (8)$$

where the various coefficients are defined as follows. The first term corresponds to particles moved to the right and the second term to particles moved to the left. In the first term, P_r is the probability of attempting to move a particle to the right, $P \left(\begin{smallmatrix} \bullet & \circ \\ x_N & x_{N+1} \end{smallmatrix} \right)$ is the density of

favourable interfaces around the rescaled position x , i.e. the probability that the site xN is occupied with $xN+1$ free, while $\delta_r \langle x^q \rangle$ is the variation of the q -th moment when a particle is moved from xN to $xN+1$. The factors of the second term have a similar definition.

We gave a compelling argument in the previous section supporting that in the strong memory limit the mean field approximation holds so that one can express the probability of interfaces as a product of observing a particle at position xN times a probability of observing a hole at position $xN+1$. Hence, we have

$$\begin{aligned} P \left(\begin{smallmatrix} \bullet & \circ \\ x_N & x_{N+1} \end{smallmatrix} \right) &= \rho(x) \left(1 - \rho \left(x + \frac{1}{N} \right) \right) \\ &= \rho(x)(1-\rho(x)) - \frac{1}{N} \rho(x) \rho'(x) + \mathcal{O}(N^{-2}), \\ P \left(\begin{smallmatrix} \circ & \bullet \\ x_N & x_{N+1} \end{smallmatrix} \right) &= (1-\rho(x)) \rho \left(x + \frac{1}{N} \right) \\ &= \rho(x)(1-\rho(x)) + \frac{1}{N} (1-\rho(x)) \rho'(x) + \mathcal{O}(N^{-2}). \end{aligned} \quad (9)$$

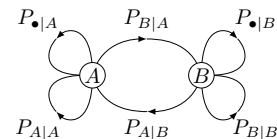
This quantity can as well be seen as the density of interfaces and it is not necessarily uniform. One also has

$$\begin{aligned} P_r &= p_r \langle \rightarrow \rangle + (1-p_\ell) \langle \leftarrow \rangle, \\ P_\ell &= p_\ell \langle \leftarrow \rangle + (1-p_r) \langle \rightarrow \rangle, \end{aligned} \quad (10)$$

where $\langle \rightarrow \rangle$ ($0 \leq \langle \rightarrow \rangle \leq 1$) is the fraction of time during which the previous move was a right jump and similarly for $\langle \leftarrow \rangle$. The factors $\delta_r \langle x^q \rangle$ and $\delta_\ell \langle x^q \rangle$ are obtained by a simple Taylor expansion, the only subtlety being that in the case of $\delta_\ell \langle x^q \rangle$ the particle is moved from $xN+1$ to xN . Putting everything together, we get

$$\begin{aligned} \Delta_q &= \frac{[p_r \langle \rightarrow \rangle + (1-p_\ell) \langle \leftarrow \rangle]}{\alpha N^2} \int_0^1 \left[\rho(1-\rho) - \frac{1}{N} \rho \rho' \right] \\ &\quad \times \left[qx^{q-1} + \frac{1}{2N} q(q-1)x^{q-2} \right] dx \\ &\quad + \frac{[p_\ell \langle \leftarrow \rangle + (1-p_r) \langle \rightarrow \rangle]}{\alpha N^2} \int_0^1 \left[\rho(1-\rho) + \frac{1}{N} (1-\rho) \rho' \right] \\ &\quad \times \left[-qx^{q-1} - \frac{1}{2N} q(q-1)x^{q-2} \right] dx. \end{aligned} \quad (11)$$

There remains to compute $\langle \rightarrow \rangle$ and $\langle \leftarrow \rangle$. In general, let us consider a process where we can choose between A , B or nothing, respectively with probability $P_{A|x}$, $P_{B|x}$ and $P_{\bullet|x}$ with $x \in \{A, B\}$ the last value chosen. Pictorially we have:



We call $\langle A \rangle$ (resp. $\langle B \rangle$) the fraction of time during which the previous choice was A (resp. B). Then we have $\langle A \rangle + \langle B \rangle = 1$ and $\langle A \rangle P_{B|A} = \langle B \rangle P_{A|B}$ so that

$$\langle A \rangle = \frac{P_{A|B}}{P_{A|B} + P_{B|A}}, \quad \langle B \rangle = \frac{P_{B|A}}{P_{A|B} + P_{B|A}}. \quad (12)$$

