Finitely Additive Beliefs and Universal Type Spaces

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Abstract

In this paper we examine the existence of a universal (to be precise: terminal) type space when beliefs are described by finitely additive probability measures. We find that in the category of all type spaces that satisfy certain measurability conditions (κ-measurability, for some fixed regular cardinal κ), there is a universal type space (i.e. a terminal object, that is a type space to which every type space can be mapped in a unique beliefs-preserving way (the morphisms of our category, the so-called type morphisms)), while, by an probabilistic adaption of the elegant sober-drunk example of Heifetz and Samet (1998a), we show that if all subsets of the spaces are required to be measurable there is no universal type space.

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1 Introduction

The seminal result of Mertens and Zamir (1985), namely the existence of a universal type space in the sense of Haransyi (1967/68) under the assumption that the underlying space of states of nature is compact and Hausdorff, has been generalized in various ways (mainly by relaxing the topological assumptions made there) by Brandenburger and Dekel (1993), Heifetz (1993), Mertens, Sorin and Zamir (1994), and finally to the general measure-theoretic case by Heifetz and Samet (1998b)[1]. However, it has always been assumed that the players' beliefs are σ-additive. This seems to be a rather strong assumption on the epistemic attitudes of the players.

Savage's Postulates (1954, 1972) imply subjective probabilities that are finitely but not countably additive. Given the importance of Savage's theory within decision theory (i.e. "one-player game theory"), it is natural to ask - and desirable to know - what happens if we describe the beliefs of players accordingly in an interactive context (games) by finitely additive probability measures? Does there still exist a universal type space? And if so, what does it look like? Is it, for example, the space of consistent hierarchies?

Still, we are dealing with (now finitely additive) measures, so we have to define a field of events. The question arises as to which measurability condition is the right one. Should the field of events be just a field, a σ-field or should we assume that all subsets of the space of states of the world are events, i.e. that the field is simply the power set? As this question does not seem to have a clear-cut answer, we analyze the existence/nonexistence of a universal type space for finitely additive beliefs for a whole class of different measurability conditions that include the three above-mentioned cases.

We introduce the κ-fields, where κ is an (often regular) infinite cardinal number, as fields that are closed under the intersection of every set of events (i.e. subset of the field) that has cardinality strictly less than κ. It follows that $\aleph_0$-fields are fields in the usual sense and $\aleph_1$-fields are σ-fields in the usual sense. Then, we define ∞-fields as fields that are closed under the intersection of every set of events (of cardinality whatsoever).

We define κ-type spaces as type spaces where the set of measurable events in the set of states of the world, as well as the set of measurable events in the set of states of nature, is a κ-field, and ∞-type spaces as type spaces where the set of measurable events in the set of states of nature is the full power set and the set of measurable events in the set of states of the world is a ∞-field. Also, we define $*$-type spaces as type spaces where the set of measurable events in the set of states of the world as well as the set of measurable events in the set of states of nature is the full power set. Furthermore, we define

[1]Relevant are also Vassilakis (1992), Battigalli-Siniscalchi (1999) and the preference-based approach of Epstein and Wang (1996).

type morphisms, i.e. structure preserving maps from one κ-type space (resp. ∞-type space or $*$-type space) to another (not necessarily different) one.

Given a nonempty set of players I, a nonempty set of states of nature S, and a κ-field Σ_S on S, we define, similar to Heifetz and Samet (1998b), a kind of modal language, the formulas of which we call κ-expressions. But if κ is uncountable, contrary to Heifetz and Samet (1998b), we allow also for formulas of infinite length (but strictly less than κ). Then, we collect the κ-descriptions (by means of κ-expressions) of all states of the world in all κ-type spaces on S for player set I. In this way, we construct in Section 3 a universal κ-type space on S for player set I to which every κ-type space on S for player set I can be mapped by a unique type morphism (Theorem 1). However, since we use all the κ-type spaces (on the same space of states of nature and for the same player set) to construct the universal κ-type space, this construction tells us little about the structure of the universal κ-type space. (As for example, in the Mertens-Zamir (1985) case, where we know in addition that the universal type space is the space of consistent hierarchies.)

As Heifetz (2001) has shown, there are consistent hierarchies of finitely additive (in fact even σ-additive) beliefs up to - but excluding - level ω (i.e. the first infinite level), that have at least two different finitely additive extensions to level ω. Does a similar phenomenon holds also on the higher transfinite levels of consistent hierarchies? Or, put differently in terms of expressions rather than hierarchies, is there, on the contrary, a regular cardinal $\widehat{\kappa}$ such that for all (regular) cardinals $\kappa > \widehat{\kappa}$, the $\widehat{\kappa}$-description of a state in a κ-type space determines already the κ-description? If this were the case, it would be unnecessary to consider κ-type spaces for $\kappa > \widehat{\kappa}$ and we could restrict ourselves to κ-type spaces for $\kappa \leq \widehat{\kappa}$. We show in Theorem 2, by using a probabilistic adaptation of the "sober-drunk"-example of Heifetz and Samet (1998a), that - with at least two players and two states of nature - this is not the case. Hence, it makes sense to consider κ-type spaces for every (regular) infinite cardinal κ.

Also, this example implies that - again, with at least two players and two states of nature - there is no (weak) universal ∞-type space and no (weak) universal $*$-type space (Theorem 3 and Corollary 1).

In Section 5, given a regular cardinal κ, a nonempty set of players I, a nonempty set of states of nature S, and a κ-field Σ_S on S, we construct the space of consistent hierarchies up to (but excluding) level κ and show that this space constitutes a product κ-type space (Theorem 4). On top of that, this space turns out to be the universal κ-type space (Theorem 5). Therefore we provide a characterization of the universal κ-type space that we were missing in Section 3, which is independent of the property of being universal.

In Section 6, we show that the universal κ-type space is beliefs complete (i.e. for each player, every finitely additive probability measure on the product of the space of states of nature and the other players' component spaces is the marginal of a type of that player). In addition, in the case $\kappa = \aleph_0$, the component space of each player is - up to isomorphism of measurable spaces - the space of finitely additive probability measures on the product of the space of states of nature and the other players' component spaces.

2 Preliminaries

First, we will define κ-measurable spaces, where κ-denotes here an (often regular) infinite cardinal number. For an introduction to ordinal and cardinal numbers, see Devlin (1993) or any other textbook on set theory. Then, we will develop parts of the theory of κ-measurable spaces needed in the sequel, collect some known facts about finitely additive (probability) measures and define the main objects of our study in this paper, the κ-, ∞- and $*$-type spaces.

An infinite cardinal number κ is called *regular*, if it is not the supremum of a set of less than κ-many ordinal numbers which are all strictly smaller than κ. For example, $\aleph_0$ and all the $\aleph_{\alpha+1}$ are regular, while $\aleph_\omega$ is *singular* (i.e. infinite and not regular) ($\aleph_\omega = \sup\{\aleph_n \mid n < \omega\}$), where ω denotes here the first infinite ordinal number. For a set M, denote by $|M|$ the cardinality of M.

Unless otherwise stated, $\alpha, \beta, \gamma, \zeta, \eta, \xi$ denote ordinal numbers, δ delta-measures, θ functions from the set of states of the world to the set of states of nature, κ cardinal numbers, λ limit ordinal numbers, μ and ν measures, π projections, φ, χ, ψ expressions, and ω, apart form above, sets of expressions.

Definition 1 Let κ be an infinite cardinal number and M a nonempty set.

A *κ-field on M* is a set $\Sigma \subseteq$ Pow(M) such that:

1. $M \in \Sigma$,
2. $A \in \Sigma$ implies $M \setminus A \in \Sigma$,
3. $\mathcal{E} \subseteq \Sigma$ and $|\mathcal{E}| < \kappa$ imply $\bigcap \mathcal{E} := \bigcap_{E \in \mathcal{E}} E \in \Sigma$.

It follows that $\mathcal{E} \subseteq \Sigma$ and $|\mathcal{E}| < \kappa$ imply $\bigcup \mathcal{E} := \bigcup_{E \in \mathcal{E}} E \in \Sigma$.

Remark 1 *According to our definition, a set of subsets of a nonempty set is a $\aleph_0$-field iff it is a field (in the usual sense) and a $\aleph_1$-field iff it is a σ-field (in the usual sense). If $\kappa' < \kappa$, then every κ-field is a κ'-field.*

Definition 2 Let M be a nonempty set.

A *∞-field*[2] *on* M is a set $\Sigma \subseteq \mathrm{Pow(M)}$ such that:

1. $M \in \Sigma$,
2. $A \in \Sigma$ implies $M \setminus A \in \Sigma$,
3. $\mathcal{E} \subseteq \Sigma$ implies $\bigcap \mathcal{E} \in \Sigma$.

It follows that $\mathcal{E} \subseteq \Sigma$ implies $\bigcup \mathcal{E} \in \Sigma$.

Definition 3 A *κ-measurable space* is a pair (M, Σ), where M is a nonempty set and Σ is a κ-field on M.

Remark 2 *Let κ be a singular cardinal number and (M, Σ) be a κ-measurable space. Then Σ is already a κ^+-field, where κ^+ denotes the successor cardinal of κ.*

Proof Let $\mathcal{E} \subseteq \Sigma$ such that $|\mathcal{E}| \leq \kappa$. So, $\mathcal{E}$ has the form $\{E_\alpha \mid \alpha < \kappa\}$. Let $\widehat{\kappa} < \kappa$ be the cofinality of κ. Then there is a function $f : \widehat{\kappa} \to \kappa$, such that $\bigcup_{\beta < \widehat{\kappa}} f(\beta) = \kappa$. Note that $|f(\beta)| < \kappa$, for $\beta < \widehat{\kappa}$. It follows that $\bigcap_{\alpha<\kappa} E_\alpha = \bigcap_{\beta<\widehat{\kappa}} \left(\bigcap_{\alpha<f(\beta)} E_\alpha \right) \in \Sigma$. Since Σ is a field, it follows that it is a κ^+-field. ■

Since κ^+ is always regular, the above remark shows that it is redundant to consider κ-fields (κ-measurable spaces, respectively) if κ is a singular cardinal.

Definition 4 A *∞-measurable space* is a pair (M, Σ), where M is a nonempty set and Σ is a ∞-field on M.

Definition 5 A *$*$-measurable space* is a pair $(M, \mathrm{Pow}(M))$, where M is a non-empty set and $\mathrm{Pow}(M)$ is the power set (i.e. the set of all subsets) of M.

[2] What we call here "∞-field", is called "universal field" in Aumann (1999).

Note that every $*$-measurable space is a ∞-measurable space and every ∞-measurable space is a κ-measurable space for every infinite ordinal κ. A ∞-measurable space (M, Σ) is a $*$-measurable space iff for all $m \neq m' \in M$ there is a $E \in \Sigma$ such that $m \in E$ and $m' \notin E$.

Example 1 Let $M = \{0, 1\}, \Sigma = \{\emptyset, M\}$. (M, Σ) is ∞-measurable, but not $*$-measurable.

Remark 3 *If M is a nonempty set and $\mathcal{E} \subseteq \mathrm{Pow(M)}$, then the intersection of all κ-fields that contain $\mathcal{E}$ (as a subset) is a κ-field and it is the smallest one that contains $\mathcal{E}$. We denote this κ-field by $\kappa(\mathcal{E})$ and call it the κ-field generated by $\mathcal{E}$.*

Note that if κ is greater than κ', then $\kappa(\mathcal{E}) \supseteq \kappa'(\mathcal{E})$ and $\kappa(\mathcal{E})$ might strictly contain $\kappa'(\mathcal{E})$. If Σ is a κ-field and $\mathcal{E} \subseteq \Sigma$ such that $\Sigma = \kappa'(\mathcal{E})$, then also $\Sigma = \kappa(\mathcal{E})$, but there might be $\mathcal{E}' \subseteq \Sigma$ such that $\kappa(\mathcal{E}') = \Sigma$ and $\kappa'(\mathcal{E}')$ is strictly contained in Σ.

The following lemma shows that to prove the measurability of a function between two κ-measurable spaces, it is enough to proof the measurability for a generating set of the κ-field of the image space.

Lemma 1 *Let (M', Σ') and (M, Σ) be κ-measurable spaces, where κ is a regular cardinal number. Let $\mathcal{E} \subseteq \Sigma$ such that $\kappa(\mathcal{E}) = \Sigma$. Then $f : M' \to M$ is $\Sigma' - \Sigma$ measurable iff for all $E \in \mathcal{E} : f^{-1}(E) \in \Sigma'$.*

Proof Define $\mathcal{E}_0 := \mathcal{E} \cup \{M\}$. If $0 < \alpha < \kappa$ and if $\mathcal{E}_\beta$ is already defined for all $\beta < \alpha$, then define

$$\mathcal{E}_\alpha := \left\{ \bigcap_{A \in \Lambda} A \;\middle|\; \Lambda \subseteq \bigcup_{\beta < \alpha} \mathcal{E}_\beta \text{ s.t. } |\Lambda| < \kappa \right\} \cup \left\{ M \backslash A \;\middle|\; A \in \bigcup_{\beta < \alpha} \mathcal{E}_\beta \right\} \cup \bigcup_{\beta < \alpha} \mathcal{E}_\beta.$$

By transfinite induction, it is obvious that $\bigcup_{\beta<\kappa} \mathcal{E}_\beta \subseteq \Sigma$. We show that $\bigcup_{\beta<\kappa} \mathcal{E}_\beta$ is a κ-field, which implies that $\bigcup_{\beta<\kappa} \mathcal{E}_\beta = \Sigma$. We have to show that $\bigcup_{\beta<\kappa} \mathcal{E}_\beta$ is closed under the intersection of less than κ-many elements. The other properties are clear. So, let $\widehat{\kappa}$ be a cardinal number with $\widehat{\kappa} < \kappa$ and $E_\gamma \in \bigcup_{\beta<\kappa} \mathcal{E}_\beta$ for $\gamma < \widehat{\kappa}$. Let $f(\gamma) := \min \{\beta < \kappa \mid E_\gamma \in \mathcal{E}_\beta\}$. By the regularity of κ, we have $\zeta := \sup \{f(\beta) \mid \beta < \widehat{\kappa}\} < \kappa$, hence $\bigcap_{\gamma<\widehat{\kappa}} E_\gamma \in \mathcal{E}_{\zeta+1} \subseteq \bigcup_{\beta<\kappa} \mathcal{E}_\beta$.

Having now established that form of Σ, the lemma follows from the fact that inverse images of maps commute with arbitrary intersections and complements. ■

Definition 6 Let $(M_j, \Sigma_j)_{j\in J}$ be a family of κ-measurable spaces. Let $\mathcal{E}$ be the set of sets of the form $\Pi_{j\in J} E_j$, such that $E_j \in \Sigma_j$, for $j \in J$, and $E_j = M_j$ for all but finitely many[3] $j \in J$.
We define the *product κ-field* of the Σ_j, $j \in J$, on $\Pi_{j\in J} M_j$ as the κ-field $\kappa(\mathcal{E})$.

Convention 1 Let κ be a (regular) infinite cardinal number, let J and L be nonempty disjoint sets and let (M_i, Σ_i) be a κ-measurable space, for $i \in J \cup L$. Let the product space $M_J := \Pi_{j\in J} M_j$ be endowed with the product κ-field Σ_J of the Σ_j, $j \in J$, let the product space $M_L := \Pi_{l\in L} M_l$ be endowed with the product κ-field Σ_L of the Σ_l, $l \in L$, and let the product space $M_J \times M_L$ be endowed with the product κ-field $\Sigma_{\{J,L\}}$ of Σ_J and Σ_L. Consider the product space $M_{J\cup L} := \Pi_{i\in J\cup L} M_i$ and let $M_{J\cup L}$ be endowed with the product κ-field $\Sigma_{J\cup L}$ of the Σ_i, $i \in J \cup L$. Now, let

$$h : M_J \times M_L \to M_{J\cup L}$$

be defined by

$$h((m_j)_{j\in J}, (m_l)_{l\in L}) := (m_i)_{i\in J\cup L}.$$

Then, h is clearly one-to-one and onto, but also, by looking at the generators of the product κ-fields (and applying Lemma 1 to h and h^{-1}), one sees that h and h^{-1} are measurable. Hence h is an isomorphism of measurable spaces and we are justified to identify these two κ-measurable spaces, in all what follows (with some abuse of notation).

Lemma 2 *Let M be a nonempty set, let κ be a regular cardinal number and, for $\alpha < \kappa$, let Σ_α be a κ-field on M such that $\alpha < \beta < \kappa$ implies $\Sigma_\alpha \subseteq \Sigma_\beta$. Then $\Sigma := \bigcup_{\alpha<\kappa} \Sigma_\alpha$ is a κ -field on M.*

[3]It is easily seen that if we would change the definition, also "for $j \in J$: $E_j \in \Sigma_j$ and $E_j = M_j$ for all but one $j \in J$" and "for $j \in J$: $E_j \in \Sigma_j$ and $E_j = M_j$ for all but less than κ-many $j \in J$" would yield the same product κ-field as the one defined above.

Proof We have just to show that the intersection of less than κ-many Σ-measurab-le sets is a Σ-measurable set, the other properties are clear. So, let $\widehat{\kappa} < \kappa$ and $E_\beta \in \Sigma$, for $\beta < \widehat{\kappa}$. Let $f(\beta) := \min\{\alpha < \kappa \mid E_\beta \in \Sigma_\alpha\}$. By the regularity of κ, it follows that $\gamma := \sup\{f(\beta) \mid \beta < \widehat{\kappa}\} < \kappa$, hence $\bigcap_{\beta<\widehat{\kappa}} E_\beta \in \Sigma_\gamma \subseteq \Sigma$. ∎

Definition 7 Let M be a nonempty set and $\mathcal{F}$ a field on M.
A *finitely additive measure on* $(M, \mathcal{F})$ is a function $\mu : \mathcal{F} \to \mathbb{R} \cup \{+\infty\}$, such that

- $0 \le \mu(F)$, for all $F \in \mathcal{F}$,
- $\mu(E \cup F) = \mu(E) + \mu(F)$, for all disjoint $E, F \in \mathcal{F}$.

μ is a *finitely additive probability measure on* $(M, \mathcal{F})$, if in addition

- $\mu(M) = 1$.

Definition 8 Let M be a nonempty set, $\mathcal{F}$ a field on M, μ a finitely additive measure on $(M, \mathcal{F})$, and $E \subseteq M$.
We define the *outer measure of* E *induced by* μ as

$$\mu^*(E) := \inf\{\mu(F) \mid F \in \mathcal{F} \text{ such that } E \subseteq F\},$$

and the *inner measure of* E *induced by* μ as

$$\mu_*(E) := \sup\{\mu(F) \mid F \in \mathcal{F} \text{ such that } F \subseteq E\}.$$

If not stated otherwise, we keep the following

Convention 2
- Products of κ-measurable spaces are endowed with the product κ-field.
- If (M, Σ) is a κ-measurable space, then $\Delta^\kappa(M, \Sigma)$ denotes the space of finitely additive probability measures on (M, Σ). We consider this space itself as a κ-measurable space endowed with the κ-field Σ_{Δ^κ} generated by all the sets $\{\mu \in \Delta^\kappa(M, \Sigma) \mid \mu(E) \ge p\}$, where $E \in \Sigma$ and $p \in [0, 1]$.

Of course, this convention depends on the particular κ chosen.

Remark 4 *Let* (M', Σ') *and* (M, Σ) *be* κ*-measurable spaces and let* $f : M' \to M$ *be measurable.*

1. *If* μ' *is a finitely additive probability measure on* (M', Σ')*, then* $\mu'\left(f^{-1}(\cdot)\right)$ *(that is* $\mu'\left(f^{-1}(E)\right)$*, for* $E \in \Sigma$*) is a finitely additive probability measure on* (M, Σ)*.*

2. *If* $\Delta_f^\kappa : \Delta^\kappa(M', \Sigma') \to \Delta^\kappa(M, \Sigma)$ *is defined by* $\Delta_f^\kappa(\mu') := \mu'\left(f^{-1}(\cdot)\right)$*, for* $\mu' \in \Delta^\kappa(M', \Sigma')$*, then* Δ_f^κ *is measurable, since we have* $\Delta_f^\kappa(\mu')(E) \geq p$ *iff* $\mu'\left(f^{-1}(E)\right) \geq p$*, for* $E \in \Sigma$*.*

Remark 5 *Let* (M', Σ') *and* (M, Σ) *be* κ*-measurable spaces and let* $f : M' \to M$ *be measurable and onto.*
Then:

1. $f^{-1}(\Sigma) := \{f^{-1}(E) \mid E \in \Sigma\}$ *is a* κ *-field on* M' *and a subset of* Σ'*.*

2. *If* μ *is a finitely additive measure on* (M, Σ)*, then* μ *induces a finitely additive measure* μ' *on* $(M', f^{-1}(\Sigma))$ *defined by* $\mu'(f^{-1}(E)) := \mu(E)$*. Furthermore, if* μ *is a finitely additive probability measure, then* μ' *is a finitely additive probability measure.*

Lemma 3 *Let* $\gamma < \alpha$ *be ordinal numbers. For* $\gamma \leq \beta < \alpha$ *let* $\left(M^\beta, \mathcal{F}^\beta\right)$ *be a* $\aleph_0$*-measurable space (i.e.* M^β *is a nonempty set and* $\mathcal{F}^\beta$ *is a field on* M^β*) and* μ^β *a finitely additive probability measure on* $\left(M^\beta, \mathcal{F}^\beta\right)$*, let* M^α *be a nonempty set, and for* $\gamma \leq \xi < \zeta \leq \alpha$ *let* $f_{\xi,\zeta} : M^\zeta \to M^\xi$ *be onto and, if* $\zeta < \alpha$*, let* $f_{\xi,\zeta}$ *be* $\mathcal{F}^\zeta$*-*$\mathcal{F}^\xi$*-measurable, such that:*

1. $f_{\xi,\beta} \circ f_{\beta,\zeta} = f_{\xi,\zeta}$*, for all* $\xi < \beta < \zeta$ *such that* $\gamma \leq \xi < \beta < \zeta \leq \alpha$*,*

2. $\mu^\beta\left(f_{\xi,\beta}^{-1}\left(E^\xi\right)\right) = \mu^\xi\left(E^\xi\right)$*, for all* $\xi < \beta$ *such that* $\gamma \leq \xi < \beta < \alpha$ *and all* $E^\xi \in \mathcal{F}^\xi$*.*

Then:

- $\bigcup_{\gamma \leq \beta < \alpha} f_{\beta,\alpha}^{-1}\left(\mathcal{F}^\beta\right)$ *is a field on* M^α*,*

- $\left(\mu^\beta\right)_{\gamma \leq \beta < \alpha}$ *induces a well-defined finitely additive probability measure* $\mu^{<\alpha}$ *on* $\left(M^\alpha, \bigcup_{\gamma \leq \beta < \alpha} f_{\beta,\alpha}^{-1}\left(\mathcal{F}^\beta\right)\right)$*, defined by* $\mu^{<\alpha}\left(f_{\beta,\alpha}^{-1}\left(E^\beta\right)\right) := \mu^\beta\left(E^\beta\right)$*, for* $E^\beta \in \mathcal{F}^\beta$*.*

Proof That $\mu^{<\alpha}$ is well-defined follows from the above conditions 1 and 2 and the fact that the $f_{\beta,\alpha}$'s are onto. In light of the preceding remark, the rest is clear. ■

Notation 1 Let M be a nonempty set, $\mathcal{F}$ a field on M, and $E \subseteq M$. Then denote by $[\mathcal{F}, E]$ the set of all subsets of M of the form $(L \cap E) \cup (N \cap (M \setminus E))$, where $L, N \in \mathcal{F}$. It is easy to check that $[\mathcal{F}, E]$ is the smallest field that extends $\mathcal{F}$ and contains E as an element.

