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Three essays on economics: information aggregation, pension schemes and economic growth.

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To Vito, Lidia and Alessandra

No' si volta chi a stella è fisso

Leonardo da Vinci

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Contents

Introduction

This dissertation can be divided into two parts. The first part is composed of one chapter focusing on how to aggregate information in a game theory model, and the second part is composed of two chapters focusing on the link between pension schemes and economic growth.

Chapter 1 started as an attempt to formalize Hayek's intuition that prices coordinate economic agents by aggregating the decentralized information (see Hayek (1945)). I initially built a game where players indirectly communicate through a public signal aggregating their private information. After I introduced Pareto efficiency, the chapter found is natural collocation in the "renegotiation-proof" literature. The main result of the chapter defines renegotiation-proof equilibria when players indirectly communicate through a public signal. The result also provides a bridge between the two parts of the dissertation, as I will now summarize.

The first part of the dissertation builds a theoretical framework useful to analyze different types of information aggregators in a market, including financial benchmarks. Financial benchmarks are reference points in numerous trading processes, particularly in over-the-counter markets. Some of them are a truncated or weighted average of the transactions in a market at a specific time (like the WM/R 4 pm for the forex market); others are the results of auctions (e.g. the gold and silver fix). In 2008 the London Interbank Offered Rate (LIBOR), a crucial financial benchmark, became the center of a scandal where financial institutions colluded and manipulated its value. The institutions benefited from the collusion through other trading positions since LIBOR is widely used in pricing different types of contracts. Pensions funds were among the counter-parties of these contracts and faced several losses due to the collusion, highlighting a new vulnerability of funded pension schemes, which are the subject of analysis in the second part of this dissertation.

Chapter 2 studies the link between pension schemes, financial intermediation, consumption shocks and economic growth. The chapter builds on an overlapping generation model with endogenous (Schumpeterian) growth where agents face consumption shocks, and where a financial intermediary offers insurance against the shocks and pools the savings. We show that a funded pension scheme increases the resources available for investments due to the economic effects of the financial intermediary. As firms can invest the resources in R&D and production, the funded pension scheme also acts as a research and production subsidy leading to a Pareto-superior growth rate of the economy.

R&D and production are not the only channels under which a pension scheme can influence the potential economic growth of a country. Most of the pension reforms moved from a pay-as-you-go scheme to a funded one. However, these reforms drop a link between different generations - and with it also a critical risk-sharing mechanism - potentially affecting the economic growth rate of a country.

Chapter 3 proposes to invert the intergenerational links of the pay-as-you-go pension schemes. Resources would flow from old generations to younger ones, as if it were a system of public bequests. We model the idea in an overlapping generation model with three periods: in the first period the consumer invests in education; in the second period he works; in the last period he retires and consumes his savings. We show that our pension system offers intergenerational risk-sharing as the pay-as-you-go scheme while contributing to capital accumulation as the funded scheme.

Part I

Information aggregation

Chapter 1

Indirect renegotiation in repeated games

Abstract The renegotiation-proof literature assumes direct communication. In this chapter, we define renegotiation-proof equilibria when players indirectly communicate through a public signal aggregating their private information. We show that renegotiation-proof equilibria are equivalent to Pareto-efficient correlated equilibria where communication influences the definition of Pareto efficiency.

Keywords: Communication Equilibria - Renegotiation-Proof Equilibria - Incomplete Information - Common Knowledge.

JEL Classification: C72, D82, D83[1]

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1.1 Introduction

In repeated games, renegotiation-proof equilibria are Pareto-efficient and self-enforcing outcomes in every continuation of the game - see Farrell and Maskin (1989a). They represent outcomes of a cheap-talk negotiation phase which is implicitly assumed to be direct[2]. However, indirect communication is of particular economic interest.

In most of economic activities prices aggregate and partially reveal the private information of each economic agent. The process is equivalent to an indirect communication phase where each player influences, and is influenced by, a public signal (the price).

In this chapter, we define renegotiation-proof equilibria in a game where players indirectly communicate. We show that the result - an indirect renegotiation-proof equilibrium - is a Pareto-efficient correlated equilibrium. The result is not obvious since the outcome needs to remain self-enforcing in any continuation of the game, and since communication complicates the analysis, as pointed out in Bergin and MacLeod (1993).

We formalize the intuitions in a Bayesian game extended with an information structure aggregating players' private information into a public signal. To define indirect renegotiation-proof equilibria, we first show in a static game how an equilibrium of the extended Bayesian game is equivalent to a communication equilibrium - see Myerson (1986) and Forges (1986). Forges (1990) substitutes a communication device with intra-play cheap talk. We follow a similar path, but we replace the device with a public signal aggregating the private information. Moreover, we show how pre-play communication influences the information available and so the definition of (ex-post) Pareto efficiency.

Then, we show that the equilibria of the static game are steady states of a repeated game where the public signal does not change the information available. Also, we show that the steady states are self-enforcing in every possible continuation of the game. By combining the results on Pareto efficiency and self-enforcement, we reach a definition of indirect renegotiation-proof equilibria.

[2]For example, in a repeated two-players Prisoner Dilemma game where players can directly communicate, a player may forgive the opponent who finks once. This strategy is a renegotiation-proof equilibrium since (i) the alternative where both players do not fink is Pareto efficient; (ii) the alternative is self-enforcing at the moment of the negotiation and in all the continuations of the game (in future nobody wants to deviate from this strategy).

We faced two main challenges in defining indirect renegotiation-proof equilibria. The first challenge was to model information about the strategies and allow players to communicate this information. To address this problem, we introduced payoff-irrelevant types, one for each pure strategy. The second challenge was to formalize the aggregation of private information. To address this other problem, we added a random variable whose realization depends on the profile of payoff -relevant and -irrelevant types. We aggregated the private information by aggregating the expectations (about the realization of the random variable and contingent on the private information) of every single player[3].

The structure of the chapter is as follows. The next section presents the Bayesian game and the information structure. Section 1.3 shows how communication influences the definition of Pareto efficiency, and Section 1.4 discusses self-enforcement in any continuation of the game. In Section 1.5 we define indirect renegotiation-proof outcomes and present an economic application. Section 1.6 concludes discussing further links to the literature and the role of rational expectations.

1.2 The model

In this section, we present the Bayesian game and the information structure. Also, we formalize the pre-play information aggregation phase - which is based on McKelvey and Page (1986) - and the equilibria of the game.

1.2.1 Bayesian setup

Define a Bayesian game $\mathcal{G}$

$$\mathcal{G} = \left((T_i, A_i, u_i)_{i \in N}, \rho\right)$$

where $N = \{1, ..., n\}$ is the set of players; T_i is the finite set of payoff relevant types for $i \in N$, with $t_i \in T_i$, $T \equiv \times_{i=1}^{n} T_i$, and $t \in T$; A_i is the finite set of actions, with $a_i \in A_i$, $A \equiv \times_{i=1}^{n} A_i$, and $a \in A$; $u_i : A \times T \to \mathbb{R}$ is the Von Neumann-Morgenstern utility function; and $\rho \in \Delta^{|T|-1}$ is the joint probability distribution over the set of types T, with Δ^z denoting the unit simplex of dimension z.

[3] In less abstract terms, suppose two firms and an inverse demand function aggregating the quantities produced. Each firm has some expectations about the total quantity produced (the random variable) given only its production (the private information). We can use the inverse demand function to aggregate the expectations, and we can therefore aggregate the private information by aggregating the expectations. We formalize this game in Example 4.

1.2.2 Information structure

Define the following information structure $\mathcal{I}$

$$\mathcal{I} = \left(\left(\hat{T}_i, \mathcal{P}_i\right)_{i\in N}, \chi, h, \pi\right)$$

The set $\hat{T}_i$ contains i's payoff-irrelevant types (or meta-types): for each player i we add further types, one for each i's strategy, to represent the information about the strategy he can play. The set of meta-types for all the players is the Cartesian product $\hat{T} \equiv \times_{i=1}^{n} \hat{T}_i$ and is finite, with $\hat{t}_i \in \hat{T}_i$ and $\hat{t} \in \hat{T}$.

The partition $\mathcal{P}_i$ of the set $T \times \hat{T}$ identifies the private information for each player $i \in N$, with $\mathcal{P}_i(t, \hat{t})$ being the element of the partition at $(t, \hat{t})$.

The random variable $\chi : T \times \hat{T} \to \mathbb{R}$ depends on the profiles of types and meta-types, and players have expectations about its realization. (We postpone the discussion on its role in aggregating the information to the next section.)

The aggregation function $h : \mathbb{R}^n \to \mathbb{R}$ maps the profile of expectations into a public signal. It differs from the random variable χ as h's realization is public, while χ's is not.

Finally, the prior π identifies the joint probability distribution over $T \times \hat{T}$. It follows that $\sum_{\hat{t}\in\hat{T}} \pi(t, \hat{t}) = \rho(t)$, for any $t \in T$.

The information structure $\mathcal{I}$ extends the Bayesian game $\mathcal{G}$ defining the following game:

$$\mathcal{G}_{\mathcal{I}} = \left(\left(T_i \times \hat{T}_i, A_i, \mathcal{P}_i, u_i\right)_{i\in N}, \chi, h, \pi\right)$$

The information structure $\mathcal{I}$ adds a pre-play phase where a public signal aggregates the private information. Next, we formalize the information aggregation phase and characterize the strategies and the equilibria of $\mathcal{G}_{\mathcal{I}}$.

1.2.3 Pre-play information aggregation

We now formalize the information aggregation phase. We show how a public signal aggregates and updates the private information of each player by aggregating their expectations about the realization of the random variable.

Each player i forms some expectations, X_i, about the realization of χ conditional on his private information $\mathcal{P}_i(t, \hat{t})$, i.e.

$$X_i(t,\hat{t}) \equiv E\left(\chi|\mathcal{P}_i(t,\hat{t})\right) \equiv \sum_{t'\in T,\hat{t}'\in\hat{T}} \chi(t',\hat{t}')\frac{\pi\left((t',\hat{t}')\cap\mathcal{P}_i(t,\hat{t})\right)}{\pi\left(\mathcal{P}_i(t,\hat{t})\right)} \tag{1.1}$$

Since each player $i \in N$ has some expectations $X_i(t,\hat{t})$, we can define the array of posteriors of all the players as the map $X : T \times \hat{T} \to \mathbb{R}^n$, or

$$X(t,\hat{t}) = \{X_i(t,\hat{t})\}_{i\in N}$$

The aggregation function h mechanically - players have no incentive to lie as we prove in the next section - aggregates the array of posteriors into a public signal. We can collect the profiles of types and meta-types leading to the same public signal by defining the following set $\mathcal{H}(t,\hat{t})$

$$\mathcal{H}(t,\hat{t}) = \{(t',\hat{t}') \in T\times\hat{T}|h\left(X(t',\hat{t}')\right) = h\left(X(t,\hat{t})\right)\} \tag{1.2}$$

Therefore, the public signal alone may not allow players to distinguish the profile $(t',\hat{t}')$ from $(t,\hat{t})$ if they both lead to the same public signal $h\left(X(t,\hat{t})\right) = h\left(X(t',\hat{t}')\right) = h^*$. But, some players may still distinguish the two profiles given their private information. The set $\mathcal{H}(t,\hat{t})$ defines an element of the public information partition

$$\mathcal{H} = \{\mathcal{H}(t,\hat{t})\}_{(t,\hat{t})\in T\times\hat{T}}$$

which is used by each player $i \in N$ to update his private information as follows

$$\mathcal{P}_i^h \equiv \mathcal{P}_i \vee \mathcal{H} \tag{1.3}$$

The new information updates the posteriors (1.1) in

$$E\left(\chi|\mathcal{P}_i^h(t,\hat{t})\right) \equiv \textstyle\sum_{t'\in T,\hat{t}'\in\hat{T}} \chi(t',\hat{t}')\frac{\pi\left((t',\hat{t}')\cap\mathcal{P}_i^h(t,\hat{t})\right)}{\pi\left(\mathcal{P}_i^h(t,\hat{t})\right)} \equiv \textstyle\sum_{t'\in T,\hat{t}'\in\hat{T}} \chi(t',\hat{t}')\pi\left((t',\hat{t}')|\mathcal{P}_i^h(t,\hat{t})\right) \tag{1.4}$$

from which, each player $i \in N$ also updates his information about other players

$$\pi((t'_{-i},\hat{t}'_{-i})|\mathcal{P}_i^h(t,\hat{t})) = \sum_{t'_i\in T_i,\hat{t}'_i\in\hat{T}_i} \pi((t'_i,t'_{-i},\hat{t}'_i,\hat{t}'_{-i})|\mathcal{P}_i^h(t,\hat{t})) \tag{1.5}$$

Therefore, the public signal partially reveals the private information aggregated, and each player infers more information about other players, including information - through the meta-types - about the strategies they will play. Since we have formalized the pre-play phase of the game, we now have all the elements to define an equilibrium of the game $\mathcal{G}_{\mathcal{I}}$.

1.2.4 Strategies and equilibria

In $\mathcal{G}_{\mathcal{I}}$ a strategy σ_i for each player $i \in N$ maps each profile of types and meta-types into a probability distribution over actions. From the pre-play information aggregation phase, the public signal updates the information each player has about his opponents through (1.5). Then, the expected payoff of choosing a strategy σ_i at $(t_i, \hat{t}_i)$ is

$$\sum_{t_{-i}\in T_{-i},\hat{t}_{-i}\in\hat{T}_{-i}} \left[\pi((t_{-i},\hat{t}_{-i})|\mathcal{P}_i^h(t,\hat{t}))\sum_{a_i\in A_i,a_{-i}\in A_{-i}}\left(\sigma_i(a_i|t_i,\hat{t}_i)\prod_{j\neq i}\sigma_j(a_j|t_j,\hat{t}_j)u_i(a_i,a_{-i},t)\right)\right]$$

where $\sigma_i(a_i|t_i, \hat{t}_i)$ is i's probability of playing a_i given his type t_i and meta-type $\hat{t}_i$.

An equilibrium of the game $\mathcal{G}_{\mathcal{I}}$ is an outcome where no player deviates from his strategy given the information revealed by the aggregation of private information.

Definition 1. *A strategy profile* $\{\sigma_i\}_{i\in N} \in \bigtimes_i \left(\Delta^{|A_i|-1}\right)^{T_i\times\hat{T}_i}$ *is an equilibrium of the game* $\mathcal{G}_{\mathcal{I}}$ *if, for any* $i \in N$, $t_i \in T_i$, $\hat{t}_i \in \hat{T}_i$ *and* σ_i'

$$\sum_{t_{-i},\hat{t}_{-i}}\left[\pi((t_{-i},\hat{t}_{-i})|\mathcal{P}_i^h(t_i,t_{-i},\hat{t}_i,\hat{t}_{-i}))\sum_a\left(\sigma_i(a_i|t_i,\hat{t}_i)\prod_{j\neq i}\sigma_j(a_j|t_j,\hat{t}_j)u_i(a_i,a_{-i},t_i,t_{-i})\right)\right]\geq$$
$$\sum_{t_{-i},\hat{t}_{-i}}\left[\pi((t_{-i},\hat{t}_{-i})|\mathcal{P}_i^h(t_i,t_{-i},\hat{t}_i,\hat{t}_{-i}))\sum_a\left(\sigma_i'(a_i|t_i,\hat{t}_i)\prod_{j\neq i}\sigma_j(a_j|t_j,\hat{t}_j)u_i(a_i,a_{-i},t_i,t_{-i})\right)\right]$$

Definition 1 is a Bayes-Nash equilibrium of $\mathcal{G}_{\mathcal{I}}$ and describes the outcome of the following sequence of events:

1. Nature draws $t \in T$ and $\hat{t} \in \hat{T}$;
2. Every player $i \in N$ forms some expectations about χ given his private information $\mathcal{P}_i(t, \hat{t})$;
3. The aggregation function h publicly selects an outcome given the array of expectations $X(t, \hat{t})$;
4. Every player $i \in N$ infers some information from the public signal;
5. Players choose an action and payoffs realize.