In the case at hand, this leads to

$$\begin{aligned} \langle \rightarrow \rangle &= \frac{(1 - p_\ell) \int_0^1 [\rho(1 - \rho) - \frac{1}{N} \rho \rho'] dx}{\text{Den}}, \\ \langle \leftarrow \rangle &= \frac{(1 - p_r) \int_0^1 [\rho(1 - \rho) + \frac{1}{N} (1 - \rho) \rho'] dx}{\text{Den}}, \end{aligned} \quad (13)$$

with

$$\begin{aligned} \text{Den} &= (1 - p_\ell) \int_0^1 \left[\rho(1 - \rho) - \frac{1}{N} \rho \rho' \right] dx \\ &+ (1 - p_r) \int_0^1 \left[\rho(1 - \rho) + \frac{1}{N} (1 - \rho) \rho' \right] dx. \end{aligned} \quad (14)$$

Expanding this expression at first order in $1/N$ leads to at order $\mathcal{O}(N^{-3})$

$$\begin{aligned} \Delta_q &= \frac{1}{\alpha N^2} \frac{p_r - p_\ell}{2 - p_r - p_\ell} \left(\int_0^1 \rho(1 - \rho) q x^{q-1} dx \right. \\ &\quad \left. + \frac{1}{2N} \int_0^1 \rho(1 - \rho) q(q-1) x^{q-2} dx \right) \\ &+ \frac{1}{\alpha N^3} \left[\frac{p_\ell - p_r}{2 - p_r - p_\ell} \int_0^1 \rho \rho' q x^{q-1} dx \right. \\ &\quad - \frac{1 - p_r}{2 - p_r - p_\ell} \int_0^1 \rho' q x^{q-1} dx \\ &\quad + \frac{2(1 - p_r)(1 - p_\ell)(1 - p_\ell - p_r)}{(2 - p_r - p_\ell)^2} \times \\ &\quad \left. \times \frac{\int_0^1 \rho' dx}{\int_0^1 \rho(1 - \rho) dx} \int_0^1 \rho(1 - \rho) q x^{q-1} dx \right]. \end{aligned} \quad (15)$$

Projection onto a maximal entropy state in the strong memory limit . – In the strong memory limit, using (15) one finds

$$\Delta_q(T) = \begin{cases} o(N^{-3}) & \text{if } q = 1 \\ \mathcal{O}(N^{-3}) & \text{if } q > 1, \end{cases} \quad (16)$$

where if $p_\ell - p_r = o\left(\frac{p_\ell + p_r}{N}\right)$,

$$\Delta_1(T) = -\frac{p_\ell + p_r}{2\alpha N^3} \int_0^1 \rho' dx, \quad (17)$$

and if $p_\ell - p_r = \mathcal{O}\left(\frac{p_\ell + p_r}{N}\right)$,

$$\Delta_1(T) = \frac{p_r - p_\ell}{2\alpha N^2} \int_0^1 \rho(1 - \rho) dx - \frac{p_\ell + p_r}{2\alpha N^3} \int_0^1 \rho' dx. \quad (18)$$

We expanded at first order in p_ℓ, p_r and used that in both cases $p_\ell - p_r$ is negligible compared to $p_\ell + p_r$. The asymmetry between the $q = 1$ and $q > 1$ cases in (16) has interesting consequences. If we rescale the time as

$$\tau = \frac{T}{N^3}, \quad (19)$$

for a fixed $\tau = \mathcal{O}(1)$, the first moment is constant $\frac{d\langle x \rangle}{d\tau} = 0$, while the other moments evolve with τ , according to

$$\begin{aligned} \frac{d\langle x^q \rangle}{d\tau} &= \frac{1}{2\alpha} \left[\frac{\int_0^1 \rho' dx}{\int_0^1 \rho(1 - \rho) dx} \int_0^1 \rho(1 - \rho) q x^{q-1} dx \right. \\ &\quad \left. - \int_0^1 \rho' q x^{q-1} dx \right]. \end{aligned} \quad (20)$$

