A Comparison of Standard Multi-Unit Auctions with Synergies *

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Abstract

In an example with two objects and four bidders, some of which have superadditive values, we characterize the equilibria of a simultaneous ascending auction and compare the revenue and efficiency generated with ones generated by the sequential, the one-shot simultaneous, and the Vickrey-Clarke-Groves auctions.

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1 Introduction

The simultaneous ascending auction was first used in 1994 by the Federal Communication Commission to sell licences to use the spectrum in the United States (see e.g. McAfee and McMillan (1996) and Cramton (1997) for a description and evaluation of the actual FCC auctions). The auction was designed to address two main concerns. First, the ascending bids would provide an extensive process of information revelation that would enhance an efficient distribution of the licences. Second, the simultaneous sale of large sets of related licences would allow bidders to exploit possible complementarities between licences.

In this paper, we study a very close variant of the simultaneous ascending auction within the framework of Krishna and Rosenthal (1996) and therefore emphasize the presence of complementarities in at least some of the bidders' preferences. Within a simple example with two objects and four bidders (two local and two global bidders), we characterize the equilibria of the variant of the ascending auction and compare the outcome in terms of revenue and efficiency with other standard auctions studied in the literature, such as the sequential the one-shot simultaneous, and the Vickrey-Clarke-Groves auctions

2 A Simultaneous Ascending Auction

2.1 Framework

We use a special case of the framework of Krishna and Rosenthal (1996). Two objects are put for sale to a set of four participants of two types: local bidders who are interested in one of the two objects and global bidders who are interested in both objects. For each of the two objects there is one local bidder with private value for the given object drawn from $[0, 1]$ according to a uniform distribution. The remaining two global bidders have a private value for both objects individually drawn from $[0, 1]$ also according to a uniform distribution; their value for both objects together is greater or equal to the sum of values of the individual objects. More specifically, if x is the value drawn for the individual objects, then the value for both objects together is $2x + \alpha$, where $\alpha \geq 0$ is publicly known and coincides across global bidders.

2.2 Auction rules

The auction mechanism we study is a simultaneously ascending, second-price auction for two objects. This constitutes a slight variant of the simultaneous ascending auction in that prices are exogenously and simultaneously risen by an auctioneer, and could therefore be viewed as a Japanese auction for multiple objects.

Prices start from zero for all objects and are simultaneously and continuously increased until only one agent is left on a given auction in which case prices on that auction stop and continue on the remaining auction. Once an agent has dropped from a given auction the exit is irrevocable. The last agent receives the object at the price at which the given auction stopped. The number and the identity of agents active on any given auction is publicly known at any time.

2.3 Equilibrium Strategies

Given that each local bidder is willing to buy at most one unit, he has a (weakly) dominant strategy to remain active until the price reaches his own valuation.

The global bidders' strategies are equally simple when they are active on only one of the two objects. If a global bidder already dropped out of say auction 1, then he will remain active on auction 2 until the price of object 2 reaches his valuation x. If, on the other hand, he bought object 1, then he will remain active on auction 2 until the price of object 2 reaches the value $x + \alpha$. These are (weakly) dominant continuation strategies.

However, these as well as the more complex exiting times for the global bidders who are still active on both auctions can be obtained more generally as the solutions to the equations:

$$\pi_0(t) = \pi_1(t), \tag{1}$$

where $\pi_0(t)$ denotes the expected payoff to a bidder of exiting from a given auction at time t, without buying the object, and (if applicable) continuing optimally on the other auction, while $\pi_1(t)$ denotes the expected payoff to a bidder of exiting from the same auction at time t, buying the object, and (again if applicable) continuing optimally on the other auction. Solutions to equation (1) are times at which the expected payoff of continuing on the auction for the given object is equal to the expected payoff of dropping out. For the local bidders and the continuation strategies of the global bidders mentioned above, equation (1) reduces to $x - t = 0$ or $x + \alpha - t = 0$ where x is the drawn value.

