Strong versions of impulsive controllability and sampled observability

P.-A. Absil [a], Irina Cojuhari [b], Ion Fiodorov [b], André L. Tits [c]

[a] *ICTEAM Institute, UCLouvain, 1348 Louvain-la-Neuve, Belgium*

[b] *Department of Software Engineering and Automatics, Technical University of Moldova, Chisinau, Republic of Moldova*

[c] *Department of Electrical and Computer Engineering and Institute for Systems Research, University of Maryland, College Park, MD 20742, USA*

Abstract

We give a simple proof of the (perhaps not so) well known fact that exponential polynomials of order k with real exponents have at most $k-1$ real zeros. We deduce several results that relate to impulsive controllability and sampled observability of finite-dimensional linear time-invariant dynamical systems. We prove that the initial state of a continuous-time linear time-invariant dynamical system of dimension n can be uniquely reconstructed from the sampled output regardless of its sampling time sequence of length n if and only if the system is observable and all the eigenvalues of the system matrix are real. This result thus characterizes sampled observability with arbitrary sampling times. Likewise, we prove that the system is controllable by means of n impulses regardless of when they occur if and only if the system is controllable and all the eigenvalues of the system matrix are real. We also show that, if the system is observable and all the eigenvalues of the system matrix are real, then there exists an initial state such that the times where the output crosses a prescribed threshold are prescribed m times with $m < n$.

Key words: Impulsive controllability; impulse controllability; sampled observability; sum of Dirac impulses; finitely sampled output; quasi-polynomials; quasipolynomials; polynomial–exponential functions; extended polynomial sums

1 Introduction

A well-known consequence of d'Alembert's theorem is that every nonzero, single-variable, degree n polynomial, with real or complex coefficients, has at most n real roots. It is also known that every nonzero linear combination of n exponentials $t \mapsto e^{\lambda_i t}$, with real λ_i's, has at most $n-1$ zeros; see [6, Corollary 3.2]. This fact can also be deduced, e.g., from [26, Proposition 110] where the nonsingularity of certain generalized Vandermonde matrices is studied.

A lesser-known result—though it is mentioned as well known in [19]—that subsumes these two properties is recalled in Theorem 2 below.

Definition 1 *A function of the form*

$$y(t) = p_1(t)e^{\lambda_1 t} + ... + p_\ell(t)e^{\lambda_\ell t}, \qquad (1)$$

where the λ_i's are distinct real numbers and the p_i's are nonzero real polynomials, is termed a purely real exponential polynomial *(in this paper,* exponential polynomial *for short) of order* $\mathrm{ord}(y) := \ell + \deg p_1 + ... + \deg p_\ell = \sum_{j=1}^{\ell}(\deg p_j + 1)$*. Moreover, in order to streamline forthcoming statements, we say that the zero function is the exponential polynomial of order* 0*.*

In the literature, exponential polynomials are also referred to as quasi-polynomials (or quasipolynomials) [20], polynomial–exponential functions [16], and extended polynomial sums [17]. In [22], the term "real exponential polynomial" indicates that the λ's and the coefficients of the p's are real, but in [19, (8)] the same term refers to the functions of the form (1) where the p's and λ's appear in complex-conjugate pairs, i.e., the functions (1) that are real-valued whenever t is real. In this paper, the term "exponential polynomial" is used to mean purely real exponential polynomials in the

Email addresses: pa.absil@uclouvain.be (P.-A. Absil), irina.cojuhari@ati.utm.md (Irina Cojuhari), ion.fiodorov@ati.utm.md (Ion Fiodorov), andre@umd.edu (André L. Tits).

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sense of Definition 1.

Recall that a C^∞ function y has a zero of *multiplicity* k at a point t_* if $y(t_*) = \cdots = \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} y(t_*) = 0$ and $\frac{\mathrm{d}^k}{\mathrm{d}t^k} y(t_*) \neq 0$; see, e.g., [6, p. 224]. In the rest of this paper, unless otherwise stated (by means of the phrase "complex zeros", resp. "distinct zeros"), *zeros* refer to real zeros and multiplicities are counted.

Theorem 2 (zeros of exponential polynomials) *Let $y(\cdot)$ be a nonzero exponential polynomial of positive order (Definition 1). Then $y(\cdot)$ has at most $\mathrm{ord}(y)-1$ zeros (in the sense specified above, i.e., real zeros counting multiplicities).*

Theorem 2 can be readily deduced from more advanced results, such as [19, Theorem 2] or [5, Theorem 3.1].