As for the first moment, we see that it evolves on the time scale

$$t = \frac{T(p_r + p_\ell)}{N^3}. \quad (21)$$

In the limit of $\tau \rightarrow \infty$ these higher moments reach their stationary states satisfying $\frac{d\langle x^q \rangle}{d\tau} = 0$ for $q > 1$. These equations are automatically satisfied if $\partial_x \rho(x) \propto \rho(x)(1 - \rho(x))$. This condition has the following interesting interpretation. If we suppose that in the $\tau \rightarrow \infty$ at fixed t limit, the microscopic configurations of the system with a fixed first moment are equiprobable, then the density of particles $\rho(x, t, \tau)$ when $\tau \rightarrow \infty$ at fixed t , reaches a τ -stationary state $\rho(x, t)$ and this stationary state has to maximize the (Shannon) entropy functional, see e.g. [29]

$$S[\rho] = - \int_0^1 \rho \log(\rho) + (1 - \rho) \log(1 - \rho) dx, \quad (22)$$

at fixed value of the first moment $\int_0^1 x \rho dx$. Such a constrain translates into the differential equation

$$\partial_x \rho(x, t) = -\lambda(t) \rho(x, t) (1 - \rho(x, t)), \quad (23)$$

with $\lambda(t)$ a Lagrange multiplier, which is indeed the equation obtained from the analysis of the time evolution of the moments.

Hence, if we now use the scaling (21) for t and choose $t = \mathcal{O}(1)$, the system is constantly projected onto a state satisfying (23). The projection process is well observed numerically as illustrated by Fig. 2.

Exact time evolution of the system. – The spatial dependence is entirely determined by eq. (23) with the condition $\int_0^1 \rho(x) dx = \alpha$. The solution is

$$\rho_\lambda(x) = \frac{1}{1 + B_\lambda e^{\lambda x}}, \quad (24)$$

with

$$B_\lambda = \frac{e^{\lambda(\alpha-1)} - 1}{1 - e^{\lambda\alpha}}. \quad (25)$$

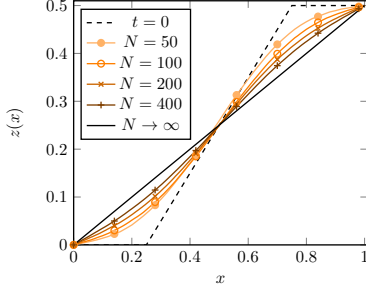


Fig. 2: Average $z(x, t)$ over 500 configurations of various sizes for a fixed time $t = 0.007$ choosing for initial configuration an interval of $N/2$ particles at the center. The stroboscopic jump probabilities are $p_r = p_\ell = 1/\sqrt{N}$. The first moment of the initial configuration vanishes, hence the system should be (instantly) projected on the state satisfying (23) with vanishing first moment (solid black line).

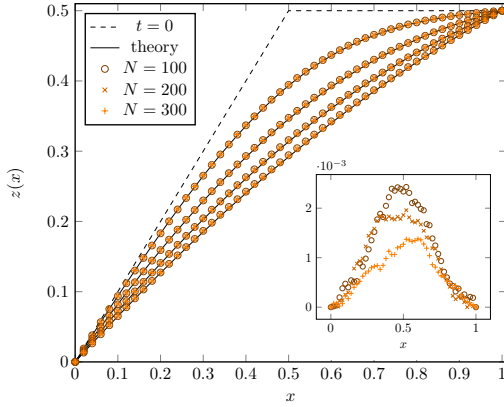


Fig. 3: Average $z(x, t)$ over 4 000 configurations of size $N = 100, 200, 300$ for $t = 0.07, 0.14, 0.21, 0.28$ choosing a step-like initial configuration with an interval of $N/2$ particles on the left. The stroboscopic jump probabilities are $p_r = p_\ell = 1/\sqrt{N}$. The inset shows the absolute value of the error at $t = 0.21$.