For further reference, we cite the following two lemmas (in a somewhat different form), which are theorems by Łoś and Marczewski (1949) and Horn and Tarski (1948).

Lemma 4 *Let M be a nonempty set, $\mathcal{F}$ a field on M, $E \subseteq M$, μ a finitely additive probability measure on $(M, \mathcal{F})$, $\mu_*(E)$ the inner measure of E, $\mu^*(E)$ the outer measure of E, and p a real number such that $\mu_*(E) \leq p \leq \mu^*(E)$. Then there exists a finitely additive probability measure ν that extends μ to the field $[\mathcal{F}, E]$ such that $\nu(E) = p$.*

Proof Follows directly from Theorem 2 of Łoś and Marczewski (1949). ■

Sometimes, we will refer to the above lemma as the "Łoś-Marczewski Theorem".

Lemma 5 *Let $\mathcal{F}_1 \subseteq \mathcal{F}_2$ be fields on the nonempty set M and let μ be a finitely additive probability measure on $(M, \mathcal{F}_1)$. Then there exists an extension of μ to a finitely additive probability measure ν on $(M, \mathcal{F}_2)$.*

Proof Follows from 4 (i) of Łoś and Marczewski (1949) and also from Horn and Tarski (1948), p. 477, Theorem 1.21. ■

For the rest of this section, unless otherwise stated, we fix a regular cardinal κ. Furthermore we fix a nonempty set of players I, a nonempty set of states of nature S, and, unless otherwise stated, a κ-field Σ_S on S, such that for all $s, s' \in S$ with $s \neq s'$ there is a $E \in \Sigma_S$ such that $s \in E$ and $s' \notin E$. Without loss of generality, we assume that $0 \notin I$ and set $I_0 := I \cup \{0\}$.

We define now κ-type spaces, ∞-type spaces and $*$-type spaces, i.e. the objects which we will study in the rest of this paper.

Definition 9 A *κ-type space on S for player set I* is a 4-tuple

$$\underline{M} := \left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle,$$

where

- M is a nonempty set,
- Σ is a κ-field on M,
- for $i \in I : T_i$ is a $\Sigma - \Sigma_{\Delta^\kappa}$-measurable function form M to $\Delta^\kappa (M, \Sigma)$, the space of finitely additive probability measures on (M, Σ), such that for all $m \in M$ and $A \in \Sigma : [T_i (m)] \subseteq A$ implies $T_i (m) (A) = 1$, where $[T_i (m)] := \{m' \in M \mid T_i (m') = T_i (m)\}$,
- θ is a Σ-Σ_S-measurable function from M to S.

We will refer to the property that for all $i \in I$, $m \in M$ and $A \in \Sigma$: $[T_i (m)] \subseteq A$ implies $T_i (m) (A) = 1$, as the "*introspection property*" of κ-type spaces (∞-type spaces and $*$-type spaces, resp., see below). Doing obvious changes, proofs (of the theorems in this paper) would went through, if we were to abandon this property, in fact, things would be easier then.

Definition 10 A *∞-type space on S for player set I* is a 4-tuple

$$\underline{M} := \left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle,$$

where

- M is a nonempty set,
- Σ is a ∞-field on M,
- for $i \in I : T_i$ is a measurable function form (M, Σ) to $\Delta^\infty (M, \Sigma)$, the space of finitely additive probability measures on (M, Σ), endowed with the ∞-field generated by all the sets $\{\mu \in \Delta^\infty (M, \Sigma) \mid \mu (E) \geq p\}$, where $E \in \Sigma$ and $p \in [0, 1]$, such that for all $m \in M$ and $A \in \Sigma$: $[T_i (m)] \subseteq A$ implies $T_i (m) (A) = 1$, where $[T_i (m)] := \{m' \in M \mid T_i (m') = T_i (m)\}$,
- θ is a Σ-$\mathrm{Pow}(S)$-measurable function from M to S.

Note that for $\mu \neq \nu \in \Delta^{\infty}(M, \Sigma)$ there is a $E \in \Sigma$ and a $p \in [0,1]$ such that $\mu(E) \geq p$ and $\nu(E) < p$. This implies that the ∞-field of $\Delta^{\infty}(M, \Sigma)$ is in fact $\mathrm{Pow}(\Delta^{\infty}(M, \Sigma))$, the full power set. Hence, by the measurability of T_i, we have $[T_i(m)] \in \Sigma$. So, in fact, the condition that $[T_i(m)] \subseteq A$ implies $T_i(m)(A) = 1$ reduces to $T_i(m)([T_i(m)]) = 1$.

By the definitions, it is obvious that every ∞-type space on S is a κ-type space on S, for every regular κ. (Set $\Sigma_S := \mathrm{Pow}(S)$ in the κ-type space.)

Definition 11 A *$*$-type space on S for player set I* is a 3-tuple

$$\underline{M} := \left\langle M, (T_i)_{i \in I}, \theta \right\rangle,$$

where

- M is a nonempty set,
- for $i \in I : T_i$ is a function form M to $\Delta(M, \mathrm{Pow}(M))$, the space of finitely additive probability measures on $(M, \mathrm{Pow}(M))$, such that for all $m \in M : T_i(m)([T_i(m)]) = 1$, where $[T_i(m)] := \{m' \in M \mid T_i(m') = T_i(m)\}$
- θ is a function from M to S.

It is obvious from the definitions, that every $*$-type space on S is a ∞-type space on S. (Set $\Sigma := \mathrm{Pow}(M)$ in the ∞-type space.) And hence, it is also a κ-type space on S.

We define now the beliefs preserving maps between type spaces.

Definition 12 Let $\underline{M}' = \left\langle M', \Sigma', (T_i')_{i \in I}, \theta' \right\rangle$ and $\underline{M} = \left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle$ be κ-type spaces (∞-type spaces, $*$-type spaces, respectively) on S for player set I.

A function $f : M' \to M$ is a *type morphism* if it satisfies the following conditions:

1. f is $\Sigma' - \Sigma$-measurable,
2. for all $m' \in M'$:
$$\theta'(m') = \theta(f(m')),$$
3. for all $m' \in M'$, $E \in \Sigma$, and $i \in I$:
$$T_i(f(m'))(E) = T_i'(m')\left(f^{-1}(E)\right).$$

Note that the above definition of a type morphism does not depend on κ, that is, if $\kappa < \kappa'$, and $\underline{M}$ and $\underline{M}'$ are κ'-type spaces (∞-type spaces, $*$-type spaces, respectively), then $f : M' \to M$ is a type morphism from $\underline{M}'$ to $\underline{M}$ viewed as κ'-type spaces (∞-type spaces, $*$-type spaces, respectively) iff it is a type morphism from $\underline{M}'$ to $\underline{M}$ viewed as κ-type spaces. Similarly, if $\underline{M}'$ and $\underline{M}$ are $*$-type spaces, then $f : M' \to M$ is a type morphism from $\underline{M}'$ to $\underline{M}$ viewed as $*$-type spaces iff it is a type morphism from $\underline{M}'$ to $\underline{M}$ viewed as ∞-type spaces. (Note that in the case of $*$-type spaces, every function $f : M' \to M$ is measurable.)

Definition 13 A type morphism is a *type isomorphism*, if it is one-to-one, onto, and the inverse function is also a type morphism.

It is easy to see that a function $f : M' \to M$ is a type isomorphism iff it is a type morphism and isomorphism of the measurable spaces (M', Σ') and (M, Σ).

Definition 14 A *product κ-type space on S for player set I* is a 4-tuple

$$\underline{M} := \langle M, \Sigma, (T_i)_{i \in I}, \theta \rangle$$

such that there are κ-measurable spaces (M_j, Σ_j), for $j \in I_0$, such that (up to type isomorphism):

- $M_0 = S$,
- $\Sigma_0 = \Sigma_S$,
- $M = \Pi_{j \in I_0} M_j$,
- Σ is the product κ-field on M which is generated by the Σ_j, $j \in I_0$,
- for $i \in I$: T_i is a $\Sigma_i - \Sigma_{\Delta^\kappa}$-measurable function from M_i to $\Delta^\kappa(M, \Sigma)$, the space of finitely additive probability measures on (M, Σ), such that for all $m_i \in M_i : \text{marg}_{M_i}(T_i(m_i)) = \delta_{m_i}$,
- $\theta : M_0 \to S$ is the identity on S.

Obviously, T_i, for $i \in I$, can be viewed as a $\Sigma - \Sigma_{\Delta^\kappa}$-measurable function from M, to $\Delta^\kappa(M, \Sigma)$ and θ can be viewed as a $\Sigma - \Sigma_S$-measurable function θ from M to S. (Take first the projection to M_i, for $i \in I$, (resp. to M_0) and then the original function.) So, every product κ-type space on S is a κ-type space on S. Note that if $i \in I$ and if m_i is the ith coordinate of m, then we have by the fifth point of the definition that $[T_i(m)] = \{m_i\} \times \Pi_{j \in I_0 \setminus \{i\}} M_j$.

An easy check shows:

Remark 6 *κ-type spaces on S for player set I (∞-type spaces, $*$-type spaces, respectively), as objects, and type morphisms, as morphisms, form a category.*

Definition 15

- A κ-type space $\underline{\Omega}$ on S for player set I (∞-type space, $*$-type space, respectively) is *weak-universal* if for every κ-type space $\underline{M}$ on S for player set I (∞-type space, $*$-type space, respectively) there is a type morphism from $\underline{M}$ to $\underline{\Omega}$.
- A κ-type space $\underline{\Omega}$ on S for player set I (∞-type space, $*$-type space, respectively) is *universal*[4] if for every κ-type space $\underline{M}$ on S for player set I (∞-type space, $*$-type space, respectively) there is a *unique* type morphism from $\underline{M}$ to $\underline{\Omega}$.

Remark 7 *Universal κ-type spaces on S for player set I (∞-type spaces, $*$-type spaces, respectively) are unique up to type isomorphism.*

Proof If $\underline{\Omega}$ and $\underline{U}$ are universal κ-type spaces (∞-type spaces, $*$-type spaces, respectively) (on the same space of states of nature and for the same player set, of course), then there are type morphisms $f : U \to \Omega$ and $g : \Omega \to U$. It is easy to check, that the composite of two type morphisms is also a type morphism and that the identity is always a type morphism form a κ-type space $\underline{\Omega}$ (∞-type space, $*$-type space, respectively) to itself. By the uniqueness , it follows that $g \circ f = \mathrm{id}_U$ and therefore f is one-to-one and g is onto, and $f \circ g = \mathrm{id}_\Omega$ and therefore g is one-to-one and f is onto. f and g are type morphisms and $f = g^{-1}$ and $g = f^{-1}$. ■

[4]We use here the term "universal type space" although, in terms of category theory the term "terminal type space" would be the adequate one, since the universal type space is a terminal object in the category of type spaces. However, we take the former notion to keep the terms of the already existing type space literature.

3 The Universal κ-Type Space in Terms of Expressions

Again, for this section, unless otherwise stated, we fix a regular cardinal κ, a nonempty player set I, and a κ-measurable space of states of nature (S, Σ_S) such that for all $s, s' \in S$ with $s \neq s'$ there is a $E \in \Sigma_S$ such that $s \in E$ and $s' \notin E$.

Given these data, we define *κ-expressions* (allowing also for infinite conjunctions) which are natural generalizations of the *expressions* defined by Heifetz and Samet (1998b). Expressions are defined in a similar fashion as, for example, the formulas of the probability logic of Heifetz and Mongin (2001). Analogous to Heifetz and Samet (1998b), given a κ-type space on S for player set I and a state of the world in this type space, we define the *κ-description* of this state as the set of those *κ-expressions* that are true in this state of the world. Then, we show that the set of all κ-descriptions constitutes a κ-type space (Proposition 3) and that this κ-type space is the universal κ-type space (Theorem 1).

Definition 16 For a κ-type space $\left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle$ on S for player set I, $i \in I$, $E \in \Sigma$, and $p \in [0,1]$ define

$$\overline{B}_i^p(E) := \{m \in M \mid T_i(m)(E) \geq p\}.$$

Note that $\overline{B}_i^p(E) = T_i^{-1}(\{\mu \in \Delta^\kappa(M, \Sigma) \mid \mu(E) \geq p\})$ and that $\overline{B}_i^p(E) \in \Sigma$, if $E \in \Sigma$.

Definition 17 Given a κ-measurable space of states of nature (S, Σ_S) and a nonempty player set I, the set Φ^κ of *κ-expressions* is the least set such that:

1. every $E \in \Sigma_S$ is a κ-expression,
2. if φ is a κ-expression, then $\neg\varphi$ is a κ-expression,
3. if φ is a κ-expression, then $B_i^p(\varphi)$ is a κ-expression, for $i \in I$ and $p \in [0,1]$,
4. if Ψ is a nonempty set of κ-expressions with $|\Psi| < \kappa$, then $\bigwedge_{\varphi \in \Psi} \varphi$ is a κ-expression.

If Ψ is a nonempty set of κ-expressions with $|\Psi| < \kappa$, then we set $\bigvee_{\varphi \in \Psi} \varphi := \neg \bigwedge_{\varphi \in \Psi} \neg\varphi$.

Since we work here with a fixed regular κ, we omit sometimes, in this section, the superscript κ.

Definition 18 Let $\underline{M} := \left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle$ be a κ-type space on S for player set I. Define

1. $E^{\underline{M}} := \theta^{-1}(E)$, for $E \in \Sigma_S$,
2. $(\neg\varphi)^{\underline{M}} := M \setminus \varphi^{\underline{M}}$, for $\varphi \in \Phi^\kappa$,
3. $(B_i^p(\varphi))^{\underline{M}} := \overline{B}_i^p\left(\varphi^{\underline{M}}\right)$, for $\varphi \in \Phi^\kappa$, $i \in I$ and $p \in [0,1]$,
4. $\left(\bigwedge_{\varphi \in \Psi} \varphi\right)^{\underline{M}} := \bigcap_{\varphi \in \Psi} \varphi^{\underline{M}}$, for Ψ such that $\emptyset \neq \Psi \subseteq \Phi^\kappa$ and $|\Psi| < \kappa$.

So, defined as above, κ-expressions define measurable subsets of M. It is easy to check that $\left(\bigvee_{\varphi \in \Psi} \varphi\right)^{\underline{M}} := \bigcup_{\varphi \in \Psi} \varphi^{\underline{M}}$, for Ψ such that $\emptyset \neq \Psi \subseteq \Phi^\kappa$ and $|\Psi| < \kappa$.

If no confusion may arise, we omit - with some abuse of notation - the superscript $\underline{M}$.

Definition 19 For a κ-type space $\underline{M} := \left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle$ on S for player set I and $m \in M$ define $D^\kappa(m)$, the *κ-description* of m, as

$$D^\kappa(m) := \left\{\varphi \in \Phi^\kappa \mid m \in \varphi^{\underline{M}}\right\}.$$

Again, we omit sometimes, in this section, the superscript κ.

Proposition 1 *Let $\left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle$ and $\left\langle N, \Sigma^N, \left(T_i^N\right)_{i \in I}, \theta^N \right\rangle$ be κ-type spaces on S for player set I and let $f : M \to N$ be a type morphism.*
Then, for all $m \in M$:

$$D^\kappa(f(m)) = D^\kappa(m).$$

Proof We show by induction on the formation of the expressions that $m \in \varphi^M$ iff $f(m) \in \varphi^N$:

- Let $E \in \Sigma_S$. We have $\theta^N(f(m)) = \theta(m)$, so $f(m) \in E^N$ iff $m \in E^M$.
- We have

$$f(m) \in (\neg\varphi)^N \text{ iff } f(m) \notin \varphi^N \text{ iff } m \notin \varphi^M \text{ iff } m \in (\neg\varphi)^M.$$

- Let Ψ be a nonempty set of expressions with $|\Psi| < \kappa$. Then:

$$f(m) \in (\bigwedge_{\varphi\in\Psi} \varphi)^N \text{ iff for all } \varphi \in \Psi : \ f(m) \in \varphi^N,$$

which is by induction hypothesis the case iff for all $\varphi \in \Psi : m \in \varphi^M$, which is the case iff $m \in (\bigwedge_{\varphi\in\Psi} \varphi)^M$.

- We have

$$f(m) \in (B_i^p(\varphi))^N \text{ iff } T_i^N(f(m))\left(\varphi^N\right) \geq p \text{ iff } T_i(m)\left(f^{-1}\left(\varphi^N\right)\right) \geq p.$$

By the induction hypothesis: $f^{-1}\left(\varphi^N\right) = \varphi^M$. Hence $T_i(m)\left(f^{-1}\left(\varphi^N\right)\right) = T_i(m)\left(\varphi^M\right)$. We have $T_i(m)\left(\varphi^M\right) \geq p$ iff $m \in (B_i^p(\varphi))^M$. It follows that

$$f(m) \in (B_i^p(\varphi))^N \text{ iff } m \in (B_i^p(\varphi))^M.$$

■

Definition 20 Define Ω^κ to be the set of all κ-descriptions of states of the world in κ-type spaces on S for player set I. For $\varphi \in \Phi^\kappa$ define

$$[\varphi] := \{\omega \in \Omega^\kappa \mid \varphi \in \omega\}.$$

Again, we omit sometimes, in this section, the superscript κ.

Obviously, we have $\Omega \setminus [\varphi] = [\neg\varphi]$ and $\bigcap_{\psi\in\Psi} [\psi] = \left[\bigwedge_{\psi\in\Psi} \psi\right]$, where φ is an κ-expression and Ψ is a nonempty set of κ-expressions with $|\Psi| < \kappa$. It follows that:

Remark 8 *The set*

$$\Sigma_\Omega := \{[\varphi] \mid \varphi \in \Phi^\kappa\}$$

is a κ-field on Ω.

Lemma 6 *For every κ-type space $\underline{M}$ on S for player set I and for every $\varphi \in \Phi^\kappa$, the κ-description map $D : M \to \Omega$ satisfies*

$$D^{-1}([\varphi]) = \varphi^{\underline{M}}.$$

Proof Clear by the definition of $[\varphi]$. ■

Note that Lemma 6 implies that D is measurable.

Proposition 2 *For every $i \in I$ there exists a function*

$$T_i^* : \Omega \to \Delta^\kappa (\Omega, \Sigma_\Omega)$$

such that for every κ-type space $\underline{M}$ on S for player set I with κ-description map D and every $m \in M$:

$$T_i^* (D (m)) = T_i (m) \circ D^{-1}.$$

Proof For $\omega \in \Omega$ chose a κ-type space $\underline{M}$ on S for player set I and $m \in M$ such that $D (m) = \omega$. For $[\varphi] \in \Sigma_\Omega$ define

$$T_i^* (\omega) ([\varphi]) := T_i (m) \circ D^{-1} ([\varphi]) .$$

We have

$$T_i (m) \circ D^{-1} ([\varphi]) = T_i (m) \left(\varphi^{\underline{M}}\right) = \sup \{p \mid B_i^p (\varphi) \in D (m)\} ,$$

so $T_i^* (\omega) ([\varphi])$ depends just on $D (m)$ and is well-defined. By Remark 4, we have

$$T_i (m) \circ D^{-1} \in \Delta^\kappa (\Omega, \Sigma_\Omega) .$$

■

Lemma 7 *There is a measurable function $\theta^* : \Omega \to S$ such that for every κ-type space $\underline{M}$ on S for player set I and every $m \in M$:*

$$\theta^* (D (m)) = \theta (m) .$$

Proof Let

$$d_0 (m) := \left\{E \in \Sigma_S \mid m \in \theta^{-1} (E)\right\} .$$

Obviously, $d_0 (m) = D (m) \cap \Sigma_S$. By the properties of (S, Σ_S) , we have for all $s \in S : \{s\} = \bigcap_{s \in E \in \Sigma_S} E$. It follows for every κ-type space $\underline{M}'$ on S for player set I and $m' \in M'$ that

$$\theta (m') = s \text{ iff } d_0 (m') = \{E \mid s \in E\} .$$

For $\omega \in \Omega$ chose a κ-type space $\underline{M}$ on S for player set I and $m \in M$, such that $D(m) = \omega$. Define now $\theta^*(\omega) := \theta(m)$. Since $\theta(m)$ just depends on $D(m)$, $\theta^*(\omega)$ is well-defined.

It remains to show that θ^* is measurable: Let $E \in \Sigma_S$. We have

$$\theta^*(D(m)) \in E \text{ iff } m \in \theta^{-1}(E) \text{ iff } E \in D(m) \text{ iff } D(m) \in [E].$$

It follows that $\theta^{*-1}(E) = [E]$. ■

Proposition 3

$$\left\langle \Omega, \Sigma_\Omega, (T_i^*)_{i \in I}, \theta^* \right\rangle$$

is a κ-type space on S for player set I.

Proof It remains to show:

1. Ω is nonempty.
2. For every $i \in I : T_i^*$ is measurable as a function from Ω to $\Delta^\kappa(\Omega, \Sigma_\Omega)$.
3. For every $i \in I$, $\omega \in \Omega$ and $A \in \Sigma_\Omega$: If
$$\{\omega' \in \Omega \mid T_i^*(\omega') = T_i^*(\omega)\} \subseteq A,$$
then $T_i^*(\omega)(A) = 1$.

To:

1. Let $M := \{m\}$ and chose $s \in S$. Set $\Sigma := \mathrm{Pow}(M)$, $T_i(m) := \delta_m$, for $i \in I$, and $\theta(m) := s$. Then
$$\left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle$$
is a κ-type space (even a $*$-type space) on S for player set I.
2. Since inverse images commute with unions, intersections and complements, it is enough to show that $T_i^{*-1}(b^p(E)) \in \Sigma_\Omega$, for
$$b^p(E) := \{\mu \in \Delta^\kappa(\Omega, \Sigma_\Omega) \mid \mu(E) \geq p\},$$
where $E \in \Sigma_\Omega$ and $p \in [0,1]$. We have
$$T_i^{*-1}(b^p(E)) = \{\omega \in \Omega \mid T_i^*(\omega)(E) \geq p\}.$$
Since $E \in \Sigma_\Omega$, there is a κ-expression φ such that $E = [\varphi]$. Note that if $p \in [0,1]$ and $p = \sup\{q \mid B_i^q(\varphi) \in \omega\}$, then $B_i^p(\varphi) \in \omega$. This implies that
$$\omega \in T_i^{*-1}(b^p([\varphi])) \text{ iff } B_i^p(\varphi) \in \omega.$$
It follows that $T_i^{*-1}(b^p(E)) = [B_i^p(\varphi)]$.

