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LIDAM Discussion Paper ISBA
2025 / 02

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Derivatives under Market Impact: Disentangling Cost and Information

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February 14, 2025

Abstract

In this paper, we examine the implications of a large trader's activity and its market impact when pricing and hedging a large derivatives position. We highlight the issue of market manipulation in this context, providing examples to illustrate how the trader can manipulate the payoff. To understand how a large trader's knowledge influences their actions, we introduce a problem reduction by considering the perspective of an insider trader, who is aware of the large trader's trading policy but does not incur transaction costs. Through this framework, we establish the existence of *information-neutral probability measures*, under which the discounted asset is a martingale process for the insider trader. We then develop a derivatives hedging policy for the large trader that accounts for both transaction costs and the permanent impact of their hedging activities, while mitigating market manipulation. The paper concludes with numerical results that showcase the optimal delta-hedging strategy for an out-of-the-money call option, comparing it to the Black-Scholes model.

Keywords: Option Pricing, Market Impact, Illiquid Markets, Transaction Costs, Stochastic Optimal Control.

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1 Introduction

The traditional option pricing framework, as outlined in the seminal paper by Black & Scholes (1973), assumes a frictionless and liquid financial market. However, this framework may not hold when dealing with large notional derivatives on illiquid assets that are subject to market impact. When a significant hedge is executed, it can consume the market's liquidity, which in turn can have an impact on the underlying asset's spot price. Therefore, a new mathematical framework is needed to effectively hedge derivatives in these illiquid markets.

The theory of market impact extensively developed in the literature typically describes two types: temporary impact and permanent impact, see Almgren & Chriss (1997).[1] The temporary impact represents the slippage or transaction costs and refers to the short-term deviation from the price due to the costs incurred in executing the trade. In contrast, the permanent impact refers to the lasting effect on the price of the underlying asset resulting from the trade.

In the field of hedging derivatives under permanent market impact, two main trends emerge. On the one hand, the full hedge approach, as considered in the paper of Liu & Yong (2005), seeks to fully offset the risk of a derivative instrument. Their approach has been further developed and expanded upon in recent papers by Loeper *et al.* (2018), Said (2019), Bordag & Frey (2008), and Abergel & Loeper (2017). Full hedge strategies can result in high option prices in illiquid markets due to the cost of adjusting the delta as the option approaches expiration or the underlying spot approaches the strikes. On the other hand, partial hedging, as in Jonsson & Sircar (2002), aims to balance the cost of hedging against the benefit of retaining some exposure to the underlying asset unhedged. A common approach is optimizing a gain function for the large trader, see Guéant & Pu (2017). However, this approach also presents the risk of market manipulation, as the large trader's ill-designed utility function may drive them to act in their self-interest.

On that note, a significant gap remains in the literature regarding the potential for market manipulation under option trading. Previous studies have primarily focused on market manipulation in the context of optimal execution strategies for non-derivative assets, such as stocks and futures (e.g. Fruth *et al.* (2014)). However, the unique complexities of options have been largely overlooked in the literature. Derivatives contain an additional layer of manipulation risk because traders can distort not only the market price of the underlying hedges, but also the payoff structure of the options, see Horst & Naujokat (2011). For instance, a large trader could manipulate the underlying asset's price to alter the value of their option positions, a scenario we will illustrate in detail in Section 2. Some existing models with a market impact component, and interestingly even models with no market impact component, can lead to payoff manipulation as the delta-hedging strategy they suggest might result in significant share executions that unintentionally distort the option's payoff, see Roch (2022) . Traders are aware of this shortcoming in standard models, and to mitigate these in practice, they often employ a method known as overhedging, which smooths out the option's payoff at the edges, reducing the likelihood of large, day-to-day

[1] More recently, transient impact has also been introduced as a decaying function over time.

delta executions for hedging the option. This practical method involves booking the option at a modified payoff, higher than the actual payoff, resulting in a higher premium.

These considerations underscore the need for further study into the problem of payoff manipulation in option strategies. In this paper, our aim is to explore how a large trader can hedge a large derivatives position while considering the potential for market manipulation. We discuss that the large trader's knowledge about the trading activities is key, as this information gives opportunities for the trader to use it for its benefit. To better understand the implications of this information, we perform a problem reduction by hypothesizing an insider trader who has access to the same information about the large trader's activities but is not large enough to generate any market impact. By positioning ourselves at the same level of perception as this insider trader, we can better understand how the market would be perceived under this information. Specifically, we study how the martingale pricing theory would apply to this insider trader. Therefore, while our research objective differs, this paper is influenced by and also is a case study for foundational works done by Hillairet (2004), who examined information levels and martingale pricing theories, by Grorud & Pontier (1998), Amendinger *et al.* (1998) who investigated how such information modifies the utility function of a trading portfolio to benefit the insider, and by Eyraud-Loisel (2005, 2013) who provided insights into filtration enlargement due to private information. Given that the large trader is also an insider, we hope this problem reduction will provide insights into the martingale pricing and hedging strategies for the large trader as well.

Given the above context, one key contribution of this paper relies on its methodology to disentangle information about the large trader's activity from other factors that influence options' price and hedging strategy. Central to this is the conceptualization of the insider trader, who possesses the same information as a large trader but cannot impact market prices. In this paper, we derive a set of martingale measures for the insider that we call information-neutral measures. Once the martingale pricing foundation for the insider trader is developed, using the information-neutral measures, along with the discussion on some necessary conditions for mitigating market manipulation, the paper derives a cost optimization function to establish a hedging policy for the large trader.

This paper is structured as follows. Section 2 discusses the challenges and the inherent risk of market manipulation within derivatives pricing and hedging. It provides a case example and introduces our problem reduction approach by hypothesizing a market with one large trader and an insider trader. Section 3 describes the stock dynamics and the market impact model used in this paper. Based on this dynamics, Section 4 explores the pricing equation for the insider trader and elaborates on information-neutral probabilities for asset valuation as a martingale. Section 5 analyzes the portfolio dynamics of the large trader and formulates the mathematics of its hedging portfolio. Section 6 explores the optimal hedging strategy for the large trader, addressing market manipulation within the stochastic control framework. Section 7 presents a numerical implementation using finite difference methods, comparing strategies between large and insider traders.

2 Challenges in Traditional Hedging Models and Motivation for our Approach

Large hedging activities in derivative markets can significantly influence the underlying asset prices, leading to distortions that can manifest as price movements or manipulated payoffs. Understanding and mitigating these effects is crucial for a trader active with large derivatives positions in a limited liquidity market. Therefore, this section discusses the first forms of market manipulation relevant to large traders and explains the challenges of a pricing model not falling into the trap of unintentionally manipulating the payoff and, lastly, our research methodology to tackle this problem.

Abusive Squeeze: An abusive squeeze occurs when a trader deliberately creates a market condition that amplifies their ability to profit from the subsequent price movements of an asset. This typically involves sequentially buying or selling a large quantity of the asset to influence its price in a manner that benefits their trading position. Such actions can lead to artificial price levels. In the context of derivatives, an abusive squeeze might involve a large trader influencing the underlying asset's price of a derivative contract tied to it. Regulatory bodies recognize this behavior as a form of market abuse (Financial Conduct Authority 2005).

Payoff Manipulation: Payoff manipulation refers to the scenario where a trader's actions directly affect the payoff structure of a derivative instrument through their market impact. For instance, by strategically buying or selling the underlying asset, a trader can manipulate the settlement price of options or other derivatives to their advantage. This is particularly relevant in cases where the trader holds significant positions that can influence the market price. These practices are generally prohibited under regulations designed to maintain market integrity (European Parliament and of the Council 2014; Financial Conduct Authority 2005).

One interesting consideration about payoff manipulation is that it can also be unintentional. This can occur when using models, such as Black & Scholes, for delta hedging, which do not account for the price changes caused by the hedging activities of large traders. As a result, the standard delta-hedging strategies derived from these models can inadvertently lead to market conditions that result in payoff manipulation. To further explain this, the following example illustrates how such unintentional manipulation can happen in the case of a KO (Knock-out) call option.

Example 1 (Payoff Manipulation for an Out-of-the-Money Knock-out Call Option)**.** Consider a large trader who holds a large position of a short-term, out-of-the-money Knock-out (KO) call option with a knock-out barrier set at 105, while the current stock price is 100. A KO call option is a type of barrier option that ceases to exist if the underlying asset's price reaches or exceeds a specified level, known as the knock-out barrier. If an investor holds a KO call option with a barrier at 105 and the stock price hits 105, the option becomes void and worthless. This characteristic makes KO options particularly sensitive to price movements of the underlying asset near the barrier level, exposing them to potential manipulation more than other types of derivatives.

Figure 1 illustrates the relationship between the spot price of the underlying asset and the price of the Knock-out (KO) call option. In the figure, the horizontal axis represents the spot price of the underlying asset, while the vertical axis denotes the price of the option. The solid line shows how the price of the KO call option changes as the spot price varies. As the spot price increases but remains below the KO barrier, the option price increases, reflecting its growing value. However, as the spot price approaches the KO barrier, the potential for the option to become worthless at the barrier exerts downward pressure on its price. The dashed line represents the payoff of the option at expiry. This payoff is contingent on the spot price relative to the KO barrier. If the spot price remains below the KO barrier at expiry, the option payoff reflects its intrinsic value. Conversely, if the spot price reaches or exceeds the KO barrier, the option is knocked out, resulting in no payoff. In this figure, the current spot price, depicted via a red dot, is in the money (greater than the strike price) but still far from the KO barrier.

