COMPUTING CLOSEST STABLE NONNEGATIVE MATRIX

Yurii Nesterov, Vladimir Protasov

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Computing closest stable non-negative matrix

Yu. Nesterov* and V.Yu.Protasov†

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Abstract

The problem of finding the closest stable matrix for a dynamical system has many applications. It is studied for both continuous and discrete-time systems and the corresponding optimization problems are formulated for various matrix norms. As a rule, non-convexity of these formulations does not allow finding their global solutions. In this paper, we analyse positive discrete-time systems. They also suffer from non-convexity of the stability region, and the problem in the Frobenius norm or in Euclidean norm remains hard for them. However, it turns out that for certain polyhedral norms, the situation is much better. We show, that for the distances measured in the max-norm, we can find exact solution of the corresponding nonconvex projection problems in polynomial time. For the distance measured in the operator ℓ_∞-norm or ℓ_1-norm, the exact solution is also efficiently found. To this end, we develop a modification of the recently introduced spectral simplex method. On the other hand, for all these three norms, we obtain exact descriptions of the region of stability around a given stable matrix. In the case of max-norm, this can be seen as an extension onto the class of nonnegative matrices of the Kharitonov theorem providing a stability criterion for polynomials with interval coefficients [12, 21].

Keywords: non-negative matrices, spectral radius, Schur stability, iterative optimization method, polyhedral norm, non-symmetric eigenvalue problem

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1 Introduction

We address the problem of finding the closest stable or closest unstable non-negative matrix to a given matrix A. The stability is considered in the sense of Schur: a matrix

*Center for Operations Research and Econometrics (CORE), Catholic University of Louvain (UCL) and National Research University Higher School of Economics. E-mail: Yurii.Nesterov@uclouvain.be

†DISIM of University of L'Aquila (Italy), Department of Computer Science of Higher School of Economics (Russia), E-mail: v-protassov@yandex.ru

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is stable if all its eigenvalues are strictly less than one by modulo. If the matrix A is stable, then the problem is to find the closest unstable matrix to it, i.e., the closest to A matrix X such that $\rho(X) = 1$, where ρ denotes the spectral radius. If A is unstable, i.e., $\rho(A) > 1$, then the closest stable matrix does not exist, because the set of stable matrices is open. Hence, by the closest stable matrix we understand the closest matrix X with $\rho(X) = 1$ (although X is actually unstable). Sometimes a matrix with the spectral radius one is referred to as *weakly stable*. So, if A is unstable, then the problem is to find the closest weakly stable matrix.

In all these problems, the choice of matrix norm plays a crucial role. In this paper, we consider three polyhedral norms:

- the max-norm $\|X\|_{\max} = \max\limits_{(i,j)} |X^{(i,j)}|$;
- the ℓ_1 operator norm: $\|X\|_1 = \sup\limits_{u \neq 0} \frac{\|Xu\|_1}{\|u\|_1}$, where $\|u\|_1 = \sum\limits_{i=1}^{n} |u^{(i)}|$.
- the ℓ_∞ operator norm: $\|X\|_\infty = \sup\limits_{u \neq 0} \frac{\|Xu\|_\infty}{\|u\|_\infty}$, where $\|u\|_\infty = \max\limits_{1 \leq i \leq n} |u^{(i)}|$.

Note that $\|X\|_\infty = \max\limits_{1 \leq i \leq n} \|X^T e_i\|_1$, where e_i is the ith coordinate vector in $\mathbb{R}^n$. Thus, we actually consider six problems of fining closest stable/unstable matrix in these three norms. For all problems, we characterize the optimal matrix and construct efficient algorithms for finding the solution. For the max norm, we explicitly find the closest unstable matrix and present an algorithm based on bisection for computing the closest stable matrix. For the $\ell_\infty-$ and ℓ_1-norms, we also characterize the optimal matrices. For the closest unstable matrix, the solution is found explicitly, while for finding the closest stable matrix, we use the concept of product families and apply the spectral simplex method, which can optimize the spectral radius over such families [19, 22]. To this end, we develop a modification of this algorithm, the *greedy spectral simplex method*, which may be of some independent interest.

Motivation. The problem of finding the closest stable or unstable matrix plays an important role in the analysis of differential equations, linear dynamical systems, electrodynamics, etc., see [1, 4, 7, 9, 15, 18] and references therein. This problem is notoriously hard due to properties of the spectral radius as a function of matrix: it is neither convex nor concave, it may lose Lipschitz continuity at some points, etc. That is why the majority of methods for this problem find only local minima [4, 9, 10, 20]. Nevertheless, we are going to see that for some classes of matrices and matrix norms, this problem is efficiently solvable even for absolute minima. We analyse the case of non-negative matrices. They correspond to positive linear systems arising naturally in problems of combinatorics, mathematical economics, population dynamics, etc. [1, 2, 5, 6, 14, 17, 25]. We show that on the set of non-negative matrices equipped either with the max norm (entrywise maximum), or with the ℓ_∞ or ℓ_1 operator norms, the closest stable and unstable matrices admit explicit descriptions and can be found by efficient algorithms.

Finally, let us note that in the problem of finding the closest stable/unstable *non-negative* matrix to a matrix A, the matrix A itself does not have to be non-negative. For any real-valued matrix A, this problem can be reduced to the case of non-negative A. Indeed, if we denote $A_+ = \max\{A, 0\}$ (the entrywise maximum), then we see that the closest stable non-negative matrices to the matrices A and A_+ are the same. Indeed, if

$\rho(A_+) \leq 1$, then A_+ is the closest stable non-negative matrix for A, because it is the closest non-negative matrix for it. Otherwise, if $\rho(A_+) > 1$, then we denote by A' and A'' the closest stable non-negative matrices for A_+ and for A respectively. Since increasing the entry of a non-negative matrix increases its spectral radius, see equation (3) below, it follows that $A' \leq A_+$ and $A'' \leq A_+$. If $\|A' - A_+\| < \|A'' - A_+\|$, then $\|A' - A\| < \|A'' - A\|$ which contradicts to the choice of A''. Hence $\|A' - A_+\| = \|A'' - A_+\|$, and A'' is the closest stable non-negative matrix for A_+, which concludes the proof.

Thus, in what follows we assume the initial matrix A is non-negative.

The last remark concerns possible statements of the main problems. We consider the two main problems:

1) *Destabilizing problem*: for a matrix A such that $\rho(A) < 1$ find the closest matrix X such that $\rho(X) \geq 1$.

2) *Stabilizing problem*: for a matrix A such that $\rho(A) > 1$ find the closest matrix X such that $\rho(X) \leq 1$.

Sometimes those problems are formulated in a strong sense: given a parameter $\delta > 0$, find the closest matrix such that $\rho(X) \geq 1 + \delta$ (respectively, $\rho(X) \leq 1 - \delta$). The methods developed in this paper (Sections 2, 4, and 4.5) are directly adapted to those strong problems as well, it suffices to replace everywhere the spectral radius 1 by $1 + \delta$ (respectively, $1 - \delta$).

Contents. We start with solving the problems of the closest stable/unstable non-negative matrix in the max-norm (Section 2). We show that global minima for both problems admit explicit description and can be found by polynomial algorithms. To make them more efficient, we take a close look at the problem of computing the largest eigenvalue of a non-negative matrix. In Section 3 we develop a new method with local quadratic rate of convergence and polynomial-time worst-case global performance guarantees (this result is not standard for Linear Algebra (see, for example, Chaper 7 in [8])). In Section 4 we address the problems of the closest stable/unstable non-negative matrix in the ℓ_∞ and ℓ_1 norms. We show that the closest unstable matrix admits an explicit description and can be computed within polynomial time, while for finding the closest stable matrix we develop a new *greedy spectral simplex method*. In both problems we apply the method of computing the Perron eigenvalue (see Section 3) and show that the corresponding algorithms have local quadratic convergence. Note that the greedy spectral simplex method has a much wider range of applications and, probably, is of independent interest.

In Section 5 we give some examples and discuss the complexity issue. We show that our method for finding closest stable matrices in ℓ_1 and in ℓ_∞ norms work surprisingly fast even in high dimensions. For positive matrix of dimension 100, it finds the closest stable matrix for less than one second. For dimension 1000 it does for about $3 - 4$ minutes. For sparse matrices, the algorithm works slower but still very efficient.

Notation. In what follows, we denote by $\mathbb{R}^{n \times n}$ the set of real $n \times n$-matrices, and by $\mathbb{R}^{n \times n}_+$ the set of non-negative matrices. For $A \in \mathbb{R}^{n \times n}_+$ and $x \in \mathbb{R}^n_+$, denote

$$\operatorname{supp}(A) \;=\; \{(i,j) \mid \; A^{(i,j)} > 0\}\,, \qquad \operatorname{supp}(x) \;=\; \{i \mid \; x^{(i)} > 0\}.$$

For two vectors $x, y \in \mathbb{R}^n_+$, we denote $x \geq y$ if $x - y \in \mathbb{R}^n_+$. The *active set* of this equality is $\{i \mid x^{(i)} = y^{(i)}\}$.

We denote $\Omega = \{1, \ldots, n\}$, and for any nonempty subset $\mathcal{I} \subset \Omega$, let $V_{\mathcal{I}} = \operatorname{span}\{e_i \mid i \in \mathcal{I}\}$. So, $V_{\mathcal{I}}$ is the coordinate subspace spanned by the basis vectors with indices from $\mathcal{I}$. A *support* of a vector $v \geq 0$ is the set of its positive indices. Thus, $\operatorname{supp}(v) = \{i \in \Omega \mid v^{(i)} > 0\}$. If $v > 0$, then $\operatorname{supp}(v) = \Omega$, i.e., a positive vector possesses a full support.

Finally, we use notation I_n for the unit $n \times n$-matrix, and $J_n \in \mathbb{R}^{n \times n}$ for the matrix of all ones.

Let $A \in \mathbb{R}^{n \times n}$ be a real square matrix with spectrum $\Lambda(A) \stackrel{\text{def}}{=} \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$. Denote by $\rho(A)$ its spectral radius:

$$\rho(A) \;\; = \;\; \max_{\lambda \in \Lambda(A)} |\lambda|.$$

If the matrix A is non-negative, then by Perron-Frobenius theorem, $\rho(A) \in \Lambda(A)$. So, there exists a positive eigenvalue equal to the spectral radius. This eigenvalue will be denoted by $\lambda_{\max}$ and referred to as *the leading eigenvalue*. An arbitrary non-negative eigenvector $v \neq 0$ with eigenvalue $\lambda_{\max}$ is called the *leading eigenvector*. By the same Perron-Frobenius theorem, every non-negative matrix has at least one leading eigenvector [11, chapter 8].

2 Problem with distances measured in max-norm

In this case, the problem is rather simple and admits very efficient solutions. For a non-negative $n \times n$ matrix A, we consider the following problems:

1) If $\rho(A) < 1$, then we find the closest unstable matrix:

$$\|X - A\|_{\max} \;\to\; \min : \quad \rho(X) \;=\; 1,\; X \;\geq\; 0\,. \tag{1}$$

2) If $\rho(A) > 1$, then we are interested in finding the closest stable matrix. The corresponding problem looks exactly as (1).

2.1 Closest unstable matrix

The spectral radius of a non-negative matrix A can be represented in the following minimax form:

$$\rho(A) \;\; = \;\; \inf_{x>0} \max_{1 \leq i \leq n} \tfrac{1}{x^{(i)}} \langle e_i, Ax \rangle. \tag{2}$$

For the proof, it suffices to consider positive matrices and then take a limit. For the leading eigenvector x, the right hand side is equal to $\rho(A)$, this proves the inequality $\geq$. If we have a strict inequality, then there exists a number $q < \rho(A)$ which is larger than the right hand side. Hence, for some positive vector x, we have $\max\limits_{1 \leq i \leq n} \tfrac{1}{x^{(i)}} \langle e_i, Ax \rangle \;<\; q$, i.e., $Ax < qx$. Therefore, $A^k x < q^k x$ for all k, and consequently $\|A^k\| \leq C\, q^k$. Finally, $\rho(A) = \lim_{k \to \infty} \|A^k\|^{1/k} \leq q$, which is a contradiction.

An important consequence of this representation is monotonicity of this function:

$$A \geq B \in \mathbb{R}^{n \times n}_{+} \quad \Rightarrow \quad \rho(A) \geq \rho(B). \tag{3}$$

Sometimes we need conditions for strict monotonicity.