Steps 2, 3 and 4 form the pre-play information aggregation phase where players influence the public signal which, in turn, influences them. In the next section, we show how this phase describes an indirect communication mechanism. We now conclude the section showing how to apply the information aggregation phase to a well-known game. Further examples are in Appendix C.

Exemple 1. *Suppose the following battle of the sexes game*

	L	*R*
U	*2; 1*	*0; 0*
D	*0; 0*	*1; 2*

The type set T is trivial since each player has only one type. We assume that: the set of meta-types equals the set of pure strategies, $\hat{T}_1 = \{U, D\}$ and $\hat{T}_2 = \{L, R\}$; and the following random variable χ and uniform common prior

χ	*L*	*R*
U	*1*	*0*
D	*1*	*1*

Pr	*L*	*R*
U	*1/4*	*1/4*
D	*1/4*	*1/4*

Pure strategies

We now define the private information of each player when they play pure strategies. In pure strategies, players can distinguish between their meta-types (pure strategies), but not between the ones of the other player. Then, the following partitions represent players' private information

$$\mathcal{P}_1 = \{(UL, UR), (DL, DR)\} \quad \mathcal{P}_2 = \{(UL, DL), (UR, DR)\}$$

Given that $t \in T$ is constant, the following table reports the expectations[4] *of each player and the result of an aggregation function which returns the average of the expectations*

$\hat{t}$	$X_1(\hat{t})$	$X_2(\hat{t})$	$h = \frac{X_1(\hat{t})}{2} + \frac{X_2(\hat{t})}{2}$
UL	*1/2*	*1*	*3/4*
UR	*1/2*	*1/2*	*1/2*
DL	*1*	*1*	*1*
DR	*1*	*1/2*	*3/4*

[4] Suppose the state is (UL), then for Player 1 the relevant element of the partition is $\{(UL, UR)\}$. The expected value of χ given this private information is $\pi(UL|\mathcal{P}_1(UL)) * \chi(UL) + \pi(UR|\mathcal{P}_1(UL)) * \chi(UR) = \frac{1}{2} * 1 + \frac{1}{2} * 0 = \frac{1}{2}$. Similarly for the state (UR). Now suppose the state is (DL), then for Player 1 the relevant element of the partition is $\{(DL, DR)\}$. The expected value of χ given this private information is $\pi(DL|\mathcal{P}_1(DL)) * \chi(DL) + \pi(DR|\mathcal{P}_1(DL)) * \chi(DR) = \frac{1}{2} * 1 + \frac{1}{2} * 1 = 1$. Similarly for the state (DR) and for Player 2.

The realization of the public signal defines the public information partition $\mathcal{H}$

$$\mathcal{H} = \{(UL, DR), (DL), (UR)\}$$

Also, players use $\mathcal{H}$ *to update their private information obtaining*

$$\mathcal{P}_1^h = \mathcal{P}_2^h = \{(UL), (UR), (DL), (DR)\}$$

Therefore, each player perfectly infers the strategy of the other player and the possible equilibria correspond to the Nash equilibria in pure strategies, i.e. (UL) *and* (DR)*.*

Mixed strategies

In mixed strategies, the randomization represents a player who cannot discriminate between his meta-types. So, if the set of meta-types equals the set of pure strategies, the following partitions represent the private information of the two players

$$\mathcal{P}_1 = \mathcal{P}_2 = \{(UL, UR, DL, DR)\}$$

The expected value of χ *is the same for all the possible profiles of meta-types*[5]*, so the public information partition is*

$$\mathcal{H} = \{(UL, DR, DL, UR)\}$$

which implies that players' updated private information remains unchanged

$$\mathcal{P}_1^h = \mathcal{P}_2^h = \{(UL, UR, DL, DR)\}$$

Therefore, the public signal is uninformative, players cannot indirectly communicate, and the extended game becomes a standard Bayesian game. (Therefore, our definition guarantees that an equilibrium exists as $\mathcal{G}_{\mathcal{I}}$ *becomes a Bayesian game* $\mathcal{G}$ *when the signal is uninformative.)*

1.3 Communication and Pareto efficiency

In the previous section, we defined the pre-play information aggregation phase. We now recall the definition of a communication equilibrium, show that the information structure $\mathcal{I}$ can replicate any communication equilibrium, and discuss how communication influences ex-post efficiency.

[5] $X_1(UL) = X_1(UR) = X_1(DL) = X_1(DR) = \pi(UL|\mathcal{P}_1(.)) * \chi(UL) + \pi(UR|\mathcal{P}_1(.)) * \chi(DR) + \pi(DL|\mathcal{P}_1(.)) * \chi(DL) + \pi(DR|\mathcal{P}_1(.)) * \chi(DR) = \frac{1}{4} * 1 + \frac{1}{4} * 0 + \frac{1}{4} * 1 + \frac{1}{4} * 1 = \frac{3}{4}$.

1.3.1 Definition of a communication equilibrium

A communication device q - we use the canonical definition without any loss of generality, see Forges (1986) - receives a type t_i' from each player i and delivers a probability distribution over pure strategies

$$q(.|t') \in \Delta^{|A|-1}, \forall t' \in T$$

The distribution q is an equilibrium of the Bayesian game $\mathcal{G}$ if players report their true types and follow the recommendation of the communication device

Definition 2. *A communication device q for the Bayesian game $\mathcal{G}$ is a Communication Equilibrium if for any $i \in N$, $t_i, t_i' \in T_i$, $a_i' \in A_i$*

$$\begin{aligned}\sum_{t_{-i}} \left[\rho(t_{-i}|t_i) \sum_{a_i, a_{-i}} (q(a_i, a_{-i}|t_i, t_{-i}) u_i(a_i, a_{-i}, t)) \right] \geq \\ \sum_{t_{-i}} \left[\rho(t_{-i}|t_i) \sum_{a_i, a_{-i}} (q(a_i, a_{-i}|t_i', t_{-i}) u_i(a_i', a_{-i}, t)) \right]\end{aligned} \tag{1.6}$$

Therefore, communication formalizes a system of inputs and outputs influencing the strategies played. Next, we show how our information aggregation phase is analogous to a communication device.

1.3.2 h as a communication device

The definitions of q and h are closely related. The communication device $q(.|t)$ delivers a probability distribution over pure strategies receiving players' types as input. The aggregation function h delivers a probability distribution $\pi(.|\mathcal{H}(t, \hat{t}))$ over $T \times \hat{T}$ receiving players' expectations as an input. Moreover, q adds a pre-play communication phase as h adds a pre-play information aggregation phase.

Given the analogies, it should not surprise that the information structure $\mathcal{I}$ can replicate any communication equilibrium.

Proposition 1. *Given any Bayesian game $\mathcal{G}$ and a communication device q, there exists an information structure $\mathcal{I}$ such that the extended Bayesian game $\mathcal{G}_{\mathcal{I}}$ has an equilibrium payoff-equivalent to the Communication Equilibrium of $\mathcal{G}$.*

Proof. In Appendix A.1. $\square$

To understand why the proposition holds, set $\hat{T}$ as a copy of A. Then, the public signal would generate the following correlated distribution over the set of pure strategies

$$\tilde{q}(a|\mathcal{H}(t,\tilde{a})) = \sum_{a'\in A}\left(\prod_{i\in N}\sigma_i(a_i|t_i,a'_i)\hat{\pi}(a'|\mathcal{H}(t,\tilde{a}))\right) \tag{1.7}$$

where $\hat{\pi}$ is the marginal probability distribution over the meta-types $\hat{T}$, $\sum_{t\in T}\pi(t,\hat{t}) = \hat{\pi}(\hat{t})$ for any $\hat{t}\in\hat{T}$. In equilibrium, the condition further simplifies in[6]

$$\tilde{q}(a|\mathcal{H}(t,\tilde{a}))) = \hat{\pi}(a|\mathcal{H}(t,\tilde{a})))$$

Therefore, if the set of meta-types is a copy of the set of actions, then the marginal probability distribution over the meta-types $\hat{T}$ equals the correlated distribution over actions, leading to the equivalence between information aggregation and communication.

The correlated distribution (1.7) also reveals that the information aggregation phase influences the outcome by shaping the set $\mathcal{H}(t,\tilde{a})$ through (1.2). We will now show how indirect communication influences Pareto efficiency.

1.3.3 Pareto efficiency

Holmström and Myerson (1983) proposes several efficiency definitions in incomplete information. However, Definition 1 already prescribes no deviations given the realization of the public signal. Therefore, we can define Pareto efficiency depending on the ex-post information revealed by the public signal

$$\begin{aligned}\sum_{t_{-i}}\left[\rho(t_{-i}|t_i)\sum_{a_i,a_{-i}}\big(\tilde{q}(a_i,a_{-i}|\mathcal{H}(t,\hat{t}))u_i(a_i,a_{-i},t)\big)\right] \geq \\ \sum_{t_{-i}}\left[\rho(t_{-i}|t_i)\sum_{a_i,a_{-i}}\big(\tilde{q}(a'_i,a_{-i}|\mathcal{H}(t,\hat{t}))u_i(a_i,a_{-i},t)\big)\right] \qquad \forall i\in N\end{aligned} \tag{1.8}$$

where $\tilde{q}(.|\mathcal{H}(t,\hat{t}))$ is defined as in (1.7) and is the correlated distribution over pure strategies generated by the realization of the public signal. Information aggregation and communication therefore also affects ex-post Pareto efficiency through

[6] Each player $i\in N$ chooses his own strategy $\sigma_i(a_i|t_i,a'_i)$. But, in equilibrium every player $i\in N$ follows the suggestion of the communication device (in the proof of Proposition 1), so if $a'_i\in A_i$ is suggested then $\sigma_i(a'_i|t_i,a'_i)=1$; otherwise $\sigma_i(a_i|t_i,a'_i)=0$ for every $a_i\neq a'_i\in A_i$.

$\mathcal{H}(t,\hat{t})$. The condition is not with respect to $\mathcal{P}_i^h(t,\hat{t})$ since some players may still have different information, and the set of all the players (our only coalition) cannot coordinate using information not available to everyone.

Next, we will focus on the second characteristic of a renegotiation-proof equilibrium: self-enforcement in every continuation of the game. As previously done, we conclude the section with an example applying the concepts just discussed.

Exemple 2. *We now replicate the correlated equilibria of the battle of the sexes game with an information structure where a random variable takes a positive value only in* (UL) *and* (DR) *and with uniform common prior*

	L	R
U	2; 1	0; 0
D	0; 0	1; 2

χ	L	R
U	1	0
D	0	1

Pr	L	R
U	1/4	1/4
D	1/4	1/4

Players can forecast the value of the public signal (in the next section we show how players can learn to predict the realization of the public signal). Then, the following partitions represent the private information of each player

$$\mathcal{P}_1 = \mathcal{P}_2 = \{(UL, DR), (UR, DL)\}$$

Under the same aggregation function of Example 1*, the private information leads to the following table of expected values and realizations of the public signal*

$\hat{t}$	$X_1(\hat{t})$	$X_2(\hat{t})$	$h = \frac{X_1(\hat{t})}{2} + \frac{X_2(\hat{t})}{2}$
UL	*1*	*1*	*1*
UR	*0*	*0*	*0*
DL	*0*	*0*	*0*
DR	*1*	*1*	*1*

from which the public information partition is

$$\mathcal{H} = \{(UL, DR), (UR, DL)\}$$

and players' updated private information is

$$\mathcal{P}_1^h = \mathcal{P}_2^h = \{(UL, DR), (UR, DL)\}$$

Therefore, players correlate their actions thanks to the information aggregation, and we obtain an outcome pay-off equivalent to a correlated equilibrium one. Moreover, from (1.8), (UL, DR) *is also Pareto-efficient.*

1.4 Repetition and self-enforcement

In the previous section, we modeled Pareto efficiency when players indirectly communicate in a static game. However, renegotiation-proof equilibria are defined in repeated games. We now show that the equilibria defined in the static game $\mathcal{G}_{\mathcal{I}}$ are self-enforcing in any continuation of the game, when the game is repeated.

1.4.1 The repeated game

The repetition of the static game $\mathcal{G}_{\mathcal{I}}$ changes the notation and the timing of the events. We now formalize these changes.

The game $\mathcal{G}_{\mathcal{I}}^K$ is the Bayesian game extended with the information structure aggregating the private information $\mathcal{G}_{\mathcal{I}}$ played over a series of discrete time steps, $k \in \{0, 1, 2, ...\}$. Henceforth, we use the convention z^k to denote an object z at k.

At time k agents play and influence the realization of the future public signal. At $k+1$ when the signal realizes, players infer some information and play again. Therefore, in the repeated game $\mathcal{G}_{\mathcal{I}}^K$, the timing of events becomes:

1. At time k every player $i \in N$ has some private information $\mathcal{P}_i^{h,k}(t, \hat{t})$;
2. Every player $i \in N$ chooses an action a_i^k and payoffs realize;
3. The public signal selects an outcome given the profile of expectations
$$h\left(\left\{E\left(\chi|\mathcal{P}_i^{h,k}(t,\hat{t})\right)\right\}_{i\in N}\right) \tag{1.9}$$
4. The signal realization defines $\mathcal{H}^k(t, \hat{t})$, and so the partition $\mathcal{H}^k$;
5. Information updates $\mathcal{P}_i^{h,k+1} = \mathcal{P}_i^{h,k} \vee \mathcal{H}^k$, and players start again from Point 1 under the updated partitions.

In this sequence of events we assume each player does not try to influence other players because, in expectations, he does not affect the information revealed by the public signal (from the proof of Proposition 1). Also, we assume players entirely discount future payoffs - so they are interested only in the immediate outcome - because players indirectly communicate, and so each player may potentially face different opponents at each k.