3. Let φ be a κ-expression and

$$\{\omega' \in \Omega \mid T_i^*(\omega') = T_i^*(\omega)\} \subseteq [\varphi].$$

Chose a κ-type space $\underline{M}$ on S for player set I and $m \in M$ such that $D(m) = \omega$. Let $m' \in M$. If $T_i^*(D(m')) \neq T_i^*(D(m))$, then there is a κ-expression ψ such that $T_i(m')\left(\psi^M\right) \neq T_i(m)\left(\psi^M\right)$. It follows that

$$D(\{m' \in M \mid T_i(m') = T_i(m)\}) \subseteq \{\omega' \in \Omega \mid T_i^*(\omega') = T_i^*(\omega)\},$$

which implies

$$\{m' \in M \mid T_i(m') = T_i(m)\} \subseteq D^{-1}([\varphi]) = \varphi^M.$$

So we have

$$1 = T_i(m)\left(\varphi^M\right) = T_i(m) \circ D^{-1}([\varphi]) = T_i^*(\omega)([\varphi]).$$

■

Lemma 8 *The κ-description map*

$$D : \Omega \to \Omega$$

is the identity.

Proof For $\omega \in \Omega$, we have

$$\omega = \{\varphi \in \Phi \mid \omega \in [\varphi]\}.$$

We have to show that for every κ-expression φ and every $\omega \in \Omega : \omega \in \varphi^{\underline{\Omega}}$ iff $\omega \in [\varphi]$. We know this already if $\varphi = E$, where $E \in \Sigma_S$. It is obvious that $\Omega \setminus [\varphi] = [\neg\varphi]$, and that if Ψ is a nonempty set of κ-expressions of cardinality $< \kappa$, then

$$\bigcap_{\varphi \in \Psi} [\varphi] = \left[\bigwedge_{\varphi \in \Psi} \varphi\right].$$

So it remains to show that $[\varphi] = \varphi^{\underline{\Omega}}$ implies $[B_i^p(\varphi)] = \overline{B}_i^p([\varphi])$. For $\omega \in \Omega$, chose a κ-type space $\underline{M}$ on S for player set I and $m \in M$ such that $D(m) = \omega$. We have

$$D(m) \in [B_i^p(\varphi)] \text{ iff } B_i^p(\varphi) \in D(m) \text{ iff } T_i(m)\left(\varphi^{\underline{M}}\right) \geq p.$$

But we have

$$T_i^*(\omega)([\varphi]) = T_i(m) \circ D^{-1}([\varphi]) = T_i(m)\left(\varphi^{\underline{M}}\right).$$

This implies that $[B_i^p(\varphi)] = \overline{B}_i^p([\varphi])$. ■

Theorem 1 *The space*

$$\left\langle \Omega, \Sigma_\Omega, \left(T_i^*\right)_{i\in I}, \theta^* \right\rangle$$

is a universal κ-type space on S for player set I.

Proof According to Lemma 6, for every κ-type space $\underline{M}$ on S for player set I, the κ-description map $D : M \to \Omega$ is measurable, and according to Proposition 2 and Lemma 7, D is a type morphism. It remains to show that it is the unique type morphism from $\underline{M}$ to $\underline{\Omega}$. But this clear by Proposition 1 and Lemma 8.

■

4 Spaces of Arbitrary Complexity

Is there a cardinal κ, such that the κ-descriptions determine already the κ'-descriptions, for all cardinal numbers $\kappa' > \kappa$? In the sequel, using a probabilistic adaptation of the elegant "sober-drunk" example of Heifetz and Samet (1998a) (see that paper also for the "story" interpreting the mathematical structure), we construct, for every regular cardinal κ', a κ'-type space (in fact even a $*$-type space), such that for every ordinal $\alpha < \kappa'$ there are at least two states of the world such that for every κ'-expression of depth $\leq \alpha$ this κ'-expression is true either in both states or in none of the two, and yet there is a κ'-expression of depth $\alpha + 1$ that is true in one state and not in the other. Thus, we answer the above question in the negative. Hence, it makes sense to consider κ-type spaces for every regular cardinality κ whatsoever.

In addition, this example will imply that, for at least two players and two states of nature, there is no universal $*$-type space and no universal ∞-type space (Theorem 3 and Corollary 1).

For this section, let $I := \{a, b\}$ be the set of players (the following analysis can be trivially extended to more than two players). We fix a set of states of nature $S = \{h, t\}$, consisting of the two possible outcomes of tossing a coin, h(ead) and t(ail).

To simplify the notation let us make the following

Convention 3 $\{i, j\} := \{a, b\}$, that is $j = \begin{cases} a, & \text{if} \quad i = b, \\ b, & \text{if} \quad i = a. \end{cases}$

The following three definitions of are taken form Heifetz and Samet (1998a).

Definition 21 Let α be an ordinal. A *record of length* α is a sequence $r^\alpha = (r(\beta))_{\beta<\alpha}$ of numbers "0" and "1" such that for every limit ordinal $\lambda \leq \alpha$ there is an ordinal $\gamma < \lambda$ such that $r(\beta) = 0$ for all ordinals β that satisfy $\gamma \leq \beta < \lambda$.

For every infinite ordinal γ there are a unique natural number n and a unique limit ordinal $\widehat{\lambda}$ such that $\gamma = \widehat{\lambda}+n$. We say γ is *even* or *odd* according to wether n is even or odd. If γ is a finite ordinal, i.e. a natural number, we take the usual notion of being even or odd.

Definition 22 Let α be an ordinal, r^α a record of length α, and λ a limit ordinal $\leq \alpha$. By the definition of a record, there is a minimal ordinal $o^\lambda(r^\alpha) < \lambda$ such that $r^\alpha(\beta) = 0$, for all β with $o^\lambda(r^\alpha) \leq \beta < \lambda$.
Define λ-par(r^α), the λ-*parity* of r^α, as

$$\lambda - \mathrm{par}(r^\alpha) := \begin{cases} \text{even, if } o^\lambda(r^\alpha) \text{ is even,} \\ \text{odd, if } o^\lambda(r^\alpha) \text{ is odd.} \end{cases}$$

Note that by the definition of a record, $o^\lambda(r^\alpha)$ must be either 0 or a successor ordinal (i.e. $o^\lambda(r^\alpha) = \gamma + 1$, for some ordinal γ).

Definition 23 Let α be an ordinal. Define the spaces W^α by

- $$W^0 := \{h, t\},$$

- $$W^\alpha := \{(w_0, w_a^\alpha, w_b^\alpha) \mid w_0 \in \{h, t\}, w_a^\alpha \text{ and } w_b^\alpha \text{ are records of length } \alpha\},$$

  if $\alpha \geq 1$.

Definition 24
- If $0 < \beta \leq \alpha$ and $r^\alpha = (r(\xi))_{\xi<\alpha}$ is a record of length α, then denote by $r^\alpha\lceil\beta$ the record $(r(\xi))_{\xi<\beta}$ of length β.

- If $0 \leq \alpha$ and $w^\alpha \in W^\alpha$, then define $w^\alpha \lceil 0 := w_0$.
- If $0 < \beta \leq \alpha$ and $w^\alpha \in W^\alpha$, then define $w^\alpha \lceil \beta := (w_0, w_a^\alpha \lceil \beta, w_b^\alpha \lceil \beta)$.

By the definition, it is obvious that $w^\alpha \lceil \beta \in W^\beta$, for every $\beta < \alpha$.

Definition 25 Let $0 \leq \beta \leq \alpha$. Define

$$\pi_{\beta,\alpha} : W^\alpha \to W^\beta$$

by $\pi_{\beta,\alpha}(w^\alpha) := w^\alpha \lceil \beta$.

It is obvious that $\pi_{\xi,\beta}(\pi_{\beta,\alpha}(w^\alpha)) = \pi_{\xi,\alpha}(w^\alpha)$, for $0 \leq \xi \leq \beta \leq \alpha$.

Remark 9 *Let $0 \leq \beta < \alpha$, $w^\beta \in W^\beta$, and $i \in \{a, b\}$. Then there are $w^\alpha, u^\alpha \in W^\alpha$ such that*

$$w^\alpha \lceil \beta = u^\alpha \lceil \beta = w^\beta \textit{ and } 0 = w_i^\alpha(\beta) \neq u_i^\alpha(\beta) = 1.$$

And in particular, it follows that $\pi_{\beta,\alpha} : W^\alpha \to W^\beta$ is onto.

Let $w^\alpha = (w_0, w_a^\alpha, w_b^\alpha)$ be a state in W^α. We define the element of i's partition that contains this element:

Definition 26 Let α be an ordinal > 0, and $w^\alpha \in W^\alpha$. We define:

$$P_i(w^\alpha) := \left\{ (v_0, v_a^\alpha, v_b^\alpha) \in W^\alpha \;\middle|\; \begin{array}{l} v_i^\alpha = w_i^\alpha, \\ w_i^\alpha(0) = 1 \text{ implies } v_0 = w_0, \\ \text{for all } \beta \text{ such that } \beta + 1 < \alpha : \\ \; w_i^\alpha(\beta+1) = 1 \text{ implies } v_j^\alpha(\beta) = w_j^\alpha(\beta), \\ \text{for every limit ordinal } \lambda < \alpha : \\ \; w_i^\alpha(\lambda) = 1 \text{ implies } \lambda - \text{par}\left(v_j^\alpha\right) = \lambda - \text{par}\left(w_j^\alpha\right). \end{array} \right\}.$$

Remark 10 • *Let α be an ordinal > 0. The set $\{P_i(w^\alpha) \mid w^\alpha \in W^\alpha\}$ is a partition of W^α and $w^\alpha \in P_i(w^\alpha)$.*

- *Let $0 < \beta < \alpha$ and $u^\alpha \in P_i(w^\alpha)$. Then $u^\alpha \lceil \beta \in P_i(w^\alpha \lceil \beta)$, and hence $\pi^{-1}_{\beta,\alpha}(P_i(w^\alpha \lceil \beta)) \supseteq P_i(w^\alpha)$.*

It is easy to see that if α is an infinite ordinal number, then the cardinality of W^α is the same as the cardinality of α. (To see that, in the case of an infinite α, the cardinality of W^α does not exceed that of α, note that the definition of a record implies that there are only finitely many $\beta < \alpha$ such that $r^\alpha(\beta) = 1$. (Consider, assuming the contrary, the minimal $\gamma \leq \alpha$ such that there are infinitely many $\beta < \gamma$ with $r^\alpha(\beta) = 1$.))

Lemma 9 *Let γ be an ordinal > 0, $\alpha = \gamma + 1$, $w^\alpha \in W^\alpha$, and*

$$E \in \left[\left[\bigcup_{\beta<\gamma} \pi^{-1}_{\beta,\alpha}\left(\mathrm{Pow}\left(W^\beta\right)\right), \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^\alpha\lceil\gamma\right)\right)\right], P_i\left(w^\alpha\right)\right].$$

Then there are a $\beta < \gamma$ and $A_\beta, C_\beta, D_\beta \in \mathrm{Pow}\left(W^\beta\right)$ such that

$$\begin{aligned} E &= \left(\pi^{-1}_{\beta,\alpha}(A_\beta) \cap P_i(w^\alpha)\right) \cup \left(\pi^{-1}_{\beta,\alpha}(C_\beta) \cap \pi^{-1}_{\gamma,\alpha}(P_i(w^\alpha\lceil\gamma)) \cap (W^\alpha \setminus P_i(w^\alpha))\right) \\ &\quad \cup \left(\pi^{-1}_{\beta,\alpha}(D_\beta) \cap \left(W^\alpha \setminus \pi^{-1}_{\gamma,\alpha}(P_i(w^\alpha\lceil\gamma))\right)\right). \end{aligned}$$

Proof By the definition of

$$\left[\left[\bigcup_{\beta<\gamma} \pi^{-1}_{\beta,\alpha}\left(\mathrm{Pow}\left(W^\beta\right)\right), \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^\alpha\lceil\gamma\right)\right)\right], P_i\left(w^\alpha\right)\right],$$

E has the form

$$\begin{aligned} E = &\Bigg(\Big(\left(\pi^{-1}_{\beta,\alpha}(A_\beta) \cap \pi^{-1}_{\gamma,\alpha}(P_i(w^\alpha\lceil\gamma))\right) \cup \left(\pi^{-1}_{\eta,\alpha}(B_\eta) \cap \left(\pi^{-1}_{\gamma,\alpha}(W^\gamma \setminus P_i(w^\alpha\lceil\gamma))\right)\right)\Big) \\ &\cap P_i(w^\alpha)\Bigg) \cup \\ &\Bigg(\Big(\left(\pi^{-1}_{\xi,\alpha}(C_\xi) \cap \pi^{-1}_{\gamma,\alpha}(P_i(w^\alpha\lceil\gamma))\right) \cup \left(\pi^{-1}_{\zeta,\alpha}(D_\zeta) \cap \left(\pi^{-1}_{\gamma,\alpha}(W^\gamma \setminus P_i(w^\alpha\lceil\gamma))\right)\right)\Big) \\ &\cap (W^\alpha \setminus P_i(w^\alpha))\Bigg), \end{aligned}$$

where $\beta, \eta, \xi, \zeta < \gamma$ and $A_\beta \subseteq W^\beta$, $B_\eta \subseteq W^\eta$, $C_\xi \subseteq W^\xi$, $D_\zeta \subseteq W^\zeta$.

The lemma follows from the following facts: If $\eta < \beta$, then $\pi_{\eta,\beta}^{-1}(B_\eta) \subseteq W^\beta$ and $\pi_{\eta,\alpha}^{-1}(B_\eta) = \pi_{\beta,\alpha}^{-1}\left(\pi_{\eta,\beta}^{-1}(B_\eta)\right)$, so we can assume without loss of generality that $\beta = \eta = \xi = \zeta$. We have

$$\pi_{\gamma,\alpha}^{-1}(W^\gamma \setminus P_i(w^\alpha\lceil\gamma)) = W^\alpha \setminus \pi_{\gamma,\alpha}^{-1}(P_i(w^\alpha\lceil\gamma))$$

and $P_i(w^\alpha) \subseteq \pi_{\gamma,\alpha}^{-1}(P_i(w^\alpha\lceil\gamma))$. ■

Notation 2 • For $0 \leq \alpha$ and $w_0 \in \{h, t\}$, we denote by

$$[X_0^\alpha = w_0] \text{ the set } \{u^\alpha \in W^\alpha \mid u_0 = w_0\}.$$

- For $\beta < \alpha$, $i \in \{a, b\}$ and $w_i^\alpha(\beta) \in \{0, 1\}$, we denote by

$$[X_i^\alpha(\beta) = w_i^\alpha(\beta)] \text{ the set } \{u^\alpha \in W^\alpha \mid u_i^\alpha(\beta) = w_i^\alpha(\beta)\}.$$

- For a limit ordinal $\lambda \leq \alpha$ and $\lambda\text{-par}(w_i^\alpha) \in \{\text{even, odd}\}$, we denote by

$$[\lambda\text{-par}(X_i^\alpha) = \lambda\text{-par}(w_i^\alpha)] \text{ the set } \{u^\alpha \in W^\alpha \mid \lambda\text{-par}(u_i^\alpha) = \lambda\text{-par}(w_i^\alpha)\}.$$

Remark 11 *Let $0 \leq \alpha \leq \gamma$ and $w_0 \in \{h, t\}$.*
Then:

- $$\pi_{\alpha,\gamma}^{-1}([X_0^\alpha = w_0]) = [X_0^\gamma = w_0].$$

- *If $\beta < \alpha$, $i \in \{a, b\}$, and $w^\gamma \in W^\gamma$, then*

$$\begin{array}{rcll} w_i^\gamma(\beta) & = & (w_i^\gamma\lceil\alpha)(\beta) & \textit{and} \\ \pi_{\alpha,\gamma}^{-1}([X_i^\alpha(\beta) = (w_i^\gamma\lceil\alpha)(\beta)]) & = & [X_i^\gamma(\beta) = w_i^\gamma(\beta)]. & \end{array}$$

- *If $i \in \{a, b\}$, $w^\gamma \in W^\gamma$, and if λ is a limit ordinal such that $\lambda \leq \alpha$, then*

$$\begin{array}{rcll} \lambda - \text{par}(w_i^\gamma) & = & \lambda - \text{par}(w_i^\gamma\lceil\alpha) & \textit{and} \\ \pi_{\alpha,\gamma}^{-1}([\lambda\text{-par}(X_i^\alpha) = \lambda\text{-par}(w_i^\gamma\lceil\alpha)]) & = & [\lambda\text{-par}(X_i^\gamma) = \lambda\text{-par}(w_i^\gamma)]. & \end{array}$$

For further reference, we cite here (in a slightly changed formulation and in our notation) Lemma 3.2. of Heifetz and Samet (1998a):

Lemma 10 *Let $v^\alpha, w^\alpha \in W^\alpha$, where $v^\alpha \lceil \gamma + 1 \in P_i\left(w_i^\alpha \lceil \gamma + 1\right)$, for some $\gamma < \alpha$. Then there is a $u^\alpha \in P_i\left(w^\alpha\right)$ such that $u^\alpha \lceil \gamma = v^\alpha \lceil \gamma$.*

Lemma 11 *Let λ be a limit ordinal, $\alpha = \lambda + 1$, $w^\alpha \in W^\alpha$, $w_i^\alpha\left(\lambda\right) = 0$, and $E = \pi_{\beta,\alpha}^{-1}\left(E_\beta\right)$, where $E_\beta \subseteq W^\beta$ for a $\beta < \lambda$.*
Then:

- *If $v^\alpha \in E \cap P_i\left(w^\alpha\right)$, then there is a $u^\alpha \in E \cap P_i\left(w^\alpha\right)$ such that*
$$\lambda - \text{par}\left(u_j^\alpha\right) \neq \lambda - \text{par}\left(v_j^\alpha\right).$$
- *If*
$$v^\alpha \in E \cap \pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^\alpha \lceil \lambda\right)\right) \cap \left(W^\alpha \setminus P_i\left(w^\alpha\right)\right),$$
then there is a
$$u^\alpha \in E \cap \pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^\alpha \lceil \lambda\right)\right) \cap \left(W^\alpha \setminus P_i\left(w^\alpha\right)\right)$$
such that λ-par $\left(u_j^\alpha\right) \neq \lambda$-par $\left(v_j^\alpha\right)$.
- *If*
$$v^\alpha \in E \cap \left(W^\alpha \setminus \pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^\alpha \lceil \lambda\right)\right)\right),$$
then there is a
$$u^\alpha \in E \cap \left(W^\alpha \setminus \pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^\alpha \lceil \lambda\right)\right)\right)$$
such that λ-par $\left(u_j^\alpha\right) \neq \lambda$-par $\left(v_j^\alpha\right)$.

Proof Let $v^\alpha \in E$. Since $\beta, o^\lambda\left(v_i^\alpha\right), o^\lambda\left(v_j^\alpha\right) < \lambda$, there is an ordinal ξ such that $\max\left\{\beta, o^\lambda\left(v_i^\alpha\right), o^\lambda\left(v_j^\alpha\right)\right\} \leq \xi < \lambda$ and such that the parity of $\xi + 1$ is different from λ-par $\left(v_j^\alpha\right)$. Define now $u^\alpha \in W^\alpha$ by

$$\begin{array}{lcl} u_0 & := & v_0, \\ u_i^\alpha & := & v_i^\alpha, \\ u_j^\alpha\left(\gamma\right) & := & v_j^\alpha\left(\gamma\right), \quad \text{for all } \gamma < \alpha \text{ with } \gamma \neq \xi, \\ u_j^\alpha\left(\xi\right) & := & 1. \end{array}$$

It follows that λ-par $\left(u_j^\alpha\right) \neq \lambda$-par $\left(v_j^\alpha\right)$ and $u^\alpha \lceil \beta = v^\alpha \lceil \beta$, which implies $u^\alpha \in E$.

- If $v^\alpha \in P_i(w^\alpha)$, then it is easy to check that $u^\alpha \in P_i(v^\alpha) = P_i(w^\alpha)$.
- If

$$v^\alpha \in \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha \lceil \lambda)) \cap (W^\alpha \setminus P_i(w^\alpha)),$$

then it follows that $v^\alpha \in \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha \lceil \lambda))$ and $v_i^\alpha(\lambda) = 1$. It is again easy to check that

$$u^\alpha \in \pi_{\lambda,\alpha}^{-1}(P_i(v^\alpha \lceil \lambda)) = \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha \lceil \lambda))$$

and since $u_i^\alpha(\lambda) = 1$, we have $u^\alpha \in (W^\alpha \setminus P_i(w^\alpha))$.

- If $v^\alpha \notin \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha \lceil \lambda))$, then there are four cases:

1. $v_i^\alpha \lceil \lambda \neq w_i^\alpha \lceil \lambda$:
From $u_i^\alpha \lceil \lambda = v_i^\alpha \lceil \lambda$ it follows that $u^\alpha \notin \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha \lceil \lambda))$.

2. There is a $\gamma < \lambda$ such that

$$(v_i^\alpha \lceil \lambda)(\gamma + 1) = (w_i^\alpha \lceil \lambda)(\gamma + 1) = 1 \text{ and } \left(v_j^\alpha \lceil \lambda\right)(\gamma) \neq \left(w_j^\alpha \lceil \lambda\right)(\gamma) :$$

Since $\gamma + 1 < \lambda$ and $\max\left\{\beta, o^\lambda(v_i^\alpha), o^\lambda\left(v_j^\alpha\right)\right\} > \gamma + 1$, it follows that $u^\alpha \lceil \gamma + 2 = v^\alpha \lceil \gamma + 2$ and therefore $u^\alpha \notin \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha \lceil \lambda))$.

3. There is a limit ordinal $\widehat{\lambda} < \lambda$ such that

$$(v_i^\alpha \lceil \lambda)(\widehat{\lambda}) = (w_i^\alpha \lceil \lambda)(\widehat{\lambda}) = 1 \text{ and } \widehat{\lambda} - \text{par}\left(v_j^\alpha\right) \neq \widehat{\lambda} - \text{par}\left(w_j^\alpha\right) :$$

We have $\xi > \widehat{\lambda}$, and therefore

$$(u_i^\alpha \lceil \lambda)(\widehat{\lambda}) = (v_i^\alpha \lceil \lambda)(\widehat{\lambda}) \text{ and } \widehat{\lambda} - \text{par}\left(u_j^\alpha\right) = \widehat{\lambda} - \text{par}\left(v_j^\alpha\right).$$

It follows that $u^\alpha \notin \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha \lceil \lambda))$.

4. $(v_i^\alpha \lceil \lambda)(0) = (w_i^\alpha \lceil \lambda)(0) = 1$ and $v_0 \neq w_0$:
We have

$$(u_i^\alpha \lceil \lambda)(0) = (v_i^\alpha \lceil \lambda)(0) = 1 \text{ and } u_0 = v_0.$$

It follows that $u^\alpha \notin \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha \lceil \lambda))$.