Lemma 1 *Let $A, B \in \mathbb{R}^{n\times n}_+$. If for some $\gamma > 0$ we have*

$$A^{(i,j)} \quad \leq \quad \gamma B^{(i,j)} \quad \forall i,j \in \{1,\ldots,n\}$$

then $\rho(A) \leq \gamma\rho(B)$.

Proof:
It follows immediately from the definition (2). □

Remark 1 *Assumptions of Lemma 1 ensure strict monotonicity of spectral radius only if $A + B$ is an irreducible matrix. This condition cannot be dropped, and the corresponding examples are well known.* □

Consider the set of weakly stable non-negative matrices

$$\mathcal{S}_n \quad = \quad \{A \in \mathbb{R}^{n\times n}_+ : \ \rho(A) \leq 1\},$$

and denote by $\mathcal{S}^0_n$ the set of stable matrices, for which inequality in the above definition is strict. In what follows, we often use a simple criterion for stable matrices.

Lemma 2 *Non-negative matrix A is stable if and only if the matrix $(I_n - A)^{-1}$ is well defined and non-negative.*

Indeed, if $\rho(A) < 1$, then the matrix $(I_n - A)^{-1}$ can be represented by a convergent series $\sum\limits_{k=0}^{\infty} A^k$, which is a non-negative matrix.

Let $Y \stackrel{\text{def}}{=} (I_n - A)^{-1}$ be well defined and non-negative. Since it is non-degenerate, it has the same system of eigenvectors as matrix A. Moreover, since the function $f(\lambda) = \frac{1}{1-\lambda}$ is decreasing on the interval $(0, 1)$, we see that the leading eigenvalue $\lambda_{\max} = \rho(A)$ of the matrix A (the maximal by modulus eigenvalue, which is positive by the Perron-Frobenius theorem) is mapped to the leading eigenvalue $f(\lambda_{\max})$ of Y. Consequently, $\rho(Y) = \frac{1}{1-\rho(A)}$. Hence, for the leading eigenvector $s \in \mathbb{R}^n_+$ of matrix A we have $Ys = \frac{1}{1-\rho(A)}s$. Since $Y \geq 0$, we conclude that $\rho(A) < 1$. □

As it was pointed out by the anonymous Referee, this lemma can also be quickly deduced from Theorem 2.3 of [2]. We are grateful to the referee for this remark. This simple lemma helps us in computing the distance between a stable matrix and the boundary of the set of unstable matrices. Let us prove first an auxiliary statement.

Lemma 3 *Let $A \in \mathcal{S}^0_n$ and $H \in \mathbb{R}^{n\times n}_+$, $H \neq 0$. Denote $\xi(A, H) = \rho((I_n - A)^{-1}H)$. Then*

$$A + \alpha H \quad \in \quad \mathcal{S}^0_n, \quad 0 \leq \alpha < \tfrac{1}{\xi(A,H)}, \tag{4}$$

and $\rho\left(A + \frac{H}{\xi(A,H)}\right) = 1$.

Proof:
Denote $B = (I_n - A)^{-1}H \geq 0$. Note that equality $H = (I_n - A)B$ implies that $I_n - A - \alpha H = (I_n - A) - \alpha (I_n - A)B$. Therefore,

$$W(\alpha) \stackrel{\text{def}}{=} I_n - (A + \alpha H) = (I_n - A)(I_n - \alpha B), \tag{5}$$

and $\rho(\alpha B) < 1$ for all $\alpha \in \left[0, \frac{1}{\rho(B)}\right)$. Consequently, all matrices $W(\alpha)$ are well defined for $\alpha \in \left[0, \frac{1}{\rho(B)}\right)$ and $W^{-1}(\alpha)$ are non-negative as a product of non-negative matrices. Hence, by Lemma 2, all matrices $W(\alpha)$ are stable.

Assume now that the matrix H is strictly positive. In this case its leading eigenvector v is also strictly positive and $W\left(\frac{1}{\rho(B)}\right) v \stackrel{(5)}{=} 0$. Hence $\rho\left(A + \frac{H}{\rho(B)}\right) = 1$. Now, if H is an arbitrary non-negative matrix, we perturb it to be strictly positive and prove this equality. Then by continuity of spectral radius, we conclude that for arbitrary non-negative H, we have $\rho\left(A + \frac{H}{\rho(B)}\right) = 1$. □

Corollary 1 *Let $A \in \mathcal{S}_n^0$ and $H \in \mathbb{R}_+^{n\times n}$, $H \neq 0$. Then all matrices from the set*

$$\left\{X \in \mathbb{R}^{n\times n} : \ 0 \leq X < A + \frac{H}{\rho((I_n - A)^{-1}H)}\right\}$$

are stable.

Proof:
This is a direct consequence of Lemma 3 and of monotonicity of spectral radius. □

We conclude this section by a variant of Corollary 1 for the special case $H = J_n$ (the matrix of all ones). It gives an explicit formula for the closest (in the max-norm) unstable matrix.

Theorem 1 *Let $A \in \mathcal{S}_n^0$ and $e \in \mathbb{R}^n$ be the vector of all ones. Then all matrices from the set*

$$\left\{X \in \mathbb{R}^{n\times n} : \ 0 \leq X < A + \frac{J_n}{\langle (I_n - A)^{-1}e, e\rangle}\right\}, \tag{6}$$

are stable. At the same time, $\rho\left(A + \frac{J_n}{\langle (I_n - A)^{-1}e, e\rangle}\right) = 1$.

Proof:
It is enough to note that $J_n = ee^T$. Therefore

$$\rho((I_n - A)^{-1}J_n) = \rho((I_n - A)^{-1}ee^T) = \langle (I_n - A)^{-1}e, e\rangle.$$

□

The above statement can be seen as an analogue for non-negative matrices of the well-known Kharitonov theorem, describing an ℓ_∞-neighborhood of a vector of coefficients, which belongs to the set of stable polynomials [12].

2.2 Closest stable matrix

In the previous subsection, we characterized the distance from a *stable* matrix to the boundary of stability. In this section, we consider another group of questions related to the distance from an *unstable* matrix to the nonconvex set of weakly stable matrices $\mathcal{S}_n$. As above, the distance is measured in the max-norm $\|X-A\|_{\max} = \max\limits_{1\le i,j\le n} |X^{(i,j)} - A^{(i,j)}|$.

Let $A \in \mathbb{R}^{n\times n}_+$. Consider the following parametric family of minimization problems:

$$\min_{X\in\mathbb{R}^{n\times n}_+} \{\rho(X) : \ \|X-A\|_{\max} \le \tau\}, \quad \tau \ge 0. \tag{7}$$

For a non-negative matrix A and for a given parameter $\tau > 0$, let $A[\tau]$ be the matrix with the following elements:

$$A^{(i,j)}[\tau] \ = \ \max\{0, A^{(i,j)} - \tau\}, \quad i,j = 1,\ldots,n. \tag{8}$$

Lemma 4 *The optimal solution of problem (7) is the matrix $A[\tau]$ defined in (8).*

Proof:
Indeed, matrix $A[\tau]$ is feasible for problem (7). On the other hand, for any other feasible solution X, we have $X \ge A[\tau]$. Thus, $A[\tau]$ is optimal for (7) in view of monotonicity of spectral radius (3). □

Consider now the following projection problem:

$$\tau_A \ = \ \min_{X\in\mathcal{S}_n} \|X-A\|_{\max}. \tag{9}$$

Lemma 5 *Value τ_A is the unique root of equation $\rho(A[\tau]) = 1$.*

Proof:
Indeed, in view of Lemma 1, the function $\rho(A[\tau])$ is monotonically decreasing. □

Example 1 *Consider the following Sudoku matrix:*

$$A = \left[\begin{array}{ccc|ccc|ccc} 5&3&4&6&7&8&9&1&2\\ 6&7&2&1&9&5&3&4&8\\ 1&9&8&3&4&2&5&6&7\\ \hline 8&5&9&7&6&1&4&2&3\\ 4&2&6&8&5&3&7&9&1\\ 7&1&3&9&2&4&8&5&6\\ \hline 9&6&1&5&3&7&2&8&4\\ 2&8&7&4&1&9&6&3&5\\ 3&4&5&2&8&6&1&7&9 \end{array}\right]$$

In accordance to Lemma 5, we can easily find its ℓ_∞-projection onto the set of stable matrices:

$$X^* = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

with $\tau_A = 8$. □

Before we present an algorithm for solving problem (9) let us recall the notion of *bisection by number* or *bisection by index*. Suppose we have an ordered sequence of numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ and decreasing function $f(x_i)$ on it. If for some a, we need to find the smallest i such that $f(x_i) < a$, then we compute first $f(x_k)$ with $x = [n/2]$ (the integer part of the number $n/2$). If $f(x_k) < a$, then we do the next iteration for the sequence $x_1, \cdots, x_k$, i.e., compute the function at the point with index $[k/2]$. Otherwise we do the next iteration for the sequence $x_{k+1}, \cdots, x_n$. Now we present the algorithm.

Let $A \in \mathbb{R}^{n\times n}_+$. Assume that $\rho(A) > 1$. Our strategy of solving the problem (9) is as follows.

1. Sort all elements of matrix A in an increasing order.
2. Using the criterion of Lemma 2, find by bisection in the element number the value τ_1, which is the largest between all $A^{(i,j)}$ and zero, having $\rho(A[\tau_1]) \geq 1$, and value τ_2, which is the smallest element of A with $\rho(A[\tau_2]) < 1$.
3. Form the matrix H, with elements $H^{(i,j)} = \begin{cases} 1, & \text{if } (i,j) \in \operatorname{supp}(A(\tau_1)), \\ 0, & \text{otherwise} \end{cases}$.
4. Compute the output as $\tau_A = \tau_2 - \frac{1}{\rho((I_n - A[\tau_2])^{-1}H)}$.

(10)

Theorem 2 *Algorithm (10) computes an optimal solution of problem (9).*

Proof:
First of all, let us show that the Algorithm (10) is well-defined. Indeed,

$$A\left(\max_{1\leq i,j\leq n} A^{(i,j)}\right) = 0,$$

and $\rho(A[0]) = \rho(A) > 1$. Thus, we can find two values $\tau_1 < \tau_2$ from the set

$$\{0\} \bigcup \{A^{(i,j)}\}_{i,j=1}^n$$

such that $\rho(A[\tau_1]) \geq 1$, $\rho(A[\tau_2]) < 1$, and $A[\tau]$ is linear for $\tau \in [\tau_1, \tau_2]$. Hence, for τ from this interval we have

$$A[\tau] \;=\; A[\tau_2] + \tfrac{\tau_2 - \tau}{\tau_2 - \tau_1}(A[\tau_1] - A[\tau_2]) \;=\; A[\tau_2] + (\tau_2 - \tau)H.$$

It remains to apply Lemma 3. □

Let us discuss the computational complexity of Algorithm (10). Implementation of Step 1 needs $O(n^2 \log_2 n)$ operations. Step 2 requires $O(n^3 \log_2 n)$ operations. Step 3 needs $O(n^2)$ operations. And only Step 4, at which we have to compute the spectral radius of a non-negative matrix $(I_n - A[\tau_2])^{-1}H$ needs an iterative procedure, which rate of convergence may depend on the particular data. This is the reason why we analyze in Section 3 a computational method for approaching the spectral radius of a square matrix. If this method is used at Step 4 of Algorithm (10), then the whole procedure will have polynomial-time complexity.

3 Computing the largest eigenvalue

Let A be a squared real matrix with spectrum $\Lambda(A)$. One of the most popular procedure for approaching its leading eigenvalue is the Power Method:

$$x_{k+1} \;=\; \tfrac{Ax_k}{\|Ax_k\|}, \quad k \geq 0, \tag{11}$$

where $\|\cdot\|$ is an arbitrary norm for $\mathbb{R}^n$. This method has two important advantages.

- Its iteration is very simple.
- Under some conditions, it has linear rate of convergence.

However, after a close look at this scheme, we can see that these advantages are not very convincing. Indeed, if matrix A is dense, then each iteration of method (11) needs $O(n^2)$ operations. Moreover, the rate of convergence of this method depends on the gap between the magnitudes of the leading eigenvalue and of all others. The smaller is this gap, the slower is the rate of convergence. Hence, for method (11) it is impossible to derive worst-case polynomial-time complexity bounds.[1)]

It this section, we present a scheme which has much better theoretical guarantees. It is based on the interpretation of the leading eigenvalue of matrix A as a root of the polynomial $p(\tau) = \det(\tau I_n - A)$.