In this paper, we give a detailed yet short proof of Theorem 2 that only relies on basic calculus (Section 2.1), then we present several consequences in signals and systems. Furthermore, we provide conditions (see Theorem 9) that are not only sufficient but also *necessary* for an n-dimensional continuous-time linear time-invariant (LTI) system to be impulsively controllable for every sequence of n times at which the impulses occur. Likewise, we provide conditions (see Theorem 13) that are not only sufficient but also *necessary* for every sequence of n sample times to make the system sample observable. The main findings are summarized in the conclusion (Section 5).

The required background in controllability (also termed reachability) and observability can be found, e.g., in [1].

2 Zeros of exponential polynomials

2.1 Proof of Theorem 2.

The only elementary proof of Theorem 2 that we could locate in the literature [18, p. 234, §75] uses an induction argument on the number of exponential terms. Here we provide an elementary proof by induction on the order.

If $\mathrm{ord}(y) = 1$, then $y(t) = ce^{\lambda t}$ with $c \neq 0$; it has $\mathrm{ord}(y) - 1 = 0$ zeros. Now assume that the claim holds when $\mathrm{ord}(y) = k-1 \geq 1$. Consider $y(t)$ as in (1) of order k; i.e., consider $y(t) = p_1(t)e^{\lambda_1 t} + \ldots + p_\ell(t)e^{\lambda_\ell t}$, where the λ_i's are distinct real numbers and the p_i's are nonzero real polynomials, with $\sum_{j=1}^{\ell}(\deg p_j + 1) = k$. Observe that $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\lambda_1 t} y(t)\right)$ is an exponential polynomial of order $k-1$. By the induction hypothesis, it has at most $k-2$ zeros. It is a well-known consequence (see [6, Proposition 2.1]) of Rolle's lemma that the number of zeros of a C^∞ function is at most one more than the number of zeros of its derivative. It follows that $e^{-\lambda_1 t} y(t)$, and thus $y(t)$ itself, has at most $k-2+1 = k-1$ zeros. This concludes the proof of Theorem 2.

2.2 Comments

Theorem 2 remains true when the coefficients of the polynomials are allowed to be complex. Indeed, it then follows from Theorem 2 that the real part of $y(\cdot)$ has at most $\mathrm{ord}(y) - 1$ zeros (and likewise for the imaginary part). Thus the real and imaginary parts have at most $\mathrm{ord}(y) - 1$ common zeros, and these common zeros are the zeros of $y(\cdot)$.

Clearly, Theorem 2 becomes false if the λ_i's are allowed to be complex numbers. In that case however, several results are known; see, e.g., [19]. Related results can be found in the literature on oscillation theory and disconjugacy [3].

Counting the *complex* zeros of exponential polynomials is also a more intricate matter; see, e.g., [22].

Proposition 3 *The bound in Theorem 2 is tight. In fact, letting $\mathrm{Z}(y)$ denote the number of real zeros of the exponential polynomial y, it holds that $\{\mathrm{Z}(y) \mid \mathrm{ord}(y) = k\} = \{0, \ldots, k-1\}$.*

PROOF. Given $m \in \{0, \ldots, k-1\}$, choose a polynomial p of degree k with m zeros and observe that $y(t) = p(t)e^t$, an exponential polynomial, has m zeros. □

3 Controllability-related consequences

Consider the continuous-time LTI single-input control system in state-space form:

$$\dot{x}(t) = Ax(t) + bu(t), \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^{n\times 1}$. (The multi-input case is more intricate and left for further work.) Recall that the solutions of (2) take the form $x(t_b) = e^{A(t_b - t_a)}x(t_a) + \int_{t_a}^{t_b} e^{A(t_b-\tau)} bu(\tau)\mathrm{d}\tau$ for all t_a, t_b.

We start with a matrix result that will be used repeatedly in the rest of the paper. Observe that it is not assumed that A is diagonalizable; i.e., A can be defective.

Proposition 4 *The following conditions are equivalent.*

(C1) *(A, b) is controllable and all eigenvalues of A are real.*
(C2) *Given $t_\mathrm{f} \in \mathbb{R}$, for all distinct real numbers $t_1, \ldots, t_n$, matrix*

$$\Phi := \left[e^{A(t_\mathrm{f}-t_1)}b \ \ldots \ e^{A(t_\mathrm{f}-t_n)}b\right] \tag{3}$$

is invertible.