$\blacksquare$

Lemma 12 *Let β be an ordinal, $\alpha = (\beta + 1) + 1$, $w^\alpha \in W^\alpha$, $w_i^\alpha (\beta + 1) = 0$, and $E = \pi_{\beta,\alpha}^{-1} (E_\beta)$, such that $E_\beta \subseteq W^\beta$.*
Then:

- *If $v^\alpha \in E \cap P_i (w^\alpha)$, then there is a $u^\alpha \in E \cap P_i (w^\alpha)$ such that*

$$u_j^\alpha (\beta) \neq v_j^\alpha (\beta) .$$

- *If*

$$v^\alpha \in E \cap \pi_{\beta+1,\alpha}^{-1} (P_i (w^\alpha \lceil \beta + 1)) \cap (W^\alpha \setminus P_i (w^\alpha)),$$

  then there is a

$$u^\alpha \in E \cap \pi_{\beta+1,\alpha}^{-1} (P_i (w^\alpha \lceil \beta + 1)) \cap (W^\alpha \setminus P_i (w^\alpha))$$

  such that $u_j^\alpha (\beta) \neq v_j^\alpha (\beta)$.

- *If*

$$v^\alpha \in E \cap \left(W^\alpha \setminus \pi_{\beta+1,\alpha}^{-1} (P_i (w^\alpha \lceil \beta + 1))\right),$$

  then there is a

$$u^\alpha \in E \cap \left(W^\alpha \setminus \pi_{\beta+1,\alpha}^{-1} (P_i (w^\alpha \lceil \beta + 1))\right)$$

  such that $u_j^\alpha (\beta) \neq v_j^\alpha (\beta)$.

Proof Let $v^\alpha \in E$. Define $u^\alpha \in W^\alpha$ by

$$\begin{array}{lcll} u_0 & := & v_0, & \\ u_i^\alpha & := & v_i^\alpha, & \\ u_j^\alpha (\gamma) & := & v_j^\alpha (\gamma), & \text{for all } \gamma < \alpha \text{ with } \gamma \neq \beta, \\ u_j^\alpha (\beta) & := & 1 - v_j^\alpha (\beta). & \end{array}$$

It follows that $u_j^\alpha (\beta) \neq v_j^\alpha (\beta)$ and $u^\alpha \lceil \beta = v^\alpha \lceil \beta$, which implies $u^\alpha \in E$.

- If $v^\alpha \in P_i (w^\alpha)$, then it is easy to check that $u^\alpha \in P_i (v^\alpha) = P_i (w^\alpha)$.

- If

$$v^{\alpha}\in\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right)\cap\left(W^{\alpha}\setminus P_i\left(w^{\alpha}\right)\right),$$

then it follows that $v^{\alpha}\in\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right)$ and $v_i^{\alpha}\left(\beta+1\right)=1$. It is again easy to check that

$$u^{\alpha}\in\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(v^{\alpha}\lceil\beta+1\right)\right)=\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right)$$

and since $u_i^{\alpha}\left(\beta+1\right)=1$, we have $u^{\alpha}\in\left(W^{\alpha}\setminus P_i\left(w^{\alpha}\right)\right).$

- If $v^{\alpha}\notin\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right)$, then there are four cases:

1. $v_i^{\alpha}\lceil\beta+1\neq w_i^{\alpha}\lceil\beta+1:$
From $u_i^{\alpha}\lceil\beta+1=v_i^{\alpha}\lceil\beta+1$, it follows that $u^{\alpha}\notin\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right).$

2. There is a $\gamma<\beta$ such that

$$\begin{array}{ll}\left(v_i^{\alpha}\lceil\beta+1\right)\left(\gamma+1\right)=\left(w_i^{\alpha}\lceil\beta+1\right)\left(\gamma+1\right)=1 & \text{and}\\ \left(v_j^{\alpha}\lceil\beta+1\right)\left(\gamma\right)\neq\left(w_j^{\alpha}\lceil\beta+1\right)\left(\gamma\right) & :\end{array}$$

By the definition of u^{α}, $\left(u_i^{\alpha}\lceil\beta+1\right)\left(\gamma+1\right)=1$ and, since $\gamma<\beta$,

$$\left(v_j^{\alpha}\lceil\beta+1\right)\left(\gamma\right)=\left(u_j^{\alpha}\lceil\beta+1\right)\left(\gamma\right),$$

hence $u^{\alpha}\notin\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right).$

3. There is a limit ordinal $\lambda<\beta+1$ such that

$$\left(v_i^{\alpha}\lceil\beta+1\right)\left(\lambda\right)=\left(w_i^{\alpha}\lceil\beta+1\right)\left(\lambda\right)=1\text{ and }\lambda-\text{par}\left(v_j^{\alpha}\right)\neq\lambda-\text{par}\left(w_j^{\alpha}\right):$$

We have $\left(u_i^{\alpha}\lceil\beta+1\right)\left(\lambda\right)=1$ and, since $\lambda\leq\beta$, λ-par $\left(u_j^{\alpha}\right)=\lambda$-par $\left(v_j^{\alpha}\right)$.
It follows that $u^{\alpha}\notin\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right).$

4. $\left(v_i^{\alpha}\lceil\beta+1\right)\left(0\right)=\left(w_i^{\alpha}\lceil\beta+1\right)\left(0\right)=1$ and $v_0\neq w_0:$
We have

$$\left(u_i^{\alpha}\lceil\beta+1\right)\left(0\right)=\left(v_i^{\alpha}\lceil\beta+1\right)\left(0\right)=1\text{ and }u_0=v_0.$$

It follows that $u^{\alpha}\notin\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right).$

■

Lemma 13 *Let λ be a limit ordinal, $\alpha = \lambda + 1$, $w^\alpha \in W^\alpha$, $w_i^\alpha(\lambda) = 0$, and*

$$E \in \left[\left[\bigcup_{\beta<\lambda} \pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\beta\right)\right), \pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\lambda\right)\right)\right], P_i\left(w^\alpha\right)\right].$$

Then:

- *If $v^\alpha \in E$, then there is a $u^\alpha \in E$ such that $\lambda\text{-par}\left(u_j^\alpha\right) \neq \lambda\text{-par}\left(v_j^\alpha\right)$.*
- *If $v^\alpha \in W^\alpha \setminus E$, then there is a $u^\alpha \in W^\alpha \setminus E$ such that $\lambda\text{-par}\left(u_j^\alpha\right) \neq \lambda\text{-par}\left(v_j^\alpha\right)$.*
- *If*

$$E \supseteq \left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right],$$

then $E = W^\alpha$.

- *If*

$$E \subseteq \left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right],$$

then $E = \emptyset$.

Proof The first point of the Lemma follows from Lemma 9 and Lemma 11.

The second point follows from the first and the fact that

$$\left[\left[\bigcup_{\beta<\lambda} \pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\beta\right)\right), \pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\lambda\right)\right)\right], P_i\left(w^\alpha\right)\right]$$

is a field on W^α (and therefore it is closed under complements).

The last two points of the Lemma follow directly from the first two points.

■

Lemma 14 *Let β be an ordinal, $\alpha = (\beta+1)+1$, $w^\alpha \in W^\alpha$, $w_i^\alpha(\beta+1) = 0$, and*

$$E \in \left[\left[\pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\beta\right)\right), \pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\beta+1\right)\right)\right], P_i\left(w^\alpha\right)\right].$$

Then:

- *If $v^\alpha \in E$, then there is a $u^\alpha \in E$ such that $u_j^\alpha(\beta) \neq v_j^\alpha(\beta)$.*

- *If* $v^\alpha \in W^\alpha \setminus E$, *then there is a* $u^\alpha \in W^\alpha \setminus E$ *such that* $u_j^\alpha(\beta) \neq v_j^\alpha(\beta)$.
- *If*
$$E \supseteq \left[X_j^\alpha(\beta) = w_j^\alpha(\beta)\right],$$
then $E = W^\alpha$.
- *If*
$$E \subseteq \left[X_j^\alpha(\beta) = w_j^\alpha(\beta)\right],$$
then $E = \emptyset$.

Proof Note that if $\alpha = \beta + 1$, then

$$\bigcup_{\xi < \beta+1} \pi_{\xi,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\xi\right)\right) = \pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\beta\right)\right).$$

The proof is now analogous to the proof of Lemma 13, just replace λ by $\beta+1$ and Lemma 11 by Lemma 12. ■

Lemma 15 *Let* γ *be an ordinal* > 0, $\alpha = \gamma + 1$, $w^\alpha \in W^\alpha$, *and*

$$E \in \left[\bigcup_{\beta<\gamma} \pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\beta\right)\right), \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\gamma\right)\right)\right]$$

such that $E \supseteq P_i(w^\alpha)$.

Then

$$E \supseteq \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\gamma\right)\right).$$

Proof Since $P_i(w^\alpha) \subseteq \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\gamma\right)\right)$, it follows from the definition of

$$\left[\bigcup_{\beta<\gamma} \pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\beta\right)\right), \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\gamma\right)\right)\right],$$

that there is a $\beta < \gamma$ and a $E^\beta \subseteq W^\beta$ such that

$$E \cap \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\gamma\right)\right) = \pi_{\beta,\alpha}^{-1}\left(E^\beta\right) \cap \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\gamma\right)\right) \supseteq P_i(w^\alpha).$$

Claim: $\pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right) \supseteq \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right).$
Assume to the contrary, that there is a

$$v^{\alpha} \in \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right) \setminus \pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right).$$

Since $\beta+1 \leq \gamma$, we have $v^{\alpha}\lceil\beta+1 \in P_i\left(w^{\alpha}\lceil\beta+1\right)$. By Lemma 10, there is a $u^{\alpha} \in P_i\left(w^{\alpha}\right)$ such that $u^{\alpha}\lceil\beta = v^{\alpha}\lceil\beta$. Since $E^{\beta} \subseteq W^{\beta}$ and $v^{\alpha} \notin \pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right)$, it follows that $u^{\alpha} \notin \pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right)$, a contradiction to $\pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right) \supseteq P_i\left(w^{\alpha}\right)$. ■

Lemma 16 *Let λ be a limit ordinal, $w^{\lambda} \in W^{\lambda}$, $\beta < \lambda$, and $E^{\beta} \subseteq W^{\beta}$ such that $\pi_{\beta,\lambda}^{-1}\left(E^{\beta}\right) \supseteq P_i\left(w^{\lambda}\right)$.*
Then

$$\pi_{\beta,\lambda}^{-1}\left(E^{\beta}\right) \supseteq \pi_{\beta+1,\lambda}^{-1}\left(P_i\left(w^{\lambda}\lceil\beta+1\right)\right).$$

Proof Assume that there is a

$$v^{\lambda} \in \pi_{\beta+1,\lambda}^{-1}\left(P_i\left(w^{\lambda}\lceil\beta+1\right)\right) \setminus \pi_{\beta,\lambda}^{-1}\left(E^{\beta}\right).$$

By Lemma 10, there is a $u^{\lambda} \in P_i\left(w^{\lambda}\right)$ such that $u^{\lambda}\lceil\beta = v^{\lambda}\lceil\beta$. Therefore $u^{\lambda} \notin \pi_{\beta,\lambda}^{-1}\left(E^{\beta}\right)$, a contradiction to $\pi_{\beta,\lambda}^{-1}\left(E^{\beta}\right) \supseteq P_i\left(w^{\lambda}\right)$. ■

4.1 The Construction

Now, let κ be a fixed regular cardinal.

For every $w^{\kappa} \in W^{\kappa}$ and $i \in \{a,b\}$, we will define finitely additive probability measures $T_i\left(w^{\kappa}\right)$ on $\left(W^{\kappa}, \operatorname{Pow}\left(W^{\kappa}\right)\right)$ such that:

- $T_i\left(u^{\kappa}\right) = T_i\left(w^{\kappa}\right)$, for $u^{\kappa} \in P_i\left(w^{\kappa}\right)$,
- $T_i\left(w^{\kappa}\right)\left(P_i\left(w^{\kappa}\right)\right) = 1$,
- $T_i\left(w^{\kappa}\right)\left(\left[X_0^{\kappa} = w_0\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^{\kappa}(0) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^{\kappa}(0) = 0, \end{cases}$
- for $\beta < \kappa$:
  $T_i\left(w^{\kappa}\right)\left(\left[X_j^{\kappa}(\beta) = w_j^{\kappa}(\beta)\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^{\kappa}(\beta+1) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^{\kappa}(\beta+1) = 0, \end{cases}$

- for $\lambda < \kappa$ such that λ is a limit ordinal :
  $T_i(w^\kappa)\left(\left[\lambda\text{-par}\left(X_j^\kappa\right) = \lambda\text{-par}\left(w_j^\kappa\right)\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^\kappa(\lambda) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^\kappa(\lambda) = 0. \end{cases}$

If we define

$$\theta(w^\kappa) := w_0,$$

it follows from the first two points and the fact that $T_i(w^\kappa)$ will be defined on $(W^\kappa, \text{Pow}(W^\kappa))$, that

$$\left\langle W^\kappa, (T_i)_{i\in\{a,b\}}, \theta \right\rangle$$

will be a $*$-type space on $S = \{h, t\}$ for player set $\{a, b\}$.

The construction will not be carried out at once. By a transfinite induction on $1 \leq \alpha \leq \kappa$, we will endow W^α with fields $\mathcal{F}(i, w^\alpha)$ and finitely additive probability measures $T_i^\alpha(w^\alpha)$ on $(W^\alpha, \mathcal{F}(i, w^\alpha))$ such that:

Induction Hypothesis

1. $\mathcal{F}(i, w^\alpha) := \left[\bigcup_{\beta<\alpha}\left(\pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^\beta\right)\right)\right), P_i(w^\alpha)\right],$

2. for every $\beta < \alpha$ and $E^\beta \in \mathcal{F}(i, w^\alpha\lceil\beta)$:
   $T_i^\alpha(w^\alpha)\left(\pi_{\beta,\alpha}^{-1}\left(E^\beta\right)\right) = T_i^\beta(w^\alpha\lceil\beta)\left(E^\beta\right),$
   that is $\text{marg}_{\left(W^\beta, \mathcal{F}(i, w^\alpha\lceil\beta)\right)} T_i^\alpha(w^\alpha) = T_i^\beta(w^\alpha\lceil\beta),$

3. $T_i^\alpha(w^\alpha) = T_i^\alpha(u^\alpha)$, for $u^\alpha \in P_i(w^\alpha)$,

4. $T_i^\alpha(w^\alpha)(P_i(w^\alpha)) = 1,$

5. $T_i^\alpha(w^\alpha)([X_0^\alpha = w_0]) = \begin{cases} 1, & \text{if} \quad w_i^\alpha(0) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^\alpha(0) = 0, \end{cases}$

6. for β such that $\beta + 1 < \alpha$:
   $T_i^\alpha(w^\alpha)\left(\left[X_j^\alpha(\beta) = w_j^\alpha(\beta)\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^\alpha(\beta+1) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^\alpha(\beta+1) = 0, \end{cases}$

7. for $\lambda < \alpha$ such that λ is a limit ordinal :
   $T_i^\alpha(w^\alpha)\left(\left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^\alpha(\lambda) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^\alpha(\lambda) = 0. \end{cases}$

Since inverse images commute with complements, arbitrary unions and intersections, it follows:

Remark 12 *Let $\beta + 1 \leq \alpha$. Then:*

$$\pi^{-1}_{\beta+1,\alpha}\left(\mathcal{F}\left(i, w^{\alpha}\lceil\beta+1\right)\right) = \left[\pi^{-1}_{\beta,\alpha}\left(\mathrm{Pow}\left(W^{\beta}\right)\right), \pi^{-1}_{\beta+1,\alpha}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right)\right].$$

Remark 13 *Let $1 \leq \beta \leq \alpha \leq \kappa$ and let $u^{\alpha}, w^{\alpha} \in W^{\alpha}$ such that $u^{\alpha} \in P_i\left(w^{\alpha}\right)$ (and hence $P_i\left(u^{\alpha}\right) = P_i\left(w^{\alpha}\right)$). Then:*

- *We have*

$$\begin{array}{lcll} \mathcal{F}\left(i, w^{\alpha}\right) & = & \mathcal{F}\left(i, u^{\alpha}\right) & \textit{and} \\ \mathcal{F}\left(i, w^{\alpha}\lceil\beta\right) & = & \mathcal{F}\left(i, u^{\alpha}\lceil\beta\right). & \end{array}$$

- *If*

$$\begin{array}{lcll} T_i^{\alpha}\left(w^{\alpha}\right) & = & T_i^{\alpha}\left(u^{\alpha}\right), & \\ T_i^{\beta}\left(w^{\alpha}\lceil\beta\right) & = & T_i^{\beta}\left(u^{\alpha}\lceil\beta\right) & \textit{and} \\ \mathrm{marg}_{\left(W^{\beta}, \mathcal{F}\left(i, w^{\alpha}\lceil\beta\right)\right)} T_i^{\alpha}\left(w^{\alpha}\right) & = & T_i^{\beta}\left(w^{\alpha}\lceil\beta\right), & \end{array}$$

  then we also have

$$\mathrm{marg}_{\left(W^{\beta}, \mathcal{F}\left(i, u^{\alpha}\lceil\beta\right)\right)} T_i^{\alpha}\left(u^{\alpha}\right) = T_i^{\beta}\left(u^{\alpha}\lceil\beta\right).$$

Definition and Construction

(By transfinite induction on $1 \leq \alpha \leq \kappa$.)

Step $\alpha = 1$:

We have $W^0 = \{h, t\}$. Let $w^1 \in W^1$.
Define

$$\begin{array}{lcl} T_i^{<1}\left(w^1\right)\left(\left[X_0^1 = w_0\right]\right) & := & \left\{ \begin{array}{lll} 1, & \text{if} & w_i^1(0) = 1, \\ \frac{1}{2}, & \text{if} & w_i^1(0) = 0, \end{array} \right. \\ T_i^{<1}\left(w^1\right)\left(\left[X_0^1 \neq w_0\right]\right) & := & 1 - T_i^{<1}\left(w^1\right)\left(\left[X_0^1 = w_0\right]\right), \\ T_i^{<1}\left(w^1\right)(\emptyset) & := & 0, \\ T_i^{<1}\left(w^1\right)\left(W^1\right) & := & 1. \end{array}$$

It is clear, that by this definition, $T_i^{<1}(w^1)$ is a probability measure on

$$\left(W^1, \pi_{0,1}^{-1}\left(\text{Pow}\left(W^0\right)\right)\right).$$

Let $E^0 \subseteq W^0$ such that $P_i(w^1) \subseteq \pi_{0,1}^{-1}(E^0)$.

1. <u>case:</u> $w_i^1(0) = 1$: Then $E^0 = W^0$ or $E^0 = [X_0^0 = w_0]$, hence the outer measure $T_i^{<1}(w^1)^*(P_i(w^1))$ is equal to 1.

2. <u>case:</u> $w_i^1(0) = 0$: Then $E^0 = W^0$ and the outer measure $T_i^{<1}(w^1)^*(P_i(w^1))$ is equal to 1.

For $u^1 \in P_i(w^1)$, we have in both cases that $P_i(u^1) = P_i(w^1)$ and $T_i^{<1}(u^1) = T_i^{<1}(w^1)$. For each $P_i(u^1)$ such that $u^1 \in W^1$, chose a representing element $w^1 \in P_i(u^1) = P_i(w^1)$. By the Łoś-Marczewski Theorem, we can extend $T_i^{<1}(w^1)$ to a finitely additive probability measure $T_i^1(w^1)$ on the field

$$\mathcal{F}\left(i, w^1\right) = \left[\pi_{0,1}^{-1}\left(\text{Pow}\left(W^0\right)\right), P_i\left(w^1\right)\right]$$

such that $T_i^1(w^1)(P_i(w^1)) = 1$. Define $T_i^1(u^1) := T_i^1(w^1)$, for all $u^1 \in P_i(w^1)$.
(Note that for $u^1 \in P_i(w^1)$, it is the case that $\mathcal{F}(i, u^1) = \mathcal{F}(i, w^1)$.) $T_i^1(u^1)$ and $\mathcal{F}(i, u^1)$ satisfy the conditions 1.-7. of the induction hypothesis.

Step $\alpha = (\beta + 1) + 1$:

For each $P_i(u^\alpha)$ such that $u^\alpha \in W^\alpha$, chose a representing element $w^\alpha \in P_i(u^\alpha) = P_i(w^\alpha)$.

Let $T_i^{<\alpha}(w^\alpha)$ be the finitely additive probability measure defined on the field

$$\pi_{\beta+1,\alpha}^{-1}\left(\mathcal{F}\left(i, w^\alpha\lceil\beta+1\right)\right) = \left[\pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^\beta\right)\right), \pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\beta+1\right)\right)\right],$$

which is induced by $T_i^{\beta+1}(w^\alpha\lceil\beta+1)$ (as defined in Lemma 3). According to Lemma 15 and the induction hypothesis, we have for the outer measure of $P_i(w^\alpha)$: $T_i^{<\alpha}(w^\alpha)^*(P_i(w^\alpha)) = 1$. So, by the Łoś-Marczewski Theorem, we can extend $T_i^{<\alpha}(w^\alpha)$ to a finitely additive probability measure $\widetilde{T}_i^\alpha(w^\alpha)$ defined on the field

$$\widetilde{\mathcal{F}}\left(i, w^\alpha\right) := \left[\left[\pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^\beta\right)\right), \pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\beta+1\right)\right)\right], P_i\left(w^\alpha\right)\right]$$

such that $\widetilde{T}_i^\alpha(w^\alpha)(P_i(w^\alpha)) = 1$.

1. case: $w_i^\alpha(\beta+1)=1$: Then

$$\pi_{\beta+1,\alpha}^{-1}\left(\left[X_j^{\beta+1}(\beta)=\left(w_j^\alpha\lceil\beta+1\right)(\beta)\right]\right)=\left[X_j^\alpha(\beta)=w_j^\alpha(\beta)\right]\supseteq P_i(w^\alpha).$$

By Lemma 5, extend $\widetilde{T}_i^\alpha(w^\alpha)$ to a finitely additive probability measure $T_i^\alpha(w^\alpha)$ on the field

$$\left[\pi_{\beta+1,\alpha}^{-1}\left(\mathrm{Pow}\left(W^{\beta+1}\right)\right),P_i(w^\alpha)\right]=\mathcal{F}(i,w^\alpha).$$

By the above, we have

$$T_i^\alpha(w^\alpha)\left(\left[X_j^\alpha(\beta)=w_j^\alpha(\beta)\right]\right)=1.$$

Define now $T_i^\alpha(u^\alpha):=T_i^\alpha(w^\alpha)$, for all $u^\alpha\in P_i(w^\alpha)$. (Note that, for $u^\alpha\in P_i(w^\alpha)$, we have $\widetilde{\mathcal{F}}(i,u^\alpha)=\widetilde{\mathcal{F}}(i,w^\alpha)$, $\mathcal{F}(i,u^\alpha)=\mathcal{F}(i,w^\alpha)$, and $u_i^\alpha(\beta+1)=w_i^\alpha(\beta+1)=1$ and hence, $u_j^\alpha(\beta)=w_j^\alpha(\beta)$.) It is now easy to check that $T_i^\alpha(u^\alpha)$ and $\mathcal{F}(i,u^\alpha)$ satisfy the conditions 1.-7. of the induction hypothesis.