We need to introduce the following notion.

Definition 1 *A real polynomial p has a semi-dominant real root τ_* if $p(\tau_*) = 0$ and*

$$\tau_* \;\geq\; \mathrm{Re}\,\lambda, \tag{12}$$

where $\lambda \in \mathbb{C}$ is any other root of this polynomial.

[1)] This drawback is typical for all other standard methods for approximating the eigenvalues of nonsymmetric matrices (see, for example, Section 7 in [8]).

Example 2 Let A be a real symmetric matrix. Then $\lambda_{\max}(A)$ is a a semi-dominant root of the polynomial $p(\tau) = \det(\tau I_n - A)$.

Example 3 By the Perron-Frobenius theorem (see, for instance, [11, chapter 8]), for a non-negative matrix A, its spectral radius is a semi-dominant real root of the polynomial $p(\tau) = \det(\tau I_n - A)$.

Our interest to polynomials with semi-dominant real roots can be explained by the following property.

Lemma 6 *Let a monic polynomial p of degree n have a semi-dominant real root τ_*. Then the function $p(t)$ is a strictly increasing non-negative convex function on the set $[\tau_*, +\infty)$. Moreover, on this half-line all its derivatives are non-negative and for $\tau \geq \tau_*$ we have*

$$p(\tau) \quad \geq \quad (\tau - \tau_*)^n, \tag{13}$$

$$p(\tau) \quad \geq \quad \tfrac{1}{n} p'(\tau)(\tau - \tau_*). \tag{14}$$

Proof:
Denote by $\mathcal{R}(p)$ the set of all real roots of polynomial p, and by $\mathcal{C}(p)$ the set of all its complex roots. Further, for a real root $x \in \mathbb{R}$, define function $\xi_x(\tau) = \tau - x$, and for a complex root $\lambda \in \mathbb{C}$ define function $\psi_\lambda(\tau) = (\tau - \mathrm{Re}\ \lambda)^2 + (\mathrm{Im}\ \lambda)^2$. Then

$$p(\tau) \quad = \quad \left(\prod_{x \in \mathcal{R}(p)} \xi_x(\tau) \right) \cdot \left(\prod_{\lambda \in \mathcal{C}(p)} \psi_\lambda(\tau) \right). \tag{15}$$

In view of Definition 1, the polynomial p is a product of functions, for which all derivatives are non-negative on the set $[\tau_*, +\infty)$ (we treat the value of function as its derivative of degree zero). Hence, the same is true for the polynomial itself.

Further, for $\tau \geq \tau_*$ we have

$$\xi_x(\tau) \quad \stackrel{(12)}{\geq} \quad \tau - \tau_*,\ x \in \mathcal{R}(p), \quad \psi_\lambda(\tau) \quad \stackrel{(12)}{\geq} \quad (\tau - \tau_*)^2,\ \lambda \in \mathcal{C}(p).$$

Hence, (13) follows from representation (15).

In order to prove inequality (14), note that

$$\frac{p'(\tau)(\tau - \tau^*)}{p(\tau)} \quad = \quad \sum_{x \in \mathcal{R}(p)} \frac{\tau - \tau^*}{\tau - x} + 2 \sum_{\lambda \in \mathcal{C}(p)} \frac{(\tau - \mathrm{Re}\ \lambda)(\tau - \tau_*)}{(\tau - \mathrm{Re}\ \lambda)^2 + (\mathrm{Im}\ \lambda)^2}.$$

In view of condition (12), each term in the above sums is smaller than one. Hence, (14) follows. □

Let us show that the Newton Method is especially efficient in finding the maximal roots of increasing convex univariate functions.

Consider a convex univariate function f such that

$$f(\tau_*) \quad = \quad 0, \quad f(\tau) \ > \ 0, \text{ for } \tau > \tau_*. \tag{16}$$

Let us choose $\tau_0 > \tau_*$. Consider the following Newton process:

$$\tau_{k+1} \quad = \quad \tau_k - \frac{f(\tau_k)}{g_k}, \tag{17}$$

where $g_k \in \partial f(\tau_k)$. Thus, we do not assume f to be differentiable for $\tau \geq \tau_*$.

Theorem 3 *Method (17) is well defined. For any $k \geq 0$ we have*

$$f(\tau_{k+1})g_{k+1} \;\leq\; \tfrac{1}{4} f(\tau_k) g_k \,, \tag{18}$$

where τ_k, g_k are defined in (17). Thus, $f(x_k) \leq \left(\frac{1}{2}\right)^k g_0(\tau_0 - \tau_*)$, provided $\tau_0 \geq \tau_*$.

Proof:
Denote $f_k = f(\tau_k)$. Let us assume that $f_k > 0$ for all $k \geq 0$. Since f is convex, $0 = f(\tau_*) \geq f_k + g_k(\tau_* - \tau_k)$. Thus,

$$g_k(\tau_k - \tau_*) \;\geq\; f_k \;>\; 0. \tag{19}$$

This means that $g_k > 0$ and $\tau_{k+1} \in (\tau_*, \tau_k)$. In particular, we conclude that

$$\tau_k - \tau_* \;\leq\; \tau_0 - \tau_*. \tag{20}$$

Further, for any $k \geq 0$ we have:

$$f_k \;\geq\; f_{k+1} + g_{k+1}(\tau_k - \tau_{k+1}) \overset{(17)}{=} f_{k+1} + \tfrac{f_k g_{k+1}}{g_k}.$$

Thus, $1 \geq \frac{f_{k+1}}{f_k} + \frac{g_{k+1}}{g_k} \geq 2\sqrt{\frac{f_{k+1} g_{k+1}}{f_k g_k}}$, and this is (18). Finally, since f is convex, we have

$$g_0 \overset{(19)}{\geq} \sqrt{\tfrac{f_0 g_0}{\tau_0 - \tau_*}} \overset{(18)}{\geq} 2^k \sqrt{\tfrac{f_k g_k}{\tau_0 - \tau_*}} \overset{(19)}{\geq} 2^k \sqrt{\tfrac{f_k^2}{(\tau_0 - \tau_*)(\tau_k - \tau_*)}} \overset{(20)}{\geq} 2^k \tfrac{f_k}{\tau_0 - \tau_*}.$$

□

For a polynomial with semi-dominant root, we can guarantee a linear rate of convergence in the argument.

Theorem 4 *Let a polynomial p has semi-dominant real root. Then for the sequence $\{\tau_k\}_{k \geq 0}$, generated by method (17) we have*

$$\tau_k - \tau_* \;\leq (1 - \tfrac{1}{n})^k (\tau_0 - \tau_*), \quad k \geq 0. \tag{21}$$

Proof:
Indeed, it is enough to combine inequality (14) with the step-size rule of method (17). □

In view of Theorem 4, method (17) can be equipped with a reliable stopping criterion. Indeed, if we need to achieve accuracy $\epsilon > 0$ in the argument, we can use the right-hand side of inequality

$$\tau_k - \tau^* \;\overset{(14)}{\leq}\; \tfrac{n f(\tau_k)}{f'(\tau_k)} \;\leq\; \epsilon \tag{22}$$

as a stopping rule. Since

$$\tfrac{n f(\tau_k)}{f'(\tau_k)} \overset{(17)}{=} n(\tau_k - \tau_{k+1}) \;\leq\; n(\tau_k - \tau^*) \overset{(21)}{\leq} n e^{-k/n} (\tau_0 - \tau_*),$$

this criterion will be satisfied after

$$n \left\lceil \ln \frac{n(\tau_0 - \tau^*)}{\epsilon} \right\rceil$$

iterations at most.

Thus, we have seen that method (17) has linear rate of convergence, which does not depend on the particular properties of function f. Let us show that in non-degenerate situation this method has local quadratic convergence (this never happens with the Power Method (11)).

Theorem 5 *Let convex function f be twice differentiable. Assume that it satisfies the conditions (16) and its second derivative increases for $\tau \geq \tau_*$. Then for any $k \geq 0$ we have*

$$f(\tau_{k+1}) \quad \leq \quad \frac{f''(\tau_k)}{2(f'(\tau_k))^2} \cdot f^2(\tau_k). \tag{23}$$

If the root τ_* is non-degenerate:

$$f'(\tau_*) \quad > \quad 0, \tag{24}$$

then $f(\tau_{k+1}) \leq \frac{f''(\tau_0)}{2(f'(\tau_*))^2} \cdot f^2(\tau_k)$.

Proof:
In view of conditions of the theorem, $f''(\tau) \leq f''(\tau_k)$ for all $\tau \in [\tau_{k+1}, \tau_k]$. Therefore,

$$f(\tau_{k+1}) \quad \leq \quad f(\tau_k) + f'(\tau_k)(\tau_{k+1} - \tau_k) + \tfrac{1}{2} f''(\tau_k)(\tau_{k+1} - \tau_k)^2 \quad \stackrel{(17)}{=} \quad \tfrac{1}{2} f''(\tau_k) \tfrac{f^2(\tau_k)}{(f'(\tau_k))^2}.$$

For proving the last statement, it remains to note that $f''(\tau_k) \leq f''(\tau_0)$ and $f'(\tau_k) \geq f'(\tau_*)$. □

Corollary 2 *If f is a monic polynomial of degree n with real roots, then*

$$f(\tau_{k+1}) \quad \leq \quad \tfrac{n-1}{2n} f(\tau_k), \quad k \geq 0\,, \tag{25}$$

where the sequence τ_k is defined in (17).

Proof:
Indeed, in this case $f(t) = \prod_{i=1}^{n} (t - x_i)$ with $x_i \in \mathbb{R}$, $i = 1, \ldots, n$. Therefore,

$$\begin{aligned} f'(t) \quad &= \quad f(t) \sum_{i=1}^{n} \tfrac{1}{t - x_i}, \\ f''(t) \quad &= \quad f(t) \left[\left(\sum_{i=1}^{n} \tfrac{1}{t - x_i} \right)^2 - \sum_{i=1}^{n} \tfrac{1}{(t - x_i)^2} \right] \quad \leq \quad \left(1 - \tfrac{1}{n}\right) f(t) \left(\sum_{i=1}^{n} \tfrac{1}{t - x_i} \right)^2. \end{aligned}$$

It remains to use inequality (23). □

Note that both Theorems 3 and 5 are applicable to our main object of interest, the polynomial $p(\tau) = \det(\tau I_n - A)$, where A is non-negative $n \times n$-matrix. However, the

direct application of method (17) to this polynomial is expensive since at each iteration we need to compute a determinant of $n \times n$-matrix. This computation needs $O(n^3)$ operations. However, we can significantly reduce this cost by transforming matrix A in a special *Hessenberg form*.

Recall that matrix A has a lower Hessenberg form if

$$A^{(i,j)} \;=\; 0, \quad \forall j \geq i+2, \quad i,j = 1,\ldots,n.$$

Thus, it has the following structure:

$$A_n(a,b,L) = \left(\begin{array}{c|c} a & \frac{L}{b^T} \end{array} \right),$$

where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^{n-1}$, and L is a lower-triangular $(n-1) \times (n-1)$-matrix. Any matrix can be represented in this form by transformation

$$A \;\rightarrow\; U^T A U,$$

where $U \in \mathbb{R}^{n \times n}$ is an orthogonal matrix. This transformation is standard and it can be computed in $O(n^3)$ operations. At the same time, it does not change the polynomial $p(\tau) = \det(\tau I_n - A)$. Hence, let us assume that we already have matrix A in the lower Hessenberg form.

In this case, all matrices $B(\tau) \stackrel{\text{def}}{=} \tau I_n - A$ have also the Hessenberg structure. Let us show that their determinants can be easily computed.

Lemma 7 *Let matrix $B \in \mathbb{R}^{n \times n}$ have a lower Hessenberg form:*

$$B = \left(\begin{array}{c|c|c} \alpha & \beta & 0 \ldots 0 \\ \hline a_1 & a_2 & \begin{array}{c} L \\ b^T \end{array} \end{array} \right),$$

where $a_1, a_2 \in \mathbb{R}^{n-1}$, L is a lower-triangular $(n-2) \times (n-2)$-matrix, and $b \in \mathbb{R}^{n-2}$ Then

$$\det B \;=\; \det A_{n-1}(\alpha a_2 - \beta a_1, b, L). \tag{26}$$

Proof:
For $x \in \mathbb{R}^{n-1}$, consider the function $d(x) = \det A_{n-1}(x, b, L)$. Note that this function is linear in x. Therefore, by applying Laplace formula to the first row of matrix B, we get $\det B = \alpha d(a_2) - \beta d(a_1) = d(\alpha a_2 - \beta a_1)$. □

Thus, using the recursion (26), the value of polynomial $p(\tau) = \det B(\tau)$ can be computed in $\sum\limits_{k=1}^{n-1} 2(n-k) = n(n-1)$ multiplications. Clearly, its derivative can be also computed in $O(n^2)$ operations.