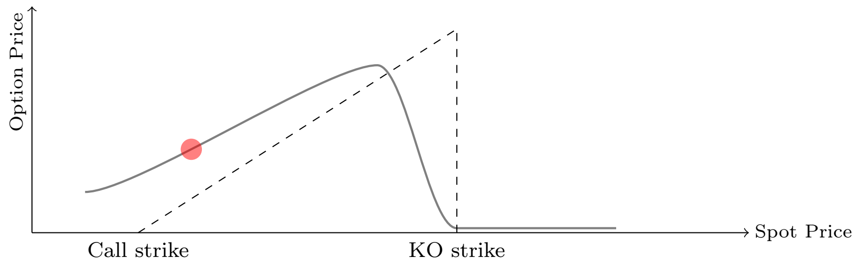

Figure 1: Spot is in the money and but still far from the KO strike.

Now, let's assume the stock price jumps closer to the barrier, but not too close, as illustrated in Figure 2. In this scenario, represented by the red dot moving further up, the stock price increases but still remains below the knock-out barrier.

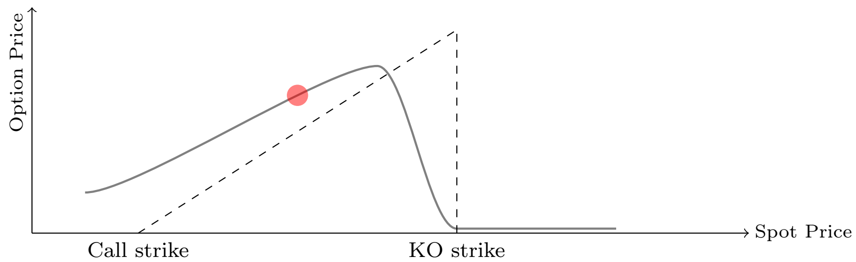

Figure 2: As the spot moves further up, and also the delta of the option is going further up.

Under a delta hedging strategy using the classical Black-Scholes model, the seller of the option must purchase more of the underlying asset to hedge their position as the option's sensitivity to the spot price, i.e. the delta, increases. This is shown in Figure 2, where the red point has moved further up the solid line, indicating a higher delta compared to the red point in Figure 1. As delta increases, it reflects the growing sensitivity of the option's price to changes in the spot price. To maintain a delta-neutral position and mitigate risk, the trader buys additional stock. This strategy offsets the option's price changes with corresponding movements in the underlying asset.

However, this aggressive hedging can drive the stock price closer to the knock-out barrier, potentially breaching it. When the barrier is breached, as shown in Figure 3, the KO condition is triggered, and the option becomes worthless, resulting in no payoff.

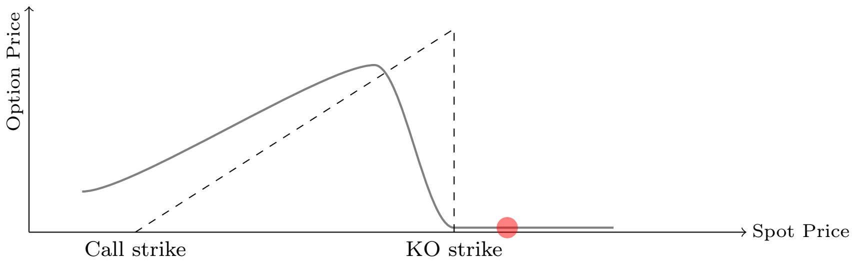

Figure 3: Large hedging activity moves the spot further, breaching the KO.

Example 1 demonstrates how a trader's hedging actions, dictated by models that do not account for market impact, can unintentionally manipulate the payoff of the option. Recognizing this oversight, risk managers and traders often adapt their strategies to mitigate the inefficiencies of these models.

In practice, savvy traders and risk managers are well aware of the potential for unintentional payoff manipulation in models that do not incorporate market impact. To mitigate this issue, they often adopt strategies to adjust the pricing and hedging of options with barriers. One common method is to modify the payoff structure by setting the barrier strike slightly above the actual KO strike and adjusting the option to function like a reverse straddle, peaking at the KO strike. This modification reduces the option's price sensitivity to minor movements in the underlying asset, thereby minimizing the likelihood of the model suggesting large delta hedging adjustments. By employing this approach, traders effectively price and hedge a hypothetical payoff that is more likely to trigger a payment than the real payoff, allowing them to sell the option at a higher premium. While a detailed explanation of this heuristic approach is beyond the scope of this paper, this note underscores that the issues illustrated in the previous example are well-recognized in practice. Traders and risk managers are aware of these model shortcomings and take proactive steps to address them.

Our approach and the ecosystem studied in this paper

The KO call option example underscores the necessity for models that account for the market impacts of trading activities and sets some criteria for preventing market manipulation. But before proposing such a model, it is essential to deepen our understanding of the problem by exploring how a large trader's knowledge of their market impact influences their behavior. Payoff manipulation can occur when a large trader, aware of their ability to affect market prices, follows a delta hedging strategy that benefits them. To explore how this knowledge influences their actions, we introduce the concept of an insider trader. This insider trader possesses the same information as the large trader but does not impact the market because their trades are too small to influence prices. By studying the option pricing and delta hedging strategies of this insider, we aim to gain insights into how information alone, without the market impact, affects pricing and hedging strategies in derivatives trading.

Therefore, we hypothesize a market characterized by limited liquidity, with two type of actors:[2]

- The ***large trader*** who has market impact and is aware of the effect of its own actions on the market.
- The ***insider trader*** who possesses the same knowledge as the large trader but does not generate any market impact because its share execution size is small.

3 The Market Impact Model

We consider a finite, deterministic time horizon $T \in \mathbb{R}^+$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $\{W_t\}_{0\leq t\leq T}$ be a standard Brownian motion and $\{N_t\}_{0\leq t\leq T}$ be a homogeneous Poisson process with intensity $\phi > 0$. We assume that the processes $\{W_t\}_{0\leq t\leq T}$ and $\{N_t\}_{0\leq t\leq T}$ are independent. The probability measure $\mathbb{P}$ is traditionally called a real-world probability measure. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the filtration jointly generated by the processes $\{W_t\}_{0\leq t\leq T}$ and $\{N_t\}_{0\leq t\leq T}$.

While leveraging a homogeneous Poisson process for its simplicity and tractability, we acknowledge the limitations this assumption may impose, particularly in more nuanced market contexts where a Cox process, with an intensity that adapts to market conditions and is $\mathbb{F}$-adapted, could offer a more refined model. Nonetheless, our focus remains on the homogeneous Poisson process to maintain model simplicity and computational feasibility.

The core of our model revolves around three $\mathbb{F}$-adapted processes: the underlying spot price process $\{S_t\}_{0\leq t\leq T}$, the process $\{\chi_t\}_{0\leq t\leq T}$ denoting the annualized rate of shares executed by a large trader, and the process $\{V_t\}_{0\leq t\leq T}$ representing the evolving volume of shares held by the large trader. These processes are governed by the following dynamics:

$$\begin{cases} \dfrac{dS_t}{S_{t^-}} = \mu_t dt + \sigma dW_t + \Lambda(\chi_t) dN_t, \\ dV_t = \chi_t dt, \\ \chi_t = \tilde{\chi}(t, S_t, V_t), \end{cases} \tag{3.1}$$

with $S_0 > 0$ and $V_0 \geq 0$.

In Equation (3.1), $\tilde{\chi} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a piecewise continuous and bounded function that models the execution speed of the large trader's orders. Moreover, $\Lambda : \mathbb{R} \to \mathbb{R}$, a non-decreasing, square-integrable function, encapsulates the permanent market impact of the large

[2] There are other types of traders such as *average traders* who are too small to influence the market and are unaware of the large trader's activities. However, this group of traders is not the focus of this paper.

trader's actions, satisfying $\Lambda(0) = 0$. The asset's drift, represented by μ_t, is assumed to be a deterministic real function, and the constant volatility $\sigma \in \mathbb{R}^+$ is also deterministic.

In financial terms, large trader executing a trading strategy $\{\chi_t\}_{t\geq 0}$, where χ_t is the (annualized) speed of the shares execution program by the large trader at time t. To conceal its order size, the large trader keeps its trading strategy hidden from the market. Therefore, the market will not know about χ_t until it is executed.[3] However, there is one insider trader who is privy to the large trader's trading strategy.

In Equation (3.1), the size of the jump is denoted by $\Lambda(\chi_t)$, as opposed to $\Lambda(\chi_{t^-})$. The notation t^- is typically used to ensure the validity of the closed-form exponential representation of the process $\{S_t\}_{0\leq t\leq T}$, which is a product of the impact of all jumps and the exponential Gaussian process, as detailed in Doléans-Dade (1970). In simpler terms, for the exponential formulation to hold, jumps need to be applied first to the spot price early at t^-, followed by other sources of variation. However, in our model, an exponential expansion is not employed. From a financial perspective, this implies that while the jump increments are applied to the price at S_{t^-} before other variations, its actual magnitude is determined at time t, not at t^-.

The trading activity of the large trader is characterized by an execution rate of χ_t shares at any given moment t. We postulate that during a brief interval $[t, t+dt)$, the market's liquidity is either sufficient to absorb the trade without any lasting impact or insufficient, leading to a permanent market impact. In scenarios where market liquidity can accommodate the trade volume, the executed orders do not result in a permanent change to the asset's price. Conversely, when market depth is inadequate, the trading activity induces a permanent market impact, quantified by the function $\Lambda(\chi_t)$. This impact is realized at the occurrence of each Poisson jump, where the asset's price adjustment is directly tied to the instantaneous trading rate.

Note that the market impact could be positive or negative depending on whether the shares are bought or sold. In the case of no trading activity by the large trader, Equation (3.1) reduces to the standard Black-Scholes dynamics.