At time k players only observe the history of all public signals realizations. Then, their information is the initial private information and what they inferred from the past realizations, since

$$\begin{aligned}
\mathcal{P}_i^{h,k+1} &= \mathcal{P}_i^{h,k} \vee \mathcal{H}^k \\
&= \mathcal{P}_i^{h,k-1} \vee \mathcal{H}^{k-1} \vee \mathcal{H}^k \\
&= \mathcal{P}_i^{h,k-2} \vee \mathcal{H}^{k-2} \vee \mathcal{H}^{k-1} \vee \mathcal{H}^k \\
&\vdots \\
&= \mathcal{P}_i^{h,0} \vee \mathcal{H}^1 \vee \mathcal{H}^2 \vee \cdots \vee \mathcal{H}^k
\end{aligned} \tag{1.10}$$

Given this information, every player $i \in N$ chooses an action a_i^k that maximizes his expected utility

$$a_i^k \in \arg\max_{a_i} \sum_{t_{-i},\hat{t}_{-i},a_{-i}^k} \left[\pi((t_{-i},\hat{t}_{-i})|\mathcal{P}_i^{h,k}(t,\hat{t}))\sigma_{-i}(a_{-i}^k|t_{-i},\hat{t}_{-i})u_i(a_i^k,a_{-i}^k,t)\right]$$

leading to the following definition of an equilibrium

Definition 3. *A sequence of strategy profiles* $\left\{\{\sigma_i^k\}_{i\in N}\right\}_{k\in\{0,1,2,\dots\}}$ *is an equilibrium of the game* $\mathcal{G}_{\mathcal{I}}^K$*, if each strategy profile* $\{\sigma_i^k\}_{i\in N} \in \bigtimes_i \left(\Delta^{|A_i|-1}\right)^{T_i\times\hat{T}_i}$ *satisfies for any* $i \in N$*,* k*,* $t_i \in T_i$*,* $\hat{t}_i \in \hat{T}_i$ *and* $\tilde{\sigma}_i^k$

$$\sum_{t_{-i},\hat{t}_{-i}} \left[\pi((t_{-i},\hat{t}_{-i})|\mathcal{P}_i^{h,k}(t_i,t_{-i},\hat{t}_i,\hat{t}_{-i}))\sum_a\left(\sigma_i^k(a_i|t_i,\hat{t}_i)\prod_{j\neq i}\sigma_j^k(a_j|t_j,\hat{t}_j)u_i(a_i,a_{-i},t_i,t_{-i})\right)\right] \geq$$
$$\sum_{t_{-i},\hat{t}_{-i}} \left[\pi((t_{-i},\hat{t}_{-i})|\mathcal{P}_i^{h,k}(t_i,t_{-i},\hat{t}_i,\hat{t}_{-i}))\sum_a\left(\tilde{\sigma}_i^k(a_i|t_i,\hat{t}_i)\prod_{j\neq i}\sigma_j^k(a_j|t_j,\hat{t}_j)u_i(a_i,a_{-i},t_i,t_{-i})\right)\right]$$

The definition acknowledges that information evolves as the game is played and that the sequence of strategic decisions constitutes the equilibrium of the repeated game $\mathcal{G}_{\mathcal{I}}^K$. We can now study the link between these equilibria and the equilibria of the static game $\mathcal{G}_{\mathcal{I}}$.

1.4.2 Self-enforcing steady states

We now show that an equilibrium in the static game is self-enforcing in every repetition of the game since information stops changing.

Since sets are finite, there exists a K such that players cannot infer further information from the public signal - McKelvey and Page (1986) presents the following updating process, extending Geanakoplos and Polemarchakis (1982).

Lemma 1. *The information inference from the public signal stops at a finite K.*

Proof. From (1.10) the updating process of the partitions follows $P_i^{h,k+1} = P_i^{h,k} \vee \mathcal{H}^k$. Then $\mathcal{H}^k \supseteq \mathcal{H}^{k-1} \supseteq \mathcal{H}^{k-2} \supseteq \mathcal{H}^{k-3} \supseteq ....$ Since we have finite elements, $\mathcal{H}^k$ cannot be finer than the join -the coarsest common refinement- of players' partitions. Then the process stops. $\square$

Therefore, players stop inferring information at K. However, without new information also the public signal (from Equation (1.9)) and the strategies stop changing. Moreover, if strategies remain unaltered after the observation of the public signal, we reach an equilibrium of the static game $\mathcal{G}_{\mathcal{I}}$.

Proposition 2. *Given any game $\mathcal{G}_{\mathcal{I}}^K$, there exists a finite time K such that an equilibrium at $k > K$ is payoff-equivalent to an equilibrium of $\mathcal{G}_{\mathcal{I}}$.*

Proof. From Lemma 1 there exists a K such that the learning process stops. Since the learning process ends, players can not further refine their information, and for any player $i \in N$, $P_i^{h,K} = P_i^{h,K+1}$. Since $P_i^{h,K} = P_i^{h,K+1}$, the best response at $K+1$ is as at K, i.e. players do not change their strategy after the observation of the public signal. $\square$

Therefore, the equilibria in the static game are steady-states of the repeated game, and from Proposition 2 it follows that

Corollary 1. *An equilibrium of $\mathcal{G}_{\mathcal{I}}$ is self-enforcing in any repetition of the game.*

Therefore, the equilibria defined in the previous sections satisfy self-enforcing in any continuation of the game when the game is repeated. Example 3 shows how to apply the information aggregation phase in a repeated game.

Exemple 3. *Suppose the battle of the sexes game as in Example 2.*

	L	R
U	2; 1	0; 0
D	0; 0	1; 2

χ	L	R
U	1	0
D	0	1

Pr	L	R
U	1/4	1/4
D	1/4	1/4

We assume that at $k = 0$ only Player 1 has some information about the signal. Then, the following partitions represent the initial private information for each player

$$\mathcal{P}_1^{k=0} = \{(UL, DR), (UR, DL)\}$$

$$\mathcal{P}_2^{k=0} = \{(UL, DL), (UR, DR)\}$$

Players maximize their expected utility choosing certain actions $a_1^{k=0}, a_2^{k=0}$. *The private information leads the following table of expected values and public signals*

$\hat{t}$	$X_1^{k=0}(\hat{t})$	$X_2^{k=0}(\hat{t})$	$h = \frac{X_1^{k=0}(\hat{t})}{2} + \frac{X_2^{k=0}(\hat{t})}{2}$
UL	1	1/2	3/4
UR	0	1/2	1/4
DL	0	1/2	1/4
DR	1	1/2	3/4

from which the public information partition is

$$\mathcal{H}^{k=0} = \{(UL, DR), (UR, DL)\}$$

and the next period private information is

$$\mathcal{P}_1^{h,k=1} = \{(UL, DR), (UR, DL)\}$$

$$\mathcal{P}_2^{h,k=1} = \{(UL), (UR), (UR), (DR)\}$$

Therefore, Player 1*'s information remains unaltered, while Player* 2 *refines his own. The table at* $k = 1$ *updates in*

$\hat{t}$	$X_1^{k=1}(\hat{t})$	$X_2^{k=1}(\hat{t})$	$h = \frac{X_1^{k=1}(\hat{t})}{2} + \frac{X_2^{k=1}(\hat{t})}{2}$
UL	1	1	1
UR	0	0	0
DL	0	0	0
DR	1	1	1

from which the public information partition is

$$\mathcal{H}^{k=1} = \{(UL, DR), (UR, DL)\}$$

Therefore, the information updating stops and players learn to correlate their actions using the public signal realization.

1.5 Indirect renegotiation-proof equilibria

We now have all the elements to define indirect renegotiation-proof equilibria.

Definition 4. *A strategy profile* $\{\sigma_i\}_{i\in N} \in \bigtimes_i \left(\Delta^{|A_i|-1}\right)^{T_i\times\hat{T}_i}$ *is a Renegotiation-Proof Equilibrium of the game* $\mathcal{G}_{\mathcal{I}}^{K}$ *if the following two conditions are satisfied for any* $i \in N$, $t_i \in T_i$, $\hat{t}_i \in \hat{T}_i$, σ_i' *and any* $k' \in \{0, ..., K\}$

$$\sum_{t_{-i},\hat{t}_{-i}} \left[\pi((t_{-i},\hat{t}_{-i})|\mathcal{P}_i^{h,k}(t_i,t_{-i},\hat{t}_i,\hat{t}_{-i}))\sum_a\left(\sigma_i(a_i|t_i,\hat{t}_i)\prod_{j\neq i}\sigma_j(a_j|t_j,\hat{t}_j)u_i(a_i,a_{-i},t_i,t_{-i})\right)\right] \geq$$
$$\sum_{t_{-i},\hat{t}_{-i}} \left[\pi((t_{-i},\hat{t}_{-i})|\mathcal{P}_i^{h,k}(t_i,t_{-i},\hat{t}_i,\hat{t}_{-i}))\sum_a\left(\sigma_i'(a_i|t_i,\hat{t}_i)\prod_{j\neq i}\sigma_j(a_j|t_j,\hat{t}_j)u_i(a_i,a_{-i},t_i,t_{-i})\right)\right]$$

and

$$\sum_{t_{-i}}\left[\rho(t_{-i}|t_i)\sum_{a_i,a_{-i}}\left(\tilde{q}(a_i,a_{-i}|\mathcal{H}^k(t,\hat{t}))u_i(a_i,a_{-i},t)\right)\right] \geq$$
$$\sum_{t_{-i}}\left[\rho(t_{-i}|t_i)\sum_{a_i,a_{-i}}\left(\tilde{q}(a_i,a_{-i}|\mathcal{H}^{k'}(t,\hat{t}))u_i(a_i,a_{-i},t)\right)\right] \qquad \forall i\in N$$

The first condition states the incentive constraint and requires an equilibrium to be immune from individual deviations, as in Definition 1. Also, in the previous section, we have seen that this equilibrium is self-enforcing in any continuation of the game. The second condition states Pareto efficiency and requires players to not collectively switch to an alternative outcome. Also, in the previous sections, we have seen that information aggregation influences this definition. Therefore, these two conditions characterize a renegotiation-proof outcome as a Pareto-efficient correlated equilibrium.

We updated Pareto efficiency from (1.8) to fit the different information at each stage of the repeated game, as players may want to coordinate on an outcome at any stage k. However, from the updating process (1.10) information at K is finer than at $k < K$. Then, whenever players prefer to coordinate on a signal of a period $k < K$, there is an information loss.

Remark. *Players can be interested in "forgetting" some information to achieve Pareto-efficient equilibria.*

In the renegotiation-proof literature, players may forgive a deviation of another player to achieve a Pareto-efficient equilibrium. The logic behind our last remark is analogous. Players may discard some information (also about types) to obtain an efficient outcome. Moreover, since a renegotiation-proof strategy must be an equilibrium in any continuation of the game, players cannot discard the information in a stage and recover it in the following one.

Neeman and Pavlov (2013) already defines renegotiation-proof equilibria in incomplete information game using an ex-post definition of Pareto efficiency. However, in our information structure Pareto efficiency depends on the information revealed by the public signal. Also, our information structure helps in formalizing economic applications, as the following example shows.

Exemple 4. *Two firms, $i = \{1, 2\}$, decide whether to compete (in a Cournot model) or collude. The cost function $c_i(q_i)$ can be either low $\underline{c}(q_i) = 10 + 2q_i$ or high $\overline{c}(q_i) = 12 + 8q_i$, with q_i representing the quantity supplied by i. It is common knowledge that Firm 2 has low production costs and that the inverse demand function is $p = 50 - 2(q_1 + q_2)$.*

In case of symmetric costs, firms try to collude, and a deviation instantly triggers a retaliation where they produce until marginal revenues allow to recover the fix costs[7]*. In case of asymmetric costs, firms either compete (still in a Cournot model) or bargain over the production quantities, with each firm pushing the production quota towards its monopolistic quantity. For simplicity we assume that Firm 2 (which has low production costs) has a high bargaining power*[8]*.*

What we described is a Bayesian game $\mathcal{G} = \big((T_i, A_i, u_i)_{i\in N}, \rho\big)$, where the sets of types identify the production costs, $T_1 = \{\underline{c}_1, \overline{c}_1\}$, $T_2 = \{\underline{c}_2\}$; the set of actions $A_i = \{c_i, m_i\}$ determines whether a firm competes or colludes; the payoffs (the profits) are in Table 1.1; and ρ is uniform.

[7] We assume instant retaliation as in Osborne (1976). Therefore, Firm i produces: $q_i = 10$ when retaliates, since $p = 50 - 2(10 + 10) = 10$; $q_i = 8$ when competes, since it solves $(50 - 2(q_i + q_{-i}))q_i - c_i(q_i)$; $q_i = 6$ when colludes, since it solves the joint profits maximization $(50 - 2Q)Q - c(Q)$ with $q_i = \frac{Q}{2}$.

[8] The cartel bargaining problem is $\max_Q[(50 - 2Q)Q - 2Q]^{\delta}[(50 - 2Q)Q - 8Q]^{1-\delta}$ where $q_i = Q/2$. The fix costs do not appear as they are also the cost of not reaching an agreement; $\delta \to 1$ as Firm 2 has low production costs and so a high bargain power. The first best solution would require the firm with lower production costs to satisfy the entire demand and make a direct money transfer to the other firm, but we want to avoid the assumption of direct transfers between firms.

$\underline{c}_1, \underline{c}_2$	c	m
c	118; 118	70; 70
	$(Q = 16)$	$(Q = 20)$
m	70; 70	134; 134
	$(Q = 20)$	$(Q = 12)$

$\overline{c}_1, \underline{c}_2$	c	m
c	60; 152	96; 134
	$(Q = 15)$	$(Q = 12)$
m	60; 152	96; 134
	$(Q = 15)$	$(Q = 12)$

Table 1.1: Payoff matrix

We can now introduce the information structure $\mathcal{I} = \left(\left(\hat{T}_i, \mathcal{P}_i\right)_{i \in N}, \chi, h, \pi\right)$. *We assume that* $\hat{T}_i = A_i = \{c_i, m_i\}$ *(as in the previous examples); and that each firm knows only its production costs and its strategy. Then, the following partitions represent firms' private information*

$$\mathcal{P}_1 = \begin{Bmatrix} \{(\underline{c}_1, c, \underline{c}_2, c), (\underline{c}_1, c, \underline{c}_2, m)\}, \{(\underline{c}_1, m, \underline{c}_2, c), (\underline{c}_1, m, \underline{c}_2, m)\}, \\ \{(\overline{c}_1, c, \underline{c}_2, c), (\overline{c}_1, c, \underline{c}_2, m)\}, \{(\overline{c}_1, m, \underline{c}_2, c), (\overline{c}_1, m, \underline{c}_2, m)\} \end{Bmatrix}$$

$$\mathcal{P}_2 = \begin{Bmatrix} \{(\underline{c}_1, c, \underline{c}_2, c), (\underline{c}_1, m, \underline{c}_2, c), (\overline{c}_1, c, \underline{c}_2, c), (\overline{c}_1, m, \underline{c}_2, c)\}, \\ \{(\underline{c}_1, c, \underline{c}_2, m), (\underline{c}_1, m, \underline{c}_2, m), (\overline{c}_1, c, \underline{c}_2, m), (\overline{c}_1, m, \underline{c}_2, m)\} \end{Bmatrix}$$

The random variable χ *represents the total quantity produced* Q

$$\chi = Q : \begin{cases} 20 & \textit{if } (\underline{c}_1, c, \underline{c}_2, m), (\underline{c}_1, m, \underline{c}_2, c) \\ 16 & \textit{if } (\underline{c}_1, c, \underline{c}_2, c) \\ 15 & \textit{if } (\overline{c}_1, c, \underline{c}_2, c), (\overline{c}_1, m, \underline{c}_2, c) \\ 12 & \textit{if } (\underline{c}_1, m, \underline{c}_2, m), (\overline{c}_1, c, \underline{c}_2, m), (\overline{c}_1, m, \underline{c}_2, m) \end{cases}$$

while the aggregation function h *mimics the price function*

$$h = 50 - X_1 + X_2$$

where X_i *is* i*'s expectations about the overall quantity produced given its private information (its own quantity). We assume firms have uniform common prior* π *over* $T \times \hat{T} = \bigtimes_i T_i \times \hat{T}_i$.