Note that the above procedure has a hidden drawback. Indeed, for a high dimension we can expect the value of polynomial $p(\tau) = \det(\tau I_n - A)$ to be very big. Therefore, the computation of its value and its derivative is computationally unstable. However, note that in the Newton method

$$\tau_{k+1} \;=\; \tau_k - \tfrac{p(\tau_k)}{p'(\tau_k)}, \quad k \geq 0, \tag{27}$$

the step size is given by a *ratio of polynomials*, which can be computed in a stable way.

Indeed, let us assume that our polynomial is represented in a multiplicative form: $p(\tau) = \prod_{k=1}^{m} f_k(\tau)$, where f_k are some functions defined in a neighborhood of $\tau \in \mathbb{R}$. Then

$$\frac{p'(\tau)}{p(\tau)} \quad = \quad \sum_{k=1}^{m} \frac{f_k'(\tau)}{f_k(\tau)}. \tag{28}$$

Thus, any multiplicative representation of polynomial p allows a direct computation of the Newton step in an additive form, which is much more stable. Let us show how this representation can be computed for a Hessenberg matrix.

Our procedure is based on the recursion described in Lemma 7. However, we introduce in it some *scaling functions*, which prevent the growth of intermediate coefficients.

We generate a sequence of Hessenberg matrices H_k of decreasing dimension. Let us choose $\tau_0 \in \mathbb{R}$ and define $H_0(\tau) = \tau I_n - A$. At iteration k, we assume that our matrix has the following structure:

$$H_k(\tau) = \left(\begin{array}{c|c|c} \alpha_k(\tau) & \beta_k(\tau) & 0 \ldots 0 \\ \hline a_k(\tau) & b_k(\tau) & \begin{array}{c} L_k(\tau) \\ c_k^T(\tau) \end{array} \end{array} \right) \in \mathbb{R}^{(n-k)\times(n-k)},$$

where $a_k(\tau), b_k(\tau) \in \mathbb{R}^{n-k-1}$, $L_k(\tau)$ is a lower-triangular $(n-k-2)\times(n-k-2)$-matrix, and $c_k(\tau) \in \mathbb{R}^{n-k-2}$. Let us define an arbitrary function of two variable $f_k(\alpha, \beta)$, which is analytic in the neighborhood of point $(\alpha_k(\tau_0), \beta_k(\tau_0))^T \in \mathbb{R}^2$. Then, for the next iteration we define

$$H_{k+1}(\tau) \quad = \quad A_{n-k-1}\left(\tfrac{\alpha_k(\tau)b_k(\tau) - \beta_k(\tau)a_k(\tau)}{f_k(\alpha_k(\tau), \beta_k(\tau))}, c_k(\tau), L_k(\tau) \right).$$

In this process, the last generated matrix will be $H_{n-2}(\tau) \in \mathbb{R}^{2\times 2}$. At this moment, we define

$$f_{n-2}(\tau) \quad = \quad \alpha_{n-2}(\tau)b_{n-2}(\tau) - \beta_{n-2}(\tau)a_{n-2}(\tau)$$

(in this case, $a_{n-2}(\tau)$ and $b_{n-2}(\tau)$ are real values). Under this convention, by Lemma 7 we have $p(\tau) = \prod_{k=0}^{n-2} f_k(\tau)$.

In the above process, it is reasonable to choose functions f_k, $0 \le k \le n-3$, in the simplest form. For example, they could be linear functions of two variables with coefficients ± 1, ensuring the condition

$$f_k(\tau_0) \quad = \quad |\alpha_k(\tau_0)| + |\beta_k(\tau_0)|.$$

In this case, all matrices $H_k(\tau)$ will be some rational functions of τ, well defined in a neighborhood of τ_0 (since in the process (27) we cannot have $|\alpha_k(\tau_0)| + |\beta_k(\tau_0)| = 0$). Therefore, the derivatives of functions f_k at τ_0 can be easily computed by forward differentiation of the recursion formulas. The total complexity of this process will be of the order $O(n^2)$.

4 Problems with distances measured in ℓ_∞- and ℓ_1-norms

As we know, the ℓ_∞-norm $\|X\|_\infty = \max\limits_{1\le i\le n} \sum_{j=1}^n |X^{(i,j)}|$ is dual to the ℓ_1-norm $\|X\|_1 = \max\limits_{1\le j\le n} \sum_{i=1}^n |X^{(i,j)}|$, in particular, $\|X\|_\infty = \|X^T\|_1$ (to be more precise, the norms ℓ_1 and ℓ_∞ are dual in $\mathbb{R}^n$, and we consider the corresponding operator norms of those dual norms). Therefore, if X is a closest stable matrix for A in the ℓ_∞-norm, then X^T is a closest stable matrix for A^T in the ℓ_1-norm. Thus, the problem in the ℓ_1-norm is equivalent to the same problem in ℓ_∞-norm with replacement of rows by columns. Therefore, we will deal with the ℓ_∞-norm only. Thus, for a non-negative $n \times n$ matrix A, we consider the following two problems.

1) If $\rho(A) < 1$, then we find the closest unstable matrix:

$$\|X - A\|_\infty \;\to\; \min : \quad \rho(X) \,=\, 1,\; X \,\ge\, 0\,. \tag{29}$$

2) If $\rho(A) > 1$, then we find the closest stable matrix. Its mathematical formulation is the same as (29).

We will solve problems 1) and 2) by applying the technique of optimizing the spectral radius over product families of matrices with row uncertainties. This is possible since any ball in the space of matrices equipped with the ℓ_∞-norm forms a product family. For implementing this strategy, we will develop a greedy spectral simplex method, which minimizes the spectral radius over the matrix sets with polyhedral row uncertainties. All necessary definitions will be given later. Now we need to prove several auxiliary results on the spectral radius of non-negative matrices.

4.1 Some inequalities for spectral radius

Lemma 8 *Let $A \in \mathbb{R}_+^{n\times n}$, $u \ge 0$ be a non-zero vector and $\lambda \ge 0$ be a real number. Then $Au \ge \lambda u$ implies that $\rho(A) \ge \lambda$. If for a strictly positive vector v we have $Av \le \lambda v$, then $\rho(A) \le \lambda$.*

Proof:
If $Au \ge \lambda u$, then $A^k u \ge \lambda^k u$ for each k. Therefore, $\|A^k\| \ge \lambda^k$ for all k, and so $\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k} \ge \lambda$. The second statement is a simple consequence of the representation (2). □

Corollary 3 *Let $A \in \mathbb{R}_+^{n\times n}$, $u \ge 0$ be a vector, and $\lambda \ge 0$ be a real number. If $Au > \lambda u$, then $\rho(A) > \lambda$. If $Au < \lambda u$, then $\rho(A) < \lambda$.*

For formulating the next auxiliary result, let us recall that the active set of a vector inequality $x \ge y$ is the set of indices for which this becomes an equality: $\mathcal{I} = \{i \mid x^{(i)} = y^{(i)}\}$.

Lemma 9 *Let $A \in \mathbb{R}_+^{n\times n}$, $\rho(A) = 1$, and $u \in \mathbb{R}^n$ be a strictly positive vector such that $Au \le u$ (or $Au \ge u$). Let $\mathcal{I}$ be the active set of this inequality. Then there is a nonempty subset $\mathcal{I}' \subset \mathcal{I}$ such that the subspace $V = V_{\mathcal{I}'}$ is invariant for A^T and $\rho(A^T|_V) = 1$.*

Proof:
Consider the case $Au \leq u$ (the proof for $Au \geq u$ is literally the same). Denote $\mathcal{B} = \{b_i = A^{(i)} - e_i \mid i \in \mathcal{I}\}$, where $A^{(i)}$ is the ith row of A. If the system of inequalities $\langle b_i, h\rangle < 0$, $b_i \in \mathcal{B}$, has a solution $h \in \mathbb{R}^n$, then for all sufficiently small numbers $t > 0$, we have $A(u + th) < u + th$. Therefore, by Corollary 3, $\rho(A) < 1$, which contradicts to the assumption. Hence, this system does not have a solution, which by Farkas lemma [23] implies that $0 \in \mathrm{Conv}\,\{\mathcal{B}\}$, where Conv denotes the convex hull. So, $\sum\limits_{i\in\mathcal{I}} \tau_i(A^{(i)} - e_i) = 0$ for some numbers $\tau_i \geq 0$, $\sum\limits_i \tau_i = 1$. If v is the vector from $\mathbb{R}^n$ such that $v^{(i)} = \tau_i$ for $i \in \mathcal{I}$, and $v^{(i)} = 0$ otherwise, then $A^T v = v$ and $\mathrm{supp}\,(v) \subset \mathcal{I}$. Then the subspace $V_{\mathcal{I}'}$ with $\mathcal{I}' = \mathrm{supp}\,(v)$ is invariant for the matrix A^T and $\rho(A^T|_{\mathcal{I}'}) \geq 1$ (Lemma 8). On the other hand, $\rho(A^T|_{\mathcal{I}'}) \leq \rho(A^T) = 1$, and therefore $\rho(A^T|_{\mathcal{I}'}) = 1$. □

4.2 Optimizing the spectral radius over product families

Consider one important class of matrices for which the problem of optimizing the spectral radius admits an efficient solution.

Definition 2 *A family $\mathcal{F}$ of non-negative $n\times n$-matrices is called a product family if there exist compact sets $\mathcal{F}^{(i)} \subset \mathbb{R}^n_+$, $i = 1, \ldots, n$, such that $\mathcal{F}$ consists of all possible matrices with i-th row from $\mathcal{F}^{(i)}$, for all $i = 1, \ldots, n$.*

The sets $\mathcal{F}^{(i)}$ are called the *uncertainty sets*. They are some compact sets of non-negative vectors. Respectively, product families are sets of matrices with independent row uncertainties: their rows are independently chosen from the sets $\mathcal{F}^{(i)}$. Topologically, they are indeed products of the uncertainty sets: $\mathcal{F} = \mathcal{F}^{(1)} \times \cdots \times \mathcal{F}^{(n)}$. Such families have been studied in the literature due to applications in spectral graph theory, asynchronous systems, mathematical economics, population dynamics, etc. (see [3, 5, 13, 17, 19, 26] and the references therein).

Product families have many remarkable properties. In particular, their joint and lower spectral radii are always attained at one matrix [3]. Moreover, the problems of minimizing and maximizing the spectral radius of a matrix over some compact set of matrices, being notoriously hard in general, becomes efficiently solvable over product sets. The recent paper [19] develops such methods in the case of *polyhedral* uncertainly sets, when each $\mathcal{F}^{(i)}$ is either a polytope given by vertices or a polyhedron given by a system of linear inequalities. Our crucial observation is

For each $A \in \mathbb{R}^{n\times n}_+$ and for each $\tau > 0$, the set $\mathcal{B}_\tau(A) = \{X \in \mathbb{R}^{n\times n}_+ \mid \|X - A\|_\infty \leq \tau\}$ is a product family with polyhedral row uncertainty sets.

Thus, the positive part of any ℓ_∞-ball $\mathcal{B}_\tau(A)$ is a product family. Therefore, using methods of optimizing the spectral radius over product families, one can solve the problem $\rho(X) \to \min / \max$ over the set $X \in \mathcal{B}_\tau(A)$ and then try to adapt τ by a bisection procedure. The minimal τ such that $\min\limits_{X\in\mathcal{B}_\tau(A)} \rho(A) \leq 1$, is the distance to the closest stable matrix, the minimal τ such that $\max\limits_{X\in\mathcal{B}_\tau(A)} \rho(A) \geq 1$, is the distance to the closest

unstable matrix. For implementing this strategy, we modify some methods from [19] and [22] and apply them to the specific polyhedral uncertainly sets

$$\mathcal{F}^{(i)} = \mathcal{B}_\tau(A^{(i)}) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : \|x - A^{(i)}\|_1 \le \tau\}, \quad i = 1, \ldots, n\,.$$

Our methods for optimizing the spectral radius over product families are based on the following simple fact. Let A be a matrix from product family $\mathcal{F}$, and $v \in \mathbb{R}^n_+$ be its leading eigenvector. We say that A is *minimal in each row* (with respect to v) if $\langle v, A^{(i)}\rangle = \min\limits_{x \in \mathcal{F}^{(i)}} \langle v, x\rangle$ for all $i = 1, \ldots, n$. A similar definition is used for maximality in each row.