It is worth noting that the permanent impact model presented in Equation (3.1) comes at random times. This contrasts classical models, such as Almgren & Chriss (2001), which consider a deterministic permanent market impact function to cumulative share amount, and later Gatheral (2010) proves that one sufficient condition for no-arbitrage is the linearity of the permanent market impact model. However, later works such as Guéant (2013) suggested other plausible forms of permanent impact, which are non-linear. Recent studies in the literature have focused on stochastic liquidity models, delving into the exploration of stochastic transient impact (as investigated by Klöck (2012); Almgren (2012); Ma *et al.* (2020)) alongside stochastic permanent impact, a subject of interest in the work of Barger & Lorig (2019). In a stochastic permanent impact model, the permanent price impact has a non-isomorphic relationship with accumulated

[3]While this goes beyond the scope of this paper, there are many algorithmic methods to hide the order size; the most simple one is a price taker, which involves placing market orders that are immediately filled at the best available price. Another one may be iceberg-type limit orders. See Christensen & Woodmansey (2013) for more details.

shares and may differ across scenario paths based on share execution time and market depth on a given day, even if the same number of cumulated shares are purchased at time t. In this paper, our modeling approach for permanent market impact utilizes a jump model.

Another aspect of our permanent impact model is that, for the sake of simplicity, it assumes that the impact rate of the order (ϕ) is independent of its size (χ_t). The relationship between ϕ and χ_t has received less attention in the empirical literature because most academic literature so far investigates the average price impact rather than its variations (Donnelly 2022). We make the point that while our model can be generalized by integrating this relationship, this may diverge our focus on this this pape, as our intention is not to define a very comprehensive market impact model but to study the role of information in an exploratory model inspired by the market practice.

4 Option Hedges for the Insider Trader

The insider trader has access to the information about the trading policy $\{V_t, \chi_t\}_{0\leq t\leq T}$ of the large trader over time. Let $\{r_t\}_{0\leq t\leq T}$ be a deterministic interest rate process. We also assume that the holder of the stock benefits from a dividend yield of $q \geq 0$, as well as the presence of a securities lending market and a future market for the underlying stock. The holder of the stock bears a cost of carry b, which is the adjustment rate between the future market and the current value of the stock. These markets, although subject to liquidity issues, enable the holder to enter into short selling positions.

In the following theorem, we demonstrate the existence of a set of probability measures under which the asset price and the option value are martingales. This result is the main outcome of this section and formulates mathematically how the insider trader's information affects the pricing of derivatives and the hedging strategies.

Theorem 2. *Under the regularity condition:*

$$\Lambda(\chi_t)(\mu_t + (q-b) - r_t) < \sigma^2, \quad \mathbb{P} - a.s., 0 \leq t \leq T, \tag{4.1}$$

the following results hold:

1. *There exists a set of probability measures $\mathcal{I}$ where for any measure $I \in \mathcal{I}$ the returns of the discounted asset*
$$S_t = \mathbb{E}^I \left[e^{-\int_t^T r_s ds} e^{(q-b)(T-t)} S_T \middle| \mathcal{F}_t \right] \tag{4.2}$$
are martingales under the measure I.

2. *The option price $\mathcal{O}$ for the insider trader*
$$\mathcal{O}_t = \mathbb{E}^I \left[e^{-\int_t^T r_s ds} \mathcal{O}_T \middle| \mathcal{F}_t \right]$$

is also a martingale under probability $I \in \mathcal{I}$.

3. *We call $\mathcal{I}$ the set of information-neutral probabilities. Then, among all the information-neutral measures, there exists a probability measure $\mathcal{I}^*$ that minimizes the quadratic variation of the difference between the hedging portfolio and option value. We call $\mathcal{I}^*$ the minimal variation information-neutral probability measure.*

4. *The discounted return of the asset is a martingale under measure $\mathcal{I}^*$.*

   Moreover, the dynamic of the spot price under I^* is described by

$$\frac{dS_t}{S_{t^-}} = \left(r_t - (q-b) - \Lambda(\chi_t)\omega_t\right) dt + \sigma dW_t^{\mathcal{I}^*} + \Lambda(\chi_t) dM_t, \tag{4.3}$$

   where $\{M_t\}_{0\leq t\leq T}$ is a doubly stochastic Poisson process whose stochastic intensity process $\{\omega_t\}_{0\leq t\leq T}$ verifies

$$\omega_t = \phi\left(\frac{\sigma^2 - \Lambda(\chi_t)\left(\mu_t + (q-b) - r_t\right)}{\sigma^2 + \phi\Lambda(\chi_t)^2}\right),$$

   for each $0 \leq t \leq T$.

5. *Furthermore, if the insider trader has entered into a transaction with terminal payoff $\mathcal{O}(S_T)$, we call $\mathcal{O}(t, S, V)$, the price of this option at time t and spot level S and the net volume of shares hold by the large trader V. Then, for all $0 \leq t \leq T$, the option price $\mathcal{O}$ can be computed via the following partial differential equation:*

$$\begin{aligned} r_t\mathcal{O} = \frac{\partial \mathcal{O}}{\partial t} + \left(r_t - (q-b) - \Lambda(\chi_t)\omega_t\right) S\frac{\partial \mathcal{O}}{\partial S} + \frac{\partial \mathcal{O}}{\partial V}\chi_t dt \\ + \frac{1}{2}\frac{\partial^2 \mathcal{O}}{\partial S^2}\sigma^2 S^2 + \left(\mathcal{O}\left(t, S + S\Lambda(\chi_t), V_t\right) - \mathcal{O}\left(t, S, V\right)\right)\omega_t. \end{aligned} \tag{4.4}$$

The demonstration of Theorem 2 employs quadratic hedging and local risk-minimization techniques, akin to the approach of Föllmer *et al.* (1990). Central to this proof is the application of the Girsanov Theorem to both the jump and Brownian processes. This application results in an adjustment of the drift term from μ_t to $r_t - (q-b)$ for the Brownian motion and modifies the intensity of jumps for the Poisson process. It's noteworthy that under the conditions where $\chi_t = 0$ throughout the interval $[0, T]$ or $\phi = 0$, the equation (4.4) simplifies to the classical Black-Scholes partial differential equation.

Proof. By Itô formula applied to a process with jumps, the option price dynamics are given by

$$\begin{aligned} d\mathcal{O}_t = \frac{\partial \mathcal{O}}{\partial t}dt + \frac{\partial \mathcal{O}}{\partial S}\left(\mu_t dt + \sigma dW_t\right) S_{t^-} + \frac{1}{2}\frac{\partial^2 \mathcal{O}}{\partial S^2}\sigma^2 S_{t^-}^2 dt \\ + \frac{\partial \mathcal{O}}{\partial V}\chi_t dt + \left[\mathcal{O}\left(t, S_{t^-} + S_{t^-}\Lambda(\chi_t), V_t\right) - \mathcal{O}\left(t, S_{t^-}, V_t\right)\right] dN_t, \end{aligned}$$

where $\mathcal{O}\left(t, S_{t^-}(1+\Lambda(\chi_t)), V_t\right) - \mathcal{O}\left(t, S_{t^-}, V_t\right)$ is the jump in the price of the option due to large traders' activity χ_t, and $S_t(\mu_t dt + \sigma dW_t)$ is the continuous part of the spot movement.

On the other hand, the insider trader's option hedging portfolio at time t is given by

$$P_t = \delta_t S_t + \overbrace{(P_t - \delta_t S_t)}^{\text{Cash Account}},$$

where the insider trader has δ_t in shares and $P_t - \delta_t S_t$ in cash amount. Given that the notional is small, the insider trader - contrary to the large trader - can hedge its position without additional transaction costs. The dynamics of the portfolio P_t are as follows:

$$dP_t = \delta_t \left(dS_t + S_{t^-} q dt - S_{t^-} b dt - S_{t^-} r_t dt\right) + r_t P_t dt.$$

In our incomplete market, the option cannot be perfectly hedged by a self-financing portfolio. Therefore, we search for a hedging strategy that satisfies the following properties:

- $\delta_t = \arg\inf_\delta d\left\langle (P - \mathcal{O})\right\rangle_t$: The second order option price movement is reduced and optimally minimized.

  The stochastic part of $\mathcal{O}_t - P_t$ can be formulated as:

  $$\begin{aligned} d(\mathcal{O}_t - P_t) = \cdots \times dt + S_{t^-}\left(\frac{\partial \mathcal{O}}{\partial S} - \delta_t\right)\sigma dW_t \\ + \left[\mathcal{O}\left(t, S_{t^-}(1+\Lambda(\chi_t)), V_t\right) - \mathcal{O}\left(t, S_{t^-}, V_t\right) - \delta_t S_{t^-}\Lambda(\chi_t)\right] dN_t. \end{aligned}$$

  Therefore, the quadratic variation of $\mathcal{O}_t - P_t$ reads as follows:

  $$\begin{aligned} d\left\langle (P_t - \mathcal{O}_t)\right\rangle = \left(\frac{\partial \mathcal{O}}{\partial S} - \delta_t\right)^2 S_{t^-}^2 \sigma^2 dt \\ + \left[\delta_t - \left[\frac{\mathcal{O}\left(t, S_{t^-}(1+\Lambda(\chi_t))\right) - \mathcal{O}\left(t, S_{t^-}\right)}{S_{t^-}\Lambda(\chi_t)}\right]\right]^2 \phi S_{t^-}^2 \Lambda(\chi_t)^2 dt. \end{aligned}$$

  If the market is complete, it is possible to find δ_t such that the quadratic variation $d\left\langle (P - \mathcal{O})\right\rangle_t$ is always zero. This would have been the case for the Black-Scholes model without any market impact. However, it is impossible due to the jump component of the market impact. One can only find a strategy that minimizes the quadratic variation, but it will not neutralize it. The insider trader minimizing the quadratic variation will obtain the following formula:

  $$\delta_t S_{t^-} = \frac{\sigma^2}{\sigma^2 + \phi\Lambda(\chi_t)^2}\frac{\partial \mathcal{O}}{\partial S} S_{t^-} + \frac{\phi\Lambda(\chi_t)}{\sigma^2 + \phi\Lambda(\chi_t)^2}\left(\mathcal{O}\left(t, S_{t^-}(1+\Lambda(\chi_t)), V_t\right) - \mathcal{O}\left(t, S_{t^-}, V_t\right)\right). \tag{4.5}$$

- $\mathbb{E}^{\mathbb{P}}\left[dP_t|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[d\mathcal{O}_t|\mathcal{F}_t\right]$: The average (first order) option price movement is fully hedged by the portfolio.