In Table 1.2 - which we derive in Appendix A.2 - we compute the expectations and the public signal realizations for two periods, from which the following results follow:

- *Given the analogies between the aggregation process and the price function, firms learn to anticipate the price realization at* $k = 1$*;*

$t, \hat{t}$	$X_1^{k=0}(t,\hat{t})$	$X_2^{k=0}(t,\hat{t})$	h	$X_1^{k=1}(t,\hat{t})$	$X_2^{k=1}(t,\hat{t})$	h	p
$\underline{c}_1, c, \underline{c}_2, c$	18	16,5	15,5	16	16	18	18
$\underline{c}_1, c, \underline{c}_2, m$	18	14	18	20	20	10	10
$\underline{c}_1, m, \underline{c}_2, c$	16	16,5	17,5	20	20	10	10
$\underline{c}_1, m, \underline{c}_2, m$	16	14	20	12	12	26	26
$\bar{c}_1, c, \underline{c}_2, c$	13,5	16,5	20	15	15	20	20
$\bar{c}_1, c, \underline{c}_2, m$	13,5	14	22,5	12	12	26	26
$\bar{c}_1, m, \underline{c}_2, c$	13,5	16,5	20	15	15	20	20
$\bar{c}_1, m, \underline{c}_2, m$	13,5	14	22,5	12	12	26	26

Table 1.2: Expectations and aggregation for two periods

- *The choice of h as the inverse demand function p limits the possible correlations. For example, a convex combination of $(\underline{c}_1, c, \underline{c}_2, c)$ and $(\underline{c}_1, m, \underline{c}_2, m)$ cannot be played despite being a correlated equilibrium;*

- *In the Bayesian game without information structure there exists an equilibrium where Firm 2 always competes regardless of Firm 1 choices.*

- *When we extend the Bayesian game with the information structure $\mathcal{I}$, at $k = 0$ the public signal $h = 20$ supports a correlated equilibrium where Firm 2 colludes if Firm 1 has low production costs, but competes if Firm 1 has high production costs;*

- *When we extend the Bayesian game with the information structure $\mathcal{I}$, at $k = 1$ the public signal $h = 26$ supports a correlated equilibrium where Firm 2 always colludes regardless of 1 choices;*

- *If we compare the equilibrium with and without information structure, Firms will use the realization of the public signal (which at $k = 1$ equals the price) to sustain an implicit collusion, where Firm 2 cannot fully infer Firm 1 production costs. This outcome is an indirect renegotiation-proof equilibrium.*

1.6 Final remarks

In this chapter, we have formalized a definition of indirect renegotiation proof equilibria and seen a possible economic application. We now conclude discussing some assumptions and further links to the literature.

1.6.1 On aggregating information

Our information aggregation phase share many analogies with the mediated talk games studied in Lehrer (1996), Lehrer and Sorin (1997), and extended in Di Tillio (2004). In (public) mediated talks players sends some random inputs to a mediator which deterministically and publicly make announcements. The literature shows how this mechanism achieves correlated and communication equilibria.

In our model, we reach similar conclusions by aggregating private information. However, we have several points of departures from the mediated talk literature. For example, we analyze expectations, partitional information, and the repeated game case. These characteristics allow us to apply our information structure to economic problems, as shown in the last example.

1.6.2 On rational expectations

McKelvey and Page (1986) already discusses the role of rational expectations in a similar information structure. Rational expectations imply that players have some beliefs about the realization of the public signal and that the public signal realization does not contradict the beliefs. We also used this assumption to define players' partitions in Example 2.

Also, Rubinstein and Wolinsky (1994) defines rationalizability when beliefs are consistent with a public signal, suggesting that the solution concept is a steady state of a recurring game where players forecast the public signal. We also found a similar result in Section 1.4.

Therefore, rational expectations play an important role in our setup. However, rational expectations only simplify our reasoning, since we have also shown how players learn to forecast the public signal. We have based the learning process on the finiteness of the sets, but it should be possible to extend the results to an infinite state space using Brandenburger and Dekel (1987) to define common knowledge and Nyarko (1994) to show that beliefs converge to correlated equilibria. However, this is left for future research.

Appendix A

Derivations and proofs

A.1 Proof of Proposition 1

From Definition 1, for any $t \in T$ and $\hat{t} \in \hat{T}$, the probability of playing the profile a after observing the public signal $\mathcal{H}(t, \hat{t})$ is

$$\sum_{t \in T, \hat{t} \in \hat{T}} \left(\prod_{i \in N} \sigma_i(a_i | t_i, \hat{t}_i) \pi(t, \hat{t} | \mathcal{H}(t, \hat{t})) \right)$$

which can be written as[1]

$$\sum_{t \in T} \left[\rho(t) \sum_{\hat{t} \in \hat{T}} \left(\prod_{i \in N} \sigma_i(a_i | t_i, \hat{t}_i) \hat{\pi}(\hat{t} | \mathcal{H}(t, \hat{t})) \right) \right]$$

where $\hat{\pi}$ is the marginal probability distribution of the meta-types $\hat{T}$, i.e. $\sum_{t \in T} \pi(t, \hat{t}) = \hat{\pi}(\hat{t})$, for any $\hat{t} \in \hat{T}$. Define the probability over the profile of actions induced by the realization of the public signal as

$$\tilde{q}(a | \mathcal{H}(t, \hat{t})) \equiv \sum_{\hat{t} \in \hat{T}} \left(\prod_{i \in N} \sigma_i(a_i | t_i, \hat{t}_i) \hat{\pi}(\hat{t} | \mathcal{H}(t, \hat{t})) \right) \tag{A.1}$$

therefore

$$\sum_{t \in T} \rho(t) \tilde{q}(a | \mathcal{H}(t, \hat{t}))$$

[1] we simplify the notation from $\sum_{t \in T} \left[\rho(t) \sum_{\hat{t} \in \hat{T}} \left(\prod_{i \in N} \sigma_i(a_i | t_i, \hat{t}_i) \hat{\pi}(\hat{t} | \mathcal{H}(t, \hat{t}), t) \right) \right]$, since $t \in \mathcal{H}(t, \hat{t})$

An equilibrium of $\mathcal{G}_{\mathcal{I}}$ is analogous to a communication equilibrium if $q(a|t) = \tilde{q}(a|\mathcal{H}(t,\hat{t}))$, players follow the strategy recommended (obedience condition) and truthfully report their expectations. Therefore, we have to prove that $\tilde{q}(a|\mathcal{H}(t,\hat{t}))$ satisfies these two conditions.

Obedience. For any $t \in T$ and $\hat{t} \in \hat{T}$, after observing the public signal $\mathcal{H}(t,\hat{t})$, a player $i \in N$ would infer the following probability about the action profile a_{-i} of other players

$$\tilde{q}(a_{-i}|\mathcal{P}_i^h(t,\hat{t})) \equiv \sum_{\hat{t}_{-i}} \left(\prod_{j\neq i} \sigma_j(a_j|t_j,\hat{t}_j)\hat{\pi}(\hat{t}_{-i}|\mathcal{P}_i^h(t,\hat{t})) \right)$$

If a player $i \in N$ deviates from the recommendation of the public device $\tilde{q}(a|\mathcal{H}(t,\hat{t}))$, then his expected utility will be

$$\sum_{t_{-i}} \left[\rho(t_{-i}|t_i) \sum_{a_i,a_{-i}} \left(\sigma_i(a_i|t_i,\hat{t}_i)\tilde{q}(a_{-i}|\mathcal{P}_i^h(t,\hat{t}))u_i(a_i,a_{-i},t) \right) \right] \geq$$
$$\sum_{t_{-i}} \left[\rho(t_{-i}|t_i) \sum_{a_i,a_{-i}} \left(\tilde{q}(a_i,a_{-i}|\mathcal{H}(t,\hat{t}))u_i(a_i,a_{-i},t) \right) \right]$$

The equation means that player i would deviate choosing a $\sigma_i(a_i|t_i,\hat{t}_i)$ that maximizes his expect utility. But, he solves the problem with respect to the same information of the communication device, $\tilde{q}(a_{-i}|\mathcal{P}_i^h(t,\hat{t}))$. Therefore, he would choose the same action recommended by the device.

Truthfully reporting. Suppose a player $i \in N$ "submits" wrong expectations. We model the wrong expectations as if i claims a different private information, $\tilde{\mathcal{P}}_i(t,\hat{t})$, and so he submits $\tilde{X}_i(t,\hat{t}) \equiv E\left(\chi|\tilde{\mathcal{P}}_i(t,\hat{t})\right)$. Define the public signal under true telling as $h^* = h\left(E\left(\chi|\mathcal{P}_i(t,\hat{t})\right), E\left(\chi|\mathcal{P}_{-i}(t,\hat{t})\right)\right)$; and under false expectations from i as $\tilde{h} = h\left(E\left(\chi|\tilde{\mathcal{P}}_i(t,\hat{t})\right), E\left(\chi|\mathcal{P}_{-i}(t,\hat{t})\right)\right)$.

Player i has no information about other players' elements of the partitions, so the expected value of the signal given i's private information equals in the two cases

$$E\left(h^*|\mathcal{P}_i(t,\hat{t})\right) = E(\tilde{h}|\tilde{\mathcal{P}}_i(t,\hat{t}))$$

From the definition of an element of the public information partition, (1.2), also the public signal partition and the updating process remain unchanged. Thus, the expected distribution over pure strategies equals in the two cases

$$\tilde{q}(a_{-i}|\mathcal{P}_i^h(t,\hat{t})) = \tilde{q}(a_{-i}|\tilde{\mathcal{P}}_i^h(t,\hat{t}))$$

Therefore, given that the other players are truthfully reporting their types and meta-types, player i has no incentive in lying because (in expectations) he will not change the information he will infer from the signal.

A.2 Derivation of Table 1.2

Prior π are uniformly distributed. Firm 1 initial partition, $\mathcal{P}_1^{h,k=0}$, and Firm 2 initial partition, $\mathcal{P}_2^{h,k=0}$, are

$$\mathcal{P}_1^{h,k=0} = \left\{ \begin{array}{l} \{(\underline{c}_1,c,\underline{c}_2,c),(\underline{c}_1,c,\underline{c}_2,m)\},\{(\underline{c}_1,m,\underline{c}_2,c),(\underline{c}_1,m,\underline{c}_2,m)\}, \\ \{(\bar{c}_1,c,\underline{c}_2,c),(\bar{c}_1,c,\underline{c}_2,m)\},\{(\bar{c}_1,m,\underline{c}_2,c),(\bar{c}_1,m,\underline{c}_2,m)\} \end{array} \right\}$$

$$\mathcal{P}_2^{h,k=0} = \left\{ \begin{array}{l} \{(\underline{c}_1,c,\underline{c}_2,c),(\underline{c}_1,m,\underline{c}_2,c),(\bar{c}_1,c,\underline{c}_2,c),(\bar{c}_1,m,\underline{c}_2,c)\}, \\ \{(\underline{c}_1,c,\underline{c}_2,m),(\underline{c}_1,m,\underline{c}_2,m),(\bar{c}_1,c,\underline{c}_2,m),(\bar{c}_1,m,\underline{c}_2,m)\} \end{array} \right\}$$

and, the random variable χ is

$$\chi = Q : \begin{cases} 20 & \text{if } (\underline{c}_1,c,\underline{c}_2,m),(\underline{c}_1,m,\underline{c}_2,c) \\ 16 & \text{if } (\underline{c}_1,c,\underline{c}_2,c) \\ 15 & \text{if } (\bar{c}_1,c,\underline{c}_2,c),(\bar{c}_1,m,\underline{c}_2,c) \\ 12 & \text{if } (\underline{c}_1,m,\underline{c}_2,m),(\bar{c}_1,c,\underline{c}_2,m),(\bar{c}_1,m,\underline{c}_2,m) \end{cases}$$

Given the private information, the expectations of each player for each possible profile of types and meta-types, $X_i^{k=0}(t,\hat{t})$, and the aggregation which mimics the price function, $h = 50 - X_1^{k=0}(t,\hat{t}) + X_2^{k=0}(t,\hat{t})$, are

$t,\hat{t}$	$X_1^{k=0}(t,\hat{t})$	$X_2^{k=0}(t,\hat{t})$	h
$\underline{c}_1, c, \underline{c}_2, c$	18	16,5	15,5
$\underline{c}_1, c, \underline{c}_2, m$	18	14	18
$\underline{c}_1, m, \underline{c}_2, c$	16	16,5	17,5
$\underline{c}_1, m, \underline{c}_2, m$	16	14	20
$\bar{c}_1, c, \underline{c}_2, c$	13,5	16,5	20
$\bar{c}_1, c, \underline{c}_2, m$	13,5	14	22,5
$\bar{c}_1, m, \underline{c}_2, c$	13,5	16,5	20
$\bar{c}_1, m, \underline{c}_2, m$	13,5	14	22,5

From which the public information partition is

$$\mathcal{H}_1^{k=0} = \left\{ \begin{array}{l} \{\underline{c}_1, c, \underline{c}_2, c\}, \{\underline{c}_1, c, \underline{c}_2, m\} \{\underline{c}_1, m, \underline{c}_2, c\}, \\ \{(\underline{c}_1, m, \underline{c}_2, m), (\bar{c}_1, c, \underline{c}_2, c), (\bar{c}_1, m, \underline{c}_2, c)\}, \{(\bar{c}_1, c, \underline{c}_2, m), (\bar{c}_1, m, \underline{c}_2, m)\} \end{array} \right\}$$

and the updated private information partitions are

$$\mathcal{P}_1^{h,k=1} = \left\{ \begin{array}{l} \{(\underline{c}_1, c, \underline{c}_2, c)\}, \{(\underline{c}_1, c, \underline{c}_2, m)\}, \{(\underline{c}_1, m, \underline{c}_2, c)\}, \{(\underline{c}_1, m, \underline{c}_2, m)\}, \\ \{(\bar{c}_1, c, \underline{c}_2, c)\}, \{(\bar{c}_1, c, \underline{c}_2, m)\}, \{(\bar{c}_1, m, \underline{c}_2, c)\}, \{(\bar{c}_1, m, \underline{c}_2, m)\} \end{array} \right\}$$

$$\mathcal{P}_2^{h,k=1} = \left\{ \begin{array}{l} \{(\underline{c}_1, c, \underline{c}_2, c)\}, \{(\underline{c}_1, c, \underline{c}_2, m)\}, \{(\underline{c}_1, m, \underline{c}_2, c)\}, \{(\underline{c}_1, m, \underline{c}_2, m)\}, \\ \{(\bar{c}_1, c, \underline{c}_2, c), (\bar{c}_1, m, \underline{c}_2, c)\}, \{(\bar{c}_1, c, \underline{c}_2, m), (\bar{c}_1, m, \underline{c}_2, m)\} \end{array} \right\}$$

Therefore, Firm 1 can distinguish all the possible states after updating his initial information, while Firm 2 still cannot discriminate between Firm 1 actions in case it has high production costs.

Suppose firms play the game a second time under the updated private information. The following table obtains

$t,\hat{t}$	$X_1^{k=1}(t,\hat{t})$	$X_2^{k=1}(t,\hat{t})$	h	p
$\underline{c}_1, c, \underline{c}_2, c$	16	16	18	18
$\underline{c}_1, c, \underline{c}_2, m$	20	20	10	10
$\underline{c}_1, m, \underline{c}_2, c$	20	20	10	10
$\underline{c}_1, m, \underline{c}_2, m$	12	12	26	26
$\bar{c}_1, c, \underline{c}_2, c$	15	15	20	20
$\bar{c}_1, c, \underline{c}_2, m$	12	12	26	26
$\bar{c}_1, m, \underline{c}_2, c$	15	15	20	20
$\bar{c}_1, m, \underline{c}_2, m$	12	12	26	26

From which the public information partition is

$$\mathcal{H}_1^{k=1} = \left\{ \begin{array}{l} \{\underline{c}_1, c, \underline{c}_2, c\}, \{(\underline{c}_1, c, \underline{c}_2, m), (\underline{c}_1, m, \underline{c}_2, c)\}, \\ \{(\underline{c}_1, m, \underline{c}_2, m), (\overline{c}_1, c, \underline{c}_2, m), (\overline{c}_1, m, \underline{c}_2, c)\}, \{(\overline{c}_1, c, \underline{c}_2, m), (\overline{c}_1, m, \underline{c}_2, c)\} \end{array} \right\}$$

and the updated private information partitions are as in the previous period, i.e. $\mathcal{P}_1^{k=1} = \mathcal{P}_1^{k=2}$ and $\mathcal{P}_2^{k=1} = \mathcal{P}_2^{k=2}$. The learning process therefore stops, and the public signal behaves as the price of the economy.