Proposition 1 *Suppose A belongs to a product family $\mathcal{F}$ and $v \in \mathbb{R}^n_+$ be its leading eigenvector. Then*

1) if A is minimal in each row with respect to v, then $\rho(A) = \min\limits_{X \in \mathcal{F}} \rho(X)$.

2) if $v > 0$ and A is maximal in each row with respect to v, then $\rho(A) = \max\limits_{X \in \mathcal{F}} \rho(X)$.

Proof:
The statement directly follows from Lemma 8. □

Thus, if a matrix from product family is optimal in each row, then it provides the global optimum for the spectral radius. For strictly positive matrices, the converse is also true:

Corollary 4 *If matrix $A \in \mathcal{F}$ is strictly positive, then it has the minimal spectral radius in $\mathcal{F}$ precisely when A is minimal in each row with respect to its (unique) leading eigenvector. The same is true for maximization.*

Proof:
If A is optimal in each row, then as we showed above, it has the minimal spectral radius. It remains to prove the converse. Assume A has the minimal spectral radius, without loss of generality let $\rho(A) = 1$. Since A is strictly positive, so is its leading eigenvector v. Thus, $Av = v$. Suppose A is not minimal is some row, say, in the first row. In this case the first row can be replaced so that for the new matrix A' the inequality $A'v \le v$ is strict in the first row. Therefore, the set of indices of active rows is $\mathcal{I} = \{2, \ldots, n\}$. On the other hand, $\rho(A')$ cannot be smaller than one, because the matrix A has the smallest spectral radius in $\mathcal{F}$. Thus, $\rho(A') = 1$. Now we apply Lemma 9 to the matrix A' and to the vector v. We conclude that there exists a subset $\mathcal{I}' \subset \mathcal{I}$ such that the subspace $V_{\mathcal{I}'}$ is invariant for the matrix A^T. The latter is impossible, because the matrix A^T does not have zeros in the last $n - 1$ columns. □

Thus, if a matrix A is strictly positive, then the converse to Proposition 1 holds. However, if A has some zero entries, this may not be true. Not every matrix from $\mathcal{F}$ with the minimal (maximal) spectral radius is minimal (respectively, maximal) in each row. Nevertheless, at least one matrix with this property always exists as the following proposition states.

Proposition 2 *In every product family, there exists a matrix, which is minimal (maximal) in each row with respect to one of its leading eigenvectors.*

Proof:
For a given $\varepsilon > 0$ consider ε-shifted uncertainty sets $\mathcal{F}_\varepsilon^{(i)} = \mathcal{F}^{(i)} + \varepsilon e$ and the corresponding product family $\mathcal{F}_\varepsilon = \mathcal{F}_\varepsilon^{(1)} \times \cdots \times \mathcal{F}_\varepsilon^{(n)} = \mathcal{F} + \varepsilon J_n$. Let $A_\varepsilon \in \mathcal{F}_\varepsilon$ be the matrix with the minimal spectral radius. Since matrix A_ε is strictly positive, Corollary 4 implies that A_ε is minimal in each row. To any ε we associate one of such matrices A_ε. By compactness, there is a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$ as $k \to \infty$, the matrices A_{ε_k} converge to a matrix $A \in \mathcal{F}$ and their leading eigenvectors converge to a nonzero vector v. Then by continuity, v is an eigenvector of A, and A is minimal in each row with respect to v.

The proof for maximization is the same. □

Propositions 1 and 2 offer a method for optimizing the spectral radius over the product families by finding the optimal matrices in each row. This strategy was used in [19] for developing two optimization algorithms. One of them is the *spectral simplex method*. It consists in consecutive increasing of the spectral radius by one-row corrections of a matrix. The main idea is the following. We take a matrix A from a product family $\mathcal{F}$ and compute its leading eigenvector v. Then, for each $i = 1, \ldots, n$, we try to maximize the scalar product of v with rows from the uncertainty set $\mathcal{F}^{(i)}$. If for all i, the maximums are attained at the rows of A, then A is maximal in each row and hence has the maximal spectral radius in $\mathcal{F}$. Otherwise, we replace one row of A, say, the ith one, with the row from $\mathcal{F}^{(i)}$ maximizing the scalar product. We obtain a new matrix. We compute its leading eigenvector, optimize the scalar products of rows with this eigenvector, etc.

The advantage of this method is that it is applicable for both maximizing and minimizing the spectral radius. However, its significant shortcoming is that it works efficiently only for strictly positive matrices. If some row from $\mathcal{F}^{(i)}$ has a zero entry, then the algorithm may cycle. Even if it does not cycle, the terminal matrix, i.e., the matrix produced by the algorithm may not provide a solution. The idea of making all matrices positive by slight perturbations may cause instability, which is difficult to control. In high dimensions, even a very small perturbation of coefficients may significantly change the spectral radius (see, for instance, [24] for the corresponding analysis). The modified spectral simplex method which avoids these troubles and is applicable for all non-negative matrices, was developed in [22]. The spectral simplex method demonstrates its exceptional efficiency even for matrices of relatively big size. In this paper, we present another modification, the *greedy spectral simplex method*, speeds up the convergence rate in the case of simply structured uncertainty sets, when the minimization problem $\langle v, x\rangle \to \min,\ x \in \mathcal{F}^{(i)}$, can be easily solved. We are going to show in Section 4.6 that the ℓ_∞-balls $B^{(i)}(\tau)$ possess this property.

4.3 Closest unstable matrix

For an arbitrary non-negative $n \times n$-matrix A with $\rho(A) < 1$ we consider the problem of finding a closest unstable matrix to A in the operator ℓ_∞-norm:

$$\|X - A\|_\infty \ \to\ \min : \quad \rho(X) = 1. \tag{30}$$

Denote the minimal norm in problem (30) by τ_*. It is shown easily that the closest unstable matrix X is non-negative and, moreover, it is elementwise larger than or equal to the matrix A. So, the matrix $X - A$ is non-negative and has the sum of elements in each row at most τ_*. Thus, for an arbitrary matrix $A \geq 0$ with $\rho(A) < 1$, the general problem (30) is equivalent to the following

$$\|X - A\|_\infty \rightarrow \min : \quad X \geq A, \ \rho(X) = 1. \tag{31}$$

For characterizing the optimal solution X, we introduce notation $E_k = e\, e_k^T$ for the matrix with kth column composed by ones and all other elements being zeros.

Theorem 6 *The optimal value τ_* of problem (30) is reciprocal to the largest component of the vector $(I - A)^{-1}e$. Let k be the index of this component. Then the optimal solution of this problem is the matrix*

$$X_* = A + \tau_* E_k. \tag{32}$$

Remark 2 *The main conclusion of Theorem 6 can be formulated as follows: if we want to increase the spectral radius of matrix A as much as possible, having the sum of changes of entries in each row do not exceeding a fixed number $\tau > 0$, then we have to change all entries in one column (add τ to each entry). This "steepest growth" column of A corresponds to the largest component of the vector $(I - A)^{-1}e$.*

Remark 3 *Theorem 6 is extended to the ℓ_1 operator norm just by applying formula (30) for the transposed matrices X and A.*

Proof:
The optimal matrix X_* in problem (31) is also a solution to the maximization problem

$$\rho(X) \rightarrow \max : \quad \|X - A\|_\infty \leq \tau, \ X \geq A \tag{33}$$

for $\tau = \tau_*$. Let us characterize this matrix for arbitrary τ. By Propositions 1 and 2, X is maximal in each row for the following product family:

$$\begin{aligned} \mathcal{B}_\tau^+(A) &= \mathcal{B}_\tau(A) \cap \{X \in \mathbb{R}^{n \times n} \mid X \geq A\} \\ &= \Big\{X \in \mathbb{R}^{n \times n} : \ X \geq A, \ \langle (X - A)e, e_i \rangle \leq \tau, \ i = 1, \ldots, n\Big\}. \end{aligned}$$

Conversely, every matrix X, which is maximal in each row with respect to a strictly positive leading eigenvector and such that $\rho(X) = 1$, is the closest unstable matrix for A.

Now let us show that $v > 0$, which will enable us to apply Proposition 1. Any matrix $X \in \mathcal{B}_\tau^+(A)$ with leading eigenvector v is optimal in the ith row if and only if the scalar product $\langle X^{(i)} - A^{(i)}, v \rangle$ is maximal under the condition $\langle X^{(i)} - A^{(i)}, e \rangle = \tau$. This maximum is equal to $r\,\tau$, where r is the maximal component of vector v. Denote the index of this component by k: $v^{(k)} = r$. Then

$$X^{(i)} - A^{(i)} = \tau\, e_k, \qquad i = 1, \ldots, n.$$

Hence, if X is maximal in each row, then $X = A + \tau\, e\, e_k^T = A + \tau E_k$. Furthermore, since each set $\mathcal{B}_\tau^+(A^{(i)})$ contains a strictly positive point, it follows that

$$v^{(i)} \;=\; (Xv)^{(i)} = \max_{x \in \mathcal{B}_\tau(A^{(i)})} \langle x, v \rangle \;>\; 0.$$

Hence, the vector v is strictly positive, and by Proposition 1, the matrix X has the largest spectral radius on the set $\mathcal{B}_\tau^+(A)$.

Thus, the optimal matrix X_* for the problem (30) has the form (32) for some k and $\|X - A\|_\infty = \tau_*$. It remains to find $k \in \{1, \ldots, n\}$ for which the value of τ is minimal under the constraint $\rho(X) = 1$. Since $\rho\big(A + \tau_*\, E_k\big) = 1$, it follows that τ_* is the smallest positive root of the equation

$$\det\big(A \;-\; I \;+\; \tau\, E_k\big) \;=\; 0\,. \tag{34}$$

Since $\rho(A) < 1$, we have $(I - A)^{-1} = \sum\limits_{j=0}^{\infty} A^j \geq 0$. Multiplying equation (34) by $\det\big(-(I - A)^{-1}\big)$, we obtain

$$\det\big(I \;-\; \tau\,(I - A)^{-1}\, E_k\big) \;=\; 0\,. \tag{35}$$

The matrix $\tau\,(I - A)^{-1}\, E_k$ has only one nonzero column. This is the kth column equal to $\tau(I - A)^{-1}e$. Hence

$$\det\big(I \;-\; \tau\,(I - A)^{-1}\, E_k\big) \quad = \quad 1 \;-\; \tau\,\big[(I - A)^{-1}e\big]^{(k)}\,.$$

Thus, τ_* is the reciprocal to the kth component of the vector $(I - A)^{-1}e$. Hence the minimal τ corresponds to the largest component of this vector. □

Remark 4 *The vector $x = (I - A)^{-1}e$ needed in Theorem 6 can be found by solving the linear system $(I - A)x = e$. It suffices to find an approximate solution, because we actually need only the index of the largest component of x. This can be done by the Power Method. Indeed, since $(I - A)^{-1}e = \sum\limits_{j=0}^{\infty} A^j e$, we see that the vector $(I - A)^{-1}e$ is the limit of the following recursive sequence: $x_0 = e$, $x_{j+1} = A\,x_j + e$, $j = 0, 1, \ldots$. This Power Method converges with the linear rate $O(\rho^N(A))$. Having an approximate value of the limiting vector, we can find k as the index of its largest component and τ_* as a reciprocal to this component. After that, the closest unstable matrix X_* can be approximated by the formula (32).*

4.4 Closest stable matrix

For an arbitrary non-negative $n \times n$-matrix A with $\rho(A) > 1$, consider the problem of finding the closest matrix to A in the operator ℓ_∞-norm, which has the spectral radius equal to one. Thus, we consider the same formulation (30), but for the case $\rho(A) > 1$. This problem can be written as follows

$$\|X \;-\; A\|_\infty \;\to\; \min : \quad X \;\geq\; 0\,,\; \rho(X) \;=\; 1. \tag{36}$$

This case is more difficult than finding the closest unstable matrix because now we have to respect the non-negativity conditions for the matrix X, which were actually redundant in the former case, but now becomes a serious restriction. That is why the optimal solution X is usually found not by a formula but by an iterative procedure. The main idea is to solve the related problem

$$\left\{ \begin{array}{l} \rho(X) \ \to \ \min : \quad \|X \ - \ A\|_\infty \ \leq \ \tau \\ 0 \ \leq \ X \ \leq \ A, \end{array} \right. \tag{37}$$

and then apply a bisection in τ for finding the value of the parameter ensuring $\rho(X) = 1$. In fact, the algorithm works much faster by using a kind of mixed strategy. After several iterations of the bisection we can find τ by an explicit formula (see Section 4.6 for details).