We can derive the following partial differential equation:

$$\frac{\mathbb{E}^{\mathbb{P}}\left[dP_t|\mathcal{F}_t\right]}{dt} = r_t P_t + \left(\mu_t + q - b - r_t + \Lambda(\chi_t)\phi\right)\delta_t S_{t^-}.$$

On the other hand, we have

$$\begin{aligned}\frac{\mathbb{E}^{\mathbb{P}}\left[d\mathcal{O}_t|\mathcal{F}_t\right]}{dt} = \frac{\partial \mathcal{O}}{\partial t} + \frac{\partial \mathcal{O}}{\partial S} S_{t^-}\mu_t + \frac{1}{2}\frac{\partial^2 \mathcal{O}}{\partial S^2}\sigma^2 S_{t^-}^2 + \frac{\partial \mathcal{O}}{\partial V}\chi_t \\ + \left[\mathcal{O}\left(t, S_{t^-} + S_{t^-}\Lambda(\chi_t), V_t\right) - \mathcal{O}\left(t, S_t, V_t\right)\right]\phi.\end{aligned}$$

Combining the two equations above with Equation (4.5) we obtain that:

$$\begin{aligned}r_t P_t &= \frac{\partial \mathcal{O}}{\partial t} + \left(\mu_t - \frac{\sigma^2\left(\mu_t + q - b - r_t + \Lambda(\chi_t)\phi\right)}{\sigma^2 + \phi\Lambda(\chi_t)^2}\right) S_t \frac{\partial \mathcal{O}}{\partial S} + \frac{1}{2}\frac{\partial^2 \mathcal{O}}{\partial S^2}\sigma^2 S^2 + \frac{\partial \mathcal{O}}{\partial V}\chi_t dt \\ &+\phi\left(1 - \frac{\Lambda(\chi_t)}{\sigma^2 + \phi\Lambda(\chi_t)^2}\left(\mu_t + q - b - r_t + \Lambda(\chi_t)\phi\right)\right)\left(\mathcal{O}\left(t, S_{t^-} + S_{t^-}\Lambda(\chi_t), V_t\right) - \mathcal{O}\left(t, S_{t^-}, V_t\right)\right).\end{aligned} \tag{4.6}$$

Let's define the variable ω_t, which represent a jump intensity, as

$$\omega_t = \phi\frac{\sigma^2 - \Lambda(\chi_t)\left(\mu_t + q - b - r_t\right)}{\sigma^2 + \phi\Lambda(\chi_t)^2}. \tag{4.7}$$

We also assume the regularity condition (4.1) holds. This condition is necessary as otherwise ω_t will not be positive almost surely, and the Feynman-Kac theorem could not be applied as the intensity of the Poisson process will not be positive.

- Since this is not a self-financing portfolio, the trader can intervene at each time step by injecting or depleting cash to ensure that the hedging portfolio's value matches the theoretical value of the option. Mathematically, the trader will inject $P_t - \mathcal{O}_t$ in cash into the hedging portfolio, ensuring that $P_t = \mathcal{O}_t$. Therefore, due to these cash injections, the hedging portfolio's value is continuously adjusted to equal the option price.

  Based on the definition of ω_t and the assumption that the hedging portfolio is adjusted to satisfy $P_t = \mathcal{O}_t$, we obtain:

$$\begin{aligned}r_t\mathcal{O} = \frac{\partial \mathcal{O}}{\partial t} + \left(r_t - q + b - \Lambda(\chi_t)\omega_t\right) S_{t^-}\frac{\partial \mathcal{O}}{\partial S} + \frac{\partial \mathcal{O}}{\partial V}\chi_t dt + \frac{1}{2}\frac{\partial^2 \mathcal{O}}{\partial S^2}\sigma^2 S_{t^-}^2 \\ + \omega_t\left(\mathcal{O}\left(t, S_{t^-} + S_{t^-}\Lambda(\chi_t), V_t\right) - \mathcal{O}\left(t, S_{t^-}, V_t\right)\right).\end{aligned} \tag{4.8}$$

Having the partial differential equation above, the most common technique would be to use the Feynman-Kac formula. However, the stochastic process derived is not an exponential Levy, so one has to be careful in using the Feynman-Kac technique for non-exponential-Levy processes. Using the Feynman-Kac formula, the equivalent stochastic equation for the spot price S_t under a probability measure that we call minimal-variance information neutral probability and note it as $\mathcal{I}^*$ is given by

$$\frac{dS_t}{S_{t^-}} = (r_t + b - q - \chi_t \Lambda(\chi_t)\omega_t)\, dt + \sigma dW_t + \Lambda(\chi_t) dM_t, \tag{4.9}$$

where $\{M_t\}_{0\leq t\leq T}$ is a doubly stochastic Poisson process with stochastic intensity process $\{\omega_t\}_{0\leq t\leq T}$. Also, the following equations hold under the I^* probability:

$$S_t = \mathbb{E}^{\mathcal{I}^*}\left[e^{-\int_t^T r_s ds} e^{(q-b)(T-t)} S_T \middle| \mathcal{F}_t \right] \tag{4.10}$$

$$\mathcal{O}\left(t, S_t, V_t\right) = \mathbb{E}^{\mathcal{I}^*}\left[e^{-\int_t^T r_s ds} \mathcal{O}\left(T, S_T, V_T\right) \middle| \mathcal{F}_t \right]. \tag{4.11}$$

To validate our use of the Feynman-Kac technique, we prove that having the stochastic process of Equation (4.9) and the function $\mathcal{O}$ of Equation (4.11) would result in the partial differential equation in (4.8).

On the one hand, based on Equation (4.11), we have

$$\mathbb{E}^{\mathcal{I}^*}\left[d\mathcal{O}_t \middle| \mathcal{F}_t \right] = r_t \mathcal{O}_t\, dt. \tag{4.12}$$

On the other hand, applying Itô's lemma to the function $\mathcal{O}$:

$$\begin{aligned} \mathbb{E}^{\mathcal{I}^*}\left[d\mathcal{O}_t \middle| \mathcal{F}_t \right] = {} & \frac{\partial \mathcal{O}}{\partial t}\, dt + \frac{\partial \mathcal{O}}{\partial S} \mathbb{E}^{\mathcal{I}^*}[dS_t^C] + \frac{\partial^2 \mathcal{O}}{\partial S^2} \mathbb{E}^{\mathcal{I}^*}[d\langle S^C \rangle_t] \\ & + \frac{\partial \mathcal{O}}{\partial V} \chi_t\, dt + S_t \mathbb{E}^{\mathcal{I}^*}[dM_t] \left(\mathcal{O}\left(t, S_{t^-} + S_{t^-}\Lambda(\chi_t), V_t\right) - \mathcal{O}\left(t, S_{t^-}, V_t\right) \right), \end{aligned} \tag{4.13}$$

where S^C is the continuous part of the diffusion of S in the dynamics (4.9). Now, expanding the dynamics of (4.9) in (4.12) and (4.13), we obtain again the partial differential equation (4.8). Therefore, our usage of the Feynman-Kac technique was valid.

Note that we have proved the existence of at least one element in the set of information-neutral probability measures, which is the $\mathcal{I}^*$. The existence of $\mathcal{I}^*$ is sufficient proof for the existence of the set of information neutral measures $\mathcal{I}$ and the fact that it is not empty.

□

It is worth noting that due to market incompleteness, the condition for the uniqueness of the martingale measure is not satisfied. As further elaborated in Tankov & Voltchkova (2009), when jumps coexist with a Brownian motion, the market becomes incomplete. This means that one cannot perfectly hedge all sources of variability using a conventional hedging portfolio comprising the underlying asset and a risk-free instrument. This also indicates the existence of numerous martingale measures.

5 Large Trading Program

Before delving into the derivative hedging policy for a large trader, it is essential to develop a mathematical formulation for the large trading program and its profit and loss (P&L). This section provides this foundation.

Definition 3. We define the hedging strategy account by the couplet $\{H_t, V_t\}_{0\leq t\leq T}$, with $\{V_t\}_{0\leq t\leq T}$ representing the number of shares holding account, and $\{H_t\}_{0\leq t\leq T}$ an $\mathbb{F}$-adapted process representing the cash account, with V_0 as a constant and $H_0 = 0$. The dynamics of these processes are described as follows:

$$\begin{aligned} dV_t &= \chi_t dt, \\ dH_t &= H_t r_t dt + V_t S_t (q-b)dt - S_t dV_t - d\mathcal{C}_t, \end{aligned} \tag{5.1}$$

where the process $\{\mathcal{C}_t\}_{0\leq t\leq T}$ represents the transaction costs process by the large trader and is defined by $\mathcal{C}_0 = 0$ and $d\mathcal{C}_t = g(\chi_t)dt$, where g is a sufficiently regular function (we provide further details on the possible parametric modeling of g in Section 7.1).

Definition 4. We define the P&L of trading activity, noted as $P\&L^{\text{trading}}_{t,T}$, as the product-sum of V_u, which is the number of shares held at time u, and the discounted asset return over the interval du.

$$P\&L^{\text{trading}}_{t,T} = \int_t^T V_u d\left(e^{-\int_t^u (r_s-(q-b))ds} S_u\right) \tag{5.2}$$

Essentially, this can be seen as the profit-and-loss resulting from delta rebalancing trading activities. Under any information-neutral measure $I \in \mathcal{I}$, the expectation is given by

$$\mathbb{E}^I\left[P\&L^{\text{trading}}_{t,T} \mid \mathcal{F}_t\right] = 0. \tag{5.3}$$

This is because the discounted asset returns function as a martingale under such a probability measure and its corresponding filtration. However, this relationship does not hold under the historical measure $\mathbb{P}$, implying that:

$$\mathbb{E}^{\mathbb{P}}\left[P\&L^{\text{trading}}_{t,T} \mid \mathcal{F}_t\right] \neq 0. \tag{5.4}$$

If the $P\&L^{\text{trading}}_{t,T}$ is positive in expectation in the historical measure, it suggests that there are dynamic arbitrage opportunities in the market that can be exploited with the right knowledge. This kind of practice is flagged as "abusive squeeze," as discussed in Section 2, which manifests when an individual or entity exerts notable control over the supply, demand, or delivery processes of a qualifying or related investment.