2. case: $w_i^\alpha(\beta+1)=0$: By Lemma 14 and the induction hypothesis, we have for the outer measure of $\left[X_j^\alpha(\beta)=w_j^\alpha(\beta)\right]$:

$$\widetilde{T}_i^\alpha(w^\alpha)^*\left(\left[X_j^\alpha(\beta)=w_j^\alpha(\beta)\right]\right)=1,$$

and for the inner measure

$$\widetilde{T}_i^\alpha(w^\alpha)_*\left(\left[X_j^\alpha(\beta)=w_j^\alpha(\beta)\right]\right)=0.$$

By the Łoś-Marczewski Theorem, we can extend $\widetilde{T}_i^\alpha(w^\alpha)$ to a finitely additive probability measure $\widehat{T}_i^\alpha(w^\alpha)$ on the field

$$\left[\widetilde{\mathcal{F}}(i,w^\alpha),\left[X_j^\alpha(\beta)=w_j^\alpha(\beta)\right]\right]$$

such that

$$\widehat{T}_i^\alpha(w^\alpha)\left(\left[X_j^\alpha(\beta)=w_j^\alpha(\beta)\right]\right)=\frac{1}{2}.$$

Finally, by Lemma 5, extend $\widehat{T}_i^\alpha(w^\alpha)$ to a finitely additive probability measure $T_i^\alpha(w^\alpha)$ on $\mathcal{F}(i,w^\alpha)$. Define now $T_i^\alpha(u^\alpha):=T_i^\alpha(w^\alpha)$, for all $u^\alpha\in P_i(w^\alpha)$. It is easy to check that $T_i^\alpha(u^\alpha)$ and $\mathcal{F}(i,u^\alpha)$ satisfy the conditions 1.-7. of the induction hypothesis.

Step $\alpha = \lambda$, λ limit ordinal.

For each $P_i(u^\alpha)$ such that $u^\alpha \in W^\alpha$, chose a representing element $w^\alpha \in P_i(u^\alpha) = P_i(w^\alpha)$.

Let $T_i^{<\alpha}(w^\alpha)$ be the finitely additive probability measure defined on the field

$$\bigcup_{\beta<\alpha} \pi_{\beta,\alpha}^{-1}(\mathcal{F}(i, w^\alpha \lceil \beta)) = \bigcup_{\beta<\alpha} \pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\beta\right)\right)$$

which is induced by $\left(T_i^\beta(w^\alpha \lceil \beta)\right)_{1\leq\beta<\alpha}$ (as defined in Lemma 3).

Let $\beta < \alpha$ and $E^\beta \subseteq W^\beta$ such that $\pi_{\beta,\alpha}^{-1}\left(E^\beta\right) \supseteq P_i(w^\alpha)$. By Lemma 16, we have

$$\pi_{\beta,\alpha}^{-1}\left(E^\beta\right) \supseteq \pi_{\beta+1,\alpha}^{-1}(P_i(w^\alpha \lceil \beta+1)).$$

(Note that $\beta < \alpha$ implies $\beta + 1 < \alpha$.) Since, by the definition of $T_i^{<\alpha}(w^\alpha)$,

$$T_i^{<\alpha}(w^\alpha)\left(\pi_{\beta+1,\alpha}^{-1}(P_i(w^\alpha \lceil \beta+1))\right) = T_i^{\beta+1}(w^\alpha \lceil \beta+1)(P_i(w^\alpha \lceil \beta+1)) = 1,$$

the outer measure $T_i^{<\alpha}(w^\alpha)^*(P_i(w^\alpha))$ is equal to 1. By the Łoś-Marczewski Theorem, we can extend $T_i^{<\alpha}(w^\alpha)$ to a finitely additive probability measure $T_i^\alpha(w^\alpha)$ on the field

$$\mathcal{F}(i, w^\alpha) = \left[\bigcup_{\beta<\alpha} \pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\beta\right)\right), P_i(w^\alpha)\right]$$

such that $T_i^\alpha(w^\alpha)(P_i(w^\alpha)) = 1$. For $u^\alpha \in P_i(w^\alpha)$ define $T_i^\alpha(u^\alpha) := T_i^\alpha(w^\alpha)$. It is easy to check that $T_i^\alpha(u^\alpha)$ and $\mathcal{F}(i, u^\alpha)$ satisfy the conditions 1.-7. of the induction hypothesis.

Step $\alpha = \lambda + 1$, λ limit ordinal:

For each $P_i(u^\alpha)$ such that $u^\alpha \in W^\alpha$, chose a representing element $w^\alpha \in P_i(u^\alpha) = P_i(w^\alpha)$.

Let $T_i^{<\alpha}(w^\alpha)$ be the finitely additive probability measure defined on the field

$$\pi_{\lambda,\alpha}^{-1}(\mathcal{F}(i, w^\alpha \lceil \lambda)) = \left[\bigcup_{\beta<\lambda} \pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^\beta\right)\right), \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha \lceil \lambda))\right],$$

which is induced by $T_i^{\lambda}(w^{\alpha}\lceil\lambda)$ (as defined in Lemma 3). According to Lemma 15 and the induction hypothesis, we have for the outer measure of $P_i(w^{\alpha})$: $T_i^{<\alpha}(w^{\alpha})^*(P_i(w^{\alpha})) = 1$. So, by the Łoś-Marczewski Theorem, we can extend $T_i^{<\alpha}(w^{\alpha})$ to a finitely additive probability measure $\widetilde{T}_i^{\alpha}(w^{\alpha})$ defined on the field

$$\widetilde{\mathcal{F}}(i, w^{\alpha}) := \left[\left[\bigcup_{\beta<\lambda} \pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^{\beta}\right)\right), \pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\lambda\right)\right)\right], P_i\left(w^{\alpha}\right)\right],$$

such that $\widetilde{T}_i^{\alpha}(w^{\alpha})(P_i(w^{\alpha})) = 1$.

1. case: $w_i^{\alpha}(\lambda) = 1$. Then

$$\begin{aligned} \pi_{\lambda,\alpha}^{-1}\left(\left[\lambda\text{-par}\left(X_j^{\lambda}\right) = \lambda\text{-par}\left(w_j^{\alpha}\lceil\lambda\right)\right]\right) &= \left[\lambda\text{-par}\left(X_j^{\alpha}\right) = \lambda\text{-par}\left(w_j^{\alpha}\right)\right] \\ &\supseteq P_i\left(w^{\alpha}\right). \end{aligned}$$

   By Lemma 5, extend $\widetilde{T}_i^{\alpha}(w^{\alpha})$ to a finitely additive probability measure $T_i^{\alpha}(w^{\alpha})$ on the field

$$\left[\pi_{\lambda,\alpha}^{-1}\left(\mathrm{Pow}\left(W^{\lambda}\right)\right), P_i\left(w^{\alpha}\right)\right] = \mathcal{F}\left(i, w^{\alpha}\right).$$

   By the above, we have

$$T_i^{\alpha}\left(w^{\alpha}\right)\left(\left[\lambda\text{-par}\left(X_j^{\alpha}\right) = \lambda\text{-par}\left(w_j^{\alpha}\right)\right]\right) = 1.$$

   Define now $T_i^{\alpha}(u^{\alpha}) := T_i^{\alpha}(w^{\alpha})$, for all $u^{\alpha} \in P_i(w^{\alpha})$. (Note that $\mathcal{F}(i, u^{\alpha}) = \mathcal{F}(i, w^{\alpha})$, $u_i^{\alpha}(\lambda) = w_i^{\alpha}(\lambda) = 1$ and hence, $\lambda\text{-par}\left(w_j^{\alpha}\right) = \lambda\text{-par}\left(u_j^{\alpha}\right)$.) It is now easy to check that $T_i^{\alpha}(u^{\alpha})$ and $\mathcal{F}(i, u^{\alpha})$ satisfy the conditions 1.-7. of the induction hypothesis.

2. case: $w_i^{\alpha}(\lambda) = 0$. By Lemma 13, we have for the outer measure of $\left[\lambda\text{-par}\left(X_j^{\alpha}\right) = \lambda\text{-par}\left(w_j^{\alpha}\right)\right]$:

$$\widetilde{T}_i^{\alpha}\left(w^{\alpha}\right)^*\left(\left[\lambda\text{-par}\left(X_j^{\alpha}\right) = \lambda\text{-par}\left(w_j^{\alpha}\right)\right]\right) = 1,$$

   and for the inner measure

$$\widetilde{T}_i^{\alpha}\left(w^{\alpha}\right)_*\left(\left[\lambda\text{-par}\left(X_j^{\alpha}\right) = \lambda\text{-par}\left(w_j^{\alpha}\right)\right]\right) = 0.$$

By the Łoś-Marczewski Theorem, we can extend $\widetilde{T}_i^{\alpha}(w^{\alpha})$ to a finitely additive probability measure $\widehat{T}_i^{\alpha}(w^{\alpha})$ on the field

$$\left[\widetilde{\mathcal{F}}(i, w^{\alpha}), \left[\lambda\text{-par}\left(X_j^{\alpha}\right) = \lambda\text{-par}\left(w_j^{\alpha}\right)\right]\right]$$

such that

$$\widehat{T}_i^{\alpha}(w^{\alpha})\left(\left[\lambda\text{-par}\left(X_j^{\alpha}\right) = \lambda\text{-par}\left(w_j^{\alpha}\right)\right]\right) = \frac{1}{2}.$$

Finally, by Lemma 5, extend $\widehat{T}_i^{\alpha}(w^{\alpha})$ to a finitely additive probability measure $T_i^{\alpha}(w^{\alpha})$ on $\mathcal{F}(i, w^{\alpha})$. Define now $T_i^{\alpha}(u^{\alpha}) := T_i^{\alpha}(w^{\alpha})$, for all $u^{\alpha} \in P_i(w^{\alpha})$. It is easy to check that $T_i^{\alpha}(u^{\alpha})$ and $\mathcal{F}(i, u^{\alpha})$ satisfy the conditions 1.-7. of the induction hypothesis.

Remaining Step:

We have to extend $T_i^{\kappa}(u^{\kappa})$ to a finitely additive probability measure defined on the field $\text{Pow}(W^{\kappa})$:

By the inductive construction, $T_i^{\kappa}(u^{\kappa})$ is defined on

$$\left[\bigcup_{\beta<\kappa} \pi_{\beta,\kappa}^{-1}\left(\text{Pow}\left(W^{\beta}\right)\right), P_i(w^{\kappa})\right]$$

such that 1.-7. of the induction hypothesis are satisfied. For each $P_i(u^{\kappa})$ such that $u^{\kappa} \in W^{\kappa}$, chose a representing element

$$w^{\kappa} \in P_i(u^{\kappa}) = P_i(w^{\kappa}).$$

Now, by Lemma 5, extend $T_i^{\kappa}(w^{\kappa})$ to a finitely additive probability measure $T_i(w^{\kappa})$ on the field $\text{Pow}(W^{\kappa})$ and define

$$T_i(u^{\kappa}) := T_i(w^{\kappa}),$$

for $u^{\kappa} \in P_i(w^{\kappa})$.

Finally, let

$$\theta(w^{\kappa}) := w_0 \in S,$$

for $w^{\kappa} \in W^{\kappa}$.

$$\underline{W}^{\kappa} := \langle W^{\kappa}, T_a, T_b, \theta\rangle$$

has now all the desired properties.

Next, by induction on the formation of κ-expressions, we define the *depth* of a κ-expression:

Definition 27
- If $E \in \Sigma_S$, then

$$\mathrm{dp}\,(E) := 0,$$

- if $0 \leq p \leq 1$, $i \in I$ and if φ is a κ-expression, then

$$\mathrm{dp}\,(B_i^p\,(\varphi)) := \mathrm{dp}\,(\varphi) + 1,$$

- if φ is a κ-expression, then

$$\mathrm{dp}\,(\neg\varphi) := \mathrm{dp}\,(\varphi)\,,$$

- if Ψ is a set of κ-expressions such that $|\Psi| < \kappa$, then

$$\mathrm{dp}\Big(\bigwedge_{\varphi\in\Psi}\varphi\Big) := \sup\left\{\mathrm{dp}\,(\varphi) \mid \varphi \in \Psi\right\}.$$

It is easy to see that, since κ is regular, the depth of a κ-expression is always strictly smaller than κ.

Lemma 17 *Let $\alpha \leq \kappa$, $w^\kappa, u^\kappa \in W^\kappa$ and $w^\kappa\lceil\alpha = u^\kappa\lceil\alpha$.*
Then, for all κ-expressions φ such that $\mathrm{dp}\,(\varphi) \leq \alpha$:

$$w^\kappa \in \varphi^{W^\kappa} \textit{ iff } u^\kappa \in \varphi^{W^\kappa}.$$

Proof We prove the Lemma by induction on the formation of κ-expressions.

- Let $\varphi = E$, where $E \in \Sigma_{\{h,t\}} = \mathrm{Pow}\,(\{h,t\})$, and let $w^\kappa, u^\kappa \in W^\kappa$ such that $w^\kappa\lceil 0 = u^\kappa\lceil 0$. By definition, $v^\kappa \in E^{W^\kappa}$ iff $\theta\,(v^\kappa) \in E$, for $v^\kappa \in W^\kappa$. But we have $\theta\,(v^\kappa) = v_0 = v^\kappa\lceil 0$, for $v^\kappa \in W^\kappa$. It follows that $u^\kappa \in E^{W^\kappa}$ iff $w^\kappa \in E^{W^\kappa}$.

- Let $\varphi = \neg\psi$ such that $\operatorname{depth}(\varphi) \leq \alpha$ and let $w^\kappa, u^\kappa \in W^\kappa$ such that $w^\kappa\lceil\alpha = u^\kappa\lceil\alpha$. It follows that $\operatorname{depth}(\psi) \leq \alpha$ and $u^\kappa \in \varphi^{W^\kappa}$ iff $u^\kappa \notin \psi^{W^\kappa}$ iff- by the induction assumption - $w^\kappa \notin \psi^{W^\kappa}$, which is the case iff $w^\kappa \in \varphi^{W^\kappa}$.

- Let $p \in [0,1]$, $i \in \{a,b\}$, $\varphi = B_i^p(\psi)$, $\operatorname{dp}(\psi) + 1 = \beta + 1 \leq \alpha$, and $w^\kappa, u^\kappa \in W^\kappa$ such that $w^\kappa\lceil\alpha = u^\kappa\lceil\alpha$. By the induction assumption, there is a $E^\beta \subseteq W^\beta$ such that $\psi^{W^\kappa} = \pi_{\beta,\kappa}^{-1}\left(E^\beta\right)$. By the definition,

$$\begin{aligned} T_i(u^\kappa)\left(\psi^{W^\kappa}\right) &= T_i^\alpha(u^\kappa\lceil\alpha)\left(\pi_{\beta,\alpha}^{-1}\left(E^\beta\right)\right) \\ &= T_i^\alpha(w^\kappa\lceil\alpha)\left(\pi_{\beta,\alpha}^{-1}\left(E^\beta\right)\right) \\ &= T_i(w^\kappa)\left(\psi^{W^\kappa}\right). \end{aligned}$$

  It follows that $u^\kappa \in \left(B_i^p(\psi)\right)^{W^\kappa}$ iff $w^\kappa \in \left(B_i^p(\psi)\right)^{W^\kappa}$.

- Let $|\Psi| < \kappa$, $\varphi = \bigwedge_{\psi\in\Psi}\psi$ such that $\operatorname{depth}(\varphi) \leq \alpha$, and let $w^\kappa, u^\kappa \in W^\kappa$ such that $w^\kappa\lceil\alpha = u^\kappa\lceil\alpha$. Then $\operatorname{depth}(\psi) \leq \alpha$, for $\psi \in \Psi$. By the induction assumption, $u^\kappa \in \psi^{W^\kappa}$ iff $w^\kappa \in \psi^{W^\kappa}$, for $\psi \in \Psi$. It follows that $u^\kappa \in \varphi^{W^\kappa}$ iff $w^\kappa \in \varphi^{W^\kappa}$.

■

Lemma 18 *In the $*$-type space $\left\langle W^\kappa, (T_i)_{i\in\{a,b\}}, \theta\right\rangle$, we have:*

$$\begin{aligned} [X_i^\kappa(0) = 1] &= \overline{B}_i^1\left([X_0^\kappa = h]\right) \cup \overline{B}_i^1\left([X_0^\kappa = t]\right). \\ [X_i^\kappa(\beta+1) = 1] &= \overline{B}_i^1\left(\left[X_j^\kappa(\beta) = 1\right]\right) \cup \overline{B}_i^1\left(\left[X_j^\kappa(\beta) = 0\right]\right), \\ &\qquad \textit{for all ordinals } \beta < \kappa. \\ [X_i^\kappa(\lambda) = 1] &= \overline{B}_i^1\left(\left[\lambda\text{-par}\left(X_j^\kappa\right) = \textit{even}\right]\right) \cup \overline{B}_i^1\left(\left[\lambda\text{-par}\left(X_j^\kappa\right) = \textit{odd}\right]\right), \\ &\qquad \textit{for all limit ordinals } \lambda < \kappa. \end{aligned}$$

Proof Follows directly from the construction of $\left\langle W^\kappa, (T_i)_{i\in\{a,b\}}, \theta\right\rangle$. ■

Lemma 19 *In the $*$-type space $\left\langle W^\kappa, (T_i)_{i\in\{a,b\}}, \theta\right\rangle$, we have:*

- $$\begin{aligned} \{h\}^{W^\kappa} &= [X_0^\kappa = h], \\ \{t\}^{W^\kappa} &= [X_0^\kappa = t], \\ \operatorname{dp}(\{h\}) &= \operatorname{dp}(\{t\}) = 0. \end{aligned}$$

- *For every $i \in \{a,b\}$ and β such that $0 \leq \beta < \kappa$, there are κ-expressions $\varphi_i^0(\beta)$ and $\varphi_i^1(\beta)$ with*

$$\begin{array}{rcll} \mathrm{dp}\left(\varphi_i^0(\beta)\right) & = & \mathrm{dp}\left(\varphi_i^1(\beta)\right) & \\ & = & \beta+1 & \textit{such that} \\ \left(\varphi_i^1(\beta)\right)^{W^\kappa} & = & \left[X_i^\kappa(\beta)=1\right] & \textit{and} \\ \left(\varphi_i^0(\beta)\right)^{W^\kappa} & = & \left[X_i^\kappa(\beta)=0\right]. & \end{array}$$

- *For every $i \in \{a,b\}$ and limit ordinal $\lambda < \kappa$, there are κ-expressions $\varphi_i^{even}(\lambda)$ and $\varphi_i^{odd}(\lambda)$ with*

$$\begin{array}{rcll} \mathrm{dp}(\varphi_i^{even}(\lambda)) & = & \mathrm{dp}(\varphi_i^{odd}(\lambda)) & \\ & = & \lambda & \textit{such that} \\ (\varphi_i^{even}(\lambda))^{W^\kappa} & = & [\lambda\text{-par}\left(X_i^\kappa\right) = \textit{even}] & \textit{and} \\ (\varphi_i^{odd}(\lambda))^{W^\kappa} & = & [\lambda\text{-par}\left(X_i^\kappa\right) = \textit{odd}]. & \end{array}$$

Proof The first point is clear. We show the second and the third point by a transfinite induction on $0 \leq \beta < \kappa$.

- $\beta = 0$: According to Lemma 18 and the first point of this Lemma, we

$$\begin{array}{lrcl} \text{have} & \left[X_i^\kappa(0)=1\right] & = & \left(B_i^1(\{h\}) \vee B_i^1(\{t\})\right)^{W^\kappa} \\ \text{and} & \left[X_i^\kappa(0)=0\right] & = & \left(\neg\left(B_i^1(\{h\}) \vee B_i^1(\{t\})\right)\right)^{W^\kappa}. \end{array}$$

- $\beta = \gamma + 1$: According to the induction assumption, there are κ-expressions $\varphi_j^0(\gamma)$ and $\varphi_j^1(\gamma)$ such that

$$\begin{array}{rcll} \left(\varphi_j^0(\gamma)\right)^{W^\kappa} & = & \left[X_j^\kappa(\gamma)=0\right], & \\ \left(\varphi_j^1(\gamma)\right)^{W^\kappa} & = & \left[X_j^\kappa(\gamma)=1\right] & \text{and} \\ \mathrm{dp}\left(\varphi_j^0(\gamma)\right) & = & \mathrm{dp}\left(\varphi_j^1(\gamma)\right) & \\ & = & \beta. & \end{array}$$

  Define

$$\varphi_i^1(\beta) := B_i^1\left(\varphi_j^0(\gamma)\right) \vee B_i^1\left(\varphi_j^1(\gamma)\right)$$

  and

$$\varphi_i^0(\beta) := \neg\varphi_i^1(\beta).$$

  $\varphi_i^1(\beta)$ and $\varphi_i^0(\beta)$ are κ-expressions of depth $\beta+1$. We have

$$\left[X_i^\kappa(\beta)=0\right] = W^\kappa \setminus \left[X_i^\kappa(\beta)=1\right].$$

By Lemma 18 and the induction assumption, it follows that

$$\begin{array}{rcl} [X_i^\kappa(\beta)=1] & = & (\varphi_i^1(\beta))^{W^\kappa} \quad \text{and} \\ [X_i^\kappa(\beta)=0] & = & (\varphi_i^0(\beta))^{W^\kappa}. \end{array}$$

- Let $\lambda<\kappa$ be a limit ordinal: For $i\in\{a,b\}$ and $\beta<\lambda$ define in $\underline{W}^\kappa$:

$$\begin{array}{rcl} \left[Y_i^\lambda(\beta)\right] & := & [X_i^\kappa(\beta)=1]\cap\bigcap_{\beta<\alpha<\lambda}[X_i^\kappa(\alpha)=0], \\ \left[Z_i^\lambda\right] & := & \bigcap_{0\le\alpha<\lambda}[X_i^\kappa(\alpha)=0]. \end{array}$$

According to the induction assumption and the fact that $|\lambda|<\kappa$, it follows that

$$\begin{array}{rcl} \psi_i^\lambda(\beta) & := & \varphi_i^1(\beta)\wedge\bigwedge_{\beta<\alpha<\lambda}\varphi_i^0(\alpha) \quad \text{and} \\ \chi_i^\lambda & := & \bigwedge_{0\le\alpha<\lambda}\varphi_i^0(\alpha) \end{array}$$

are κ-expressions such that

$$\begin{array}{rcl} \mathrm{dp}\left(\psi_i^\lambda(\beta)\right) & = & \max\{\beta+1,\sup\{\mathrm{dp}(\varphi_i^0(\alpha))\mid\beta<\alpha<\lambda\}\} \\ & = & \sup\{\alpha+1\mid\alpha<\lambda\} \\ & = & \lambda, \end{array}$$

and, similarly,

$$\mathrm{dp}\left(\chi_i^\lambda\right)=\lambda.$$

It follows from the induction assumption that

$$\left[Y_i^\lambda(\beta)\right]=\left(\psi_i^\lambda(\beta)\right)^{W^\kappa} \text{ and } \left[Z_i^\lambda\right]=\left(\chi_i^\lambda\right)^{W^\kappa}.$$

Since $o^\lambda(w_i^\kappa)$, for $w_i^\kappa\in W_i^\kappa$, can never be a limit ordinal, we have

$$\begin{array}{rcl} [\lambda\text{-par}(X_i^\kappa)=\text{even}] & = & \left[Z_i^\lambda\right]\cup\bigcup_{\beta<\lambda,\ \beta\text{ odd}}\left[Y_i^\lambda(\beta)\right] \quad \text{and} \\ [\lambda\text{-par}(X_i^\kappa)=\text{odd}] & = & \bigcup_{\beta<\lambda,\ \beta\text{even}}\left[Y_i^\lambda(\beta)\right]. \end{array}$$