Our main goal now is to solve (37) for a particular τ. For this we develop a *greedy spectral simplex method*, which is a natural extension of the spectral simplex method presented and studied in [19, 22]. Let us start with some notation and auxiliary results.

Matrix $A \geq 0$ is called *irreducible* if it does not have a nontrivial invariant coordinate subspace, i.e., a subspace spanned by some elements e_i of the canonical basis. The irreducibility is actually a combinatorial notion and can be explained in terms of a graph of a matrix A: a digraph G with n vertices $\{1, \ldots, n\}$ such that there is an arc from a vertex i to a vertex j if and only if $A_{ji} > 0$. A matrix A is irreducible if and only if its graph G is strongly connected, i.e., for every pair of vertices i, j, there is a path from i to j.

Reducibility means that there is a proper nonempty subset $\Lambda \subset \Omega$ such that for each $i \in \Lambda$, the support of the ith column of A is contained in Λ.

For every matrix $A \geq 0$, there exists a suitable permutation P of the basis of $\mathbb{R}^n$, after which A gets a block upper triangular form with $r \geq 1$ diagonal blocks A_j of sizes d_j, $j = 1, \ldots, r$, called the *Frobenius factorization*:

$$P^{-1}AP \quad = \quad \left(\begin{array}{cccc} A_1 & * & \ldots & * \\ 0 & A_2 & * & \vdots \\ \vdots & & \ddots & * \\ 0 & \ldots & 0 & A_r \end{array} \right) . \tag{38}$$

For each $j = 1, \ldots, r$, the matrix A_j in the jth diagonal block is irreducible. Any non-negative matrix possesses a unique Frobenius factorization up to a permutation of blocks (see [11, chapter 8]).

The following fact of the Perron-Frobenius theory is well-known (e.g. [11, chapter 8]).

Lemma 10 *An irreducible matrix has a simple leading eigenvalue.*

The converse is not true: a matrix with a simple leading eigenvalue can be reducible.

Let A be $n \times n$ non-negative matrix. Its leading eigenvector v is called *minimal* if there is no other leading eigenvector that possesses a strictly smaller (by inclusion) support. A minimal leading eigenvector can be found by Frobenius factorization (38). For this, we need to take the biggest m such that $\rho(A_m) = \rho(A)$ (i.e., the "lowest" block with the maximal spectral radius), then the minimal eigenvector is the leading eigenvector of the submatrix with blocks $A_1, \ldots, A_m$.

The case of strictly positive leading eigenvector, when v possesses a full support, is characterized by the following statement.

Proposition 3 *If a non-negative $n \times n$ matrix A has a strictly positive minimal leading eigenvector v, then the leading eigenvalue $\lambda_{\max}$ is simple and there exists a permutation P of the basic vectors such that A gets the block upper triangular form*

$$P^{-1}AP \quad = \quad \begin{pmatrix} B & * \\ 0 & C \end{pmatrix}, \tag{39}$$

where B and C are square matrices such that C is irreducible with $\rho(C) = \lambda_{\max}$, and $\rho(B) < \lambda_{\max}$ (the block B may be empty, in which case $P = I_n$ and $C = A$).

Proof:
Without loss of generality it can be assumed that $\lambda_{\max} = 1$. Since $A^k v = v$ for all k, and v is positive, it follows that the sequence $\|A^k\|$, $k \in \mathbb{N}$, is bounded and hence the eigenvalue 1 has only one-element Jordan blocks. If there are at least two of those blocks, then A has at least two leading eigenvectors v_1 and v_2. Denoting $\alpha = \min\left\{\frac{v_1^{(i)}}{v_2^{(i)}} \;\middle|\; v_2^{(i)} > 0\right\}$ we see that $v_1 - \alpha v_2$ is a leading eigenvector, which has a zero component. This contradicts to the minimality of v. Therefore, the leading eigenvalue has a unique one-elements Jordan block, i.e., it is simple. Further, consider the Frobenius factorization of A generated by a suitable permutation matrix P (called also Frobenius normal form, see [11]). In this factorization, matrix $P^{-1}AP$ has an upper triangular block form with irreducible blocks. Since the leading eigenvalue $\lambda_{\max}$ is simple, there exists a unique block with this leading eigenvalue. Since the leading eigenvector of A is strictly positive, it follows that this block takes the last position in the diagonal (i.e. in the lower right corner of the matrix). It remains to denote this block by C and the union of all other blocks by B. □

The basis vectors corresponding to the block C in factorization (39) span an invariant coordinate subspace of matrix A^T, on which this matrix is irreducible with spectral radius equal to $\lambda_{\max}$. Thus, we obtain the following consequence.

Corollary 5 *If a non-negative $n \times n$-matrix A has a minimal leading eigenvector $v > 0$, then there exists a unique nonempty subset $\mathcal{H} \subset \Omega$ such that $V_{\mathcal{H}}$ is an invariant subspace of A^T on which this matrix is irreducible and has the spectral radius equal to $\rho(A)$.*

One can note that this statement is similar to Lemma 9. However, in Lemma 9 the vector v is arbitrary and does not have to be an eigenvector of A. Moreover, in contrast to Lemma 9, Corollary 5 states the existence and uniqueness of the common invariant subspace of A^T on which A is irreducible and has the spectral radius equal to $\rho(A)$.

We call the subset $\mathcal{H} \subset \Omega$ from Corollary 5 the *basic set* of the matrix A and $V_{\mathcal{H}}$ the basic subspace. Thus, matrix with a strictly positive minimal leading eigenvector possesses a unique basic set. By Proposition 3, the permutation P maps the set $\{n - |\mathcal{H}| + 1, \ldots, n\}$ to the set $\mathcal{H}$.

4.5 Greedy spectral simplex method to find the closest stable matrix in ℓ_∞ norm

For every $\tau > 0$, problem (37) can be solved by the greedy spectral simplex method presented in this section. We describe and analyse the algorithm for a more general

problem of minimizing the spectral radius over a product family $\mathcal{F} = \mathcal{F}^{(1)} \times \cdots \times \mathcal{F}^{(n)}$ with arbitrary polyhedral uncertainty sets $\mathcal{F}^{(i)} \subset \mathbb{R}^n_+$:

$$\rho(X) \;\to\; \min : \quad X \in \mathcal{F}. \tag{40}$$

For finding the closest stable matrix, we set $\mathcal{F}^{(i)} = \mathcal{B}^-_\tau(A^{(i)}) = \mathcal{B}_\tau(A^{(i)}) \cap \mathbb{R}^{n\times n}_+$ and obtain problem (37).

The idea of the greedy spectral simplex method naturally follows from Propositions 1 and 2. Let us take arbitrary matrix $X_0 \in \mathcal{F}$ and start the iterative scheme. In the beginning of kth iteration, $k \geq 0$, we have a matrix X_k. Let us find its leading eigenvector v_k and for every $i = 1, \ldots, n$, solve the problem $\langle x, v_k\rangle \to \min\limits_{x\in\mathcal{F}^{(i)}}$. This can be done using the standard linear programming technique. In particular, the solution x is always attained at a vertex of the polyhedron $\mathcal{F}^{(i)}$. Denote this solution (vertex of $\mathcal{F}^{(i)}$) by $X^{(i)}_{k+1}$ and compose the next matrix A_{k+1} by the optimal rows $X^{(i)}_{k+1}$, $i = 1, \ldots, n$. Then compute the leading eigenvector of the new matrix, do the next iteration, etc. The algorithm terminates when the matrix X_k is optimal in each row. In this case, we can set $X_{N+1} = X_N$. By Proposition 1, X_N provides a global minimum to the problem (40).

Applying Corollary 4, we come to the following conclusion:

Corollary 6 *If all the uncertainty sets $\mathcal{F}^{(j)}$ are strictly positive, then the spectral radius $\rho(X_k)$ of the sequence of matrices, arising in the greedy spectral simplex method, decreases in k. In particular, the algorithms never cycles.*

On the other hand, since each row $X^{(i)}_k$ is a vertex of the polyhedron $\mathcal{F}^{(j)}$, the total number of states is finite. Hence the algorithm finds the global minimum in a finite number of iterations. Thus, we have proved the following

Theorem 7 *If all the uncertainty sets $\mathcal{F}^{(j)}$ are strictly positive, then the greedy spectral simplex method finds the optimal solution in finite number of iterations.*

However, if some vectors from $\mathcal{F}^{(j)}$ have zero entries, then the spectral radius $\rho(X_k)$ may not be strictly decreasing in k. In this case, $\rho(X_k)$ may stay unchanged for many iterations and the algorithm may cycle [22]. Moreover, without the positivity assumption, matrices X_k may have multiple leading eigenvalues, which complicates their computation and causes an uncertainty in choosing the leading eigenvector v_k from the corresponding root subspace. This is the reason why the greedy spectral simplex method needs to be modified for avoiding these issues. We present below its modified version, which works efficiently for all non-negative polyhedral uncertainty sets including the case of sparse matrices.

Notation for Algorithm 1. We denote $\Omega = \{1, \ldots, n\}$, $\mathcal{S}$ is the support of the minimal eigenvector, $\mathcal{H}$ is the basic set. For an arbitrary $K \subset \Omega$, $V_K = \operatorname{span}\{e_j \mid j \in K\}$ is the corresponding coordinate subspace, $Y|_K$ is the restriction of a matrix Y to the subspace V_K, i.e., the principal submatrix of Y corresponding to indices in K. Let Y be an $n \times n$ matrix from the product family $\mathcal{F}$, v be its leading eigenvector and $K \subset \operatorname{supp}(v)$. We use the same notation $X^{(i)}$ for the ith row of matrix $X = Y|_K$ (this is a vector of dimension $|K|$) and for the corresponding row $Y^{(i)} \in \mathcal{F}_i$ of the matrix Y (of dimension n). Thus, we identify $Y^{(i)}$ with $X^{(i)} = Y^{(i)}|_K$. Since $K \subset \operatorname{supp}(v)$, we have $\langle v, X^{(i)}\rangle = \langle v, Y^{(i)}\rangle$, and this identification will not cause any confusion.

Algorithm 1: Algorithm 1 for minimusing the spectral radius over a product family

Data: $\mathcal{F}^{(i)} \subset \mathbb{R}^n_+$, $i = 1, \dots, n$, are the polyhedral uncertainty sets;
$\mathcal{F} = \mathcal{F}^{(1)} \times \cdots \times \mathcal{F}^{(n)}$ is the corresponding matrix family;
Each $\mathcal{F}^{(i)}$ is given either by a finite set of vertices or by a system of linear inequalities.
Result: $\bar{X} \in \mathcal{F}$ such that $\rho(\bar{X}) = \min\limits_{X \in \mathcal{F}} \rho(X)$.

begin

Choose arbitrary $X_1 \in \mathcal{F}$.