Appendix B

Additional results

B.1 Existence for a given h

From Proposition 1 a well-designed information structure can replicate any communication equilibrium. However, we do not know whether an equilibrium exists if we fix an information structure a priori. Therefore, we now determine the properties of the aggregation function h such that an equilibrium always exists.

Define $\mathcal{M}_{\mathcal{P}^h}(t,\hat{t})$ as the cell of the meet[1] of the partitions $\mathcal{P}_i^h, ..., \mathcal{P}_n^h$ containing $(t,\hat{t})$. A function h satisfies additively separability into monotonic components, if $h(x) = \sum_{i=1}^n h_i(x_i)$, where x_i is the i-th coordinate of x, and $h_i : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Nielsen et al. (1990) uses monotonicity and additive separability to extend Aumann (1976) and McKelvey and Page (1986) as follows

Theorem [Nielsen et al. (1990)]. *Suppose that $\mathcal{H}(t,\hat{t})$ is common knowledge at some $(t,\hat{t})$, and that h satisfies additive separability into monotonic components, then the conditional expectations at $(t,\hat{t})$ are equal, i.e. $E(\chi|\mathcal{P}_1^h(t,\hat{t})) = ... = E(\chi|\mathcal{P}_n^h(t,\hat{t})) = E(\chi|\mathcal{M}_{\mathcal{P}^h}(t,\hat{t}))$.*

The result identifies the conditions under which players' beliefs are equal, from which we can derive the conditions under which an equilibrium of $\mathcal{G}_\mathcal{I}$ always exists.

Proposition 3. *Given any Bayesian game $\mathcal{G}$ and an information structure $\mathcal{I}$, if the information aggregation function h is linear, then an equilibrium of the game $\mathcal{G}_\mathcal{I}$ exists, and it is payoff-equivalent to a communication equilibrium with support restricted to the meet of players' partitions, $\mathcal{M}_{\mathcal{P}^h}(t,\hat{t})$.*

Proof. If h is additive separable into strict monotonic components, then Nielsen et al. (1990) holds, and

[1] the finest common coarsening with respect to the "is refined by" relation

$$E(\chi|\mathcal{P}_1^h(t,\hat{t})) = ... = E(\chi|\mathcal{P}_N^h(t,\hat{t})) = E(\chi|\mathcal{M}_{\mathcal{P}^h}(t,\hat{t}))$$

From (1.4), conditional expectations are equal if posteriors are equal, i.e.

$$\pi(.|\mathcal{P}_1^h(t,\hat{t})) = ... = \pi(.|\mathcal{P}_N^h(t,\hat{t})) = \pi(.|\mathcal{M}_{\mathcal{P}^h}(t,\hat{t})) \tag{B.1}$$

From Proposition 1, the posteriors generate a correlated distribution satisfying obedience and truthfully reporting. Therefore, player $i \in N$ would infer the following probability about the action profile a

$$\tilde{q}(a|\mathcal{P}_i^h(t,\hat{t})) \equiv \sum_{\hat{t}} \left(\prod_{j\in N} \sigma_j(a_j|t_j,\hat{t}_j)\hat{\pi}(\hat{t}_j|\mathcal{P}_i^h(t,\hat{t})) \right)$$

where $\hat{\pi}$ is the marginal probability distribution of the meta-types $\hat{T}$.

From Equation (B.1) all the players have the same posteriors, then for every $i \in N$ the correlated equilibrium is

$$\tilde{q}(a|\mathcal{M}_{\mathcal{P}^h}(t,\hat{t})) \equiv \sum_{\hat{t}} \left(\prod_{j\in N} \sigma_j(a_j|t_j,\hat{t}_j)\hat{\pi}(\hat{t}_j|\mathcal{M}_{\mathcal{P}^h}(t,\hat{t})) \right)$$

where due to obedience, $supp(\hat{\pi}) = supp\,(\tilde{q}) = \mathcal{M}_{\mathcal{P}^h}(t,\hat{t}) \subseteq T \times \hat{T}$. □

The linear aggregation function is interesting because widely used in many markets. Financial and commodities markets have several benchmarks, see Duffie et al. (2017), which are often a simple weighted average (e.g. LIBOR and WM/R 4 pm London fix), and which can be modeled as public signals aggregating agents' private information.

B.2 Geometry

Figure B.1 illustrates the geometric intuition behind Proposition 3 in a tridimensional simplex - see also Nau et al. (2004). The figure represents Example 5, where vertexes (UL) and (DR) represent the two Nash equilibria in pure strategies; the point in the middle of the simplex represents the equilibrium in mixed strategies; the polytope represents the set of correlated equilibria of the game.

Moreover, when $\hat{T} = A$, we can represent players' beliefs π inside the simplex. The observation of the public signal can drive the posteriors on a specific area. In our example, the realization of the public signal defines a common knowledge face

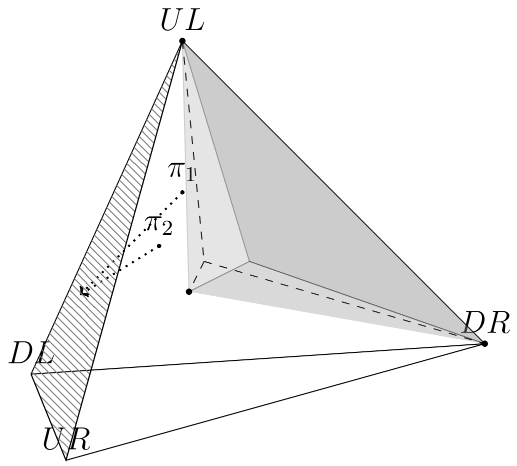

Figure B.1: Geometric representation of common knowledge as a selection mechanism.

of the simplex - meaning that players assign probability 0 to (DR) - and selects part of the polytope and the equilibrium (UL).

Exemple 5. *Battle of the sexes.*

	L	R
U	2; 1	0; 0
D	0; 0	1; 2

The type set T is trivial since each player has only one type. We assume that the set of meta-types is equal to the set of pure strategies, $\hat{T}_1 = \{U, D\}$ and $\hat{T}_2 = \{L, R\}$. We also assume the following random variable χ and common prior

χ	L	R
U	1	1
D	1	0

Pr	L	R
U	1/4	1/4
D	1/4	1/4

If players play pure strategies, the following partitions identify players' private information

$$\mathcal{P}_1^{k=0} = \{(UL, UR), (DL, DR)\} \quad \mathcal{P}_2^{k=0} = \{(UL, DL), (UR, DR)\}$$

Given the private information, the following table reports the expectations $X_i(\hat{t})$ of each player and their aggregation $h = \frac{X_1(\hat{t})}{2} + \frac{X_2(\hat{t})}{2}$

$\hat{t}$	$X_1^{k=0}(\hat{t})$	$X_2^{k=0}(\hat{t})$	$h^{k=0}$
UL	*1*	*1*	*1*
UR	*1*	*1/2*	*3/4*
DL	*1/2*	*1*	*3/4*
DR	*1/2*	*1/2*	*1/2*

From which, the public information partition is

$$\mathcal{H}^{k=0} = \{(UL), (UR, DL), (DR)\}$$

and the updated private information partitions are

$$\mathcal{P}_1^{k=1} = \mathcal{P}_2^{k=1} = \{(UL), (UR), (DL), (DR)\}$$

Agents play a second time under the new partitions, then the previous table updates in

$\hat{t}$	$X_1^{k=1}(\hat{t})$	$X_2^{k=1}(\hat{t})$	$h^{k=1}$
UL	*1*	*1*	*1*
UR	*1*	*1*	*1*
DL	*1*	*1*	*1*
DR	*0*	*0*	*0*

From which, the public information partition is

$$\mathcal{H}^{k=1} = \{(UL, UR, DL), (DR)\}$$

and players correctly predict the realization of the public signal.

Figure B.1 illustrates players' beliefs when they observe $h^{k=1} = 1$*: the set* $\mathcal{H}^{k=1}(t, \hat{t}) = \{(UL, UR, DL)\}$ *becomes common knowledge and players coordinate on* UL*.*

B.3 Remarks

Properties of h

McKelvey and Page (1986) requires h to be stochastically monotone[2]. However, Bergin and Brandenburger (1990) shows that h satisfies stochastic monotonicity

[2]Let μ and ν be probability measures on the Borel subsets of $\mathbb{R}^n$; and let μ_i be the ith marginal of μ, i.e. for any Borel subset B of $\mathbb{R}$, $\mu_i(B) = \mu(\{(x_1, ..., x_n) \in \mathbb{R}^n : x_i \in B\})$. If μ_i stochastically dominates ν_i for every $i = 1, ..., n$, i.e. $\mu_i((-\infty, x]) \leq \nu_i((-\infty, x])\ \forall x \in \mathbb{R}$, then μ stochastically dominates ν. Given μ and ν, if $\int h d\mu > \int h d\nu$, whenever μ stochastically dominates ν, then the function h is stochastically monotone.

if the map is additively separable into monotonic components, and Nielsen et al. (1990) simplifies the setup. Behavioral considerations justify monotonicity, while additive separability can be dropped when partitions are independently drawn, see Bergin (2001).

Noisy information and global games

Nielsen (1995) extends Nielsen et al. (1990) adding a multivariate random vector and errors, finding that the results do not change when the covariance between the error and the statistic is non-negative. Therefore, under those assumptions, our results are still valid even if we introduce noise in the model.

Noise brings our setup closer to the global games literature, e.g. Carlsson and van Damme (1993), Morris and Shin (1998) and Morris and Shin (2000). However, in global games, the public signal only modifies the shape of the beliefs distribution; while in our model the public signal also changes the support of such distribution, as Proposition 3 and the geometric intuition show.

Appendix C

Additional examples

C.1 Convex hull and indirect renegotiation-proof equilibria

Suppose the following game

	L	R
U	5; 1	0; 0
D	4; 4	1; 5

The game has two Nash Equilibria in pure strategies, (UL) and (DR), and one in mixed strategies where agents play each action with probability 1/2.

We now introduce an information structure $\mathcal{I}$. The type set T is trivial since each player has only one type; the set of meta-types equals the set of pure strategies $\hat{T}_1 = \{U, D\}$ and $\hat{T}_2 = \{L, R\}$; the random variable and the uniform common prior are as follows

χ	L	R
U	1	0
D	1	1

Pr	L	R
U	1/4	1/4
D	1/4	1/4

If players play pure strategies, the following partitions give agents' private information

$$\mathcal{P}_1^{h,k=0} = \{(UL, UR), (DL, DR)\} \quad \mathcal{P}_2^{h,k=0} = \{(UL, DL), (UR, DR)\}$$

Given the private information, the following table defines the expectations $X_i(\hat{t})$ of each player and their aggregation $h = \frac{X_1(\hat{t})}{2} + \frac{X_2(\hat{t})}{2}$

$\hat{t}$	$X_1^{k=0}(\hat{t})$	$X_2^{k=0}(\hat{t})$	$h^{k=0}$
UL	$1/2$	1	$3/4$
UR	$1/2$	$1/2$	$1/2$
DL	1	1	1
DR	1	$1/2$	$3/4$

From which, the public information partition is

$$\mathcal{H}^{k=0} = \{(UL, DR), (UR), (DL)\}$$

and private information partitions update in

$$\mathcal{P}_1^{h,k=1} = \mathcal{P}_2^{h,k=1} = \{(UL), (UR), (DL), (DR)\}$$

If agents play a second time under the new partitions, the previous table updates in

$\hat{t}$	$X_1^{k=1}(\hat{t})$	$X_2^{k=1}(\hat{t})$	$h^{k=1}$
UL	1	1	1
UR	1	1	1
DL	1	1	1
DR	0	0	0

From which the public information partition is

$$\mathcal{H}^{k=1} = \{(UL, DL, DR), (UR)\}$$

and the updated private information partitions are

$$\mathcal{P}_1^{h,k=2} = \mathcal{P}_2^{h,k=2} = \{(UL), (UR), (DL), (DR)\}$$

Therefore, the learning process already stops at $k = 1$.

At $k = 1$, we have $\mathcal{H}^{k=1}(t, \hat{t}) = \{(UL, DL, DR)\}$ meaning that players can correlate their behavior by randomizing over the profitable pair of actions, excluding the bad outcome $(0, 0)$. Therefore, players are in a correlated equilibrium outside the convex hull and with expected payoff $(3\frac{1}{3}, 3\frac{1}{3})$. This is also the unique indirect renegotiation-proof equilibrium of the game.

C.2 Prisoner dilemma and indirect renegotiation-proof equilibria

	L	R
U	4; 4	0; 5
D	5; 0	1; 1

We now introduce an information structure $\mathcal{I}$. The type set T is trivial since each player has only one type; the set of meta-types equals the set of pure strategies $\hat{T}_1 = \{U, D\}$ and $\hat{T}_2 = \{L, R\}$; the random variable and the uniform common prior are as follows

χ	L	R
U	1	1
D	1	0

Pr	L	R
U	1/4	1/4
D	1/4	1/4

If players play pure strategies, the following partitions identify the private information of each player

$$\mathcal{P}_1^{h,k=0} = \{(UL, UR), (DL, DR)\} \quad \mathcal{P}_2^{h,k=0} = \{(UL, DL), (UR, DR)\}$$

Given the private information, the expectations of each player $X_i(\hat{t})$ and the result of an aggregation $h = \frac{X_1(\hat{t})}{2} + \frac{X_2(\hat{t})}{2}$ are

$\hat{t}$	$X_1^{k=0}(\hat{t})$	$X_2^{k=0}(\hat{t})$	$h^{k=0}$
UL	1	1	1
UR	1	1/2	3/4
DL	1/2	1	3/4
DR	1/2	1/2	1/2

From which, the public information partition is

$$\mathcal{H}^{k=0} = \{(UL), (UR, DL), (DR)\}$$

and the updated private information partitions are

$$\mathcal{P}_1^{h,k=1} = \mathcal{P}_2^{h,k=1} = \{(UL), (UR), (DL), (DR)\}$$

If agents play a second time given the updated information, the previous table updates in

$\hat{t}$	$X_1^{k=1}(\hat{t})$	$X_2^{k=1}(\hat{t})$	$h^{k=1}$
UL	1	1	1
UR	1	1	1
DL	1	1	1
DR	0	0	0

From which the public information partition is

$$\mathcal{H}^{k=1} = \{(UL, UR, DL), (DR)\}$$

Therefore, from $k = 2$ players cannot infer further information and the public signal realization will not change. (The information revealed is as in Example 5, since the information structure is the same.)

The realizations of the public signal in $k = 0$ or $k = 1$ prevent cooperation in the Prisoner Dilemma. The only possible correlation and equilibrium is (DR) as in all the other cases individual rationality does not apply. This result is in line with the literature about renegotiation-proof equilibria in the Prisoner Dilemma game, van Damme (1989) and Farrell and Maskin (1989b), where cooperation depends on the discounting factor, which is here absent.