Using this solution, one finds the relation between λ and the value of the first moment to be

$$\langle x \rangle_\lambda = \frac{1}{2\alpha} - \frac{(\alpha-1)^2}{2\alpha} + \frac{\alpha-1}{\lambda\alpha} \ln \left(\frac{1}{B_\lambda} + 1 \right) + \frac{1}{\lambda^2\alpha} \left[\text{Li}_2 \left(\frac{1}{1+B_\lambda} \right) - \text{Li}_2 \left(\frac{1}{1+B_\lambda e^\lambda} \right) \right]. \quad (26)$$

Using this relation and the differential equation for the evolution of the first moment deduced from (18) gives a differential equation for $\lambda(t)$

$$\frac{d\lambda}{dt} = -\frac{1}{2\alpha} \left(\frac{\beta}{\lambda} + 1 \right) \frac{\frac{1}{1+B_\lambda e^\lambda} - \frac{1}{1+B_\lambda}}{\partial_\lambda \langle x \rangle_\lambda}, \quad (27)$$

with

$$\beta = \lim_{N \rightarrow \infty} N \frac{p_r - p_\ell}{p_r + p_\ell}. \quad (28)$$

The initial condition for λ depends on the initial configuration of the system described by $\rho(x, T=0)$. The initial

condition $\lambda(t=0) = \lambda_0$ is a solution to

$$\langle x \rangle_{\lambda_0} = \frac{1}{\alpha} \int_0^1 x \rho(x, t=0) dx, \quad (29)$$

with $\langle x \rangle_\lambda$ given by (26). Hence, in the scaling limit $N \rightarrow \infty$, starting from any initial state $\rho(x, 0)$, the *exact* density $\rho(x, t)$ at time t is given by (24), with $\lambda(t)$ satisfying the differential equation (27), with the initial condition $\lambda(0)$ satisfying (29).

At the τ scale some kind of continuity equation should exist, as happens in standard exclusion processes [23]. At the t scale however, due to the projection process (illustrated in Fig 2), we see there could be no continuity equation for $\rho(x, t)$ describing the dynamics for any initial condition. After the projection process, for $t > 0$, one can show that $\partial_t \rho$ is proportional to $\partial_x \rho$, but the coefficient cannot be expressed simply in terms of ρ .

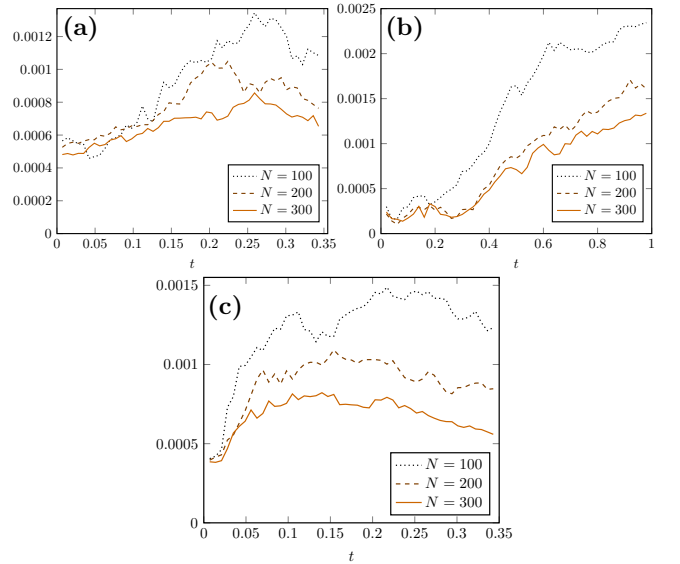


Fig. 4: Absolute value of the error between $z(x, t)$ and the theoretical prediction, averaged over x , as a function of t . The functions $z(x, t)$ are computed by averaging over 4 000 configurations. The stroboscopic jump probabilities are $p_r = p_\ell = 1/\sqrt{N}$ for (a) and (c) and $p_r = 1/\sqrt{N} + 1/N^{3/2}$ and $p_\ell = 1/\sqrt{N} - 1/N^{3/2}$ for (b). At $t=0$ all the particles are to the left. In (a) and (b) we have $\alpha = 1/2$ and in (c) we have $\alpha = 1/4$.