1 (*) k`th iteration.` For a non-negative $n \times n$-matrix X_k, compute its minimal leading eigenvector v (take any of them, if there are several ones), set $\mathcal{S} = \mathcal{S}_k = \operatorname{supp}(v)$, $X = X_k|_{\mathcal{S}}$, and go to (**);

2 (**) `Main loop.` We have a set $\mathcal{S} \subset \Omega$, a square non-negative matrix X of size $|\mathcal{S}|$, which is the principal submatrix of X_k on the set $\mathcal{S}$, and the minimal leading eigenvector $v > 0$ of X. Denote by $\mathcal{H}$ the basic set of X. For each $i \in \mathcal{S}$ solve

$$\langle x, v \rangle \ \to \ \min : \quad x \ \in \ \mathcal{F}^{(i)}. \tag{41}$$

Denote by $\mathcal{I}$ the set of indices i such that the ith row of matrix X provides the global minimum for this problem: $\mathcal{I} = \{i \in \mathcal{S} : \ \langle X^{(i)}, v \rangle = \min\limits_{x \in \mathcal{F}^{(i)}} \langle x, v \rangle\}$.;

3 **if** $\mathcal{I} = \mathcal{S}$ **then**

$\rho(X) = \min\limits_{Y \in \mathcal{F}} \rho(Y)$, and **STOP**. Algorithm 1 terminates. Go to **Return**;

else

4 Define the next matrix X' as follows:

$$X^{(i)\prime} \quad = \quad \begin{cases} X^{(i)} & , \quad i \in \mathcal{I} \\ \arg\min\limits_{x \in \mathcal{F}^{(i)}} \langle x, v \rangle & , \quad i \notin \mathcal{I} \end{cases} \qquad i = 1, \dots, n. \tag{42}$$

Thus, we leave all optimal rows of X untouched and replace all other rows by solutions of problem (42);

5 **if** $\mathcal{H} \subset \mathcal{I}$ **then**

$\rho(X') = \rho(X)$, the leading eigenvalue of X' is simple and is attained on $V_{\mathcal{H}}$;

6 We compute the leading eigenvector v' of X'. The set $\mathcal{S}$ is not changed;

7 **if** $v' > 0$ **then**

set $X = X', v = v'$. Go to (**);

else

8 set $\mathcal{S} = \operatorname{supp}(v')$, $X = X'|_{\mathcal{S}}$, and $v = v'|_{\mathcal{S}}$. Go to (**).

else

9 we have $\mathcal{H} \not\subset \mathcal{I}$ and $\rho(X') < \rho(X)$. Define the next matrix X_{k+1} as follows:

$$(X_{k+1})^{(i)} \quad = \quad \begin{cases} X^{(i)\prime} & , \quad i \in \mathcal{S} \\ X_k^{(i)} & , \quad i \notin \mathcal{S}. \end{cases} \qquad i = 1, \dots, n. \tag{43}$$

and go to the next $(k+1)$st iteration (*);

10 **return** *Define the $n \times n$ matrix $\bar{X}$ as follows: $\bar{X}^{(i)} = X^{(i)}$ for $i \in \mathcal{S}$ and $\bar{X}^{(i)} = X_k^{(i)}$ for $i \notin \mathcal{S}$. Then $\bar{X}$ is a solution;*

Comments. In each iteration, the algorithm deals with the following sets of indices: Ω is the set of all indices $\{1, \ldots, n\}$, $\mathcal{S}$ is the support of the minimal leading eigenvector, and $\mathcal{H}$ is the basic set. We have $\mathcal{H} \subset \mathcal{S} \subset \Omega$. The relation between these sets is seen from the following picture:

$$X_k = \overbrace{\begin{array}{|c|c|}\hline X & * \\ \hline 0 & * \\ \hline \end{array}}^{\Omega} \quad ; \qquad X = \overbrace{\begin{array}{|c|c|}\hline B & * \\ \hline 0 & C \\ \hline \end{array}}^{\mathcal{H}} \qquad (44)$$

Here $\rho(X_k) = \rho(X)$ and X has a positive minimal leading eigenvector; $\rho(X) = \rho(C)$ and C is irreducible, as in Proposition 3.

In Algorithm 1 we have three main components:

1) *Invariants.* In each iteration we have a matrix $X \in \mathcal{F}$, its minimal leading eigenvector v and a set of indices $\mathcal{S} = \mathrm{supp}\,(v)$;

2) *Progress measure.* After each step of the algorithm, we have either $\rho(X_k) < \rho(X_{k-1})$ or $\rho(X_k) = \rho(X_{k-1})$ and $|\mathcal{S}_k| < |\mathcal{S}_{k-1}|$. The index set $\mathcal{S}$ is always non-increasing unless $\rho(X_k) < \rho(X_{k-1})$. When ρ strictly decreases, we recompute the set $\mathcal{S}$ for the new matrix X_k and start with this set $\mathcal{S}_k = \mathcal{S}$.

Inside one step of the algorithm (in the inner loop), the progress measure is the spectral radius of the perturbed matrix: $X_\varepsilon = X + \varepsilon J_m$, where $m = |\mathcal{S}|$, J_m is the $m \times m$ matrix of ones and $\varepsilon > 0$ is a small number. During one step, the algorithm produces a sequence of matrices with the same $\rho(X)$ and with the same set $\mathcal{S}$, but the value $\rho(X_\varepsilon)$ strictly decreases (see the proof of Theorem 8).

3) *Stopping criterion.* The algorithm stops when the current matrix X_k is optimal in every row, in which case X_k is the matrix with the minimal spectral radius in $\mathcal{F}$.

Theorem 8 below provides the theoretical base of the algorithm.

Theorem 8 *Algorithm 1 is well-defined. It finds the global solution of problem (40) in a finite number of steps.*

The well-definedness means that at each iteration matrix X' has a leading eigenvector, which is unique up to a normalization . We are proving more: X' has a simple leading eigenvalue. The finite-time termination means that the algorithm does not cycle. For proving both properties, we need one auxiliary result.

Proposition 4 *Let a non-negative matrix A have a minimal leading eigenvector $v > 0$ and let $\mathcal{H}$ be the corresponding basic set. Let a non-negative matrix A' and a set $\mathcal{I} \subset \Omega$ be such that*

$$\begin{cases} A^{(i)\prime} = A^{(i)} & , \quad i \in \mathcal{I} \\ \langle A^{(i)\prime}, v\rangle < \langle A^{(i)}, v\rangle & , \quad i \notin \mathcal{I}. \end{cases}$$

Then $\rho(A') \le \rho(A)$, and the equality $\rho(A') = \rho(A)$ holds if and only if $\mathcal{H} \subset \mathcal{I}$. In this case, matrix A' has a block upper triangular form (39) in the same basis with diagonal blocks B' and C' such that $C' = C$ and $\rho(B') < \rho(A)$.

Proof:

Without loss of generality we assume $\rho(A) = 1$. Thus, $Av = v$. Since $A'v \le Av$ and therefore $A'v \le v$ with $v > 0$, it follows that $\rho(A') \le 1$.

If $\mathcal{H} \subset \mathcal{I}$, then ${A^{(i)}}' = A^{(i)}$ for all $i \in \mathcal{H}$. Therefore, the restrictions of the matrices A^T and $(A')^T$ on the subspace $V_{\mathcal{H}}$ coincide. By definition of $\mathcal{H}$, the spectral radius of the matrix A^T on this subspace is equal to one. Hence, so is the spectral radius of $(A')^T$ on this subspace. Thus, the the spectral radius of the restriction of the matrix $(A')^T$ to the subspace $V_{\mathcal{H}}$ is one. Therefore, $\rho(A') \ge 1$, and hence $\rho(A') = 1$.

Assume now that $\rho(A') = 1$ and show that in this case $\mathcal{H} \subset \mathcal{I}$. The set of active inequalities in the system $A'v \le v$ coincides with $\mathcal{I}$. Applying Lemma 9 to the matrix A' and to the vector v, we see that if $\rho(A') = 1$, then there is a subset $\mathcal{I}' \subset \mathcal{I}$ such that the subspace $V = V_{\mathcal{I}'}$ is invariant for the matrix $(A')^T$ and $\rho\big((A')^T|_V\big) = 1$. However, all columns of the matrix $(A')^T$ with indices from $\mathcal{I}$ are the same as in the matrix A^T. Therefore, the space V is invariant for the matrix A^T and $\rho\big(A^T|_V\big) = 1$.

By Corollary 5, we see that $\mathcal{I}'$ contains $\mathcal{H}$ and hence $\mathcal{H} \subset \mathcal{I}$. Therefore, ${A^{(i)}}' = A^{(i)}$ for all $i \in \mathcal{I}$. So, matrix A' has the block upper triangular form (39) in the same basis with $C' = C$.

It remains to prove that $\rho(B') < 1$. Denote by u the part of the vector v supported on the set $\Omega \setminus \mathcal{H}$. Since $\langle {B^{(i)}}', u\rangle \le \langle {A^{(i)}}', v\rangle$ for all i, we see that the set of active inequalities for $Bu \le u$ is a subset of $\mathcal{I}$, which does not intersect $\mathcal{H}$. Applying Lemma 9 again, we see that this subset must contain $\mathcal{H}$. This contradiction completes the proof. □

Proof of Theorem 8:

We need to establish two properties.

1) (well-definiteness) Every matrix X has a unique simple leading eigenvalue.

2) (finite termination) The algorithm does not cycle.

The first statement follows directly from Proposition 3. For proving non-cyclicity, we note that the spectral radii $\rho(X_k)$ strictly decrease in k. Hence, it suffices to show that the algorithm cannot cycle within one iteration. Furthermore, the sets $\mathcal{S}$ form a non-increasing embedded sequence. Therefore, cycling may happen only within one set $\mathcal{S} = \mathcal{S}_k$ on kth iteration. In this case, the greedy spectral simplex method generates a sequence of matrices X on the set $\mathcal{S}$. Denote these matrices by $X_{k,1},, X_{k,2}, \ldots$. Each of these matrices $X_{k,j}$ has a simple leading eigenvalue $\lambda_{\max}$, same for all j.

If this sequence is cycling, then the algorithm for a perturbed family $\mathcal{F}_\varepsilon = \{Y + \varepsilon J_n \mid Y \in \mathcal{F}\}$ is also cycling, whenever $\varepsilon > 0$ is small enough ($J_n \in \mathbb{R}^{n\times n}$ is the matrix of all ones). Indeed, all the rows $X^{(i)}_{k,j}$, $j \in \mathbb{N}$, run over the finite set of vertices of the polytope $\mathcal{F}^{(i)}$. Hence, all $X_{k,j}$, $j \in \mathbb{N}$ run over a finite set of matrices $\operatorname{extr}(\mathcal{F})$. The same is true for the perturbed family $\mathcal{F}_\varepsilon$: the matrices run over the finite set of vertices $\operatorname{extr}\mathcal{F}_\varepsilon$.

Furthermore, the leading eigenvector of $X_{k,j}$ corresponds to the simple eigenvalue $\lambda_{\max}$ and hence it depends continuously on the coefficients of $X_{k,j}$. Since the total set of matrices $X_{k,j}$ is finite, all their leading eigenvectors $v_{k,j}$ are uniformly close to the leading

eigenvectors of the perturbed matrices $X_{k,j,\varepsilon}$, whenever ε is small enough. Therefore, all strict inequalities $\langle X'^{(i)}, v\rangle < \langle X^{(i)}, v\rangle$, for $X = X_{k,j}$, $X' = X_{k,j+1}$, $v = v_{k,j}$, involved in the construction of matrix $X' = X_{k,j+1}$ by formula (43), remain strict after the ε-perturbation. Hence, the perturbed algorithm runs over the same sequence of perturbed matrices $X_{j,\varepsilon}$. If the algorithm cycles, it follows that $X_j = X_{j+m}$ for some j and m, and hence $X_{j,\varepsilon} = X_j + \varepsilon J_n = X_{j+m} + \varepsilon J_n = X_{j,\varepsilon}$. However, the algorithm, as applied to strictly positive matrices, does not cycle (Theorem 7). Hence, the equality $X_{j+m,\varepsilon} = X_{j,\varepsilon}$ is impossible. □

4.6 Implementation details of Algorithm 1

Each step of Algorithm 1 involves one computation of the minimal leading eigenvector of a square matrix X and the solution of minimization problem (41) for each row of X. The size m of this matrix is equal to $|\mathcal{S}|$, where $\mathcal{S}$ is the support of the leading eigenvector of the matrix obtained at the previous step. Let us look at these operations.

`Computing the leading eigenvector of` X is the most expensive operation. It can be done in two steps: Frobenius factorization of X ($O(m^2)$ operations) and computing the leading eigenvalues of the blocks. Note that by the construction of the algorithm, the leading eigenvalue $\lambda_{\max}$ of X is simple and hence $\lambda_{\max}$ is Lipschitz continuous in matrix coefficients. The computation of $\lambda_{\max}$ can be done as suggested in Section 3.

`Solving the problem (41) in each row of` X can be implemented for every polyhedral set $\mathcal{F}^{(i)}$ as a usual linear programming problem, or just by inspection of the finite number of vertices. If $\mathcal{F}^{(i)}$ is ℓ_∞-ball, then it can be done much more simply. We show this below.