C.3 Stag hunt

	L	R
U	2; 2	0; 1
D	1; 0	1; 1

The type set T is trivial since each player has only one type. Suppose that the set of meta-types is equal to the set of pure strategies: $\hat{T}_1 = \{U, D\}$ and $\hat{T}_2 = \{L, R\}$. We also assume the following random variable χ and uniform common prior

χ	L	R
U	2	0
D	0	1

Pr	L	R
U	1/4	1/4
D	1/4	1/4

If players play pure strategies, the following partitions represent the private information of each player at $k = 0$

$$\mathcal{P}_1^{h,k=0} = \{(UL, UR), (DL, DR)\} \quad \mathcal{P}_2^{h,k=0} = \{(UL, DL), (UR, DR)\}$$

The type set T is constant, therefore everything depends on the profile of meta-types $\hat{t}$. The following table reports the expectations $X_i^{k=0}(\hat{t})$ of each player, and the realization of a public signal which takes the average of expectations, i.e $h = \frac{X_1^{k=0}(\hat{t})}{2} + \frac{X_2^{k=0}(\hat{t})}{2}$

$\hat{t}$	$X_1^{k=0}(\hat{t})$	$X_2^{k=0}(\hat{t})$	h
UL	1	1	1
UR	1	1/2	3/4
DL	1/2	1	3/4
DR	1/2	1/2	1/2

From which, the public information partition is

$$\mathcal{H}^{k=0} = \{(UL), (UR, DL), (DR)\}$$

and the updated private information at $k = 1$ is

$$\mathcal{P}_1^{h,k=1} = \mathcal{P}_2^{h,k=1} = \{(UL), (UR), (DL), (DR)\}$$

At $k = 1$, expectations and public signal realizations are

$\hat{t}$	$X_1^{k=1}(\hat{t})$	$X_2^{k=1}(\hat{t})$	h
UL	2	2	2
UR	0	0	0
DL	0	0	0
DR	1	1	1

from which the information partition is

$$\mathcal{H}^{k=1} = \{(UL), (UR, DL), (DR)\}$$

Therefore, information does not change after $k = 1$ and $\mathcal{H}^{k=0} = \mathcal{H}^{k=1}$. At $k = 1$ players can realize that the expected payoff given $\mathcal{H}(t, \hat{t}) = \{(UL)\}$ is higher than the expected payoff under $\mathcal{H}(t, \hat{t}) = \{(DR)\}$. Then,

$$\mathcal{P}_1^{h,k=1} = \mathcal{P}_2^{h,k=1} = \mathcal{H}^{k=0} = \{(UL)\}$$

The constraints in Definition 4 are satisfied and (DR) is an indirect renegotiation-proof equilibrium.

C.4 Incomplete information battle of the sexes

π	$\overline{L}$	$\overline{R}$
U	2,1	0,0
D	0,0	1,2

$1 - \pi$	$\underline{L}$	$\underline{R}$
U	2,0	0,2
D	0,1	1,0

The Bayes-Nash strategies of the game are: if $\pi > \frac{1}{3}$, then $(U, (\overline{L}, \underline{R}))$ is an equilibrium; if $\pi > \frac{2}{3}$, then $(D, (\overline{R}, \underline{L}))$ is an equilibrium; if $\pi < \frac{1}{3}$, then players randomize.

We now introduce an information structure for three cases: (i) the public signal is uninformative; (ii) the public signal reveals the types; (iii) the public signal reveals the meta-types (and so the pure strategies).

Case I

We assume the following random variable χ and uniform common prior

χ	$\overline{L}$	$\overline{R}$
U	2	0
D	0	2

	$\underline{L}$	$\underline{R}$
U	1	1
D	1	1

Pr	$\overline{L}$	$\overline{R}$
U	1/8	1/8
D	1/8	1/8

	$\underline{L}$	$\underline{R}$
U	1/8	1/8
D	1/8	1/8

If players play pure strategies, the following partitions give agents' private information

$$\mathcal{P}_1 = \{(U\overline{L}, U\overline{R}, U\underline{L}, U\underline{R}), (D\overline{L}, D\overline{R}, D\underline{L}, D\underline{R})\}$$
$$\mathcal{P}_2 = \{(U\overline{L}, D\overline{L}), (U\underline{L}, D\underline{L}), (U\overline{R}, D\overline{R}), (U\underline{R}, D\underline{R})\}$$

where Player 1 cannot distinguish the type of Player 2.

Given the private information, the expectations of each player $X_i(t, \hat{t})$ and their average $h = \frac{X_1(t,\hat{t})}{2} + \frac{X_2(t,\hat{t})}{2}$ are

$t, \hat{t}$	$X_1(t, \hat{t})$	$X_2(t, \hat{t})$	h
$U, \overline{L}$	1	1	1
$U, \overline{R}$	1	1	1
$D, \overline{L}$	1	1	1
$D, \overline{R}$	1	1	1
$U, \underline{L}$	1	1	1
$U, \underline{R}$	1	1	1
$D, \underline{L}$	1	1	1
$D, \underline{R}$	1	1	1

From which the public information partition is

$$\mathcal{H} = \{\big((U,\overline{L}),(U,\overline{R}),(D,\overline{L}),(D,\overline{R}),(U,\underline{L}),(U,\underline{R}),(D,\underline{L}),(D,\underline{R})\big)\}$$

Therefore, there is no information exchange, and we go back to the Bayes-Nash equilibrium analysis.

Case II

Suppose the following random variable

χ	$\overline{L}$	$\overline{R}$
U	1	0
D	0	1

	$\underline{L}$	$\underline{R}$
U	1	1
D	1	1

We assume common prior and private information are as in the previous case. Given the private information, the expectations of each player $X_i(t,\hat{t})$ and their average $h = \frac{X_1(t,\hat{t})}{2} + \frac{X_2(t,\hat{t})}{2}$ are

$t,\hat{t}$	$X_1(t,\hat{t})$	$X_2(t,\hat{t})$	h
$U,\overline{L}$	3/4	1/2	5/8
$U,\overline{R}$	3/4	1/2	5/8
$D,\overline{L}$	3/4	1/2	5/8
$D,\overline{R}$	3/4	1/2	5/8
$U,\underline{L}$	3/4	1	7/8
$U,\underline{R}$	3/4	1	7/8
$D,\underline{L}$	3/4	1	7/8
$D,\underline{R}$	3/4	1	7/8

From which the public information partition is

$$\mathcal{H} = \{\big((U,\overline{L}),(U,\overline{R}),(D,\overline{L}),(D,\overline{R})\big),((U,\underline{L}),(U,\underline{R}),(D,\underline{L}),(D,\underline{R}))\}$$

Therefore, Player 2 can communicate his type, each game can be considered separately, and the standard Nash equilibrium notion in complete information applies.

Case III

Suppose the following random variable

χ	$\overline{L}$	$\overline{R}$
U	0	0
D	0	1

	$\underline{L}$	$\underline{R}$
U	1	1
D	1	1

Given the private information, the expectations of each player $X_i(t, \hat{t})$ and their average $h = \frac{X_1(t,\hat{t})}{2} + \frac{X_2(t,\hat{t})}{2}$ are

$t, \hat{t}$	$X_1(t, \hat{t})$	$X_2(t, \hat{t})$	h
$U, \overline{L}$	$1/2$	0	$1/4$
$U, \overline{R}$	$1/2$	$1/2$	$1/2$
$D, \overline{L}$	$3/4$	0	$3/8$
$D, \overline{R}$	$3/4$	$1/2$	$5/8$
$U, \underline{L}$	$1/2$	1	$3/2$
$U, \underline{R}$	$1/2$	1	$3/2$
$D, \underline{L}$	$3/4$	1	$7/4$
$D, \underline{R}$	$3/4$	1	$7/4$

From which the public information partition is

$$\mathcal{H} = \{(U, \overline{L}), (U, \overline{R}), (D, \overline{L}), (D, \overline{R}), ((U, \underline{L}), (U, \underline{R}), (D, \underline{L}), (D, \underline{R}))\}$$

Therefore, players can exchange information about their strategies. The public signal forms a correlated equilibrium where players coordinate on $(D, \overline{R})$, if Player 2 wants to meet, and randomize otherwise.

Part II

Pension schemes and economic growth

Chapter 2

Pensions as research and production subsidies under consumption shocks

Abstract In this chapter we study the relation between pensions and financial intermediation in an overlapping generations model with Schumpeterian growth and consumption shocks. We show that a financial intermediary may be unable to adequately isolate consumers from the shocks making a funded pension scheme equivalent to a research and production subsidy for the economy, and leading to a Pareto-superior economic growth.

Keywords: Schumpeterian Growth - Overlapping Generations - Consumption Shocks - Social Security - Financial Intermediary.

JEL Classification: D90, E21, H55, O40[1]

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2.1 Introduction

Financial intermediaries and the pension schemes are critical elements of an economy. A vast theoretical and empirical literature has separately analyzed the effects of financial intermediation and pension schemes on economic growth - see Levine (2004) and Feldstein and Liebman (2002) for a survey. However, we know very little about the effects of pension schemes in an economy with financial intermediaries.

In this chapter we propose a general equilibrium model where agents face consumption shocks, a financial intermediary provides insurance against the shocks, and firms compete in a Schumpeterian framework. We show that a funded pension scheme acts as a research and production subsidy - increasing the economic growth rate - due to the presence of the shocks.

The result is not obvious since we would expect the financial intermediary to neutralize the uncertainty due to the shocks. We do find that financial intermediation makes the optimal savings when a consumer retires independent from the shocks. However, financial intermediation also creates a wedge between savings and capital rate of return to insure consumers against the shocks. In this setting, a funded pension scheme offers a rate of return higher than the savings rate - and equal to the capital rate - increasing savings and consumption. Moreover, since firms can borrow more resources for investments in production and R&D, economic growth increases.

We formalize the intuition in a two-period overlapping generations model with a Schumpeterian production sector. To introduce the shocks, we divide each of the two periods in a continuum of points, and we assume that a shock makes consumption singular in one of these points when the consumer is young (while the consumer smooths his consumption across all the points when old).

Our financial structure builds on Bencivenga and Smith (1991), which studies financial intermediation and economic growth. We model the shocks as in von Thadden (1998), which extends Diamond and Dybvig (1983) in continuous time. Our model presents a mix of discrete and continuous time useful to model the effects of financial intermediation and pension schemes in a two-stage optimal control problem, Tomiyama (1985). To the best of our knowledge, this chapter is one of the few works applying this optimization technique in economics - for another example see Boucekkine et al. (2004).

The modeling of the financial structure and the consumer problem is also our main contribution. The Schumpeterian production function is based on Aghion and Howitt (2007), and we modify it only to link the production sector to the overlapping generation model. The Schumpeterian framework plays two roles in our analysis. First, it links a change in savings (due to the pension schemes) to a change in economic growth. Second, it justifies the introduction of the pension scheme since in a Schumpeterian economy firms under-produce and under-invest in R&D.

There exists a vast literature on the economic effects of pensions schemes. For example, Samuelson (1975) shows that pay-as-you-go pensions improve the allocation of resources in the presence of capital over-accumulation; Kubler and Krueger (2006) proves that social security enhances the economy when financial markets are incomplete; Gollier (2008) shows that a funded pension improves the economy when the pension fund provides intergenerational risk-sharing.

Similarly, there exists a vast literature on the economic effects of financial intermediation. For example, King and Levine (1993) finds evidence that the growth rate of an economy is positively correlated with financial development; Levine et al. (2000) studies the causality effects between financial intermediation and economic growth; Fisman and Love (2008) analyzes how financial intermediaries improve economic growth by providing trade credit.

However, we are not aware of connections between financial intermediation and pension schemes, as studied in this chapter, other than financial market incompleteness, which is not our case.

The structure of the chapter is as follows. In the next section, we present the model. Section 2.3 presents the effects of a pension scheme on the growth rate of the economy. We verify our results are robust to the introduction of a mortality rate in Section 2.4. Section 2.5 concludes presenting some empirical evidence fitting our theoretical findings.

2.2 The model

Our economy is composed of L_τ individuals of generation τ, with $\tau \in \mathbb{N}$. Population grows at a constant rate $n \geq 0$, $L_\tau = e^{n\tau} L_0$ with $L_0 = 1$ for simplicity. Each consumer lives for two periods. In the first period, he is endowed with one unit of labor, which he supplies inelastically in a competitive labor market. In the second period, he retires and consumes his savings.

Each consumer earns a wage ω_τ and receives a dividend π_τ from the firms; he also faces a consumption shock when young.

The financial intermediary offers insurance against the shocks and pools the savings s_τ, which he lends to the firms. After production has taken place, the financial intermediary receives e^{r_τ} from the firms, with $r_\tau > 0$ being the capital rate of return, and then remunerates the savings of the consumers.

Production takes place at the beginning of every period τ and consists of a final good, used as numeraire, and a continuum of intermediate goods, whose number is normalized to 1. Each firm can also use the resources borrowed to invest in R&D and eventually become a monopolist in one of the intermediate sectors. A monopolistic firm distributes its dividends to its shareholders: the young consumers.

We now formalize the economy just described.

2.2.1 Shocks and financial intermediation

To introduce consumption shocks, we divide each of the "two periods" of a consumer life in a continuum of time-points t. A young consumer faces a shock which makes consumption singular at $t = t'$. An old consumer smooths his consumption across all the time-points t.

The cumulative distribution function of the consumption shocks is $\Phi : (\tau, \tau + 1] \rightarrow [0, 1]$, with $\Phi'(t) \equiv \phi(t)$ and $\Phi(1) > \Phi(t)$ for any $t \in [\tau, \tau + 1)$. To simplify the algebra, we assume that the consumption shocks follow a Poisson process with rate z:

$$\phi(t) = ze^{-z} \qquad \Phi(t) = 1 - e^{-z}$$

Consumers of generation τ can only invest in an illiquid asset[2] yielding after one period e^{r_τ}. But, each of them faces a consumption shock at $t' \in [\tau, \tau + 1)$, precluding the investment possibility. Therefore, suppose the existence of a financial intermediary.

The financial intermediary makes consumer's wealth available at the time of the shock in exchange for a premium paid at the beginning of the period. We

[2]The assumption is without any loss of generality. A consumer would never hold the liquid asset (normalizing its return to 0) if he can invest in an illiquid asset yielding a positive return, pay for insurance to cover himself against the shock, and still have a positive net return.

assume the sector is competitive so that the premiums paid equal the payments made to the consumers, formally

$$L_\tau p \int_\tau^{\tau+1} s(t)dt = L_\tau \int_\tau^{\tau+1} \frac{\phi(t)}{1-\Phi(t)} s(t)dt$$

The left-hand side of the equation represents the premiums paid with p being the premium rate and $\int_\tau^{\tau+1} s(t)dt$ the wealth (the savings) insured during the entire period. The right-hand side instead represents the payments made to the consumers with $\frac{\phi(t)}{1-\Phi(t)}$ being the probability of facing a shock for a consumer who has not faced one yet. The assumption of a Poisson distribution simplifies the zero-profit condition in

$$p = z \equiv \frac{\phi(t)}{1-\Phi(t)} \tag{2.1}$$

implying that in equilibrium the premium rate equals the instantaneous risk of being subject to a shock, as in Blanchard (1985).

Therefore, during a period of investment $[\tau, \tau+1]$, the savings net rate of return equals the rate of return of the illiquid asset minus the premium paid

$$\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon = r_\tau - z \tag{2.2}$$

where $\hat{r}_\tau(t)$ is the intra-period rate of return of capital at t. Therefore, the financial intermediary offers a savings rate of return lower than the capital rate, making consumers implicitly pay for insurance.

2.2.2 Consumers

Each young consumer still does not know when he will consume. Therefore, we now assume that the financial intermediary also pools the savings of all the consumers, taking advantage of the law of large numbers to represent the consumption-savings problem.