Let us specialize this general case to $\alpha = 1/2$ and a step initial condition where all the particles are disposed to the left of the lattice. One obtains

$$\begin{aligned} \rho(x, t) &= \frac{1}{1 + e^{\lambda(t)(x - \frac{1}{2})}} \\ \frac{d\lambda}{dt} &= -\frac{1}{4} \left(1 + \frac{\beta}{\lambda} \right) \lambda^3 \tanh(\lambda/4) \left[\text{Li}_2 \left(\frac{1}{1+e^{-\lambda/2}} \right) - \text{Li}_2 \left(\frac{1}{1+e^{\lambda/2}} \right) + \frac{\lambda^2}{4} \frac{1}{1+e^{-\lambda/2}} - \frac{\lambda}{2} \log(1 + e^{\lambda/2}) \right]^{-1} \\ \lambda(t=0) &= +\infty. \end{aligned} \quad (30)$$

This differential equation can be integrated numerically. The comparison with numerics is presented in Figs. 3 and 4 in terms of the counting function z defined in (1), for jump probabilities that are taken to vanish as $1/\sqrt{N}$. The choice $1/\sqrt{N}$ is convenient for numerics as it limits the number of time steps T needed to reach a given t . We observe that the agreement with the numerics is excellent, with an error between 10^{-3} and 10^{-4} for the sizes considered. This error is also seen to decrease as the system size increases.

Away from the strong memory limit. – Let us now consider the more generic case where the jump probabilities are fixed to be

$$p_r = p \left(1 + \frac{a}{N}\right), \quad p_\ell = p \left(1 - \frac{a}{N}\right), \quad (31)$$

for some values $0 < p < 1$ and real a . We note that if we set $p = 1/2$, we recover the usual Weakly Asymmetric Exclusion Process (WASEP) and if we further impose $a = 0$ we get the Symmetric Simple Exclusion Process (SSEP). For finite p , a separation of time scales does not occur anymore. There is now only one macroscopic time scale, that is proportional to N^3 . Hence, we cannot determine the exact evolution of the model as before. However, since the strong memory limit corresponds to the limit $p \rightarrow 0$, one can consider solving (15) iteratively in p . This would yield a series expansion

$$\rho(x, t) = \sum_{j \geq 0} p^j \rho_j(x, t), \quad (32)$$

with $\rho_0(x, t)$ satisfying the differential equation (23). Then (15) for $q = 1$ yields

$$\partial_t \langle x \rangle = \frac{1}{2\alpha} \left(\frac{a}{1-p} \int_0^1 \rho(1-\rho) dx - \int_0^1 \rho' dx \right) \quad (33)$$

with the rescaled time

$$t = \frac{2Tp}{N^3}, \quad (34)$$

as in (21). We define an expansion of the full-time evolution $\rho(x, t)$ of the exclusion process with jump probabilities (31), whose n -th term is obtained by retaining only ρ_j for $j \leq n$ in (32), but by keeping the full p dependence in (33). Performing a similar analysis as before, we obtain that the leading term of this expansion is of the form (24) with $\lambda(t)$ satisfying the differential equation

$$\frac{d\lambda}{dt} = -\frac{1}{2\alpha} \left(\frac{a}{\lambda(1-p)} + 1 \right) \frac{\frac{1}{1+B_\lambda e^\lambda} - \frac{1}{1+B_\lambda}}{\partial_\lambda \langle x \rangle}. \quad (35)$$

The comparisons of the first term in the expansion with the numerical simulation of the WASEP and SSEP are reported in Fig. 5 and provide a very good agreement at short times $t \lesssim 0.1$ and near the equilibrium state $t \gtrsim 0.8$ and an approximate agreement otherwise. This suggests

that, if the mean field approximation still holds perturbatively in p , the full time evolution of the SSEP and the WASEP could be expressed as a series in p from this exclusion process. Notice that in the non-strong memory limit the choice of a step-like initial configuration is crucial to make $\rho_0(x, t)$ a good approximation since the immediate projection on the states (24) does not occur in this case, but instead is expected to take a time $\sim p$.