Let us look now at the implementation details for the problem of finding the closest non-negative stable matrix (36). Assume that $A \geq 0$ and $\rho(A) > 1$. We set $\tau_0 = \frac{1}{2}\|A\|$ and start the bisection method in τ. For each τ, we solve problem (40) for the uncertainty sets being positive parts of ℓ_∞-balls of radius τ centered at the rows of matrix A. Thus, $\mathcal{F}^{(i)} = \mathcal{B}_\tau^-(A^{(i)}) = \mathcal{B}_\tau(A^{(i)}) \cap \mathbb{R}_+^{n\times n}$.

We apply Algorithm 1 for this problem. Its implementation is basically the same as for the usual polyhedral sets. However, there are some simplifications.

`1. Realization of Algorithm 1 for` ℓ_∞`-balls` $\mathcal{F}^{(i)} = \mathcal{B}_\tau^-(A^{(i)})$. Solving minimization problem (41) at each iteration can be done explicitly. Firstly, we order the entries of the leading eigenvector v with indices from $\mathcal{S}$: $v^{(j_1)} \geq \ldots \geq v^{(j_m)}$, where $\{j_1, \ldots, j_m\} = \mathcal{S}$. Then the problem (41) becomes as follows:

$$\sum_{k=1}^m v^{(j_k)} x^{(j_k)} \;\to\; \min : \quad \sum_{k=1}^m x^{(j_k)} \;\geq\; -\tau + \sum_{k=1}^m A^{(i,j_k)}. \tag{45}$$

Define by $\ell = \ell(\tau)$ the minimal index such that $\sum_{s=1}^{\ell} A^{(i,j_s)} > \tau$. If $\sum_{s=1}^{m} A^{(i,j_s)} \leq \tau$, then we set $\ell = m+1$. The solution to the problem (45) is then

$$x^{(j_k)} \;=\; \begin{cases} 0 & k < \ell\,, \\ -\tau + \sum_{s=1}^{\ell} A^{(i,j_s)} & k = \ell\,, \\ A^{(i,j_k)} & k > \ell\,, \end{cases} \tag{46}$$

and for all $j \notin \mathcal{S}$ we set $x^{(j)} = A^{(i,j)}$. If $\ell = m+1$, then $x = 0$.

Applying bisection, we produce a sequence $\{\tau_i\}_{i\geq 0}$ converging to the optimal point. For each i, we minimize the spectral radius $\rho(X)$ on the ℓ_∞-ball $\mathcal{B}^-_{\tau_i}(A)$ by applying Algorithm 1. Denote by $X_i \in \mathcal{B}^-_{\tau_i}(A)$ the solution of this problem. When the step length of the bisection $|\tau_{i+1} - \tau_i|$ becomes small enough, we can stop it and find the exact solution in one step. The following method can be applied when either the step length of the bisection becomes small or when the ordering of entries of v stays unchanged for several τ_i. Suppose we stop at some τ_j. It can always be assumed that the values $\rho(X_k)$ and $\rho(X_{k+1})$ are on opposite sides of 1, otherwise we do several further iterations of the bisection. So, let $\rho(X_k) > 1$ and $\rho(X_{k+1}) < 1$, the opposite case is considered in the same way.

2. `Finding` $\min\tau$ `for which` $\rho(X) = 1$, $\|X - A\|_\infty = \tau$. We assume that the ordering of entries of the current leading eigenvector v coincides with the ordering for the final v (of the optimal matrix X). Consequently, we try to obtain the exact value of τ_* within one iteration by assuming $\rho(X) = 1$ for the matrix X constructed by the formula (46). We have $X = C - \tau R$, where

$$C^{(i,j_k)} \;=\; \begin{cases} 0 & k < \ell_i \\ \sum_{s=1}^{\ell_i} A^{(i,j_s)} & k = \ell_i \\ A^{(i,j_k)} & k > \ell_i \end{cases} \quad ; \qquad i,\, j_k \in \mathcal{S},\; k = 1, \ldots, m\,. \tag{47}$$

For $j \notin \mathcal{S}, i \in \mathcal{S}$, we set $C^{(i,j)} = A^{(i,j)}$; for $i \notin \mathcal{S}$, we set $C^{(i,j)} = X_k^{(i,j)}$ (let us recall that we assume $\rho(X_k) > 1$). Here ℓ_i is the smallest index such that $\sum_{s=1}^{\ell_i} A^{(i,j_s)} > \tau$ (if $\sum_{s=1}^{m} A^{(i,j_s)} \leq \tau$, then we set $\ell_i = m + 1$, and $C^{(i,j_k)} = 0$ for all $k = 1, \ldots, m$), R is a Boolean matrix, which has in ith row ($i \in \mathcal{S}$) all zeros except a single 1 at position ℓ_i (provided that $\ell_i \leq m$), and all zeros otherwise. If $i \notin \mathcal{S}$, then $R^{(i)} = 0$.

Denote $\tau_1 = \min\limits_{i\in\mathcal{S}, \ell_i \leq m} \sum_{s=1}^{\ell_i} A^{(i,j_s)}$. By construction, we have $C - \tau_1 R \geq 0$ and $\tau_1 > \tau$. Hence $\rho(C - \tau_1 R) < 1$. Since $1 = \rho\big(C - \tau R\big) = \rho\left(C - \tau_1 R + (\tau_1 - \tau)R\right)$, it follows that $\det\left(I - (C - \tau_1 R) - (\tau_1 - \tau)R\right) = 0$, and consequently

$$\det\left(\tfrac{1}{\tau_1 - \tau} I - \left[\, I - (C - \tau_1 R)\,\right]^{-1} R\right) \;=\; 0. \tag{48}$$

Note that the value $\rho(C - \tau R)$ decreases in τ and hence there is a unique $\tau > 0$ such that $\rho(C - \tau R) = 1$. Therefore, equation (48) has a unique solution τ. Note also that $\left[\, I - (C - \tau_1 R)\,\right]^{-1} \;=\; \sum_{k=0}^{\infty} (C - \tau_1 R)^k \;\geq\; 0$. Hence, equation (48) means that the number $\lambda = \frac{1}{\tau_1 - \tau}$ is the leading eigenvalue of the non-negative matrix $\left[\, I - (C - \tau_1 R)\,\right]^{-1} R$. Hence, λ can be found numerically by the method described in Section 3. Then we set $\tau_* = \tau = \tau_1 - \frac{1}{\lambda}$, and the optimal value τ_* is found.

Thus, we can shortly describe the step 2:
First we construct the Boolean matrix R by the current τ_k such that $\rho(X_k) > 1$ and $\rho(X_{k+1}) < 1$. Then we solve equation (48), find τ and check that this τ produces the same Boolean matrix R (all ℓ_i stay the same). If this is the case, then $\tau_* = \tau$ is the optimal value.

5 Numerical examples and the complexity issue in ℓ_∞ and ℓ_1 norms

Since the cases of ℓ_∞ and ℓ_1 norms are similar, we focus on the ℓ_∞ norm.

The closest unstable matrix is found by the explicit formula (4.3), where τ_* is the reciprocal to the largest component of the vector $(I - A)^{-1}e$ and k is the index of this component. Hence, finding the closest unstable matrix to a matrix A is fully reduced to solving the linear system $(I - A)x = e$. In particular, the complexity of this problem does not exceed the complexity of a linear system solver. As we know, the solution can be efficiently found in dimensions of several thousands.

For the *closest stable matrix*, the situation is more difficult. We have no theoretical results estimating the number of iterations of Algorithm 1. Complexity of each iteration can easily be estimated because this is actually the complexity of computing the Perron eigenvector for a non-negative matrix, all other operations are much cheaper. This can be realised in a standard laptop for dimensions of several thousands. So, the total complexity estimate is reduced to the number of iterations k of Algorithm 1 times the number of steps of the bisection. The latter is known to be logarithmic. As for the number of iterations k, we can estimate it only empirically, by numerical experiments. They show a surprising efficiency of Algorithm 1. Table 1 demonstrates the results for randomly generated matrices of dimensions from 50 to 1000. We consider strictly positive randomly generated matrices. The first line is the dimension, the second is the number of iterations, i.e., the total number of computations of the leading eigenvector, the third line is the computer time in a standard laptop. For each n, we made five experiments and show the average results.

n	50	100	250	500	750	1000
#	7.6	8.2	10.4	10.8	18	14.2
t	0.15s	0.75s	9.08s	44.02s	215.03s	302.43s

Table 1: Average number of iterations and computing time for finding the closest stable matrix in ℓ_∞ norm.
Positive matrices.

As we see, for positive matrices, the closest stable matrix is found in dimension 100 for less than one second, in dimension 1000 is for about 302 seconds.

Table 2 show the numerical results for sparse matrices. We see that the computation time is bigger, but the method still works very fast.

n	50	100	250	500	750	1000
#	4.4	8.2	9	11.6	11.6	12.8
t	0.15s	3.31s	20.26s	140.9s	290.06s	717.25s

Table 2: Average number of iterations and computing time for finding the closest stable matrix in ℓ_∞ norm.
Sparse matrices.

Example 4 . *Computing the closest stable matrix for a positive matrix A.* We consider an integer 10×10 matrix A with random independent components from the set $\{1, \ldots, 9\}$:

$$A = \begin{pmatrix} 3 & 3 & 3 & 6 & 6 & 4 & 1 & 3 & 5 & 4 \\ 5 & 6 & 9 & 8 & 5 & 7 & 6 & 4 & 7 & 9 \\ 8 & 9 & 2 & 1 & 2 & 1 & 2 & 3 & 7 & 6 \\ 1 & 5 & 2 & 3 & 7 & 2 & 8 & 2 & 8 & 9 \\ 6 & 8 & 9 & 7 & 3 & 5 & 7 & 1 & 8 & 2 \\ 9 & 3 & 5 & 7 & 8 & 5 & 8 & 7 & 3 & 1 \\ 4 & 4 & 8 & 3 & 2 & 4 & 4 & 9 & 2 & 4 \\ 8 & 9 & 6 & 5 & 6 & 2 & 9 & 5 & 1 & 3 \\ 4 & 5 & 4 & 6 & 7 & 1 & 9 & 4 & 1 & 6 \\ 5 & 8 & 8 & 9 & 1 & 7 & 7 & 2 & 2 & 8 \end{pmatrix}$$

We have $\rho(A) = 50.458$, so A is highly unstable. The algorithm computes the closest stable non-negative matrix:

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 9 & 0 & 0 & 0 & 6 & 2 & 7 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 6 & 0 & 9 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 9 & 0 & 5 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 6 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 5 & 0 & 8 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \end{pmatrix}$$

The computation takes 0.13 sec.

Example 5 . *Computing the closest stable matrix for a sparse matrix A.* We consider a sparse integer 10×10 matrix A with at most four positive components (randomly chosen) in each row. The positive components are from the set $\{1, \ldots, 9\}$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 2 \\ 0 & 0 & 7 & 5 & 2 & 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 & 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & 7 & 0 & 8 & 0 & 0 & 0 \\ 0 & 6 & 0 & 7 & 4 & 0 & 0 & 0 & 0 & 5 \\ 8 & 5 & 1 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 1 & 0 & 0 & 9 & 0 & 0 & 0 & 5 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 & 5 & 3 & 0 & 9 \\ 1 & 0 & 0 & 1 & 0 & 9 & 0 & 0 & 0 & 6 \end{pmatrix}$$

We have $\rho(A) = 17.59411$, so A is unstable. The algorithm computes the closest stable non-negative matrix:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 5 \\ 8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 9 \\ 1 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The computation takes 0.32 sec.

6 Conclusion

We have presented methods of numerical solution for finding the closest stable or closest unstable non-negative matrix to a given matrix A. Three possible cases of measuring distances are considered: the matrix max-norm (the maximal absolute value of entries), the ℓ_∞ operator norm (the maximal sum of elements of rows), and the ℓ_1 operator norm (the maximal sum of elements of columns). We show that in all those cases the absolute minimum can be found efficiently. The closest unstable matrix is computed by explicit formulas; the closest stable matrix can be found by an iterative relaxation scheme that makes use of recent "spectral simplex method".

From the practical point of view, we arrived at a curious conclusion: to increase the spectral radius so that the sum of entries in each row of the matrix increases by at most a, one needs to change by a all elements of one column . That "most sensitive" column corresponds to the maximal component of the vector $(I - A)^{-1}e$. In the Leontief input-output model [15], this principle means that the economy suffers the worst way because of an appreciation in one sector. Moreover, that sector can be easily identified. In the matrix models of population dynamics (see, for instance, [17]), the same principle means that if an ecological system (say, a forest) is dying, then to improve the situation, one needs to support only one type of plants and to not touch the others.

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