Consider a global bidder who is active on both auctions at time t and fix the auction for say object 1. Then $\pi_0(t), \pi_1(t)$ can be computed as:

$$\pi_0(t) = \int_t^x (x - p)\, f_0(p, H_t)\, dp,$$

$$\pi_1(t) = \int_t^{x+\alpha} (2x + \alpha - t - p)\, f_1(p, H_t)\, dp + \int_{x+\alpha}^{\infty} (x - t)\, f_1(p, H_t)\, dp,$$

where $f_0(\cdot, H_t), f_1(\cdot, H_t)$ are the relevant density functions of the price of object 2 conditional on all information H_t available at time t. Since each object is worth x individually to the global bidder, he will remain active on both objects until the price reaches at least x. But because the optimal exiting time will never be before x, whenever he does drop out of auction 1 without buying he will also drop out of auction 2. Therefore (1) reduces to $\pi_1(t) = 0$, and the optimal time to exit the auction for object 1 is characterized by the smallest time $t \in [0, 1 + \alpha]$ that solves:

$$\int_t^{x+\alpha} (2x + \alpha - t - p)\, f_1(p, H_t)\, dp + \int_{x+\alpha}^{\infty} (x - t)\, f_1(p, H_t)\, dp = 0, \quad (2)$$

When bidders adopt optimal (continuation) strategies when bidding on one object, the solutions to (2) reduce to $x + \alpha/2$ if H_t is such that all local bidders dropped out before t, and otherwise to:

$$t^* = \begin{cases} \frac{1}{3}\left(\alpha + 2x + 1 - \sqrt{(\alpha - x + 1)^2 - 3\alpha^2}\right), & x \in [0, 1 - \alpha] \\ \frac{1}{3}\left(2\alpha + 4x - 1\right), & x \in]1 - \alpha, 1] \end{cases} .$$

In summary we have:

Proposition 1 *The exiting times for both local and global bidders conditional on information until time t, $\{t^*(H_t)\}$, which are obtained as the smallest solutions to the equations (1) constitute the (essentially unique) Perfect Bayesian Equilibrium of the simultaneously ascending auction game.*

Uniqueness of the equilibria follows from the fact that, if bidders use their dominant (continuation) strategies, equations (1) are necessary and sufficient for optimality of the exiting times and moreover always have unique solutions with probability one.

3 Comparison with Alternative Auctions

We compare the simultaneously ascending auction to the sequential, the one-shot simultaneous, and the Vickrey-Clarke-Groves auctions studied respectively in Branco (1997), Krishna and Rosenthal (1996), and Krishna and Perry (1997). As is shown in Krishna and Perry (1997), the Vickrey-Clarke-Groves (VCG) auction, at least in the present setting, is ex post efficient and moreover is the auction that achieves the highest revenue among all ex post efficient auctions.

Therefore, we use it as a benchmark and compute the expected efficiency in the sense of the value generated and the expected revenue of the ascending, sequential, and simultaneous auctions as a fraction of the expected efficiency and revenue generated by the VCG auction. This gives rise to the Figures 1 and 2, which depict the relative expected efficiency and revenues respectively of the three auctions as the synergy parameter α varies from 0 to 1.

Figure 1: Efficiency of ascending —, sequential $\cdots$, and simultaneous - - auctions relative to VCG.

None of the three auctions are ex post efficient, except for the case $\alpha = 0$, where they all coincide with VCG. They can be clearly ranked in terms of efficiency, with the ascending auction the most efficient, followed by the sequential and the simultaneous. This is due to the information revealed in the course of the auctions.More strikingly, one sees that the ascending auction looses at most 0.1% over the value generated by VCG, while it gains up to 3.5% more revenue than VCG, i.e., the slight loss in efficiency is more than compensated by higher revenues. The simultaneous auction raises an even slightly higher revenue relative to the ascending auction, which is due to the more aggressive bidding on the part of the global bidders in order to exploit their synergies; to some extent it is the same aggressive bidding that is responsible for the larger loss in efficiency. Finally, the lower revenues generated by the sequential auction are due to the lack of simultaneous bidding, which is present in the other auctions.

Figure 2: Revenue of ascending —, sequential $\cdots$, and simultaneous - - auctions relative to VCG.

References

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