Proposition 1. At any time $0 \leq s \leq T$, we define $H_s + V_s S_s$ as the *accounting value* of the trading activity, the sum of the shares accounts $V_s S_s$ and the cash account H_s. The discounted accounting value of the trading activity at the terminal date T follows the following equation:

$$e^{\int_t^T -r_s ds}(H_T + S_T V_T) = (H_t + V_t S_t) + P\&L_{t,T}^{\text{trading}} - \int_t^T e^{-\int_t^u r_s ds} S_u d\mathcal{C}_u \tag{5.5}$$

The left-hand side of Equation (5.5) represents the discounted value of the shares and cash holding portfolio by the large trader at time T minus the same value at time t; it is the accounting value of profit and loss from the hedging activities portfolio between t and T. This value is equal to the $P\&L$ of trading activities, as defined earlier, minus the transaction costs, which financially make sense.

Proof. Let $b_u = e^{\int_t^u -r_s ds}$ denote the discount factor at time t with maturity T. or the zero coupon bond price, $\bar{H}_u$ the discounted cash $\bar{H}_u = b_u H_u$, and $\bar{S}_u = b_u S_u$ the discounted asset. Therefore:

$$\begin{aligned} d\bar{H}_u &= -r_u \bar{H}_u du + b_u dH_u = V_u \bar{S}_u (q-b)du - \bar{S}_u dV_u - b_u S_u d\mathcal{C}_u \\ d\bar{S}_u &= -r_u \bar{S} du + b_u dS_u. \end{aligned} \tag{5.6}$$

On the other hand, given the fact that $\{V_t\}_{0\leq t\leq T}$ - defined in Equation 5.1 - is a finite variation process, we do have the following equation:

$$d(V_u \bar{S}_u) = V_u d\bar{S}_u + \bar{S}_u dV_u. \tag{5.7}$$

Therefore, we have:

$$d\left(V_u \bar{S}_u + \bar{H}_u\right) = V_u d\bar{S}_u + V_u \bar{S}_u (q-b)du - b_u d\mathcal{C}_u(\chi_u). \tag{5.8}$$

Let's now call $\hat{S}_t = e^{\int_t^u (q-b)ds} S_u$, then one can write:

$$d\left(V_u \bar{S}_u + \bar{H}_u\right) = V_u d\hat{S}_u - b_u d\mathcal{C}_u. \tag{5.9}$$

Taking the integration, we obtain:

$$e^{-\int_t^T r_s ds}(S_T V_T + H_T) = H_t + S_t V_t + \int_t^T V_u d\left(e^{-\int_t^u (r_s - (q-b))ds} S_u\right) - \int_t^T e^{-\int_t^u r_s ds} d\mathcal{C}_u,$$

which ends the proof of Proposition 1. ∎

6 Physical Delivery Options and Hedging Strategies for Large Traders

In this section, we consider a large trader hedging an option with a terminal payoff of cash and physical delivery of shares. Physical delivery is the most prevalent transaction type for stock options, wherein the buyer receives the underlying asset upon expiration. This differs from cash settlement, where the payoff is settled in cash. For further details on option types and settlement methods, we refer to Cboe Inc. (2022). From now on, all stochastic processes are defined up to a deterministic time horizon $\widetilde{T} > 0$, split in two time periods, T the expiry date and τ the settlement time such that $T + \tau = \widetilde{T}$.

Definition 5. We assume a physical delivery European option with an expiry date of T and a settlement date $T + \tau$. Hereafter, $Q(S_T)$ denotes the terminal number of shares to be either delivered or received, and $\mathcal{H}(S_T)$ is the terminal cash transaction.

It is important to recall that V_T represents the quantity of shares held by the large trader at time T. As such, the collective sum of $Q(S_T)$ and V_T reflects the remaining shares in the large trader's account at the terminal date. For effective hedging, $Q(S_T)$ and V_T should have opposite signs. Specifically, if $Q(S_T)$ is positive, then V_T should be negative, and conversely.

Any residual shares, represented as $Q(S_T) + V_T$, need to be executed between the expiry and settlement dates. This execution cost is denoted as $\bar{C}(Q(S_T) + V_T, \tau)$. Notably, at time T, the cost associated with executing these shares between T and $T + \tau$ is considered known. Hence, $\bar{C}(Q(S_T) + V_T, \tau)$ is $\mathcal{F}_T$-adapted.

To hedge their position, the large trader maintains a portfolio consisting of H_t in cash and V_t shares at time t, using the same notation as in the previous section. Given the high hedging costs, the large trader may opt to leave some positions unhedged, anticipating favorable market movements to offset any potential issues arising from the unhedged portion. The final profit and loss at maturity seen from t is then given by

$$P\&L_{t,T} = \overbrace{\mathcal{H}(S_T) + S_T Q(S_T)}^{\text{Payoff}} + \overbrace{H_T + S_T V_T}^{\text{Accounting value of hedges}} - \overbrace{S_T \bar{C}(Q(S_T) + V_T, \tau)}^{\text{Terminal execution cost of missing shares}}. \tag{6.1}$$

Equation (6.1) has three main components: the option payoff, the accounting value of the hedging portfolio, and the terminal cost of executing any missing shares between the option expiry time T and the settlement date $T + \tau$ for physical delivery options.

By combining Equations (5.5) and (6.1), we can obtain the following formulation for the terminal $P\&L_{t,T}$:

$$
\begin{aligned}
P\&L_{t,T} &= e^{\int_t^T -r_s ds}(\mathcal{H}(S_T) + S_T Q(S_T)) + P\&L_{t,T}^{\text{trading}} \\
&\quad + \left(\int_t^T e^{-\int_t^u r_s ds} S_u d\mathcal{C}_u + e^{\int_t^T -r_s ds} S_T \bar{C}\left(Q(S_T) + V_T, \tau\right) \right) + H_t + S_t V_t.
\end{aligned} \tag{6.2}
$$

On the right-hand side of Equation (6.2), the first component represents the option payoff, and the second term is the profit and loss from trading activities, as defined in Equation (5.2), due to changes in the value of the underlying asset during the option's lifetime. The third term represents the temporary transaction costs of the delta-hedging program, while the last term represents the accounting value of the hedging portfolio at expiry.

6.1 Manipulating the Payoff Constraints and Problem Reduction

The trader's incentive is to ensure the P&L value is zero or higher. In other words, without considering any market abuse restrictions, the large trader's goal would be to optimize the P&L of the following equation by solving:

$$
\sup_{\{\chi_s\}_{s \geq t}} \mathbb{E}\left[P\&L_{t,T} \middle| \mathcal{F}_t \right].
$$

However, one must exercise caution to avoid market abuse being inherent to the optimization problem. There are two key considerations discussed hereafter.

First, consider under which probability measure the expectation of the P&L should be calculated. A natural choice might be the historical probability measure, as it reflects real-world probabilities. However, using historical probabilities could inadvertently lead to an abusive squeeze arbitrage. As discussed in 4, the expected $P\&L_{t,T}^{\text{trading}}$ is not zero under a historical probability measure. In contrast, under information-neutral measures, the expected $P\&L_{t,T}^{\text{trading}}$ of trading is zero, which is a desirable property. Therefore, using the information-neutral probability measure satisfies a condition that the trader and their model have no incentive to manipulate the stock to gain a better expected $P\&L_{t,T}^{\text{trading}}$ of trading. Specifically, among the information-neutral probability measures, we deploy the minimum variance probability measure, which we developed and defined in detail in the previous section.

Second, to address constraints related to payoff manipulation, we refine the stochastic optimization problem and exclude the payoff component from Equation (6.2) in the optimization. This step is justified because it directly removes the incentive for manipulating the option's payoff.

It is worth noting that these two points are necessary conditions but not sufficient to avoid market abuse. While these measures do not guarantee a market manipulation-proof model, they address essential aspects to minimize the risk of falling into model-induced market abuse.

Consequently, we reduce the problem to Equation (6.3), to an optimization of the cost function

J acting as its solution:

$$J(t, V_t, S_t) = \inf_{\{\chi_s\}_{s \geq t}} \mathbb{E}^{I^*} \left[e^{-\int_t^T r_s ds} S_T \bar{\mathcal{C}} \left(Q\left(S_T\right) + V_T, \tau \right) + \int_t^T e^{-\int_t^s r_u du} S_s d\mathcal{C}_s \middle| \mathcal{F}_t \right]. \tag{6.3}$$

Equation (6.3) consists of two parts: the transaction costs of executing missing shares between the expiry date T and the settlement date and the transaction costs of executing shares before the expiry date. Both terms act against each other, and the trader needs to strike a balance between buying shares too early and getting closer to a full delta hedge with fewer missing shares to recover in the future but paying extra costs because of executing the shares too quickly, or leaving its position partially unhedged. In the favorable scenario, the market will move in the trader's direction, and the partial hedge will be closer to the full hedge. Otherwise, the trader will have some time between the expiry date T and the settlement date $T + \tau$ to execute the missing shares. Note that at the expiry date, the number of required shares is fixed and is no longer uncertain.

Theorem 6 establishes the relationship between the cost function J and the partial differential equation (PDE) that governs it. The theorem states that J can be deduced from the PDE expressed in Equation (6.4), subject to the terminal condition in Equation (6.5).