Again, since $|\lambda|<\kappa$, it follows from the above that

$$\begin{array}{rcl} \varphi_i^{\text{even}}(\lambda) & := & \chi_i^\lambda\vee\bigvee_{\beta<\lambda,\ \beta\text{odd}}\psi_i^\lambda(\beta) \quad \text{and} \\ \varphi_i^{\text{odd}}(\lambda) & := & \bigvee_{\beta<\lambda,\ \beta\text{even}}\psi_i^\lambda(\beta) \end{array}$$

are κ-expressions such that

$$\mathrm{dp}(\varphi_i^{\text{even}}(\lambda))=\max\left\{\mathrm{dp}(\chi_i^\lambda),\ \sup\left\{\mathrm{dp}\left(\psi_i^\lambda(\beta)\right)\mid\beta<\lambda,\ \beta\text{ odd}\right\}\right\}=\lambda,$$

and $\quad \mathrm{dp}(\varphi_i^{\text{odd}}(\lambda))=\lambda.$

By the definitions and the induction assumption, we have

$$\begin{array}{rcl} (\varphi_i^{\text{even}}(\lambda))^{W^\kappa} & = & [\lambda\text{-par}(X_i^\kappa)=\text{even}] \quad \text{and} \\ (\varphi_i^{\text{odd}}(\lambda))^{W^\kappa} & = & [\lambda\text{-par}(X_i^\kappa)=\text{odd}]. \end{array}$$

- $\beta = \lambda$, λ limit ordinal $< \kappa$: By Lemma 18, and the above we have

$$\begin{array}{lcl} \left(B_i^1(\varphi_j^{\text{even}}(\lambda)) \vee B_i^1(\varphi_j^{\text{odd}}(\lambda))\right)^{W^\kappa} & = & [X_i^\kappa(\lambda) = 1], \\ \left(\neg\left(B_i^1(\varphi_j^{\text{even}}(\lambda)) \vee B_i^1(\varphi_j^{\text{odd}}(\lambda))\right)\right)^{W^\kappa} & = & [X_i^\kappa(\lambda) = 0], \end{array}$$

  and

$$\begin{array}{lcl} \mathrm{dp}\left(B_i^1(\varphi_j^{\text{even}}(\lambda)) \vee B_i^1(\varphi_j^{\text{odd}}(\lambda))\right)^{W^\kappa} & = & \\ \mathrm{dp}\left(\neg\left(B_i^1(\varphi_j^{\text{even}}(\lambda)) \vee B_i^1(\varphi_j^{\text{odd}}(\lambda))\right)\right)^{W^\kappa} & = & \lambda + 1. \end{array}$$

■

Theorem 2 *For every ordinal $\alpha < \kappa$ there are $u^\kappa, w^\kappa \in W^\kappa$ such that:*

1. *For all κ-expressions φ with $\mathrm{dp}(\varphi) \leq \alpha$:*

$$u^\kappa \in \varphi^{W^\kappa} \text{ iff } w^\kappa \in \varphi^{W^\kappa}.$$

2. *There is a κ-expression ψ with $\mathrm{dp}(\psi) = \alpha + 1$ such that*

$$u^\kappa \in \psi^{W^\kappa} \text{ and } w^\kappa \in (\neg\psi)^{W^\kappa}.$$

Proof Let $\alpha < \kappa$ and $i \in \{a, b\}$. By the definition of W^κ, there are $u^\kappa, w^\kappa \in W^\kappa$ such that $u^\kappa \lceil \alpha = w^\kappa \lceil \alpha$ and $1 = u_i^\kappa(\alpha) \neq w_i^\kappa(\alpha) = 0$. The first point follows now by Lemma 17. By Lemma 19, it follows that $u^\kappa \in \varphi_i^1(\alpha)^{W^\kappa}$, $w^\kappa \in (\neg\varphi_i^1(\alpha))^{W^\kappa}$, and $\mathrm{dp}(\varphi_i^1(\alpha)) = \alpha + 1$. ■

Note that Lemma 17 and the proof of Theorem 2 show that the α-th levels of u_i^κ and u_j^κ determine the κ-expressions of depth $\alpha + 1$.

Theorem 3 *Let $|I| \geq 2$ and $|S| \geq 2$. Then, there is no weak-universal ∞-type space on S for player set I and there is no weak-universal $*$-type space on S for player set I.*

Proof Assume there is a weak-universal ∞-type space (a weak-universal $*$-type space, respectively)

$$\underline{U} = \left\langle U, (T_i)_{i \in I}, \Sigma, \theta^U \right\rangle$$

on S for player set I. Then, the underlying set U has a cardinality $|U|$. There is a regular cardinal number $\kappa > |U|$.

$$\underline{W}^{\kappa} = \left\langle W^{\kappa}, (T_i)_{i \in \{a,b\}}, \mathrm{Pow}(W^{\kappa}), \theta \right\rangle$$

is a $*$-type space on $\{h, t\}$ (and therefore a ∞-type space). Since $|\{h,t\}| = 2$, we can assume without loss of generality that $\{h,t\} \subseteq S$, and since $\Sigma_S = \mathrm{Pow}(S)$, that $\Sigma_S \supseteq \mathrm{Pow}(\{h,t\})$. Also, since $|I| \geq 2$, we can assume without loss of generality that $\{a,b\} \subseteq I$. For $i \in I \setminus \{a,b\}$ define $T_i(w^{\kappa}) := \delta_{w^{\kappa}}$, and view θ as a function from W^{κ} to S. Then

$$\underline{W}_I^{\kappa} := \left\langle W^{\kappa}, (T_i)_{i \in I}, \mathrm{Pow}(W^{\kappa}), \theta \right\rangle$$

is a $*$-type space on S (with player set I). Since every $*$-type space is a ∞-type space, $\underline{W}_I^{\kappa}$ is also a ∞-type space. According to the assumption, there is a type morphism $f : W^{\kappa} \to U$. Since both spaces are in particular κ-type spaces, this morphism preserves κ-descriptions. If φ is a κ -expression in the 'language' corresponding to the set of states of nature $\{h,t\}$ and the player set $\{a,b\}$, then φ is also a κ-expression in the 'language' corresponding to the set of states of nature S and the player set I, and it is easy to check that for $w^{\kappa} \in W^{\kappa}$ we have $w^{\kappa} \in \varphi^{\underline{W}_I^{\kappa}}$ iff $w^{\kappa} \in \varphi^{\underline{W}^{\kappa}}$. So, by Lemma 19, it is still the case that two different states of $\underline{W}_I^{\kappa}$ have different κ-descriptions. Hence, since by Proposition 1, f preserves κ-descriptions, f is one-to-one. It follows that $|U| \geq |\underline{W}_I^{\kappa}| \geq \kappa$, which is a contradiction to $|U| < \kappa$. ■

Corollary 1 *Let $|I| \geq 2$ and $|S| \geq 2$. Then there is no universal ∞-type space on S for player set I and there is no universal $*$-type space on S for player set I.*

5 The Universal κ-Type Space as a Space of Consistent Hierarchies

For the rest of this paper, fix a nonempty player set I (of arbitrary cardinality), a nonempty set of states of nature S, and a κ-field Σ_S on S such that for all $s, s' \in S$ with $s \neq s'$ there is a $E \in \Sigma_S$ such that $s \in E$ and $s' \notin E$. We assume, without loss of generality, that $0 \notin I$ and define $I_0 := I \cup \{0\}$, where "0" stands for nature. For $i \in I$, we define $-i := I_0 \setminus \{i\}$.

Given the above data, we construct the space of consistent hierarchies up to (but excluding) level κ. We show that the set of all such consistent hierarchies generates a product κ-type space (Theorem 4) and that this space is the universal κ-type space (Theorem 5). It is easy to see that levels of the consistent hierarchies correspond to depths of κ-expressions. So, Theorem 2 (together with Proposition 1) shows that the hierarchies up to some lower level than κ would not suffice to construct the universal κ-type space.

The remarkable fact of the construction in this paper is that, unlike the construction in Section 3, it does not use other κ-type spaces. Therefore, it provides an independent characterization of the universal κ-type space as the space of consistent hierarchies. So, we learn here what the states of the world in the universal κ-type space 'are'.

Definition 28 Define

- $C_0^\alpha := S$, for $\kappa \geq \alpha \geq 0$,
- $\Sigma_0^\alpha := \Sigma_S$, for $\kappa \geq \alpha \geq 0$,
- C_i^0 to be a singleton, for $i \in I$,
- $\Sigma_i^0 := \{\emptyset, C_i^0\}$, for $i \in I$,
- $C^\alpha := \prod_{i \in I_0} C_i^\alpha$, for $\kappa \geq \alpha \geq 0$,
- Σ^α to be the product κ-field of the Σ_i^α, $i \in I_0$, on C^α, for $\kappa \geq \alpha \geq 0$,
- $C_{-i}^\alpha := \prod_{j \in -i} C_j^\alpha$, for $\kappa \geq \alpha \geq 0$ and $i \in I$,
- Σ_{-i}^α to be the product κ-field of the Σ_j^α, $j \in -i$, on C_{-i}^α, for $\kappa \geq \alpha \geq 0$ and $i \in I$,
- $C_i^\alpha :=$
$$\left\{ (t_i^\beta)_{\beta<\alpha} \in \prod_{\beta<\alpha} \Delta^\kappa\left(C^\beta\right) \;\middle|\; \begin{array}{rcl} \forall \beta < \gamma < \alpha & : & \operatorname{marg}_{C^\beta}\left(t_i^\gamma\right) = t_i^\beta \quad \text{and} \\ \forall 0 < \gamma < \alpha & : & \operatorname{marg}_{C_i^\gamma}\left(t_i^\gamma\right) = \delta_{\left(\left(t_i^\beta\right)_{\beta<\gamma}\right)} \end{array} \right\},$$
for $\kappa \geq \alpha \geq 1$ and $i \in I$,
- Σ_i^α to be the κ-field on C_i^α inherited from the product κ-field of the $\Sigma_{\Delta^\kappa\left(C^\beta\right)}$, $\beta < \alpha$, on $\prod_{\beta<\alpha} \Delta^\kappa\left(C^\beta\right)$, for $\kappa \geq \alpha \geq 1$ and $i \in I$,
- $\pi_{\gamma,\alpha}^0 : C_0^\alpha \to C_0^\gamma$ to be the identity on S, for $\kappa \geq \alpha \geq \gamma \geq 0$,
- $\pi_{0,\alpha}^i : C_i^\alpha \to C_i^0$ to be the obvious map, for $\kappa \geq \alpha \geq 0$ and $i \in I$,
- $\pi_{\gamma,\alpha}^i : C_i^\alpha \to C_i^\gamma$ to be the canonical projection $\pi_{\gamma,\alpha}^i((t_i^\beta)_{\beta<\alpha}) := (t_i^\beta)_{\beta<\gamma}$, for $\kappa \geq \alpha \geq \gamma \geq 1$ and $i \in I$,

- $\pi_{\gamma,\alpha} := (\pi^i_{\gamma,\alpha})_{i\in I_0}$, for $\kappa \geq \alpha \geq \gamma \geq 0$,
- $\pi^{-i}_{\gamma,\alpha} := (\pi^j_{\gamma,\alpha})_{j\in -i}$, for $\kappa \geq \alpha \geq \gamma \geq 0$ and $i \in I$.

Notation 3 Let $0 \leq \gamma < \alpha \leq \kappa$ and $c_i^\alpha = (t_i^\beta)_{\beta<\alpha} \in C_i^\alpha$. Then we define

$$\begin{array}{lll} c_i^\alpha(\gamma) & := & t_i^\gamma, \\ c_i^\alpha\lceil 0 & := & c_i^0, \qquad \text{where } C_i^0 = \{c_i^0\}, \\ c_i^\alpha\lceil\gamma & := & (t_i^\beta)_{\beta<\gamma}, \quad \text{for } \gamma > 0. \end{array}$$

Lemma 20 *1. For every $i \in I_0$ and for every ordinal α such that $0 \leq \alpha \leq \kappa$:*

$$C_i^\alpha \quad \textit{is nonempty.}$$

2. For every $i \in I_0$ and for all ordinals α, β such that $0 \leq \beta < \alpha \leq \kappa$:

$$\pi^i_{\beta,\alpha} : C_i^\alpha \to C_i^\beta$$

is onto and measurable.

Proof For $i = 0$, 1. and 2. are clear by the definitions.

For $i \in I$ and every $\alpha \leq \kappa$, the measurability of $\pi^i_{0,\alpha} : C_i^\alpha \to C_i^0$ is clear, since C_i^0 is a singleton. For $i \in I$ and $1 \leq \gamma < \alpha \leq \kappa$, the measurability of $\pi^i_{\gamma,\alpha}$ follows from the fact that the inverse images of the generators of the product κ-field of the $\Sigma_{\Delta^\kappa\left(C^\beta\right)}$, $\beta < \gamma$, on $\Pi_{\beta<\gamma}\Delta^\kappa\left(C^\beta\right)$ are among the generators of the product κ-field of the $\Sigma_{\Delta^\kappa\left(C^\beta\right)}$, $\beta < \alpha$, on $\Pi_{\beta<\alpha}\Delta^\kappa\left(C^\beta\right)$. Note that by the nonemptyness of C_i^0, for $i \in I$, 2. implies 1. Note furthermore that, if $0 \leq \beta < \gamma \leq \kappa$ and if $\pi^i_{\beta,\gamma} : C_i^\gamma \to C_i^\beta$, for $i \in I_0$, is onto and measurable, then $\pi_{\beta,\gamma} : C^\gamma \to C^\beta$ is onto and measurable.

It remains to show by transfinite induction on $\alpha \leq \kappa$ that $\pi^i_{\beta,\alpha} : C_i^\alpha \to C_i^\beta$ is onto, for $i \in I$ and $0 \leq \beta < \alpha \leq \kappa$:

- $\alpha = 0$: Since there is no ordinal $\beta < 0$, there is nothing to show.
- $\alpha = 1$: Since S and the C_i^0, for $i \in I$, are nonempty, C^0 is nonempty. It follows that $\Delta^\kappa\left(C^0\right)$ is nonempty: Take for example $c^0 \in C^0$, and define for $E \in \Sigma^0$:

$$\delta_{c^0}(E) := \begin{cases} 1 & \text{if } c^0 \in E, \\ 0 & \text{if } c^0 \notin E. \end{cases}$$

δ_{c^0} is the so-called δ-measure at c^0 and it is a (even σ-additive) probability measure on (C^0, Σ^0). Since C_i^0, for $i \in I$, is a singleton, we have $\operatorname{marg}_{C_i^0}(\delta_{c^0}) = \delta_{c_i^0}$, for $i \in I$, where $\{c_i^0\} = C_i^0$. By definition, we have $C_i^1 = \Delta^\kappa(C^0)$, for $i \in I$. Since C_i^0, for $i \in I$, is a singleton, it follows that $\pi_{0,1}^i$ is onto, for $i \in I$.

- Let $i \in I$ and let $\alpha < \kappa$ be a successor ordinal, i.e. $\alpha = \gamma + 1$, for an ordinal $\gamma > 0$. Since for every $\zeta < \gamma$ and for every $c_i^\zeta \in C_i^\zeta$ there is, by the induction assumption, a $c_i^\gamma \in C_i^\gamma$ with $\pi_{\zeta,\gamma}^i(c_i^\gamma) = c_i^\zeta$, it is enough to show that for every $c_i^\gamma \in C_i^\gamma$ there is a $c_i^\alpha \in C_i^\alpha$ with $\pi_{\gamma,\alpha}^i(c_i^\alpha) \in c_i^\gamma$. So, let $c_i^\gamma = (t_i^\beta)_{\beta<\gamma} \in C_i^\gamma$. We have to find a $t_i^\gamma \in \Delta^\kappa(C^\gamma)$ such that $\operatorname{marg}_{C^\beta}(t_i^\gamma) = t_i^\beta$, for all $\beta < \gamma$ and $\operatorname{marg}_{C_i^\gamma}(t_i^\gamma) = \delta_{\left(\left(t_i^\beta\right)_{\beta<\gamma}\right)}$. By the induction assumption, and in the case of $j = 0$, by the definition, $\pi_{\beta,\xi}^j : C_j^\xi \to C_j^\beta$ is onto and measurable, for all $j \in I_0$ and all ordinals β, ξ such that $0 \le \beta \le \xi \le \gamma$. This and the fact that $\operatorname{marg}_{C^\beta}(t_i^\xi) = t_i^\beta$, for $\beta < \xi < \gamma$, imply by Lemma 3 that c_i^γ induces a well-defined finitely additive probability measure $t_i^{<\gamma}$ on the field

$$\mathcal{F} := \left\{E \subseteq C^\gamma \mid \exists \beta < \gamma : E = \pi_{\beta,\gamma}^{-1}(F) \text{ for a } F \in \Sigma^\beta\right\} \subseteq \Sigma^\gamma$$

by $t_i^{<\gamma}\left(\pi_{\beta,\gamma}^{-1}(F)\right) := t_i^\beta(F)$. We show now that the outer measure $\left(t_i^{<\gamma}\right)^*\left(\{c_i^\gamma\} \times C_{-i}^\gamma\right)$ of $\{c_i^\gamma\} \times C_{-i}^\gamma$ is equal to 1. Clearly, it cannot be greater, since $t_i^\beta\left(C^\beta\right) = 1$, for $\beta < \gamma$. Let now $\beta < \gamma$, $E^\beta \in \Sigma^\beta$ and $\pi_{\beta,\gamma}^{-1}\left(E^\beta\right) \supseteq \{c_i^\gamma\} \times C_{-i}^\gamma$. Then $E^\beta \supseteq \left\{\pi_{\beta,\gamma}^i(c_i^\gamma)\right\} \times C_{-i}^\beta$ and therefore $t_i^\beta\left(E^\beta\right) = 1$. This implies that $t_i^{<\gamma}\left(\pi_{\beta,\gamma}^{-1}\left(E^\beta\right)\right) = 1$. By the Łoś-Marczewski Theorem, we can extend $t_i^{<\gamma}$ to a finitely additive probability measure μ_0 on the field $\left[\mathcal{F}, \{c_i^\gamma\} \times C_{-i}^\gamma\right]$ on C^γ such that $\mu_0\left(\{c_i^\gamma\} \times C_{-i}^\gamma\right) = 1$. By Lemma 5, we can extend μ_0 to a finitely additive probability measure μ_1 on the field $\left[\Sigma^\gamma, \{c_i^\gamma\} \times C_{-i}^\gamma\right]$. Define now t_i^γ as the restriction of μ_1 to Σ^γ. t_i^γ has the desired properties.

- Let $i \in I$, let α be a limit ordinal $\le \kappa$, let $\gamma < \alpha$, and let $c_i^\gamma \in C_i^\gamma$. Consider the set

$$Y := \left\{ a_i^\zeta \in \bigcup_{\gamma \le \zeta \le \alpha} C_i^\zeta \;\middle|\; \pi_{\gamma,\zeta}^i(a_i^\zeta) = c_i^\gamma \right\}.$$

Y is nonempty, because it contains c_i^γ. Y is partially ordered by:

$$a_i^\zeta \sqsubseteq a_i^\eta \text{ iff } \eta \ge \zeta \text{ and } \pi_{\zeta,\eta}^i(a_i^\eta) = a_i^\zeta.$$

Note that $a_i^\zeta \sqsubseteq a_i^\eta$ and $a_i^\eta \sqsubseteq a_i^\zeta$ imply $a_i^\zeta = a_i^\eta$. Now, let A be a nonempty and totally $\sqsubseteq$-ordered subset of Y. Define

$$\alpha(A) := \sup\{\zeta \mid \exists a_i^\zeta \in A \cap C_i^\zeta\}.$$

Obviously, we have $\alpha(A) \leq \alpha$. For every $\beta < \alpha(A)$ there is a $\zeta > \beta$ such that there is a $a_i^\zeta \in A \cap C_i^\zeta$. Define

$$\sqcup A := (t_i^\beta)_{\beta<\alpha(A)},$$

such that $t_i^\beta = a_i^\zeta(\beta)$ for a $a_i^\zeta \in A \cap C_i^\zeta$ with $\beta < \zeta$. Note that if $\beta < \zeta$ and $\beta < \eta$, and if $a_i^\zeta \in A \cap C_i^\zeta$ and $a_i^\eta \in A \cap C_i^\eta$, then we have $\eta \leq \zeta$ and $\pi_{\eta,\zeta}^i(a_i^\zeta) = a_i^\eta$ or $\zeta \leq \eta$ and $\pi_{\zeta,\eta}^i(a_i^\eta) = a_i^\zeta$. In both cases, we have $a_i^\eta(\beta) = a_i^\zeta(\beta)$, and hence $\sqcup A$ is well-defined. Note furthermore that, for $0 < \beta < \alpha(A)$ and $a_i^\zeta \in A \cap C_i^\zeta$ with $\beta < \zeta$, we have

$$\operatorname{marg}_{C_i^\beta}(\sqcup A(\beta)) = \operatorname{marg}_{C_i^\beta}(t_i^\beta) = \operatorname{marg}_{C_i^\beta}(a_i^\zeta(\beta)) = \delta_{a_i^\zeta\lceil\beta} = \delta_{\sqcup A\lceil\beta},$$

and for $\xi < \beta < \alpha(A)$ and $a_i^\zeta \in A \cap C_i^\zeta$ with $\beta < \zeta$, we have

$$\operatorname{marg}_{C^\xi}(\sqcup A(\beta)) = \operatorname{marg}_{C^\xi}(t_i^\beta) = \operatorname{marg}_{C^\xi}(a_i^\zeta(\beta)) = a_i^\zeta(\xi) = \sqcup A(\xi).$$

Hence, $\sqcup A \in C_i^{\alpha(A)}$, and by construction (since $a_i^\zeta\lceil\gamma = c_i^\gamma$, for $a_i^\zeta \in A$): $\pi_{\gamma,\alpha(A)}^i(\sqcup A) = c_i^\gamma$. Therefore, $\sqcup A \in Y$ and, by construction, $\sqcup A$ is an $\sqsubseteq$-upper bound of the set A in Y.
Therefore we can apply Zorn's Lemma, hence there is a $\sqsubseteq$-maximal element $a \in Y$. Assume $a \in C_i^\zeta$, for a $\zeta < \alpha$. Since α is a limit number, there is a η with $\zeta < \eta < \alpha$. The induction assumption implies that there is a $a_i^\eta \in C_i^\eta$, such that $\pi_{\zeta,\eta}^i(a_i^\eta) = a$. Since $a \in Y$, it follows that $\pi_{\gamma,\eta}^i(a_i^\eta) = \pi_{\gamma,\zeta}^i(a) = c_i^\gamma$, and hence $a_i^\eta \in Y$. Therefore a is not maximal, which is a contradiction. It follows that $a \in C_i^\alpha$ and $\pi_{\gamma,\alpha}^i(a) = c_i^\gamma$, as desired.