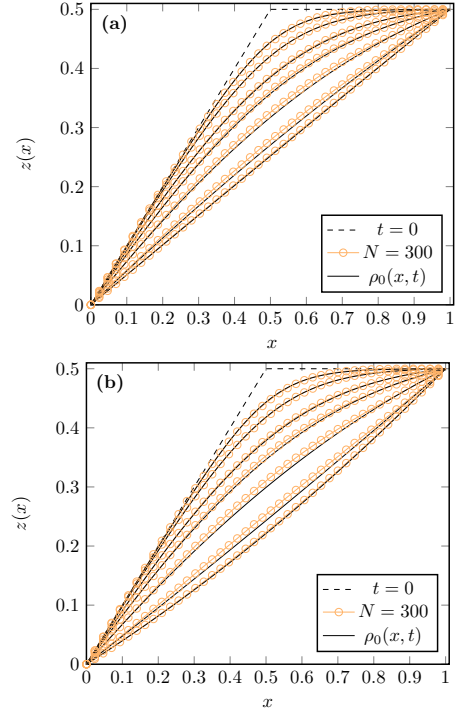


Fig. 5: Average over 1000 configurations of $z(xN, t)$ for $t = 0.02, 0.04, 0.08, 0.12, 0.2, 0.4, 0.8$ and for $N = 300$, starting from a step-like initial configuration with an interval of $N/2$ particles on the left. (a) SSEP case with jump probabilities $p_r = p_\ell = 1/2$. (b) WASEP case with $p_r = 1/2 + N/4$ and $p_\ell = 1/2 - N/4$. In both cases, the leading order prediction was obtained by solving numerically eq. (35).

Perspectives. – A natural perspective opened by this work is to study whether the approach outlined in the previous section can be pursued to describe the full time evolution of non-stroboscopic processes such as the SSEP and the WASEP. The fact that the first term of the expansion is numerically observed to approximately describe the relaxation dynamics of the SSEP and the WASEP after a step-like initial configuration is indeed promising.

Another possible direction is to study the extension of this approach to other types of boundary conditions. Indeed, in case of periodic boundary conditions for example and without anisotropy, the first moment would be necessarily constant in time, which should induce important modifications. The case of open boundaries with reservoirs would also be very relevant.

Besides, the study of this stroboscopic dynamics was inspired [29] by an attempt to describe the time evolu-

tion (according to an update rule) toward equilibrium of a directed lattice path with elementary steps $(1, 0)$, $(0, 1)$. Other choices of elementary steps lead to explicitly solvable Euler-Lagrange equations analogous with (23). This is the case for paths with elementary steps $(1, 0)$, $(0, 1)$ and $(1, 1)$, for which the corresponding exclusion process consists of two types of particles. Defining an appropriate stroboscopic dynamic on this system might be interesting.