Theorem 6. *Let $\mathcal{J}^*$ be the solution to the Equation (6.3), it is a function that depends on the optimal number of shares to be executed during a period, denoted by χ_t^*. The solution to the PDE, given by $\mathcal{J}_\chi(t, V, S)$, provides the optimal strategy for the large trader to hedge its option position.*

The cost function J can be deduced from the following partial differential equation:

$$\frac{\partial \mathcal{J}^*}{\partial t} + rS \frac{\partial \mathcal{J}^*}{\partial S} + \frac{1}{2} \frac{\partial^2 \mathcal{J}^*}{\partial S^2} \sigma^2 S^2 + \frac{\mathcal{J}^* - J}{dt} \bigg|_{(t,V,S)} = rJ\left(t, V, S\right), \tag{6.4}$$

with the terminal condition

$$J\left(T, V, S\right) = S_T \mathcal{C}\left(Q\left(S_T\right) - V_T, \tau\right), \tag{6.5}$$

where $\mathcal{J}^*$ is defined as follows:

$$\begin{aligned} \mathcal{J}_\chi(t, V, S) &= J\left(t, V + \chi dt, S + S\Lambda(\chi_t)\right) \omega_t dt + J\left(t, V + \chi dt, S\right)(1 - \omega_t dt) + Sd\mathcal{C}_t \\ \mathcal{J}^*(t, V, S) &= \inf_\chi \left\{\mathcal{J}_\chi(t, V, S)\right\}, \end{aligned} \tag{6.6}$$

and χ_t^* is the optimal number of shares to be executed during the period dt and obtained from

$$\chi_t^* = \arg\inf_\chi \left\{\mathcal{J}_\chi(t, V, S)\right\}. \tag{6.7}$$

Proof. Knowing that $dV_t = \chi_t dt$, Equation (6.3) can be reformulated as

$$J\left(t, V_t, S_t\right) + rJ\left(t, V_t, S_t\right) dt = \inf_{\chi} E^{I^*} \left[\ J\left(t+dt, V_t + \chi dt, S_t + dS_t\right) + Sd\mathcal{C}_t(\chi) \ \right]. \tag{6.8}$$

Let us define $\mathcal{J}_\chi$ as the following equation:

$$\mathcal{J}_\chi(t, V_t, S_t) = J(t, V_t + \chi dt, S(1 + \Lambda(\chi)dM_t)) + d\mathcal{C}_t.$$

Applying the Itô formula with jumps leads to the following equation, where dS^C represents the non-jump part of the asset return:

$$\mathcal{J}_\chi\left(t+dt, V_t, S_t + dS_t^C\right) = \mathcal{J}_\chi + \frac{\partial \mathcal{J}_\chi}{\partial t} dt + \frac{\partial \mathcal{J}_\chi}{\partial S} dS^C + \frac{1}{2}\frac{\partial^2 \mathcal{J}_\chi}{\partial S^2}\left(dS^C\right)^2\bigg|_{(t,V_t,S_t)}. \tag{6.9}$$

Let us also define the optimal χ^* such that:

$$\chi_t^* = \arg\inf_{\chi} \mathbb{E}\left[J\left(t, V_t + \chi\, dt, S_t\left(1 + \Lambda(\chi_t)\, dN_t\right)\right) + d\mathcal{C}_t\right].$$

In this case, Equation (6.8) becomes:

$$rJ(t, V_t, S_t)\, dt = \mathcal{J}_{\chi^*} - J + \frac{\partial \mathcal{J}_{\chi^*}}{\partial t}\, dt + \frac{\partial \mathcal{J}_{\chi^*}}{\partial S}\, \mathbb{E}[dS^C] + \frac{1}{2}\frac{\partial^2 \mathcal{J}_{\chi^*}}{\partial S^2}\, \mathbb{E}\left[\left(dS^C\right)^2\right]\bigg|_{(t,V_t,S_t)}. \tag{6.10}$$

Hence, expanding the expectation of (6.10), we obtain the partial differential equations as formulated in (6.4) and (6.5).

□

It is important to note that ω_t is a function that also depends on χ, which in turn represents the optimal speed of execution during the infinitesimal period dt and is obtained through Equation (6.7). Additionally, when there is no market impact, Λ is equal to 0, which results in Equation (6.4) reducing to the standard Black-Scholes equation.

7 Numerical Results

In this section, we present numerical results focusing on the large trader hedging an out-of-the-money call option. We analyze the option costs, hedging dynamics for the large trader, and the implications for the insider trader. To achieve this, we define a model and parameterization for transaction costs and provide an algorithm to solve the problem using a dynamic programming approach for the associated partial differential equations. Additionally, we perform a sensitivity analysis on key model parameters.

7.1 Transaction Costs Parametrization

Definition 7. Let $\mathcal{C}_t$ be a stochastic process with the dynamics defined by Equation (7.1). The process $\{\mathcal{C}_t\}_{0\leq t\leq T}$ represents the transaction costs beared by the large trader :

$$\begin{aligned} \mathcal{C}_0 &= 0 \\ d\mathcal{C}_t &= \eta \left(\frac{\chi_t}{\chi_t + A}\right)^{\alpha} \chi_t dt, \end{aligned} \tag{7.1}$$

where $A \in \mathbb{R}$ is the (annualized) average volume, $\eta \in \mathbb{R}^+$ is the impact factor, and $\alpha > 0$ is the exponential factor.

This formulation different, but still consistent with previous parameterization by Grinold & Kahn (2000) and Zarinelli *et al.* (2015) where a square root formula is deployed.

For longer periods between the expiry and settlement date, an optimal purchase/liquidation algorithm can be applied, as explored in the literature (e.g. Almgren & Chriss (1997)). However, in this study, we use a constant rate program for simplicity. In this case, the expected value of function $\bar{C}(x, \tau)$ - the terminal cost of executing x at time τ between the expiry and settlement - can be expressed as Equation (7.2) below.

Proposition 2. Assume the trader needs to execute X shares within a period of τ and chooses to do so at a constant execution speed of $\frac{X}{\tau}$.[4] Moreover, we assume that there are no dividends and lending costs during the time period from T to $T+\tau$. Under these conditions, the expected value of the cost under an information-neutral measure $I \in \mathcal{I}$ is described by the following equation:

$$\mathbb{E}^{\mathcal{I}}\left[\bar{C}(x,\tau)\,|\mathcal{F}_T\right] = \eta \times \frac{\left(\frac{x}{\tau}\right)^{\alpha}}{\left(\frac{x}{\tau}+A\right)^{\alpha}} x. \tag{7.2}$$

Overall, our cost model allows us to capture the trade-off between the size of the trade (x) and the time until the settlement date (τ). Specifically, as x increases, so does the cost of execution, but at a decreasing rate governed by the exponent α. In contrast, as τ increases, the cost of execution decreases due to the larger time frame for trading.

Proof. Having Equation (7.1) for the short-term cost, and with the strong assumptions that all shares requested will be executed, the long-term cost of execution of X shares in τ time is

$$\bar{C}(X,\tau) = \int_0^{\tau} S_t d\mathcal{C}_t.$$

[4] While optimal purchase/liquidation algorithms for longer purchase periods are well-researched topics, as seen in works like Almgren & Chriss (1997), our focus here is on the constant rate program. There are compelling reasons for selecting a constant purchase program over a more intricate one. Notably, most derivatives desks do not prioritize optimizing their execution cost, placing greater emphasis on volatility instead. Thus, even if optimal algorithms are available, they might not be employed in derivatives trading. Our goal here is not to suggest how the purchasing should be executed but to understand the behavior in the most prevalent scenario. This reasoning drives our choice for a constant rate program.

Note that in our case, $\frac{X}{\tau}$ is the constant execution rate. In case we have Equation (7.1) for the short-term cost, the long-term cost of execution is as follows:

$$\int_T^{T+\tau} S_t d\mathcal{C}_t = \eta \times \frac{\left(\frac{X}{\tau}\right)^{\alpha+1}}{\left(\frac{X}{\tau}+A\right)^{\alpha}} \int_T^{T+\tau} e^{-rt}\left(1+\Lambda\left(\frac{X}{\tau}\right)\right)^t S_t dt,$$
$$E^I\left[\int_T^{T+\tau} S_t d\mathcal{C}_t\right] = \eta \frac{\left(\frac{X}{\tau}\right)^{\alpha+1}}{\left(\frac{X}{\tau}+A\right)^{\alpha}} S_T \tau.$$

□

7.2 Solving the Partial Differential Equation

To numerically solve the two-dimensional partial differential Equation (6.4), we can use the dynamic programming approach described in Algorithm 1. First, we initialize the cost function $J(T,S,V)$ to the terminal condition $\bar{C}(Q(S)-V,\tau)$ at the expiration time T. Then, we discretize time into Δt intervals and iterate backward in time from T to 0, updating $\mathcal{J}$ at each time step using Equations (6.4), (6.6) and (6.7). We repeat this process for n iterations to ensure convergence.

Algorithm 1 Methodology for solving the large trader's cost function

$J(T,S,V) \leftarrow \bar{C}(Q(S)+V,\tau)$;
Discretize the time to $\triangle t$ intervals;
for $t=T; t \leftarrow t+\Delta t; t=0$ **do**
  Using $J(t+\Delta t,.,.)$ apply Equation (6.7) and solve $\chi_t^\star$;
  Using χ^*, apply Equation (6.6) and calculate $\mathcal{J}^*(t+\Delta t,.,.)$;
  Using $\mathcal{J}^*(t+\Delta t,.,.)$, solve Equation (6.4) and obtain $J(t,.,V)$.;
  for $iter=0; iter \leftarrow iter+1; iter < n = numiter$ **do**
    Using $J(t,.,.)$ apply Equation (6.7) and solve $\chi_t^\star$;
    Using χ^*, apply Equation (6.6) and recalculate $\mathcal{J}^*(t+\Delta t,.,.)$;
    Using $\mathcal{J}^*(t+\Delta t,.,.)$, solve Equation (6.4) and obtain $J(t,.,V)$;
  end for
  Given χ_t^*, solve the insider's trader option price $\mathcal{O}$ using a finite difference scheme of the partial differential in Equation (4.4);
end for
Obtain $J(0,S_0,0)$ and $\mathcal{O}(0,S_0,V_0)$

Once we have obtained the optimal hedging strategy χ^*, we can use it to solve for the insider's trader option price $\mathcal{O}$ using a finite difference scheme for the partial differential in Equation (4.4). Finally, we obtain the values of $\mathcal{J}$ and $\mathcal{O}$ at time 0, denoted by $J(0,S_0,0)$ and $\mathcal{O}(0,S_0,V_0)$, respectively. More details on the finite difference scheme used to solve Equation (6.4) are provided in the Appendix.