Remark 5 *In Proposition 4, condition (C2) is equivalent to* $\left[e^{A\tau_1}b \ \dots \ e^{A\tau_n}b\right]$ *invertible for all distinct* $\tau_1,\dots,\tau_n$*. The form of* Φ *in* (3) *is slightly more convenient for forthcoming results.*

PROOF of Proposition 4. (C1) $\Rightarrow$ (C2). Proceeding by contradiction, suppose that Φ is singular (i.e., not invertible). Then there is $\mu \in \mathbb{R}^n$, with $\mu \neq 0$, such that $\mu^T\Phi = 0$. Hence $t \mapsto \mu^T e^{A(t_\mathrm{f}-t)}b$, which is a (purely real) exponential polynomial of order at most n (this follows from the exponential of the Jordan form, see [1, p. 184], and the assumption that all eigenvalues of A are real), has zeros at $t_1,\dots,t_n$; this implies, in view of Theorem 2, that it is identically zero. But then, given an arbitrary $u(\cdot)$ and $t_0 < t_\mathrm{f}$, it holds that $\mu^T(x(t_\mathrm{f}) - e^{A(t_\mathrm{f}-t_0)}x(t_0)) = \mu^T\int_{t_0}^{t_\mathrm{f}} e^{A(t_\mathrm{f}-\tau)}bu(\tau)\mathrm{d}\tau = \int_{t_0}^{t_\mathrm{f}} \mu^T e^{A(t_\mathrm{f}-\tau)}b\,u(\tau)\mathrm{d}\tau = 0$, in contradiction with the controllability assumption.

(C1) $\Leftarrow$ (C2). First, if (A,b) is not controllable, then Φ is singular. Indeed, if (A,b) is not controllable, then there exists $\mu \neq 0$ such that $\mu^T A^i b = 0$, $i = 0,\dots,n-1$. By the Cayley–Hamilton theorem, $e^{A\tau}$ can be expressed as an $n-1$st order polynomial in A. Hence $\mu^T\Phi = 0$. (A similar argument can be found in [12]. Observe indeed that (C2) implies "impulsive controllability", which is equivalent to controllability in view of [12, Theorem 1].)

Second, if A has at least one pair of complex conjugate eigenvalues, we show next that there is $t_1 < \dots < t_n \leq t_\mathrm{f}$ such that Φ, defined in (3), is not invertible, concluding the proof. We assume without loss of generality that A is in real Jordan form. (To see that there is no loss of generality, let $A = S\tilde{A}S^{-1}$ be a real Jordan decomposition of A, and observe that Φ is invertible if and only if

$$S^{-1}\Phi = \left[e^{\tilde{A}(t_\mathrm{f}-t_1)}S^{-1}b \ \dots \ e^{\tilde{A}(t_\mathrm{f}-t_n)}S^{-1}b\right]$$

is invertible.) Since, by the contradiction assumption, A has at least one pair of complex conjugate eigenvalues $\alpha \pm j\omega$, $\omega \neq 0$, we can arrange the real Jordan form such that the last Jordan block has the form

$$J = \begin{bmatrix} D & I_2 & & & \\ & D & I_2 & & \\ & & \ddots & \ddots & \\ & & & D & I_2 \\ & & & & D \end{bmatrix},$$

$$\text{with } D = \alpha I_2 + \tilde{D} \text{ and } \tilde{D} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}.$$

Choose $t_1 < \dots < t_n$ with $t_{i+1} - t_i = k_i 2\pi/\omega$, $i = 1,\dots,n-1$, k_i's integers. It follows that, for $i = 1,\dots,n$,

$$\begin{aligned} &e^{\tilde{D}(t_\mathrm{f}-t_i)} \\ &= \begin{bmatrix} \cos(\omega(t_\mathrm{f}-t_i)) & \sin(\omega(t_\mathrm{f}-t_i)) \\ -\sin(\omega(t_\mathrm{f}-t_i)) & \cos(\omega(t_\mathrm{f}-t_i)) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\omega(t_\mathrm{f}-t_1)) & \sin(\omega(t_\mathrm{f}-t_1)) \\ -\sin(\omega(t_\mathrm{f}-t_1)) & \cos(\omega(t_\mathrm{f}-t_1)) \end{bmatrix} \\ &= e^{\tilde{D}(t_\mathrm{f}-t_1)} \end{aligned}$$