■

Lemma 21

$$\Sigma^{\kappa} = \bigcup_{\beta<\kappa} \pi_{\beta,\kappa}^{-1}\left(\Sigma^{\beta}\right).$$

Proof By Lemma 1 and Remark 5, it follows that $\pi_{\beta,\kappa}^{-1}\left(\Sigma^{\beta}\right)$ is a κ-field on C^{κ}, for $\beta < \kappa$. If $\gamma < \alpha < \kappa$, then, by Lemma 1 it follows that $\Sigma^{\kappa} \supseteq \pi_{\alpha,\kappa}^{-1}(\Sigma^{\alpha}) \supseteq \pi_{\gamma,\kappa}^{-1}(\Sigma^{\gamma})$ and therefore $\Sigma^{\kappa} \supseteq \bigcup_{\beta<\kappa} \pi_{\beta,\kappa}^{-1}\left(\Sigma^{\beta}\right)$. By Lemma 2, $\bigcup_{\beta<\kappa} \pi_{\beta,\kappa}^{-1}\left(\Sigma^{\beta}\right)$ is a κ-field on C^{κ}.

Σ^{κ} is the product κ-field of the Σ_i^{κ}, $i \in I_0$. By Lemma 1, Remark 5, and Lemma 2, it follows that $\bigcup_{\beta<\kappa} \left(\pi_{\beta,\kappa}^{i}\right)^{-1}\left(\Sigma_i^{\beta}\right) = \Sigma_i^{\kappa}$, since every generator of Σ_i^{κ} is in $\left(\pi_{\beta,\kappa}^{i}\right)^{-1}\left(\Sigma_i^{\beta}\right)$ for a $\beta < \kappa$, and since the left hand side is already a κ-field. It follows that the sets of the form $\Pi_{j\in I_0} E_j$ such that $E_j = C_j^{\kappa}$, for all but one $i \in I_0$, and $E_i = \left(\pi_{\beta,\kappa}^{i}\right)^{-1}\left(E_i^{\beta}\right)$, for a $\beta < \kappa$ and $E_i^{\beta} \in \Sigma_i^{\beta}$, are already a generating set of Σ^{κ}. This implies that $\bigcup_{\beta<\kappa} \pi_{\beta,\kappa}^{-1}\left(\Sigma^{\beta}\right) \supseteq \Sigma^{\kappa}$. ■

And in the same way, with obvious changes, one proves:

Lemma 22 *Let $i \in I$. Then,*

$$\Sigma_{-i}^{\kappa} = \bigcup_{\beta<\kappa} \left(\pi_{\beta,\kappa}^{-i}\right)^{-1}\left(\Sigma_{-i}^{\beta}\right).$$

Definition 29
- Define θ^{κ} to be the identity on $C_0^{\kappa} = S$.
- For $i \in I$, define $T_i^{\kappa} : C_i^{\kappa} \to \Delta^{\kappa}(C^{\kappa})$ by:

$$T_i^{\kappa}((t_i^{\beta})_{\beta<\kappa})(E) := t_i^{\alpha}(E^{\alpha}),$$

for $(t_i^{\beta})_{\beta<\kappa} \in C_i^{\kappa}$ and for $E = \pi_{\alpha,\kappa}^{-1}(E^{\alpha})$, where $E^{\alpha} \in \Sigma^{\alpha}$, for a $\alpha < \kappa$.

Since for $\beta < \gamma < \kappa$: $\mathrm{marg}_{C^{\beta}}(t_i^{\gamma}) = t_i^{\beta}$, and since $\pi_{\beta,\alpha}$ is onto, for $\beta < \alpha \leq \kappa$, this definition is independent of the particular E^{α} chosen to define $T_i^{\kappa}(c_i^{\kappa})(E)$.

Theorem 4

$$\underline{C}^{\kappa} := \langle C^{\kappa}, \Sigma^{\kappa}, (T_i^{\kappa})_{i\in I}, \theta^{\kappa}\rangle$$

is a product κ-type space on S for player set I.

Proof By the definition, Lemma 1, and Lemma 3, $T_i^{\kappa}(c_i^{\kappa})$ is a finitely additive probability measure on $\bigcup_{\beta<\kappa} \pi_{\beta,\kappa}^{-1}(\Sigma^{\beta}) = \Sigma^{\kappa}$.

$T_i^{\kappa} : C_i^{\kappa} \to \Delta^{\kappa}(C^{\kappa})$ is measurable: Let $c_i^{\kappa} = (t_i^{\beta})_{\beta<\kappa} \in C_i^{\kappa}$, $E = \pi_{\alpha,\kappa}^{-1}(E^{\alpha})$, for a $\alpha < \kappa$ and a $E^{\alpha} \in \Sigma^{\alpha}$, and let $p \in [0,1]$. We have $T_i^{\kappa}(c_i^{\kappa})(E) \geq p$ iff $t_i^{\alpha}(E^{\alpha}) \geq p$. But $\{t_i^{\alpha} \in \Delta^{\kappa}(C^{\alpha}) \mid t_i^{\alpha}(E^{\alpha}) \geq p\}$ is measurable in $\Delta^{\kappa}(C^{\alpha})$, so by the definition of Σ_i^{κ}, the set

$$\left\{(t_i^{\beta})_{\beta<\kappa} \in C_i^{\kappa} \middle| \; T_i^{\kappa}((t_i^{\beta})_{\beta<\kappa})(E) \geq p\right\}$$

is measurable.

It remains to prove the introspection property. (The measurability of θ^{κ} is clear anyway.) Let $c_i^{\kappa}, a_i^{\kappa} \in C_i^{\kappa}$, with $c_i^{\kappa} \neq a_i^{\kappa}$. Then, by the definitions, there is a $\alpha < \kappa$ and a $E^{\alpha} \in \Sigma^{\alpha}$ such that

$$T_i^{\kappa}(c_i^{\kappa})\left(\pi_{\alpha,\kappa}^{-1}(E^{\alpha})\right) \neq T_i^{\kappa}(a_i^{\kappa})\left(\pi_{\alpha,\kappa}^{-1}(E^{\alpha})\right).$$

It follows that

$$[T_i^{\kappa}(c_i^{\kappa})] := \{a^{\kappa} \in C^{\kappa} \mid T_i^{\kappa}(c_i^{\kappa}) = T_i^{\kappa}(a_i^{\kappa})\} = \{c_i^{\kappa}\} \times C_{-i}^{\kappa}.$$

Let $A \in \Sigma^{\kappa}$ with $A \supseteq \{c_i^{\kappa}\} \times C_{-i}^{\kappa}$. Since $A = \pi_{\alpha,\kappa}^{-1}(A^{\alpha})$ for a $\alpha < \kappa$ and a $A^{\alpha} \in \Sigma^{\alpha}$, it follows that $A^{\alpha} \supseteq \{c_i^{\kappa}\lceil\alpha\} \times C_{-i}^{\alpha}$. By the definition of $c_i^{\kappa} = (t_i^{\beta})_{\beta<\kappa}$, we have $\mathrm{marg}_{C_i^{\alpha}}(t_i^{\alpha}) = \delta_{(t_i^{\beta})_{\beta<\alpha}} = \delta_{c_i^{\kappa}\lceil\alpha}$. Therefore $t_i^{\alpha}(A^{\alpha}) = 1$, and by the definition, $T_i^{\kappa}(c_i^{\kappa})(A) = 1$ and therefore $\mathrm{marg}_{C_i^{\kappa}}(T_i^{\kappa}(c_i^{\kappa})) = \delta_{c_i^{\kappa}}$. ∎

Definition 30 Let

$$\langle M, \Sigma, (T_i)_{i\in I}, \theta\rangle$$

be a κ-type space on S for player set I. By transfinite induction on $\alpha \leq \kappa$, we define:

1. Define

$$h_0^{\alpha} : M \to C_0^{\alpha} = S \;\text{ by } h_0^{\alpha} := \theta,$$

for $0 \leq \alpha \leq \kappa$. Note that θ is measurable.

2. Define

$$h_i^0 : M \to C_i^0$$

in the obvious way, for $i \in I$. Note that h_i^0 is uniquely defined and measurable, since C_i^0 is a singleton.

3. If $h_i^\alpha : M \to C_i^\alpha$ is already defined and measurable, for $i \in I$, then define

$$h^\alpha : M \to C^\alpha \text{ by } h^\alpha (m) := \left(h_j^\alpha (m)\right)_{j \in I_0}.$$

h^α is measurable, since by 1. h_0^α is measurable and each h_i^α, for $i \in I$, is measurable.

4. If $h^\alpha : M \to C^\alpha$ is already defined and measurable and if $\alpha < \kappa$, then define

$$g_i^\alpha (m) (\cdot) := T_i (m) \left((h^\alpha)^{-1} (\cdot)\right), \quad \text{for } i \in I \text{ and } m \in M.$$

Note that (in the notation of Remark 4) $g_i^\alpha = \Delta_{h^\alpha}^\kappa \circ T_i$, for $i \in I$. By Remark 4, it follows that

$$g_i^\alpha : M \to \Delta^\kappa (C^\alpha), \quad \text{for } i \in I.$$

By the measurability of h^α and of T_i, for $i \in I$, and by Remark 4, we have that g_i^α is measurable, for $i \in I$.

5. Define

$$h_i^1 : M \to C_i^1 = \Delta^\kappa \left(C^0\right) \text{ by } h_i^1 := g_i^0, \quad \text{for } i \in I.$$

6. Let $\alpha = \gamma + 1$, where $0 < \gamma < \kappa$, and let $h_i^\gamma : M \to C_i^\gamma$, for $i \in I$, be already defined and measurable such that $h_i^\gamma (m) = (g_i^\beta (m))_{\beta<\gamma}$, for all $m \in M$ and for $i \in I$, where $g_i^\beta : M \to \Delta^\kappa \left(C^\beta\right)$ is already defined and measurable, for $i \in I$ and $\beta < \gamma$. By 3. and 4. from above, $g_i^\gamma : M \to \Delta^\kappa (C^\gamma)$ is defined and measurable, for $i \in I$. Define

$$h_i^{\gamma+1} (m) := (g_i^\beta (m))_{\beta<\gamma+1}, \quad \text{for } m \in M \text{ and } i \in I.$$

We have to show that $h_i^{\gamma+1} : M \to C_i^{\gamma+1}$, for $i \in I$. It suffices to show that

$$\operatorname{marg}_{C_i^\gamma} (g_i^\gamma (m)) = \delta_{\left(g_i^\beta (m)\right)_{\beta<\gamma}}, \quad \text{for } m \in M \text{ and } i \in I,$$

and for all $\beta < \gamma$:

$$\operatorname{marg}_{C^\beta}\left(g_i^\gamma(m)\right) = g_i^\beta(m), \text{ for } m \in M \text{ and } i \in I.$$

Let $i \in I$ and $E^\gamma \in \Sigma^\gamma$ such that $E^\gamma \supseteq \{(g_i^\beta(m))_{\beta<\gamma}\} \times C_{-i}^\gamma$. We have to show that

$$g_i^\gamma(m)(E^\gamma) = 1.$$

First observe that for all $m, m' \in M$ and $i \in I$: if $T_i(m) = T_i(m')$, then, by the definition, $g_i^\beta(m) = g_i^\beta(m')$, for all $\beta < \gamma$. So it follows that $h_i^\gamma(m) = h_i^\gamma(m')$. Obviously, we have $h^\gamma(m) \in \{h_i^\gamma(m)\} \times C_{-i}^\gamma$. This implies that $(h^\gamma)^{-1}(E^\gamma) \supseteq [T_i(m)]$ and therefore

$$T_i(m)\left((h^\gamma)^{-1}(E^\gamma)\right) = 1.$$

By the definitions of h_i^β and h_i^η, for $i \in I$, we have

$$h_i^\beta = \pi_{\beta,\eta}^i \circ h_i^\eta \text{ and } h_0^\beta = \theta = \mathrm{id}_S \circ \theta = \pi_{\beta,\eta}^0 \circ h_0^\eta, \text{ for } \beta < \eta \leq \gamma \text{ and } i \in I.$$

Therefore, $h^\beta = \pi_{\beta,\eta} \circ h^\eta$, for $\beta < \eta \leq \gamma$. It follows that

$$\left(h^\beta\right)^{-1}\left(E^\beta\right) = (h^\eta)^{-1}\left(\pi_{\beta,\eta}^{-1}\left(E^\beta\right)\right),$$

for $\beta < \eta \leq \gamma$ and $E^\beta \in \Sigma^\beta$.
Let $E^\beta \in \Sigma^\beta$, $\beta < \gamma$, $m \in M$ and $i \in I$. It follows now:

$$\begin{aligned} g_i^\gamma(m)\left(\pi_{\beta,\gamma}^{-1}\left(E^\beta\right)\right) &= T_i(m)\left((h^\gamma)^{-1}\left(\pi_{\beta,\gamma}^{-1}\left(E^\beta\right)\right)\right) \\ &= T_i(m)\left(\left(h^\beta\right)^{-1}\left(E^\beta\right)\right) \\ &= g_i^\beta(m)\left(E^\beta\right). \end{aligned}$$

We note that h_i^α, for $i \in I$, is measurable, since every g_i^β, for $i \in I$ and $\beta < \alpha$, is measurable.

7. Let α be a limit ordinal $\leq \kappa$ and let, for every $i \in I$ and every $\gamma < \alpha$, $h_i^\gamma : M \to C_i^\gamma$ be already defined and measurable such that $h_i^\gamma(m) = (g_i^\beta(m))_{\beta<\gamma}$, for all $m \in M$, where $g_i^\beta : M \to \Delta^\kappa\left(C^\beta\right)$ is already defined and measurable, for $\beta < \gamma$.
Since α is a limit ordinal, it follows for every $\beta < \alpha$ that $\beta + 1 < \alpha$. Therefore $g_i^\beta : M \to \Delta^\kappa\left(C^\beta\right)$ is already defined and measurable, for $\beta < \alpha$ and $i \in I$. It follows that

$$h_i^\alpha(m) := (g_i^\beta(m))_{\beta<\alpha}$$

is well-defined and measurable, for $m \in M$ and $i \in I$. For $i \in I$, we have to show that $h_i^\alpha : M \to C_i^\alpha$ and that h_i^α is measurable:
Let $\beta < \gamma < \alpha$. Since α is a limit ordinal, we have $\gamma + 1 < \alpha$. By the induction assumption, we have that $h_i^{\gamma+1} : M \to C_i^{\gamma+1}$, and it follows that g_i^ξ is measurable, for $0 \leq \xi \leq \gamma$ and that

$$\operatorname{marg}_{C_i^\gamma} \left(g_i^\gamma (m)\right) = \delta_{\left(g_i^\xi (m)\right)_{\xi < \gamma}} \quad \text{and} \quad \operatorname{marg}_{C^\beta} \left(g_i^\gamma (m)\right) = g_i^\beta (m),$$

for all $m \in M$. This shows that

$$h_i^\alpha : M \to C_i^\alpha, \quad \text{for } i \in I.$$

Finally, we note that h_i^α, for $i \in I$, is measurable, since every g_i^β, for $i \in I$ and $\beta < \alpha$, is measurable.

Proposition 4 *Let $\langle M, \Sigma, (T_i)_{i \in I}, \theta \rangle$ be a κ-type space on S for player set I. Then, the κ-hierarchy description map*

$$h^\kappa : M \to C^\kappa$$

is a type morphism.

Proof We showed already in the above definition that h^κ is measurable. We have

$$\theta^\kappa \left(h^\kappa (m)\right) = h_0^\kappa (m) = \theta (m), \quad \text{for } m \in M.$$

Let $i \in I$ and $E \in \Sigma^\kappa$. Then there is an ordinal $\alpha < \kappa$ and $E^\alpha \in \Sigma^\alpha$ such that $E = \pi_{\alpha,\kappa}^{-1} (E^\alpha)$. It follows by the definitions that for $m \in M$:

$$\begin{array}{rcl} T_i^\kappa \left(h^\kappa (m)\right)(E) & = & T_i^\kappa((g_i^\beta (m))_{\beta<\kappa})(E) \\ & = & g_i^\alpha (m) (E^\alpha) \\ & = & T_i (m) \left((h^\alpha)^{-1} (E^\alpha)\right) \\ & = & T_i (m) \left((h^\kappa)^{-1} \left(\pi_{\alpha,\kappa}^{-1} (E^\alpha)\right)\right) \\ & = & T_i (m) \left((h^\kappa)^{-1} (E)\right). \end{array}$$

■

The next Lemma shows that type morphisms between κ-type spaces preserve the κ-hierarchy descriptions.

Lemma 23 *Let* $\langle M, \Sigma, (T_i)_{i\in I}, \theta\rangle$ *and* $\langle \widehat{M}, \widehat{\Sigma}, (\widehat{T}_i)_{i\in I}, \widehat{\theta}\rangle$ *be* κ*-type spaces on* S *for player set* I*, let* $h^\kappa : M \to C^\kappa$ *and* $\widehat{h}^\kappa : \widehat{M} \to C^\kappa$ *be the* κ*-hierarchy description maps, and let* $f : M \to \widehat{M}$ *be a type morphism.*
Then, for all $m \in M$ *:*

$$h^\kappa (m) = \widehat{h}^\kappa (f (m)) .$$

Proof Since C_i^0 is a singleton, for $i \in I$, we have that $h_i^0 (m) = \widehat{h}_i^0 (f (m))$, for $i \in I$ and $m \in M$.
For $0 \leq \alpha \leq \kappa$ and $m \in M$ we have

$$h_0^\alpha (m) = \theta (m) = \widehat{\theta} (f (m)) = \widehat{h}_0^\alpha (f (m)) .$$

Let $\alpha = \beta + 1 < \kappa$ and $h^\beta (m) = \widehat{h}^\beta (f (m))$, for $m \in M$. For $i \in I$ and $m \in M$, it follows that

$$\begin{aligned} g_i^\beta (m) &= T_i (m) \circ \left(h^\beta\right)^{-1} \\ &= T_i (m) \circ (f^{-1} \circ (\widehat{h}^\beta)^{-1}) \\ &= \widehat{T}_i (f (m)) \circ (\widehat{h}^\beta)^{-1} \\ &= \widehat{g}_i^\beta (f (m)) . \end{aligned}$$

This implies that $h_i^{\beta+1} (m) = \widehat{h}_i^{\beta+1} (f (m))$, for $i \in I$ and $m \in M$.

Let $i \in I$, let α be a limit ordinal $\leq \kappa$, and let $h_i^\beta (m) = \widehat{h}_i^\beta (f (m))$, for all $\beta < \alpha$ and $m \in M$. To show that $h_i^\alpha (m) = \widehat{h}_i^\alpha (f (m))$, for $i \in I$ and $m \in M$, it suffices to verify that $g_i^\beta (m) = \widehat{g}_i^\beta (f (m))$, for all $\beta < \alpha$. But this follows from the fact that $\beta < \alpha$ implies that $\beta + 1 < \alpha$ and from

$$(g_i^\zeta (m))_{\zeta\leq\beta} = h_i^{\beta+1} (m) = \widehat{h}_i^{\beta+1} (f (m)) = (\widehat{g}_i^\zeta (f (m)))_{\zeta\leq\beta}.$$

■

Proposition 5 *The* κ*-hierarchy description map*

$$h^\kappa : C^\kappa \to C^\kappa$$

is the identity.

Proof Let $c^\kappa \in C^\kappa$, let $0 \leq \alpha \leq \kappa$ and, for $i \in I_0$, let π_i be the projection

$$\pi_i : C^\kappa \to C_i^\kappa .$$

Then

- $$h_0^{\alpha}\left(c^{\kappa}\right)=\theta^{\kappa}\left(c^{\kappa}\right)=c_0^{\kappa}=\pi_0\left(c^{\kappa}\right)=\pi_{\alpha,\kappa}^{0}\left(\pi_0\left(c^{\kappa}\right)\right).$$

- We have $h_i^0\left(c^{\kappa}\right)=c_i^0$, for $i\in I$, where $\{c_i^0\}=C_i^0$, and therefore $h_i^0\left(c^{\kappa}\right)=\pi_{0,\kappa}^{i}\left(\pi_i\left(c^{\kappa}\right)\right).$

- Let $\alpha=\beta+1<\kappa$ and let $h_i^{\beta}=\pi_{\beta,\kappa}^{i}\circ\pi_i$, for $i\in I$. Let furthermore $c_i^{\kappa}=(t_i^{\zeta})_{\zeta<\kappa}$, for $i\in I$, and let $E^{\beta}\in\Sigma^{\beta}$. For $i\in I$, we have

$$\begin{aligned}
g_i^{\beta}\left(c^{\kappa}\right)\left(E^{\beta}\right) &= T_i^{\kappa}\left(c^{\kappa}\right)\left(\left(h^{\beta}\right)^{-1}\left(E^{\beta}\right)\right)\\
&= T_i^{\kappa}\left(c^{\kappa}\right)\left(\left(\left(\pi_{\beta,\kappa}^{i}\circ\pi_i\right)_{i\in I_0}\right)^{-1}\left(E^{\beta}\right)\right)\\
&= T_i^{\kappa}\left(c^{\kappa}\right)\left(\left(\pi_{\beta,\kappa}\right)^{-1}\left(E^{\beta}\right)\right)\\
&= t_i^{\beta}\left(E^{\beta}\right).
\end{aligned}$$

It follows that $g_i^{\beta}\left(c^{\kappa}\right)=t_i^{\beta}$ and therefore $h_i^{\alpha}\left(c^{\kappa}\right)=\pi_{\alpha,\kappa}^{i}\left(\pi_i\left(c^{\kappa}\right)\right).$ (Note that by the induction assumption, $h_i^{\beta}\left(c^{\kappa}\right)=\pi_{\beta,\kappa}^{i}\left(\pi_i\left(c^{\kappa}\right)\right)=(t_i^{\zeta})_{\zeta<\beta}$, for all $i\in I$.)

- Let now α be a limit ordinal $\leq\kappa$ and let for all $\beta<\alpha$, for all $i\in I$, and all $c_i^{\kappa}=(t_i^{\zeta})_{\zeta<\kappa}$:

$$h_i^{\beta}\left(c^{\kappa}\right)=\left(g_i^{\zeta}\left(c^{\kappa}\right)\right)_{\zeta<\beta}=(t_i^{\zeta})_{\zeta<\beta}.$$

Then

$$h_i^{\alpha}\left(c^{\kappa}\right)=\left(g_i^{\beta}\left(c^{\kappa}\right)\right)_{\beta<\alpha}=(t_i^{\beta})_{\beta<\alpha}=\pi_{\alpha,\kappa}^{i}\circ\pi_i\left(c^{\kappa}\right).$$

(Note that $\beta<\alpha$ implies $\beta+1<\alpha$.)

Altogether, it follows that $h_i^{\kappa}\left(c^{\kappa}\right)=\pi_i\left(c^{\kappa}\right)$, for $i\in I_0$, and therefore $h^{\kappa}=\mathrm{id}_{C^{\kappa}}$.

■

Theorem 5 *$\underline{C}^{\kappa}$ is a universal κ-type space on S for player set I.*

Proof Let $\underline{M}=\left\langle M,\Sigma,(T_i)_{i\in I},\theta\right\rangle$ be a κ-type space on S. By Proposition 4, there is a type morphism from $\underline{M}$ to $\underline{C}^{\kappa}$. By Proposition 5 and Lemma 23, this type morphism is unique. ■

6 Beliefs Completeness

In this section, we show that the universal κ-type space is beliefs complete. In the case $\kappa = \aleph_0$, we can say much more: The component space of each player is - up to isomorphism of measurable spaces - the space of finitely additive probability measures on the product of the space of states of nature and the other players' component spaces.