7.3 Numerical Results

In this section, we present numerical results for a large trader who is the seller of a call option defined in Table 1.

T	τ	Call strike	Notional
120 days	15 days	110	1e+6

Table 1: Large Trader's option.

We also make the hypothetical parameter assumptions as per Table 2. These assumptions, though hypothetical, fall within the realistic ranges commonly used in financial modeling and empirical studies. The annual drift rate, $\mu = 0.02$, is within the range of U.S. annual returns of small-cap stocks as documented by Damodaran (2025). The volatility parameter, $\sigma = 0.25$, aligns with the historical range of the CBOE Volatility Index (VIX), which typically fluctuates between 10 and 20 during stable markets and exceeds 30 during periods of high market stress VIX (2025). The risk-free rate, $r = 0.01$, reflects the low-interest-rate environment observed in the U.S. market during 2010-2020. Also, further in this section, we will provide sensitivity testing to the choice of parameters ϕ, ν, and λ. Finally, for simplicity, we assume that the stock does not pay any dividends ($q = 0.0$) and its lending yield is zero ($b = 0.0$).

μ	σ	η	ϕ	λ	S_0	A	r	α	q	b
0.02	0.25	0.1	20.0	2.0	100	5e+6	0.01	2.0	0.0	0.0

Table 2: Market parameters assumptions.

Figure 4 depicts the terminal cost function J for the large trader's call option. The graph illustrates different scenarios depending on the final stock price and the number of shares V_T to be delivered at expiration. When the option is in the money, i.e., the stock price is above the strike price, the cost function is zero since the large trader delivers the shares at no additional cost. Conversely, when the option is out of the money, i.e., the stock price is below the strike price, the large trader needs to purchase the notional amount of shares before the settlement date, leading to higher costs. As shown in the graph, the cost function varies across different corners but gradually flattens as we move backward in time, as observed in Figure 5. These results are consistent with our cost model in Equation (6.5), which assumes a linear relationship between the cost and the spot price of the stock.

Figures 6 and 7 show the optimal amount of shares to be executed (χ^*) at time 0 and just before expiry, respectively. Our findings suggest that the optimal χ^* is much lower than the Black-Scholes Delta due to the high cost of a full hedge strategy. As shown in Figure 8, the difference in Delta between the two strategies is significant. The large trader executes only a small percentage of χ^* at time 0, whereas the Black-Scholes Delta is 100% for the same spot levels. However, as the expiry date approaches, the large trader's optimal strategy gets closer to

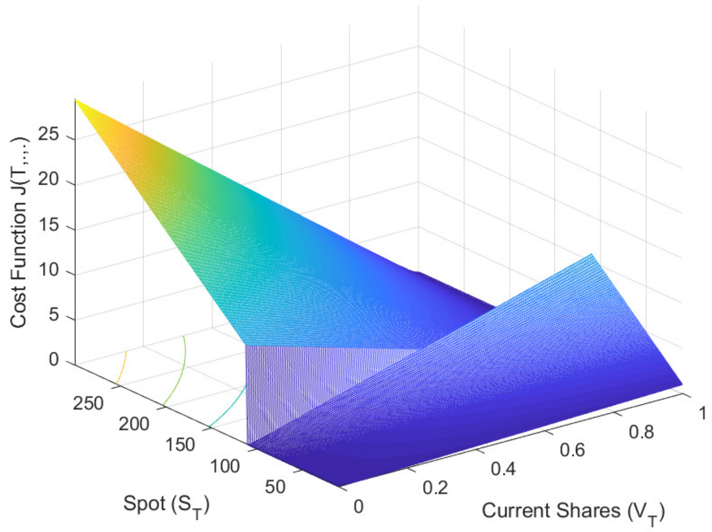

Figure 4: The terminal cost function J for a hedged option. V_T shall be read as a multiple of the notional.

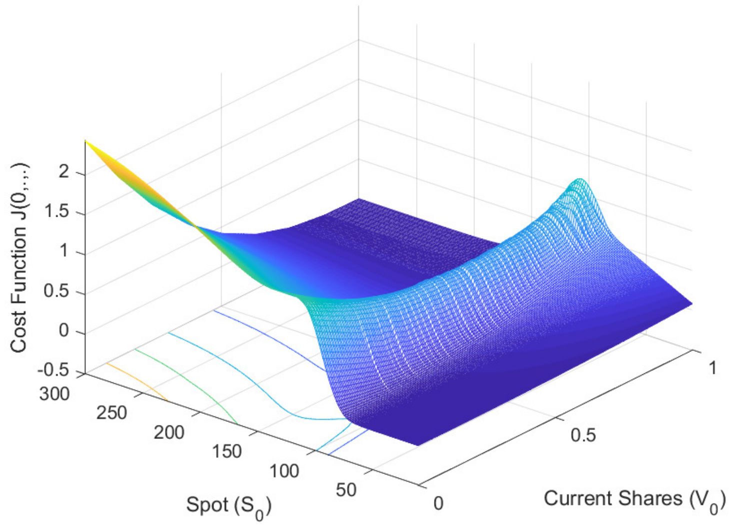

Figure 5: The cost function J at time 0 for a hedged call option. Data source: Matlab simulation.

the Black-Scholes Delta, as illustrated in Figure 9. The number of shares to be purchased by the large trader is still smaller than the Black-Scholes Delta, but it reaches 50% of the B-S Delta for high spot prices. The limited time frame near expiry means that the large trader has to overcome a significant stock shortage, making the execution of the remaining shares challenging.

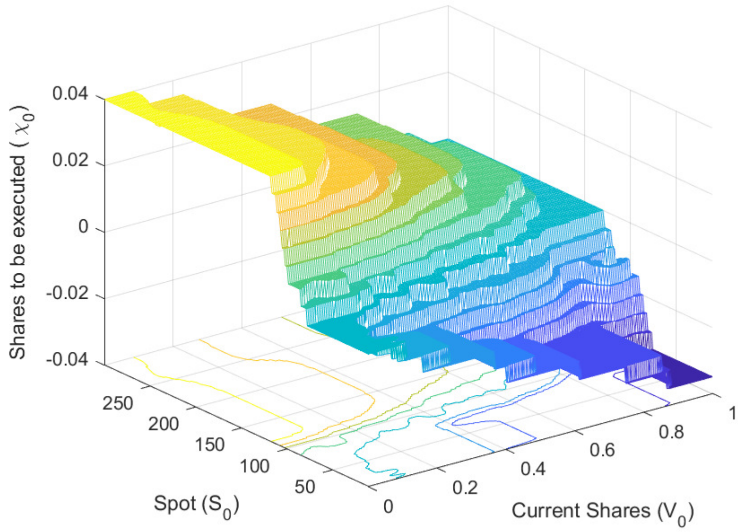

Figure 6: The optimal shares strategy function at time 0 for a call option. Data source: Matlab simulation.

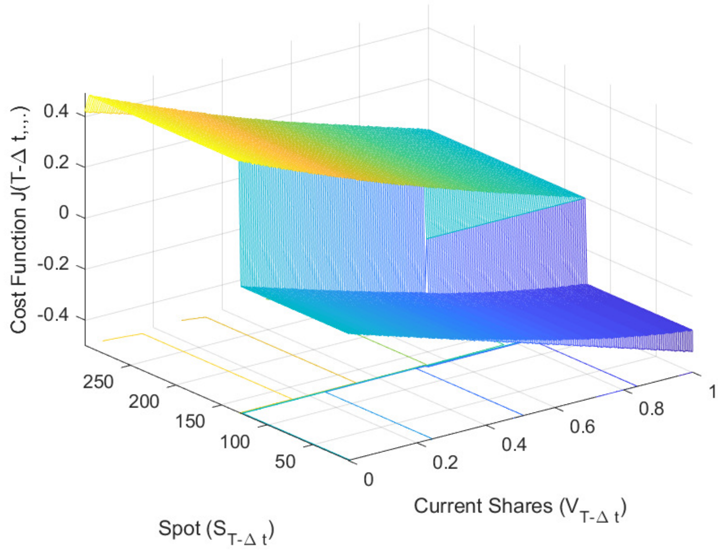

Figure 7: The optimal shares strategy function at time $T - dt$ for a call option. Data source: Matlab simulation.

Examining the impact of the large trader's hedging strategy on the insider trader's option price, as defined in Table 3, provides valuable insights. Figure 10 compares the insider trader's option price with the Black-Scholes model for the same call option. The graph highlights the

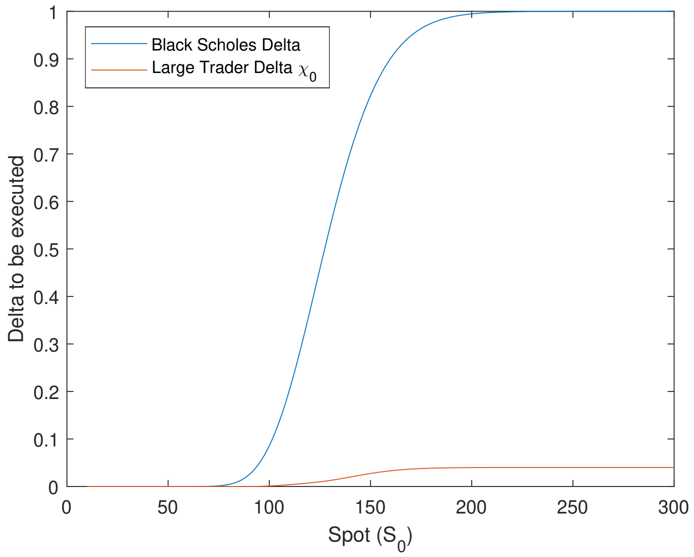

Figure 8: Comparison between BS delta and optimal shares for the large trader at inception, assuming $V_0 = 0$.