Hence, denoting by $b_{n-1:n}$ the last two elements of b (recall that b is a column vector), we obtain that the last two rows of Φ are the rows of

$$\begin{aligned} &\left[e^{D(t_\mathrm{f}-t_1)}b_{n-1:n} \ \dots \ e^{D(t_\mathrm{f}-t_n)}b_{n-1:n}\right] \\ &= \left[e^{(\alpha I_2+\tilde{D})(t_\mathrm{f}-t_1)}b_{n-1:n} \ \dots \ e^{(\alpha I_2+\tilde{D})(t_\mathrm{f}-t_n)}b_{n-1:n}\right] \\ &= \left[e^{\alpha(t_\mathrm{f}-t_1)}e^{\tilde{D}(t_\mathrm{f}-t_1)}b_{n-1:n} \ \dots \ e^{\alpha(t_\mathrm{f}-t_n)}e^{\tilde{D}(t_\mathrm{f}-t_n)}b_{n-1:n}\right] \\ &= \left[e^{\alpha(t_\mathrm{f}-t_1)}e^{\tilde{D}(t_\mathrm{f}-t_1)}b_{n-1:n} \ \dots \ e^{\alpha(t_\mathrm{f}-t_n)}e^{\tilde{D}(t_\mathrm{f}-t_1)}b_{n-1:n}\right] \\ &= e^{\tilde{D}(t_\mathrm{f}-t_1)}b_{n-1:n}\left[e^{\alpha(t_\mathrm{f}-t_1)} \ \dots \ e^{\alpha(t_\mathrm{f}-t_n)}\right] \end{aligned}$$

which has rank at most one as the outer product of two vectors. Matrix Φ is thus not invertible. □

Remark 6 *The above short proof of (C1)* $\Rightarrow$ *(C2) is solely based on Theorem 2 and basic system theory. The implication also readily follows from the finite-horizon observability result in [25, Theorem 2, "Furthermore" part].*

Remark 7 *By duality between controllability and observability, the "Second" part of the proof of (C1)* $\Leftarrow$ *(C2) can also be deduced from the following fact, alluded to without proof after [21, Definition 1]: If* $c \in \mathbb{R}^{1\times n}$ *and not all eigenvalues of* $A \in \mathbb{R}^{n\times n}$ *are real, then there exists* $x(0) \neq 0$ *such that* $y(t) := ce^{At}x(0)$ *has infinitely many zeros.*

The remainder of this section focuses on a connection between invertibility of Φ and impulsive controllability. Consider impulsive control signals $u(t) = \sum_{i=1}^n \gamma_i\delta(t-t_i)$ with $t_1 < \dots < t_n$ and $\gamma \in \mathbb{R}^n$, where δ is the Dirac distribution. Then (2) reads

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq t_k \\ x(t_k^+) = x(t_k^-) + b\gamma_k, & k = 1,\dots,n. \end{cases} \tag{4}$$

Control questions for systems such as (4) have been investigated, e.g., in [2,8,23,14,4,10]. In particular, (4) is a single-input time-invariant version of the linear impulsive control systems considered in [12]; or see [24, §2.3].

The next definition is a more restrictive version of the notion of impulsive controllability [12, Definition 1]: it imposes the existence of an impulsive control for *all* $t_1 < \cdots < t_n$.

Definition 8 *We say that system* (2) *is* strongly impulsively controllable *if for every* $x_0, x_\mathrm{f} \in \mathbb{R}^n$*, every open interval* (t_0, t_f)*, and every* $t_1 < \cdots < t_n$ *in* $[t_0, t_\mathrm{f}]$*, there exists* $\gamma \in \mathbb{R}^n$ *such that* (4) *has a solution* $x(t)$ *satisfying* $x(t_0^-) = x_0$ *and* $x(t_\mathrm{f}^+) = x_\mathrm{f}$*.*

Theorem 9 *Conditions (C1) and (C2) in Proposition 4 are further equivalent to:*

(C3) System (2) *is strongly impulsively controllable.*

In particular, system (2) *is strongly impulsively controllable if and only if it is controllable and all eigenvalues of* A *are real.*