A young consumer of generation τ faces a shock which makes him consume $c_{1,\tau}(t')$ at a certain t', and 0 in any other $t \in [\tau, \tau+1]$, with $t \neq t'$. By pooling the savings of all the consumers, we can represent consumption at each time $t \in [\tau, \tau+1]$ as a mass of consumers $L_\tau\phi(t-\tau)$ consuming $c_{1,\tau}(t)$. Therefore, the aggregate savings dynamics is

$$\dot{S}_\tau(t) = \hat{r}_\tau(t)s_\tau(t)L_\tau - c_{1,\tau}(t)L_\tau\phi(t-\tau)$$

Since population is constant within the period, the per-capita savings dynamics is $\dot{s}_\tau(t) = \dot{S}_\tau(t)/L_\tau$. We assume the financial intermediary is in a competitive environment, so that it maximizes the utility of a consumer of generation τ solving

$$\begin{aligned}
\max_{c_{1,\tau}(t),c_{2,\tau}(t),s_\tau(t)} \quad & \int_\tau^{\tau+1} \ln\left(c_{1,\tau}(t)\right)\phi(t-\tau)dt + \int_{\tau+1}^{\tau+2} e^{-\rho(t-(\tau+1))}\ln\left(c_{2,\tau}(t)\right)dt \\
\text{subject to} \quad & \dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau) \text{ for } t\in[\tau,\tau+1] \\
& \dot{s}_\tau(t) = \hat{r}_{\tau+1}(t)s_\tau(t) - c_{2,\tau}(t) \text{ for } t\in[\tau+1,\tau+2] \\
& s_\tau(\tau) = \omega_\tau + \pi_\tau, \qquad s_\tau(\tau+2) = 0
\end{aligned} \tag{2.3}$$

where $c_{1,\tau}(t)$ and $c_{2,\tau}(t)$ are respectively the consumption in the first and in the second period of life at time t; $\int_\tau^{\tau+1} \ln\left(c_{1,\tau}(t)\right)\phi(t-\tau)dt$ is the expected utility when young; $\ln\left(c_{2,\tau}(t)\right)$ is the utility from consumption when old at t; $\rho > 0$ is the discount rate; $s_\tau(\tau) = \omega_\tau + \pi_\tau$ is the initial condition consisting of the wage earned and the dividends distributed by the firms; and $s_\tau(\tau+2) = 0$ is the no waste condition.

The several assumptions in the consumer problem are necessary to obtain closed-form solutions and so understand the economic effects of the pension schemes in the next section[3].

We solve the consumption-savings problem in three steps: (I) solve for $t \in [\tau+1, \tau+2]$ using the no waste condition $s_\tau(\tau+2) = 0$; (II) solve for $t \in [\tau, \tau+1]$ using the initial condition $s_\tau(\tau) = \omega_\tau + \pi_\tau$; (III) compute the optimal savings stock when the individual retires, $s_\tau(\tau+1)$, at the switching point between the two life periods.

Proposition 4. *The allocation of resources for each individual of generation τ is*

$$c_{1,\tau}(t) = \frac{\rho e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}}{1-e^{-\rho}+\rho}\left(\omega_\tau + \pi_\tau\right) \tag{2.4}$$

$$c_{2,\tau}(t) = \frac{\rho e^{\int_\tau^t \hat{r}_{\tau+1}(\epsilon)d\epsilon - \rho(t-(\tau+1)) + \int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}}{1-e^{-\rho}+\rho}\left(\omega_\tau + \pi_\tau\right) \tag{2.5}$$

[3]Some assumptions are without loss of generality, e.g. only young consumers face the shocks. We need the other assumptions - logarithm utility function, lack of altruism and lack of a stock market - to simplify the algebra of the problem. For example, if only the young generation receives a dividend, then dividends do not change the maximization problem since they are a function of the savings of the previous generation.

$$s_\tau(\tau+1) = \frac{1-e^{-\rho}}{1-e^{-\rho}+\rho} e^{\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon} (\omega_\tau + \pi_\tau) \tag{2.6}$$

Proof. In Appendix D.1. □

From Equation (2.6) the financial intermediary, by pooling the savings together, makes the optimal savings when a consumer retires independent from the shocks. Therefore, financial intermediation transforms resources subjects to shocks into a stock which can be lent for investments.

However, transforming uncertain savings into secure capital presents complications. From the savings equation, the financial intermediary offers $e^{\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}$ making consumers implicitly pay for insurance and offering a rate of return lower than capital.

Also, consumption levels in the two periods are similarly affected by the shocks. When the savings return rate $\hat{r}_\tau(\epsilon)$ increases, consumers are willing to substitute consumption when old with consumption when young. The logarithm utility function cancels out income and substitution effects due to risk-aversion. Therefore, the substitution effect is due to the presence of shocks: from (2.1) and (2.2), $\hat{r}_\tau(\epsilon)$ increases when the arrival rate of a shock z decreases.

Therefore, financial intermediation plays an essential role in transforming resources subject to a shock into a secure capital stock. However, the process does not adequately isolate consumers from the uncertainty. In the next section, we will show the economic effects of social security in this setting but, we first complete the description of the economy.

2.2.3 Production

Final good sector

In each period τ, firms produce a homogeneous final good in a competitive framework using some intermediate goods and labor. The production function of the final good is

$$F_\tau = L_\tau^{1-\alpha} \int_0^1 A_{i,\tau} Y_{i,\tau}^\alpha di \qquad 0 < \alpha < 1 \tag{2.7}$$

where F_τ is the production of final good; L_τ is the labor supplied; $Y_{i,\tau}$ is the amount of intermediate good i used; $A_{i,\tau}$ is the productivity of the intermediate good i; and α is the input elasticity.

The final good can be stored as capital (and for simplicity capital stock fully depreciates after its use) or used for consumption (wages are paid in final good).

Intermediate goods sector

Each intermediate sector i is monopolized by a firm which produces the intermediate good using capital. The production function of the intermediate good $Y_{i,\tau}$ is

$$Y_{i,\tau} = \frac{K_{i,\tau}}{A_{i,\tau}} \tag{2.8}$$

where $K_{i,\tau}$ is the amount of capital used in sector i; and the ratio represents a production sector which is increasingly capital-intensive as the good becomes more technologically advanced and sophisticated.

We define the per-capita level of capital as $k_{i,\tau} \equiv \frac{K_{i,\tau}}{L_\tau}$, and the per-capita production of intermediate good i at τ as $y_{i,\tau} \equiv \frac{k_{i,\tau}}{A_{i,\tau}}$.

The demand curve - and so the price $p_{i,\tau}$ - of an intermediate good i follows from (2.7)

$$p_{i,\tau} = dF_\tau/dy_{i,\tau} = L_\tau^{1-\alpha}\alpha A_{i,\tau}Y_{i,\tau}^{\alpha-1} = \alpha A_{i,\tau}y_{i,\tau}^{\alpha-1} \tag{2.9}$$

Each monopolist borrows capital at the beginning of each period τ and pays $R_\tau(\tau+1) = e^{r_\tau}$ as interest, where $R_\tau(\tau+1)$ is the gross return of capital after one period and $r_\tau > 0$ is the rate of return of capital at τ.

The marginal revenue of a monopolist is $p_{i,\tau}$, and his marginal cost is $e^{r_\tau}A_{i,\tau}$[4]. Therefore, the profits of an intermediate good monopolist are $\pi_{i,\tau} = p_{i,\tau}y_{i,\tau} - e^{r_\tau}A_{i,\tau}y_{i,\tau}$; the monopolist maximizes his profits solving

$$\max_{y_{i,\tau}} \alpha A_{i,\tau}y_{i,\tau}^{\alpha} - e^{r_\tau}A_{i,\tau}y_{i,\tau} \tag{2.10}$$

The first order conditions give $\alpha^2 y_{i,\tau}^{\alpha-1} = e^{r_\tau}$, from which the optimal level of intermediate good production follows

[4]From (2.8) $A_{i,\tau}$ measures the quantity of capital necessary to produce one unit of intermediate good.

$$y_{i,\tau} = \left(\frac{\alpha^2}{e^{r_\tau}}\right)^{\frac{1}{1-\alpha}} \tag{2.11}$$

Therefore, every firm i supplies the same quantity of intermediate good $y_{i,\tau}$ and, using the production function (2.8), the intermediate good production is the same in all sectors

$$y_{i,\tau} = \frac{k_{i,\tau}}{A_{i,\tau}} = \frac{k_\tau}{A_\tau} \equiv \hat{k}_\tau$$

where $A_\tau = \int_0^1 A_{i,\tau} di$ is the average productivity across sectors; $k_\tau = \int_0^1 k_{i,\tau} di$ is the aggregate demand of per-capita capital; and $\hat{k}_\tau$ represents the productivity adjusted capital-labor ratio.

Factor prices

Firms' optimality conditions require marginal costs to equal marginal revenues. Therefore, capital returns and wages are

$$R_\tau(\tau+1) = e^{r_\tau} = \alpha^2 y_{i,\tau}^{\alpha-1} = \alpha^2 \hat{k}_\tau^{\alpha-1} \tag{2.12}$$

$$\omega_\tau = dF_\tau/dL_\tau = (1-\alpha)L_\tau^{-\alpha} \int_0^1 A_{i,\tau} Y_{i,\tau}^\alpha di = (1-\alpha)A_\tau \hat{k}_\tau^\alpha \tag{2.13}$$

Using (2.12), monopolist's profits become

$$\pi_{i,\tau} = A_{i,\tau}\alpha(1-\alpha)\hat{k}_\tau^\alpha \tag{2.14}$$

The monopolists equally redistribute the profits only among young consumers, who receive the average profit of the intermediate good sector together with their wage

$$\pi_\tau = A_\tau \alpha(1-\alpha)\hat{k}_\tau^\alpha = A_\tau^{1-\alpha}\alpha(1-\alpha)k_\tau^\alpha$$

2.2.4 R&D

Innovation in sector i increases its productivity in the next period from $A_{i,\tau}$ to $\gamma A_{i,\tau+1}$, with $\gamma > 1$. We define the probability μ of having an innovation in sector i as

$$\mu(M_{i,\tau}, \eta) = \eta M_{i,\tau}/A_{i,\tau} \tag{2.15}$$

where $M_{i,\tau}$ are the resources invested in R&D; η is the research productivity; and the ratio represents a research sector which requires more investments as progress advances.

The value of an innovation $V_{i,\tau}$ is the net present value of an asset yielding the profits from becoming a monopolists minus the expected capital loss (the investment in R&D) of being replaced by another innovator, i.e. $V_{i,\tau} = e^{-r_\tau}(\pi_{i,\tau} - M_{i,\tau})$. Also, we assume free entry in the research sector implying an expected zero profit flow, $\mu(M_{i,\tau}, \eta)V_{i,\tau} - M_{i,\tau} = 0$, from which the following representation of the value of an innovation follows

$$e^{r_\tau} V_{i,\tau} = \pi_{i,\tau} - \mu(M_{i,\tau}, \eta)V_{i,\tau} \tag{2.16}$$

Therefore, the net present value of an asset equals the profits from becoming a monopolist discounted by the probability of being replaced by another innovator. (The same equation can also be derived from a Bellman equation determining the stream of profits, see Aghion and Howitt (1992).)

The monopolist profits (2.14) and the probability of an innovation (2.15) simplify (2.16) in[5]

$$\eta \frac{\gamma\alpha(1-\alpha)\hat{k}_\tau^\alpha}{e^{r_\tau} + \mu(M_{i,\tau}, \eta)} = 1 \tag{2.17}$$

which is the characteristic Research Arbitrage equation of Schumpeterian growth models. Therefore, R&D increases as $\hat{k}_\tau$ increases because more capital implies more demand for intermediate goods and more profits for the (innovative) monopolist, stimulating the innovation rate.

2.2.5 Feasibility constraint and equilibrium

From the budget constraints in the consumer optimization problem, we obtain the following feasibility constraint (the derivation is in Appendix D.2)

$$\begin{aligned} F_\tau/L_\tau &= \int_0^1 A_{i,\tau} y_{i,\tau}^\alpha di \\ &= \int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t-\tau)e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt + m_\tau + y_\tau + e^{-n}\int_\tau^{\tau+1} c_{2,\tau-1}(t)e^{r_\tau - \int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt \end{aligned} \tag{2.18}$$

[5]The probability of an innovation (2.15) simplifies the zero profits condition from $\mu(M_{i,\tau}, \eta)V_{i,\tau} - M_{i,\tau} = 0$ to $V_{i,\tau} = A_{i,\tau}/\eta$

where $m_\tau = \int_0^1 m_{i,\tau} di = \frac{\int_0^1 M_{i,\tau} di}{L_\tau}$ represents the per-capita resources invested in R&D, and $y_\tau = \int_0^1 y_{i,\tau} di$ represents the per-capita resources invested in intermediate goods production. The constraint states that the sum of the resources consumed and used for the production of intermediate goods and investments in R&D should be equal to the quantity of final good produced. The old generation discount rate (which from (2.2) equals e^z) reflects the absence of shocks in the second period of life, so premiums can now be compounded and consumed.

An equilibrium for the overlapping generation economy with consumption shocks and Schumpeterian production function is defined as follows

Definition. *A competitive equilibrium is a sequence of individual consumption and savings choices, $\{c_{1,\tau}(t), c_{2,\tau}(t), s_\tau(\tau+1)\}_{\tau=0}^\infty$, a sequence of intermediate goods and R&D investments $\{Y_\tau, M_\tau\}_{\tau=0}^\infty$, and factor prices, $\{R_\tau, \omega_\tau\}_{\tau=0}^\infty$, such that: (i) Consumers of each generation solve (2.3); (ii) Monopolistic firms optimize according to (2.10) and (2.17); (iii) Factor prices satisfy (2.12) and (2.13); (iv) The feasibility condition (2.18) is satisfied.*

2.2.6 Social Planner

The Schumpeterian production has two sources of inefficiencies: underproduction of intermediate goods and underinvestment in R&D.

About underproduction, from the feasibility condition, a social planner would maximize consumption solving

$$\max_{y_i} \int_0^1 A_{i,\tau} y_{i,\tau}^\alpha - y_{i,\tau} - m_{i,\tau} di$$

The first order condition gives $\alpha A_{i,\tau} y_{i,\tau}^\alpha = 1$, or $y_{i,\tau} = \left(\frac{1}{\alpha A_{i,\tau}}\right)^{\frac{1}{\alpha}}$ which is higher than the monopolistic production (2.11).

About underinvestment in R&D, from the feasibility condition, a social planner would maximize consumption solving

$$\max_\mu \mu\gamma F_\tau + (1-\mu)F_\tau - Y_\tau - M_\tau(\mu)$$

The first order condition gives $A_\tau = \eta(\gamma - 1)F_\tau$. Instead, the monopolistic condition is $A_\tau = \eta V_\tau$. Therefore, from (2.15), the social planner would invest more in R&D than the monopolist does.

The inefficiencies leave room for policy intervention and, in the next section, we analyze how a funded pension scheme improves the market allocation.

2.3 Pensions and economic growth

2.3.1 Pensions

Given the market inefficiencies, suppose the government introduces a pension scheme. A pension scheme is financed through income taxation and introduces a benefit when consumers retire. Therefore, a pension scheme introducing a tax ψ_τ and a benefit ζ_τ would lead to an initial condition $s_\tau(\tau) = \omega_\tau + \pi_\tau - \psi_\tau$ and to the savings level $s_\tau(\tau+1) + \zeta_\tau$ when a consumer retires.