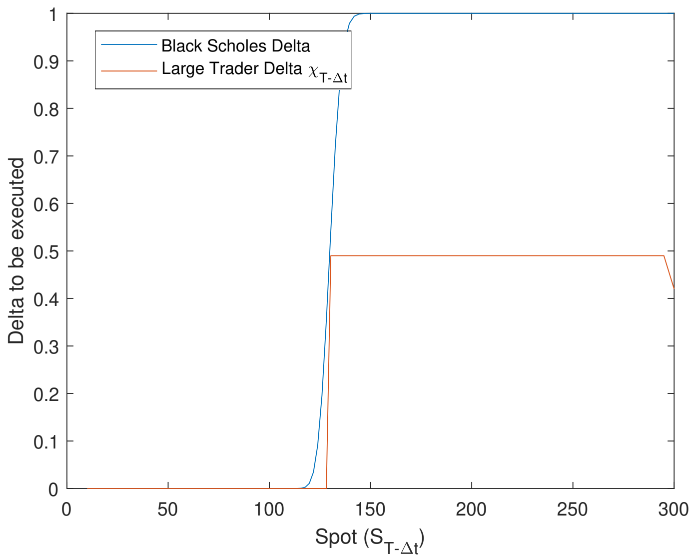

Figure 9: Comparison between BS delta and optimal shares for the large trader at expiry, assuming $V_T = 0$. Data source: Matlab simulation.

option's increased convexity around the money and the higher prices than the Black-Scholes model when the option is in the money, surpassing even the intrinsic value. This phenomenon is due to the directional market impact of the large trader. If the spot price is in the money, the large trader will purchase shares, driving the spot price deeper into the money. The insider trader's option price, therefore, reflects the market impact of the large trader, resulting in a higher option price than predicted by the standard Black-Scholes model.

T	Call strike
120 days	110

Table 3: Small trader's option.

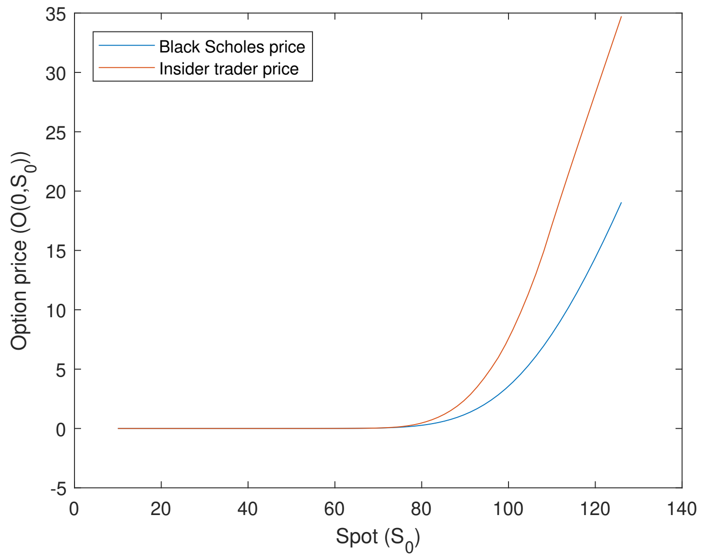

Figure 10: Comparison between Black and Scholes price and insider trader price for a call option. Data source: Matlab simulation.

We would also like to highlight the fact that the insider option price is sensitive to parameters ϕ, η , λ as shown in Table 4. Economically speaking, an increase in the market impact parameters ϕ and λ results in a higher call option price for the insider trader (for the option defined in Table 1). Recall that η is the coefficient of the transaction costs (temporary impact); the higher this coefficient is, the less conservative the large trader will become and the closer the option price will be to the one with no market impact (Black & Scholes price). We recall also that λ is the market impact factor, and ϕ is the frequency of the market impact function. Higher values of these two parameters result in a higher market impact function and, accordingly, more divergence from the Black & Scholes price.

ϕ	λ	η	Option price
10.0	0.5	0.2	4.55
10.0	1.0	0.5	4.53
20.0	1.0	0.2	5.58
20.0	0.5	0	5.46
20.0	0.0	0.2	3.59
0.0	0.0	0	3.59

Table 4: Call option price for insider trader at $t = 0$, sensitivity to different market impact parameters.

8 Conclusion

In conclusion, this paper examines the implications of a large trader's activities on option pricing and hedging. We addressed an important but overlooked problem in the literature: how a large trader's knowledge of their own trading activity can influence their actions. Therefore, a key contribution of this paper is its methodology to disentangle this knowledge from other factors influencing options' hedging strategies. To achieve this, we introduced the concept of an insider trader, who has the same information but does not impact the market.

Our analysis established the existence of information-neutral probability measures, under which the discounted asset prices are martingales for the insider trader. Using these information-neutral measures, along with necessary conditions for mitigating market manipulation, we derived a cost optimization function to establish a hedging policy for the large trader. This framework allows us to develop a robust hedging policy that accounts for transaction costs and market impact while taking into account some necessary conditions for non-manipulative practices.

The paper concludes with numerical results that showcase the optimal delta-hedging strategy for an out-of-the-money call option. Our analysis indicates that the model proposes a more relaxed delta-hedging strategy compared to the Black and Scholes model. Additionally, the price of the call option for the insider trader is observed to be higher under market impact than the Black and Scholes model. These findings offer a practical and effective approach to pricing derivatives under market impact, which can be readily implemented by practitioners.

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Appendix

Numerical Implementation of the Large Trader PDE

In terms of log returns, and assuming that χ^* is already obtained, Equation (6.4) can be written as

$$
\begin{aligned}
x &= log(S),\\
\Upsilon(t,x,V) &= J(t,V,S),\\
\Upsilon^*(t,x,V) &= \mathcal{J}^*(t,V,S),\\
\bar{\Lambda}(.) &= \ln(1+\Lambda(.)),\\
rJ &= \frac{\partial \Upsilon^*}{\partial t} + \left(r-(q-b)-\frac{1}{2}\sigma^2\right)\frac{\partial \Upsilon^*}{\partial x} + \frac{1}{2}\frac{\partial^2 \Upsilon^*}{\partial x^2}\sigma^2\\
&\quad + \left(\Upsilon^*\left(t,V,x+\bar{\Lambda}(\chi^\star)\right) - \Upsilon\left(t,V,x\right)\right)\frac{1}{dt}.
\end{aligned}
$$

We define the boundary levels x_{min}, x_{max} for the log spot and d_{min}, d_{max} for the shares. We define the discretization $\{(t_i, x_l, d_k,)\}$ with

$$
\begin{aligned}
(0,0,0) &\leq (i,l,k) \leq (I,L,K)\,,\\
(0,x_{min},d_{min}) &\leq (0,x_k,d_l) \leq (T,x_{max},d_{max})\,.
\end{aligned}
$$

Here, we assume a uniform grid, with intervals of triplet $(\triangle t, \triangle x, \triangle d)$. At the maturity level, we define the following equation:

$$
\Upsilon_{I,l.k} = C\left(d_k - Q\left(e^{x_l}\right),\tau\right)
$$

At t_{i+1}, we first calculate the optimal value in discretized terms. Assuming that we have the optimal $\chi^\star$ we define $\iota^* = \frac{\chi^\star}{\Delta d}$ and $L^\star_{i,l.k}$ as below equation for the gridpoint (t_i, x_l, d_k).

$$
L^\star_{i,l.k} = \left(\Upsilon^*_{i+1,l,k+\iota^*} - \Upsilon_{i+1,l,k}\right)(1-\omega^*\Delta t) + \left(\Upsilon^*_{i+1,l+\left\lfloor\frac{\Lambda(\iota^*\Delta d)}{\Delta x}\right\rfloor,k+\iota^*} - \Upsilon_{i+1,l,k}\right)\omega^*\Delta t + e^{x_l}d\mathcal{C}\left(\iota\Delta d\right). \tag{.1}
$$

We apply a fully implicit scheme, and the discretization will be as of below:

$$
\begin{aligned}
&\left(1+r\triangle t+\tfrac{\triangle t}{(\triangle x)^2}\sigma^2\right)\Upsilon_{i,l,k} + \left((r-(q-b)-\tfrac{1}{2}\sigma^2)\tfrac{\triangle t}{2\triangle x} - \tfrac{1}{2}\tfrac{\triangle t}{(\triangle x)^2}\sigma^2\right)\Upsilon_{i,l-1,k}\\
&\quad + \left(-(r-(q-b)-\tfrac{1}{2}\sigma^2)\tfrac{\triangle t}{2\triangle x} - \tfrac{1}{2}\tfrac{\triangle t}{(\triangle x)^2}\sigma^2\right)\Upsilon_{i,l+1,k} = \Upsilon^*_{i+1,l,k} + L^\star_{i,l.k}.
\end{aligned}
$$

With Neumann boundary condition, we have:

$$
\begin{aligned}
\Upsilon_{i,l,k} = \Upsilon_{i,0,k} = C\left(N_T - Q\left(e^{x_{min}}\right),\tau\right) &\approx -C\left(d_k - Q\left(e^{x_{min}}\right),\tau+T-t_i\right)\\
\Upsilon_{i,L-1,k} = \Upsilon_{i,L,k} &\approx C\left(d_k - Q\left(e^{x_{max}}\right),\tau+T-t_i\right).
\end{aligned}
$$