PROOF. The solution of (4) for $x(t_0^-) = x_0$ satisfies

$$\begin{aligned} x(t_\mathrm{f}^+) &= e^{A(t_\mathrm{f}-t_0)}x_0 + \int_{t_0^-}^{t_\mathrm{f}^+} e^{A(t_\mathrm{f}-\tau)} b \sum_{i=1}^n \gamma_i \delta(\tau - t_i) \mathrm{d}\tau \\ &= e^{A(t_\mathrm{f}-t_0)}x_0 + \sum_{i=1}^n \gamma_i e^{A(t_\mathrm{f}-t_i)} b \\ &= e^{A(t_\mathrm{f}-t_0)}x_0 + \Phi\gamma, \end{aligned}$$

with Φ as in (3). Hence (2) is strongly impulsively controllable if and only if (C2) of Proposition 4 holds. □

4 Observability-related consequences

Now consider the following continuous-time LTI single-output system in state-space form:

$$\dot{x}(t) = Ax(t) \qquad (5a)$$
$$y(t) = cx(t) \qquad (5b)$$

where $A \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^{1\times n}$. (Note that c is a row vector, much like b was a column vector in Section 3). Recall that the solutions of (5) are of the form $y(t) = ce^{At}x(0)$.

It is well known that the pair (A, b) is controllable if and only if the pair (b^T, A^T) is observable. By this duality, Proposition 4 becomes:

Proposition 10 *The following conditions are equivalent.*

(O1) (c, A) *is observable and all eigenvalues of* A *are real.*

(O2) For all distinct real numbers $t_1, \ldots, t_n$*, matrix*

$$\Psi = \begin{bmatrix} ce^{At_1} \\ \vdots \\ ce^{At_n} \end{bmatrix} \qquad (6)$$

is invertible.

PROOF. Observability of (c, A) is equivallent to controllability of (A^T, c^T). Invertibility (here, of Ψ) is preserved by transposition, and so are eigenvalues (here of A). Finally, given $t_\mathrm{f} \in \mathbb{R}$, $t_1, \ldots, t_n$ are distinct and arbitrary if and only if $t_\mathrm{f} - t_1, \ldots, t_\mathrm{f} - t_n$ are. This shows that Proposition 10 is equivalent to Proposition 4. □

Definition 11 *We say that system* (5) *is* strongly sampled observable *if for every distinct* $t_1, \ldots, t_n$*, it is possible to uniquely reconstruct the initial state* $x(0)$ *from* $y(t_1), \ldots, y(t_n)$*.*

Remark 12 *Observability of dynamical systems based on sampled observations has been investigated, e.g., in [15,21,25,11,7,13]; see also [9] for discrete-time systems. Strong sampled observability, as defined above, relates to Lebesgue-sampled observability [15]; the former departs from the latter in its* for every *quantifier. System* (5) *is strongly sampled observable if and only if it is* N_T*-sample observable—in the sense of [21, Definition 1]—for* $N = n$ *and all* $T > 0$*. The "only if" part is direct. For the "if" part, observe that if* (5) *is not strong sampled observable, then there exists* $t_1, \ldots, t_n$ *such that the unique reconstruction fails, hence* n_T*-sample observability fails whenever* $T \geq t_n$*. The same finite-horizon notion of sampled observability also appears in [25, Theorem 2, "Furthermore" part].*

Theorem 13 *The conditions in Proposition 10 are further equivalent to:*

(O3) System (5) *is strongly sampled observable.*

In particular, system (5) *is strongly sampled observable if and only if it is observable and all eigenvalues of* A *are real.*

PROOF. It follows from

$$\begin{bmatrix} y(t_1) \\ \vdots \\ y(t_n) \end{bmatrix} = \underbrace{\begin{bmatrix} ce^{At_1} \\ \vdots \\ ce^{At_n} \end{bmatrix}}_{=:\Psi} x(0) \qquad (7)$$

that (O2) in Proposition 10 and (O3) are equivalent. □

The fact that (O1) implies (O3) also follows from [25, Theorem 2].

Proposition 14 *The conditions in Proposition 10 are further equivalent to:*

(O4) For all distinct $t_1, \ldots, t_n$ and all $\hat{y}_1, \ldots, \hat{y}_n \in \mathbb{R}$, there exists $x(0)$ such that the output y of (5) satisfies $y(t_i) = \hat{y}_i$, $i = 1, \ldots, n$.