Theorem 6 *Let $\langle C^\kappa, \Sigma^\kappa, (T_i^\kappa)_{i \in I}, \theta^\kappa \rangle$ be the universal κ-type space on S for player set I constructed in the previous section.*
Then, for every $i \in I$:

-
$$T_i^\kappa : C_i^\kappa \to \Delta^\kappa \left(C^\kappa\right)$$
is one-to-one and measurable.

-
$$\operatorname{marg}_{C_{-i}^\kappa} \circ T_i^\kappa : C_i^\kappa \to \Delta^\kappa \left(C_{-i}^\kappa\right)$$
is onto and measurable.

-
$$\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0} : C_i^{\aleph_0} \to \Delta^{\aleph_0} \left(C_{-i}^{\aleph_0}\right)$$
is an isomorphism of measurable spaces.[5]

Proof The first point was already proved in the proof of Theorem 4.

Fix $i \in I$. According to the first point of this theorem, $T_i^\kappa : C_i^\kappa \to \Delta^\kappa \left(C^\kappa\right)$ is measurable. Since all the inverse images (under the projection) of the

[5]Although, for $\kappa > \aleph_0$, it is no longer true (contrary to the case $\kappa = \aleph_0$) that the values of a finitely additive probability measure on the measurable rectangles in a product of two κ -measurable spaces determine this finitely additive probability measure, we believe that this is true in the special case where the marginal on one of the factors is a delta-measure. Therefore we tend to conjecture that the third point of Theorem 6 holds also for $\kappa > \aleph_0$, but it seems to be considerably harder to prove.

generators of Σ_{-i}^{κ} are among the generators of Σ^{κ}, the projection from C^{κ} to C_{-i}^{κ} is measurable. Remark 4 implies now that

$$\operatorname{marg}_{C_{-i}^{\kappa}} : \Delta^{\kappa}\left(C^{\kappa}\right) \to \Delta^{\kappa}\left(C_{-i}^{\kappa}\right)$$

is measurable, and hence

$$\operatorname{marg}_{C_{-i}^{\kappa}} \circ T_i^{\kappa} : C_i^{\kappa} \to \Delta^{\kappa}\left(C_{-i}^{\kappa}\right)$$

is measurable.

Let $\mu \in \Delta^{\kappa}\left(C_{-i}^{\kappa}\right)$. Now, we construct a product κ-type space on S for player set I

$$\underline{U} := \left\langle \left(U_j\right)_{j\in I_0}, \left(\Sigma_j^U\right)_{j\in I_0}, \left(T_j^U\right)_{j\in I_0}, \theta^U \right\rangle$$

such that there is a point $u \in U_i$ with $\operatorname{marg}_{C_{-i}^{\kappa}} \circ T_i^{\kappa}\left(h_i^{\kappa}(u)\right) = \mu$.
Without loss of generality, let $u \notin C_i^{\kappa}$ and define

$$\begin{array}{lll}
U_i & := & C_i^{\kappa} \cup \{u\}, \\
U_j & := & C_j^{\kappa}, \quad \text{for } j \in I_0 \backslash \{i\}, \\
\Sigma_i^U & := & \Sigma_i^{\kappa} \cup \{E \cup \{u\} \mid E \in \Sigma_i^{\kappa}\}, \\
\Sigma_j^U & := & \Sigma_j^{\kappa}, \quad \text{for } j \in I_0 \backslash \{i\}, \\
U & := & \Pi_{j\in I_0} U_j, \\
\Sigma^U & := & \text{the product } \kappa\text{-field of the } \Sigma_j^U,\ j \in I_0, \\
U_{-i} & := & \Pi_{j\in I_0\backslash\{i\}} U_j, \\
\Sigma_{-i}^U & := & \text{the product } \kappa\text{-field of the } \Sigma_j^U,\ j \in I_0 \backslash \{i\}.
\end{array}$$

It is obvious that Σ_i^U is a κ-field on U_i. Note that $\Sigma^{\kappa} \subseteq \Sigma^U$.

Let $E = \Pi_{j\in I_0} E_j$ such that $E_j = U_j$ for all but one $j \in I_0$. The sets of this form generate Σ^U. If E is of this form, we have $E \cap C^{\kappa} \in \Sigma^{\kappa}$. Since $\left(\bigcap_{F\in\Psi} F\right) \cap C^{\kappa} = \bigcap_{F\in\Psi}\left(F \cap C^{\kappa}\right)$, for $\Psi \subseteq \Sigma^U$, and since $(U \backslash F) \cap C^{\kappa} = C^{\kappa} \backslash \left(F \cap C^{\kappa}\right)$, for $F \subseteq U$, it follows by the proof of Lemma 1 that $F \cap C^{\kappa} \in \Sigma^{\kappa}$, for $F \in \Sigma^U$.

Now define for $j \in I$, $c_j^{\kappa} \in C_j^{\kappa}$ and $E \in \Sigma^U$:

$$T_j^U(c_j^{\kappa})(E) := T_j^{\kappa}(c_j^{\kappa})\left(E \cap C^{\kappa}\right).$$

It is obvious that $T_j^U\left(c_j^{\kappa}\right)$ is a finitely additive probability measure on $\left(U, \Sigma^U\right)$, for $j \in I$ and $c_j^{\kappa} \in C_j^{\kappa}$. Clearly, $T_j^U : U_j \to \Delta^{\kappa}(U)$ is Σ_j^U-$\Sigma_{\Delta^{\kappa}(U)}$ measurable, for $j \in I \backslash \{i\}$.
Let $j \in I$, $c_j^{\kappa} \in C_j^{\kappa}$, $E \in \Sigma^U$, and $E \supseteq \left\{c_j^{\kappa}\right\} \times U_{-j}$. We have to show that $T_j^U(c_j^{\kappa})(E) = 1$. We have $\left\{c_j^{\kappa}\right\} \times U_{-j} \supseteq \left\{c_j^{\kappa}\right\} \times C_{-j}^{\kappa}$, so, $C^{\kappa} \cap E \supseteq \left\{c_j^{\kappa}\right\} \times C_{-j}^{\kappa}$ and it follows that

$$T_j^U(c_j^{\kappa})(E) = T_j^{\kappa}(c_j^{\kappa})\left(C^{\kappa} \cap E\right) = 1.$$

Σ_i^U and Σ_{-i}^U are fields on U_i and $U_{-i} := \Pi_{j\in I_0\setminus\{i\}}U_j = C_{-i}^\kappa$. The finite disjoint unions of sets of the form $E_i \times E_{-i}$, where $E_i \in \Sigma_i^U$ and $E_{-i} \in \Sigma_{-i}^U$, form a field $\mathcal{F}$ on U that is contained in Σ^U. Note that for every finitely additive probability measure ν on $\left(U, \Sigma^U\right)$, the values of ν on $\mathcal{F}$ determine already the marginal of ν on $\left(U_i, \Sigma_i^U\right)$ and on $\left(U_{-i}, \Sigma_{-i}^U\right)$. Now, define for disjoint $E_i^1 \times E_{-i}^1, \ldots, E_i^n \times E_{-i}^n \in \mathcal{F}$:

$$\widehat{\mu}(\bigcup_{l=1}^{n} E_i^l \times E_{-i}^l) := \delta_u\left(E_i^1\right) \cdot \mu\left(E_{-i}^1\right) + \delta_u\left(E_i^2\right) \cdot \mu\left(E_{-i}^2\right) + \ldots + \delta_u\left(E_i^n\right) \cdot \mu\left(E_{-i}^n\right).$$

Obviously $\widehat{\mu}$ is well-defined (because of the additivity of δ_u and μ) and it is a finitely additive probability measure on $\mathcal{F}$. Next, by the Łoś-Marczewski Theorem, extend $\widehat{\mu}$ to a finitely additive probability measure $\widetilde{\mu}$ on Σ^U. By the definitions, it is clear that $\mathrm{marg}_{U_i}\left(\widetilde{\mu}\right) = \delta_u$ and that $\mathrm{marg}_{C_{-i}^\kappa}\left(\widetilde{\mu}\right) = \mu$. Note also that

$$\widetilde{\mu}\left(\{u\} \times C_{-i}^\kappa\right) = 1 \neq T_i^U\left(c_i^\kappa\right)\left(\{u\} \times C_{-i}^\kappa\right) = 0, \quad \text{for } c_i^\kappa \in C_i^\kappa.$$

(By definition, we have $\{u\} \times C_{-i}^\kappa \in \Sigma^U$.) Define now

$$T_i^U\left(u\right) := \widetilde{\mu}.$$

It follows that $T_i^U\left(u\right)\left(\{u\} \times U_{-i}\right) = 1$.

Finally, define

$$\theta^U\left(c_0^\kappa\right) := c_0^\kappa.$$

To check that $\underline{U}$ is a κ-type space, it remains to show that

$$T_i^U : U_i \to \Delta^\kappa\left(U\right)$$

is measurable. We have for $p \in [0,1]$ and $E \in \Sigma^U$:

$$\left(T_i^U\right)^{-1}\left(\{\nu \in \Delta^\kappa\left(U\right) \mid \nu\left(E\right) \geq p\}\right) =$$
$$\begin{cases} \left(T_i^\kappa\right)^{-1}\left(\{\nu \in \Delta^\kappa\left(C^\kappa\right) \mid \nu\left(E \cap C^\kappa\right) \geq p\}\right) \in \Sigma_i^\kappa \subseteq \Sigma_i^U, & \text{if} \quad \widetilde{\mu}\left(E\right) < p, \\ \left(T_i^\kappa\right)^{-1}\left(\{\nu \in \Delta^\kappa\left(C^\kappa\right) \mid \nu\left(E \cap C^\kappa\right) \geq p\}\right) \cup \{u\} \in \Sigma_i^U, & \text{if} \quad \widetilde{\mu}\left(E\right) \geq p. \end{cases}$$

Since $\underline{U}$ is a product κ-type space on S for player set I, g_j^β is defined on U_j and is Σ_j^U-$\Sigma_{\Delta^\kappa\left(C^\beta\right)}$-measurable, for $j \in I_0$ and $0 \leq \beta < \kappa$, and h_j^α is defined on U_j and is Σ_j^U-Σ_j^α-measurable, for $j \in I_0$ and $0 \leq \alpha \leq \kappa$.

Since $U_0 = S$ and $\theta^U = \mathrm{id}_S$, we have $h_0^\alpha = \theta^U = \pi_{\alpha,\kappa}^0$, for $0 \leq \alpha \leq \kappa$.

For $j \in I$, h_j^0 is uniquely defined, since the C_j^0 are singletons. It follows that $h_j^0 = \pi_{0,\kappa}^j$, for $j \in I \setminus \{i\}$ and $h_i^0 \lceil C_i^\kappa = \pi_{0,\kappa}^i$.

Let $j \in I$, $\alpha = \beta + 1 < \kappa$, and $h_l^\beta \lceil C_l^\kappa = \pi_{\beta,\kappa}^l$, for all $l \in I_0$, and furthermore, let $c_j^\kappa = (t_j^\zeta)_{\zeta<\kappa} \in C_j^\kappa$ and $E^\beta \in \Sigma^\beta$. Then

$$\begin{array}{rcl} g_j^\beta(c_j^\kappa)\left(E^\beta\right) & = & T_j^U(c_j^\kappa)\left(\left(h^\beta\right)^{-1}\left(E^\beta\right)\right) \\ & = & T_j^\kappa(c_j^\kappa)\left(C^\kappa \cap \left(h^\beta\right)^{-1}\left(E^\beta\right)\right) \\ & = & T_j^\kappa(c_j^\kappa)\left((\pi_{\beta,\kappa})^{-1}\left(E^\beta\right)\right) \\ & = & t_j^\beta\left(E^\beta\right). \end{array}$$

It follows that $g_j^\beta(c_j^\kappa) = t_j^\beta$ and therefore $h_j^\alpha \lceil C_j^\kappa = \pi_{\alpha,\kappa}^j$.

Let $j \in I$, let $\alpha = \lambda \leq \kappa$ be a limit ordinal and for all $\beta < \alpha$: let $h_j^\beta \lceil C_j^\kappa = \pi_{\beta,\kappa}^j$, and let $c_j^\kappa = (t_j^\zeta)_{\zeta<\kappa}$. By the definition and the induction assumption (note that $\beta < \alpha$ implies $\beta + 1 < \alpha$),

$$h_j^\alpha(c_j^\kappa) = (g_j^\beta(c_j^\kappa))_{\beta<\alpha} = (t_j^\beta)_{\beta<\alpha} = \pi_{\alpha,\kappa}^j(c_j^\kappa).$$

Altogether, we showed now that $h_j^\kappa \lceil C_j^\kappa = \mathrm{id}_{C_j^\kappa}$, for $j \in I_0$.

Let now $E_{-i} \in \Sigma_{-i}^\kappa$. By the definition and Proposition 4,

$$\begin{array}{rcl} T_i^\kappa\left(h_i^\kappa(u)\right)\left(C_i^\kappa \times E_{-i}\right) & = & T_i^U(u)\left((h^\kappa)^{-1}\left(C_i^\kappa \times E_{-i}\right)\right) \\ & = & T_i^U(u)\left(U_i \times E_{-i}\right) \\ & = & \delta_u\left(U_i\right) \cdot \mu\left(E_{-i}\right) \\ & = & \mu\left(E_{-i}\right) \end{array}$$

and the second part of the theorem is proved.

For the third point, it remains to prove that

1. $$\mathrm{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0} : C_i^{\aleph_0} \to \Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)$$

   is one-to-one, and

2. for every $E_i \in \Sigma_i^{\aleph_0}$:

   $$\mathrm{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(E_i\right) \in \Sigma_{\Delta^{\aleph_0}(C_{-i}^{\aleph_0})}.$$

To 1 : Let $(t_i^n)_{n<\aleph_0}, (r_i^n)_{n<\aleph_0} \in C_i^{\aleph_0}$ and

$$\mathrm{marg}_{C_{-i}^{\aleph_0}}\left(T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\right) = \mathrm{marg}_{C_{-i}^{\aleph_0}}\left(T_i^{\aleph_0}\left((r_i^n)_{n<\aleph_0}\right)\right).$$

We show by induction on $n < \aleph_0$ that $t_i^n = r_i^n$:

Step $n = 0$: Let $E^0 \in \Sigma^0$. Since C_i^0 is a singleton, there is a $E_{-i}^0 \in \Sigma_{-i}^0$ such that $E^0 = C_i^0 \times E_{-i}^0$. We have $\pi_{0,\aleph_0}^{-1}(E^0) = C_i^{\aleph_0} \times \left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right)$. Since $\pi_{0,\aleph_0}^j$ is measurable, for $j \in I_0$, we have, by definition,

$$\begin{aligned} t_i^0(E^0) &= T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\left(C_i^{\aleph_0} \times \left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right)\right) \\ &= T_i^{\aleph_0}\left((r_i^n)_{n<\aleph_0}\right)\left(C_i^{\aleph_0} \times \left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right)\right) \\ &= r_i^0(E^0). \end{aligned}$$

Step $n \to n+1$: Since Σ^{n+1} is the product field of Σ_i^{n+1} and Σ_{-i}^{n+1} and since finite direct sums of measurable rectangles form already the product field of Σ_i^{n+1} and Σ_{-i}^{n+1}, it is enough to show that $t_i^{n+1}(F^{n+1}) = r_i^{n+1}(F^{n+1})$, for sets of the form $F^{n+1} = E_i^{n+1} \times E_{-i}^{n+1}$, where $E_i^{n+1} \in \Sigma_i^{n+1}$ and $E_{-i}^{n+1} \in \Sigma_{-i}^{n+1}$. Since $\left(t_i^k\right)_{0\le k\le n} = \left(r_i^k\right)_{0\le k\le n}$, we have $\mathrm{marg}_{C_i^{n+1}}\left(\left(t_i^{n+1}\right)\right) = \mathrm{marg}_{C_i^{n+1}}\left(\left(r_i^{n+1}\right)\right) = \delta_{\left(\left(t_i^k\right)_{0\le k\le n}\right)}$, and since

$$\begin{aligned} t_i^{n+1}\left(C_i^{n+1} \times E_{-i}^{n+1}\right) &= T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\left(C_i^{\aleph_0} \times \left(\left(\pi_{n+1,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^{n+1}\right)\right)\right) \\ &= T_i^{\aleph_0}\left((r_i^n)_{n<\aleph_0}\right)\left(C_i^{\aleph_0} \times \left(\left(\pi_{n+1,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^{n+1}\right)\right)\right) \\ &= r_i^{n+1}\left(C_i^{n+1} \times E_{-i}^{n+1}\right), \end{aligned}$$

we have

$$\begin{aligned} & t_i^{n+1}\left(E_i^{n+1} \times E_{-i}^{n+1}\right) \\ &= \delta_{\left(\left(t_i^k\right)_{0\le k\le n}\right)}\left(E_i^{n+1}\right) \cdot \left(\mathrm{marg}_{C_{-i}^{\aleph_0}}\left(T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\right)\right)\left(\left(\pi_{n+1,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^{n+1}\right)\right) \\ &= \delta_{\left(\left(r_i^k\right)_{0\le k\le n}\right)}\left(E_i^{n+1}\right) \cdot \left(\mathrm{marg}_{C_{-i}^{\aleph_0}}\left(T_i^{\aleph_0}\left((r_i^n)_{n<\aleph_0}\right)\right)\right)\left(\left(\pi_{n+1,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^{n+1}\right)\right) \\ &= r_i^{n+1}\left(E_i^{n+1} \times E_{-i}^{n+1}\right). \end{aligned}$$

It follows that $t_i^{n+1} = r_i^{n+1}$.

To 2 : Since $\Sigma_i^{\aleph_0}$ is the field on $C_i^{\aleph_0}$ inherited from the product field of the $\Sigma_{\Delta^{\aleph_0}(C^n)}$, $n < \aleph_0$, and since

$$\mathrm{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0} : C_i^{\aleph_0} \to \Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)$$

is one-to-one and onto, it remains to show - by an induction on $m < \aleph_0$ - that

$$(*) \qquad \operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0} \left(\left\{ (t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \middle| \; t_i^m \in A^m \right\} \right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)},$$

for $A^m \in \Sigma_{\Delta^{\aleph_0}(C^m)}$:

Again, since $\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}$ is one-to-one and onto and since the sets $\left\{ \mu \in \Delta^{\aleph_0}(C^m) \mid \mu(E^m) \geq p \right\}$, where $E^m \in \Sigma^m$ and $p \in [0,1]$, generate $\Sigma_{\Delta^{\aleph_0}(C^m)}$, to show $(*)$, it is enough to verify that

$$\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0} \left(\left\{ (t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \middle| \; t_i^m (E^m) \geq p \right\} \right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)},$$

for all $E^m \in \Sigma^m$ and $p \in [0,1]$.

Step $m = 0$: Let $E^0 \in \Sigma^0$ and $p \in [0,1]$. Since C_i^0 is a singleton, there is a $E_{-i}^0 \in \Sigma_{-i}^0$ such that $E^0 = C_i^0 \times E_{-i}^0$. We have

$$\pi_{0,\aleph_0}^{-1}\left(E^0\right) = C_i^{\aleph_0} \times \left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right) \right)$$

and by definition, for $(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0}$,

$$\begin{aligned} t_i^0\left(E^0\right) &= T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\left(\pi_{0,\aleph_0}^{-1}\left(E^0\right)\right) \\ &= \left(\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\right)\left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right). \end{aligned}$$

Hence

$$\begin{aligned} &\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0} \left(\left\{ (t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \middle| \; t_i^0\left(E^0\right) \geq p \right\} \right) \quad = \\ &\left\{ \mu \in \Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right) \middle| \; \mu\left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right) \geq p \right\} \quad \in \quad \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)}. \end{aligned}$$

(Note that $\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}$ was onto.)

Step $m \to m+1$: Let $E^{m+1} \in \Sigma^{m+1}$ and $p \in [0,1]$. Σ^{m+1} is the product field of Σ_i^{m+1} and Σ_{-i}^{m+1}. E^{m+1} can be written as a finite direct sum of measurable rectangles. Furthermore, E^{m+1} can be written as

$$E^{m+1} = F_i^1 \times F_{-i}^1 + \ldots + F_i^k \times F_{-i}^k,$$

for some $k \geq 1$ and $F_{-i}^l \in \Sigma_{-i}^{m+1}$, for $l = 1, \ldots, k$, and $F_i^l \in \Sigma_i^{m+1}$, for $l = 1, \ldots, k$, such that the F_i^l are disjoint.

According to the induction assumption, we have

$$\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, t_i^q \in A^q\right\}\right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)},$$

for $q = 0, \ldots, m$ and $A^q \in \Sigma_{\Delta^{\aleph_0}(C^q)}$. Recall that Σ_i^{m+1} is the field on C_i^{m+1} inherited from the product field of the $\Sigma_{\Delta^{\aleph_0}(C^q)}$, $q = 0, \ldots, m$, on $\Pi_{q\leq m}\Delta^{\aleph_0}(C^q)$. In particular, the projection from C_i^{m+1} to $\Delta^{\aleph_0}(C^q)$ is measurable, for $q = 0, \ldots, m$. Since $\pi_{m+1,\aleph_0}^i$ is onto and measurable, and since $\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}$ is one-to-one and onto, it follows that

$$\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, (t_i^q)_{q\leq m} \in F_i^l\right\}\right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)}$$

and that

$$\begin{aligned}&\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, (t_i^q)_{q\leq m} \in F_i^{l_0}\right\}\right)\\ &\cap\ \operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, (t_i^q)_{q\leq m} \in F_i^{l_1}\right\}\right) \quad = \quad \emptyset,\end{aligned}$$

for $0 \leq l_0 \neq l_1 \leq k$. As argued in 1, we have

$$t_i^{m+1}\left(F_i^l \times F_{-i}^l\right) = \delta_{\left((t_i^q)_{q\leq m}\right)}\left(F_i^l\right) \cdot \left(\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\right)\left(\left(\pi_{m+1,\aleph_0}^{-i}\right)^{-1}\left(F_{-i}^l\right)\right).$$

Since all the F_i^l are disjoint, we have $t_i^{m+1}(E^{m+1}) \geq p$ iff there is a l such that $(t_i^q)_{q\leq m} \in F_i^l$ and $\left(\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\right)\left(\left(\pi_{m+1,\aleph_0}^{-i}\right)^{-1}\left(F_{-i}^l\right)\right) \geq p$. Hence

$$\begin{aligned}&\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, t_i^{m+1}(E^{m+1}) \geq p\right\}\right) \quad =\\ &\bigcup_{l=1}^k \ \left(\left(\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, (t_i^q)_{q\leq m} \in F_i^l\right\}\right)\right)\right.\\ &\qquad \left.\cap \left\{\mu \in \Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right) \,\middle|\, \mu\left(\left(\pi_{m+1,\aleph_0}^{-i}\right)^{-1}\left(F_{-i}^l\right)\right) \geq p\right\}\right) \qquad \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)}.\end{aligned}$$

■

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