* * *

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REFERENCES

- [1] T. M. Liggett, *Stochastic interacting systems: contact, voter and exclusion processes*, Phys. Rev. Die Grundlehren der mathematischen Wissenschaften in Einzel-darstellungen. Springer (1999)
- [2] T. Chou, K. Mallick and R. K. P. Zia, *Non-equilibrium statistical mechanics: from a paradigmatic model to biological transport*, Reports on Progress in Physics **74** 116601 (2011).
- [3] M. R. Evans, P. A. Ferrari and K. Mallick, *Matrix representation of the stationary measure for the multispecies TASEP*, J. Stat. Phys. **135** 217 (2009).
- [4] S. Prohlač, M. R. Evans and K. Mallick, *The matrix product solution of the multispecies partially asymmetric exclusion process*, J. Phys. A: Math. Gen. **42** 165004 (2009).
- [5] J. de Gier and F. H. L. Essler, *Bethe ansatz solution of the asymmetric exclusion process with open boundaries*, Phys. Rev. Lett. **95** 240601 (2005).
- [6] C. Boutillier, P. Francois, K. Mallick and S. Mallick, *A matrix ansatz for the diffusion of an impurity in the asymmetric exclusion process*, J. Phys. A: Math. Gen. **35** 9703 (2002).
- [7] B. Derrida and M. R. Evans, *Bethe ansatz solution for a defect particle in the asymmetric exclusion process*, J. Phys. A: Math. Gen. **32** 4833 (1999).
- [8] T. Sasamoto, *One-dimensional partially asymmetric simple exclusion process on a ring with a defect particle*, Phys. Rev. E **61** 4980 (2000).
- [9] B. Derrida, S. A. Janowsky, J. L. Lebowitz and E. R. Speer, *Exact solution of the totally asymmetric simple exclusion process: shock profiles*, J. Stat. Phys. **73** 813 (1993).
- [10] K. Mallick, *Shocks in the asymmetry exclusion model with an impurity*, J. Phys. A: Math. Gen. **29** 5375 (1996).
- [11] M. R. Evans, N. Rajewsky and E. R. Speer, *Exact solution of a cellular automaton for traffic*, J. Stat. Phys. **95** 45 (1999).
- [12] N. Rajewsky, L. Santen, A. Schadschneider and M. Schreckenberg, *The asymmetric exclusion process: comparison of update procedures*, J. Stat. Phys. **92** 151 (1998).
- [13] M. R. Evans, Y. Kafri, K. E. P. Sugden and J. Tailleur, *Phase diagrams of two-lane driven diffusive systems*, J. Stat. Mech. **2011** P06009 (2011).
- [14] M. R. Evans, R. Juhász and L. Santen, *Shock formation in an exclusion process with creation and annihilation*, Phys. Rev. E **68** 026117 (2003).
- [15] R. Harris and R. Stinchcombe, *Disordered asymmetric simple exclusion process: Mean-field treatment*, Phys. Rev. E **70** 016108 (2004).
- [16] R. Harris and R. Stinchcombe, *Shock formation in an exclusion process with creation and annihilation*, Phys. Rev. E **70** 016108 (2004).
- [17] R. Stinchcombe and S. L. A. de Queiroz, *Smoothly varying hopping rates in driven flow with exclusion*, Phys. Rev. E **83** 061113 (2011).
- [18] S. A. Janowsky and J. L. Lebowitz, *Exact results for the asymmetric simple exclusion process with a blockage*, J. Stat. Phys. **77** 35 (1994).
- [19] S. A. Janowsky and J. Lebowitz, *Finite-size effects and shock fluctuations in the asymmetric simple-exclusion process*, Phys. Rev. A **45** 618 (1992).
- [20] V. Popkov and G. Schütz, *Steady-state selection in driven diffusive systems with open boundaries*, EPL **1** 1 (1999).
- [21] S. Katz, J. L. Lebowitz and H. Spohn, *Phase transitions in stationary nonequilibrium states of model lattice systems*, Phys. Rev. B **28** 1655 (1983).
- [22] M. Vanicat, *Integrable Floquet dynamics, generalized exclusion processes and “fused” matrix ansatz*, Nucl. Phys. B **929** 298 (2018).
- [23] F. Rezakhanlou, *Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$* , Comm. Math. Phys. **140** 417 (1991).
- [24] R. B. Stinchcombe and G. M. Schütz, *Operator algebra for stochastic dynamics and the Heisenberg chain*, Europhys. Lett. **29** 663 (1995).
- [25] J. de Gier and F. H. L. Essler, *Exact spectral gaps of the Asymmetric Exclusion Process with open boundaries*, J. Stat. Mech. P12011 (2006).
- [26] J. de Gier and F. H. L. Essler, *Slowest relaxation mode of the partially asymmetric exclusion process with open boundaries*, J. Phys. A: Math. Theor. **41** 485002 (2008).
- [27] G. Carinci, A. De Masi, C. Giardinà and E. Presutti, *Super-hydrodynamic limit in interacting particle systems*, J. Stat. Phys. **5** 155 (2013).
- [28] R. Pemantle and M. C. Wilson, *Twenty combinatorial examples of asymptotics derived from multivariate generating functions*, SIAM Review **50** (2008).
- [29] B. Debin, E. Granet and P. Ruelle, *Concavity analysis of the tangent method*, J. Stat. Mech. **28** 113104 (2019).
- [30] J. de Gier and F. Essler, *Exact spectral gaps of the asymmetric exclusion process with open boundaries*, J. Stat. Mech.: Theory Exp. **2006(12)** (2006) 12011.
- [31] N. G. Van Kampen, *Composite stochastic processes*, Physica A **96(3)** (1979) 435-453.
- [32] P. A. Ferrari and A. Galves, *Rate of convergence to equilibrium of symmetric simple exclusion processes*, Markov Proc. Rel. Fields **6** (2000).