PROOF. It follows from (7) that (O2) and (O4) are equivalent. □

The fact that (O1) implies (O4) relates to results in disconjugacy theory; see [3, Chapter 0 Proposition 1 and Chapter 3 Proposition 16].

The next result, which relates to Lebesgue sampling [15], may be of independent interest. We consider zero initial time and unit final time for notational simplicity, but the result readily generalizes to any initial and final times. Observe that the next result is about *the* set of zeros, not simply *a* set of zeros.

Proposition 15 *Assume that* (5) *is observable and that all eigenvalues of A are real. Consider $t_1 < \cdots < t_m$ in $(0, 1)$ with $m < n$ and a threshold $\delta \in \mathbb{R}$. Then there exists $x(0)$ such that $\{t_1, \ldots, t_m\}$ is the set of zeros of $y(\cdot) - \delta$ in $(0, 1)$ and $y(\cdot) - \delta$ changes sign at those zeros with a prescribed initial sign.*

PROOF. Pick $t_{m+1}, \ldots, t_n$ distinct in $(1, 2)$. For $\epsilon \in \mathbb{R}$, let $x(0; \epsilon)$ denote the solution (existence and uniqueness follow from the invertibility of Ψ defined in (6)) of

$$\begin{bmatrix} \epsilon\delta \\ \vdots \\ \epsilon\delta \\ \pm 1 \end{bmatrix} = \begin{bmatrix} ce^{At_1} \\ \vdots \\ ce^{At_{n-1}} \\ ce^{At_n} \end{bmatrix} x(0; \epsilon), \tag{8}$$

and let $y(t; \epsilon) = ce^{At}x(0; \epsilon)$ denote the corresponding output of (5), so that, from (8),

$$y(t_i, \epsilon) = \epsilon\delta, \quad i = 1, \ldots, n - 1. \tag{9}$$

In view of Theorem 2, $y(\cdot; 0)$ has at most $n - 1$ zeros (counting multiplicities), since it is a nonzero (in view of the ± 1 in (8)) exponential polynomial of order at most n. On the other hand, in view of (9), $t_1, \ldots, t_{n-1}$ are zeros of $y(\cdot; 0)$. It follows that the zeros of $y(\cdot; 0)$ are $t_1, \ldots, t_{n-1}$ with multiplicity one, i.e., they are all simple. In (8), pick the sign in the "± 1" entry such that $y(0; 0)$ has the prescribed initial sign. Then there exists a neighborhood $\mathcal{E}$ of 0 such that, for all $\epsilon \in \mathcal{E}$, the sign of $y(0; \epsilon) - \epsilon\delta$ remains the prescribed initial sign (because $\epsilon \mapsto y(0; \epsilon) - \epsilon\delta$ is continuous) and the number of zeros of $y(\cdot; \epsilon) - \epsilon\delta$ in $(0, 1)$ remains m (by an application of the implicit function theorem, using the fact that the zeros of $y(\cdot; 0)$, and thus obviously those of $y(\cdot; \epsilon) - \epsilon\delta$ for $\epsilon = 0$, are simple). In view of (9), these m zeros are $t_1, \ldots, t_m$. Pick $\epsilon > 0$ in $\mathcal{E}$, let $x(0) = \frac{1}{\epsilon}x(0; \epsilon)$, and let $y(\cdot) = \frac{1}{\epsilon}y(\cdot; \epsilon)$ denote the corresponding output of (5). Then $y(\cdot) - \delta = \frac{1}{\epsilon}(y(\cdot; \epsilon) - \epsilon\delta)$. Its zeros and signs are thus those of $y(\cdot; \epsilon) - \epsilon\delta$. □

5 Conclusion

We have introduced a strong notion of impulsive controllability (Definition 8) for n-dimensional continuous-time LTI control systems that requires an impulsive control law to exist not only for every initial and final state, but also for every prescribed sequence of n (distinct) time instants at which the impulses occur. In the single input case, we have shown that strong impulsive controllability is equivalent to (standard) controllability together with the condition that all eigenvalues of the system matrix are real.

In a dual way, we have introduced a strong notion of sampled observability (Definition 11), according to which the initial state can be uniquely reconstructed from the output measured along every prescribed sequence of n (distinct) time instants. In the single output case, we have shown that strong sampled observability is equivalent to (standard) observability together with the condition that all eigenvalues of the system matrix are real.

The extension of the results to multi-input and multi-output systems would deserve further investigation.

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