In a funded case the return of the benefit corresponds to the market capital return, $\zeta_\tau = e^{r_\tau}\psi_\tau$; while in a pay-as-you-go case it corresponds to the population growth $\zeta_\tau = e^{n}\psi_\tau$. The following proposition summarizes the effects of the pension schemes on the stylized economy studied in the previous section

Proposition 5. *A funded social security increases both capital stock and consumption. A pay-as-you-go scheme decreases consumption when $n < \int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon$ and always decreases capital stock.*

Proof. In Appendix D.3. □

The pension benefit is equivalent to a new asset yielding either r_τ or n, respectively in the funded and pay-as-you-go case. Without a pension scheme, consumers do not have any asset with these returns since their savings yield $\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon$.

A funded pension scheme increases savings and capital because the rate of return of the pension fund is higher than the savings return. Each consumer can use the extra resources to increase both consumption and savings (and so capital). Also, each consumer cannot invest only in the pension fund since he still faces consumption shocks when young.

A pay-as-you-go scheme has similar effects, but consumption does not improve if the new asset return n is lower than the one already available (the savings rate). Moreover, a pay-as-you-go scheme always displaces savings and capital as the old generation immediately consumes the resources transferred.

Therefore, financial intermediation plays an important role, especially for the funded scheme. In canonical overlapping generations models, e.g. Samuelson

(1975), funded schemes have a neutral economic effect except when credit markets are imperfect. However, in our model credit markets are perfect. The funded pension scheme effect is not neutral because the financial intermediary is implicitly making consumers pay for insurance, creating a wedge between savings and capital rate of return.

Now that we have obtained the economic effects of the pension schemes, we can link these effects to changes in economic growth.

2.3.2 Growth

In our stylized economy, we have an innovation ($A_{\tau+1} = \gamma A_\tau$) with probability $\mu(M_\tau, \eta)$ and we do not have an innovation ($A_{\tau+1} = A_\tau$) with probability $1 - \mu(M_\tau, \eta)$. Then, the growth rate of the economy at τ is

$$g_\tau = \frac{A_{\tau+1} - A_\tau}{A_\tau} = \frac{[\mu(M_\tau, \eta)\gamma A_\tau + (1 - \mu(M_\tau, \eta))A_\tau] - A_\tau}{A_\tau} = (\gamma - 1)\mu(M_\tau, \eta)$$

Using the equilibrium factor price for capital (2.12) and the Research Arbitrage (2.17), we can explicit how growth relies on the adjusted capital $\hat{k}_\tau$

$$g_\tau = (\gamma - 1)[\eta\gamma\alpha(1 - \alpha)\hat{k}_\tau^\alpha - \alpha^2\hat{k}_\tau^{\alpha-1}] \tag{2.19}$$

Therefore, the growth rate increases as the productivity-adjusted capital-labor ratio increases. Also, Proposition 5 states that a funded pension scheme increases capital, while a pay-as-you-go one decreases it. Therefore, it follows that

Proposition 6. *A funded social security increases the growth rate of the economy. A pay-as-you-go pension decreases the growth rate of the economy.*

Proof. From Equation (2.19) the growth rate is positively correlated with the level of capital. Also, from Proposition 5, the funded social security increases the capital available with respect to the competitive allocation, while a pay-as-you-go one decreases it. □

To see the implications of social security on growth, we draw the following curves in Figure 2.1

$$\begin{aligned} &GG : g_\tau = (\gamma - 1)[\gamma\alpha(1 - \alpha)\hat{k}_\tau^\alpha - \alpha^2\hat{k}_\tau^{\alpha-1}] \\ &KK : d\hat{k}_\tau/dt = \eta\hat{k}_\tau^{2\alpha-1} - g_\tau \end{aligned}$$

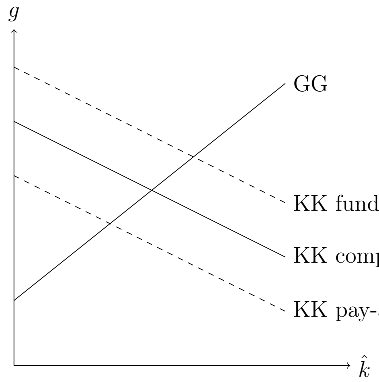

Figure 2.1: Pension schemes effects on growth

where $d\hat{k}/dt$ follows from differentiating $\hat{k} \equiv k_\tau/A_\tau$ and $\eta = \frac{1-e^{-\rho}}{1-e^{-\rho}+\rho}\frac{e^{-z}}{1+n}\frac{\alpha^2(1-\alpha)^2}{A_\tau}$ is a factor[6] with positive sign.

The intersection of the two curves identifies the growth rate and the adjusted capital-labor stock in equilibrium. From the previous propositions, the introduction of a funded pension scheme increases the capital stock. The change shifts the KK curve upwards, increasing the growth rate of the economy. On the other hand, a pay-as-you-go scheme reduces the capital stock. In this case, the KK curves moves downwards, resulting in a lower growth rate.

As remarked in Aghion and Howitt (2007) there are two reasons behind the positive correlation between capital and growth in this Schumpeterian model: an increase in capital means higher wages, more consumption and so more profits for the monopolist which pushes innovation; an increase in capital also means a lower rate of capital return which stimulates innovations as well.

Also, we have already shown how the market imperfections lead to underproduction and underinvestment. Therefore, the increase of capital due to the funded scheme is analogous to a Pareto-improving production and a research subsidy.

Therefore, a financial intermediary plays a crucial role in enhancing the welfare of the economy by making a funded pension scheme not neutral, and by transforming resources subject to uncertainty into a capital stock which can be invested.

[6]Since $\hat{k} \equiv k_\tau/A_\tau$, then $d\hat{k}_\tau/dt = dk_\tau/k_\tau - dA_\tau/A_\tau$, where $dA_\tau/A_\tau = g_\tau$. The feasibility constraint requires $k_{\tau+1} = \frac{s_\tau(\tau+1)}{1+n}$. From (2.6) we have the stock of savings $s_\tau(\tau+1)$. Using the factor prices (2.12) and (2.13), and the profits (2.14), we can derive dk_τ/k_τ.

2.4 Robustness to mortality rate

Pension schemes may have different economic effects depending on consumers' life expectancy. Therefore, we now check whether our results are robust to the introduction of a mortality rate. We already used Blanchard (1985) to model the consumption shocks; we will now use it to model life expectancy as in the original paper.

If the probability of dying is $d \in (0,1)$, then the probability of being alive at t' computed at t is $e^{-d(t'-t)}$. As in Blanchard (1985), each consumer buys an insurance which pays at each t an amount $ds_\tau(t)$ in exchange for his wealth after he dies. The profits for the insurance companies are zero, and the consumption problem becomes

$$\begin{aligned}
\max_{c_{1,\tau}(t),c_{2,\tau}(t),s_\tau(t)} \quad & \int_\tau^{\tau+1} \ln\left(c_{1,\tau}(t)\right)\phi(t-\tau)dt + \int_{\tau+1}^{\infty} e^{-(\rho+d)(t-(\tau+1))}\ln\left(c_{2,\tau}(t)\right)dt \\
\text{subject to} \quad & \dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau) \text{ for } t \in [\tau,\tau+1] \\
& \dot{s}_\tau(t) = (\hat{r}_{\tau+1}(t)+d)s_\tau(t) - c_{2,\tau}(t) \text{ for } t \in [\tau+1,\infty] \\
& s_\tau(\tau) = \omega_\tau + \pi_\tau
\end{aligned} \tag{2.20}$$

Therefore, the mortality rate updates the discount rate and the second-period constraint of (2.3). The mortality rate also substitutes the previous terminal condition $s_\tau(\tau+2)=0$ with the following transversality condition

$$\lim_{t\to\infty} s_\tau(t)e^{-d(t-(\tau+1))-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} = 0$$

Solving the model, it follows that

Proposition 7. *There exists no mortality rate d such that a pay-as-you-go and a funded scheme have similar economic effects.*

Proof. In Appendix D.4. □

The result follows from the different effects on capital accumulation of the funded and pay-as-you-go pension schemes. A pay-as-you-go scheme already reduces capital, and the mortality rate further discounts its rate of return, which now is $n-d$. A pay-as-you-go pension should increase capital to reach an economic effect analogous to a funded pension scheme. However, given the effect of the mortality rate on the rate of return of the pay-as-you-go pension scheme, a pay-as-you-go pension would increase capital only if the mortality rate d were negative, which is impossible. Therefore, our results are robust to the introduction of a mortality rate.

2.5 Final remarks

In this chapter, we presented an overlapping generations model with endogenous Schumpeterian growth and consumption shocks. A financial intermediary makes consumers implicitly pay for insurance against the shocks by offering a savings rate lower than the capital one. In this setting, a funded pension scheme becomes analogous to a new asset with returns higher than the ones available to the consumers, increasing savings and consumption. Since the financial intermediary also transforms the resources subjects to uncertainty into a secure capital stock, firms can borrow this capital and use it for production and R&D investments, enhancing the growth rate of the economy.

The results of the model fit the evidence from the Chilean pension reform in 1981 which introduced a funded pension scheme. Corsetti and Schmidt-Hebbel (1995) finds that the pension reform increased capital productivity, in line with our model. The same paper also finds an increase in wages, and a decrease in bank deposit interest rates, which jumped from 12% in 1979-1981, to 9.1% in 1981-1985. Our model suggests that the reduction of interest rates may be the result of the increase in capital available; while the increase in wages may be due to the higher economic growth rate. There is no consistency in the empirical evidence about the effects of the pension reform on savings. Corsetti and Schmidt-Hebbel (1995) finds an increase in private savings, while Holzmann (1997) finds a reduction. Our model suggests that the increase in the economic growth rate may explain the positive effect on savings; while the decrease in interest rates may explain the negative effect.

Appendix D

Derivations and proofs

D.1 Proof of Proposition 4

Solve

$$\begin{aligned}
\max_{c_{1,\tau}(t),c_{2,\tau}(t),s_\tau(t)} \quad & \int_{\tau}^{\tau+1} \ln\left(c_{1,\tau}(t)\right)\phi(t-\tau)dt + \int_{\tau+1}^{\tau+2} e^{-\rho(t-(\tau+1))}\ln\left(c_{2,\tau}(t)\right)dt \\
\text{subject to} \quad & \dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau) \text{ for } t\in[\tau,\tau+1] \\
& \dot{s}_\tau(t) = \hat{r}_{\tau+1}(t)s_\tau(t) - c_{2,\tau}(t) \text{ for } t\in[\tau+1,\tau+2] \\
& s_\tau(\tau) = \omega_\tau + \pi_\tau, \qquad s_\tau(\tau+2) = 0
\end{aligned}$$

Solve $t \in [\tau+1, \tau+2]$

The problem becomes

$$\begin{aligned}
\max_{c_{2,\tau}(t),s_\tau(t)} \quad & \int_{\tau+1}^{\tau+2} e^{-\rho(t-(\tau+1))}\ln\left(c_{2,\tau}(t)\right)dt \\
\text{subject to} \quad & \dot{s}_\tau(t) = \hat{r}_{\tau+1}(t)s_\tau(t) - c_{2,\tau}(t) \\
& s_\tau(\tau+1) \text{ given}, s_\tau(\tau+2) = 0
\end{aligned}$$

The corresponding Hamiltonian is

$$\mathcal{H} : e^{-\rho(t-(\tau+1))}\ln\left(c_{2,\tau}(t)\right) + \lambda_2(t)\left(\hat{r}_{\tau+1}(t)s_\tau(t) - c_{2,\tau}(t)\right)$$

from which the first order necessary (and sufficient) conditions are

$$\begin{aligned}
&\partial\mathcal{H}/\partial c = 0: \quad e^{-\rho(t-(\tau+1))}/c_{2,\tau}(t) - \lambda_2(t) = 0 \\
&\partial\mathcal{H}/\partial s = -\dot{\lambda}_2(t): \quad \lambda_2(t)\hat{r}_{\tau+1}(t) = -\dot{\lambda}_2(t) \\
&\lambda_2(\tau+2) \geq 0, \qquad s_\tau(\tau+2)\lambda_2(\tau+2) = 0
\end{aligned}$$

Integrating the second condition we obtain

$$\lambda_2(t) = \frac{1}{a} e^{-\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon}$$

which together with the first condition gives

$$c_{2,\tau}(t) = a e^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon - \rho(t-(\tau+1))}$$

At $\tau + 1$, we obtain $c_{2,\tau}(\tau + 1) = a$. So the terminal condition and the budget constraint imply that the present value consumption is equal to the consumer wealth (the resources he has saved)

$$\begin{aligned}
s_\tau(\tau+1) &= \int_{\tau+1}^{\tau+2} \frac{c_{2,\tau}(t)}{e^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon}} dt \\
&= \int_{\tau+1}^{\tau+2} \frac{a e^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon - \rho(t-(\tau+1))}}{e^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon}} dt \\
&= a \int_{\tau+1}^{\tau+2} e^{-\rho(t-(\tau+1))} dt \\
&= \frac{a}{\rho}(1 - e^{-\rho})
\end{aligned}$$

and hence

$$a = \frac{\rho s_\tau(\tau+1)}{1 - e^{-\rho}}$$

Using $c_{2,\tau}(t) = a e^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon - \rho(t-(\tau+1))}$, we obtain the optimal consumption value as a function of the savings stock when the individual retires

$$U_2^*(s_\tau(\tau+1)) \equiv \int_{\tau+1}^{\tau+2} e^{-\rho(t-(\tau+1))} \ln\left(\frac{\rho s_\tau(\tau+1)}{1 - e^{-\rho}} e^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon - \rho(t-(\tau+1))} \right) dt$$

Solve $t \in [\tau, \tau + 1]$

The problem reduces to

$$\begin{aligned}
\max_{c_{1,\tau}(t), s_\tau(t)} \quad & \int_{\tau}^{\tau+1} \ln\left(c_{1,\tau}(t)\right) \phi(t-\tau) dt + U_2^*(s_\tau(\tau+1)) \\
\text{subject to} \quad & \dot{s}_\tau(t) = \hat{r}_\tau(t) s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau) \\
& s_\tau(\tau) = \omega_\tau + \pi_\tau
\end{aligned}$$

The Hamiltonian of the problem is

$$\mathcal{H} : \ln\left(c_{1,\tau}(t)\right)\phi(t-\tau) + U_2^*(s_\tau(\tau+1)) + \lambda_1(t)(\hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau))$$

from which the first order conditions are

$$\begin{aligned}
&\partial\mathcal{H}/\partial c = 0 : \quad \phi(t-\tau)/c_{1,\tau}(t) - \lambda_1(t)\phi(t-\tau) = 0\\
&\partial\mathcal{H}/\partial s = -\dot{\lambda}_1(t) : \quad \lambda_1(t)\hat{r}_\tau(t) = -\dot{\lambda}_1(t)\\
&\lambda_1(\tau+1) \geq 0
\end{aligned}$$

Integrating the second expression we obtain

$$\lambda_1(t) = \frac{1}{b}e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}$$

which together with the first one gives

$$c_{1,\tau}(t) = be^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}$$

By continuity $\lambda_1(\tau+1) = \lambda_2(\tau+1)$, from which we have

$$b = ae^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon} = \frac{\rho}{1-e^{-\rho}}e^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}s_\tau(\tau+1)$$

Therefore,

$$c_{1,\tau}(t) = \frac{\rho e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon - \int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}}{1-e^{-\rho}}s_\tau(\tau+1) \tag{D.1}$$

Compute $s_\tau(\tau+1)$

From the two previous sub-problems, we obtained the two optimal consumption paths as a function of the optimal retirement savings. The last element we need, and that we now compute.

The first-period budget constraint is

$$\dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau)$$

We guess that its integration leads to

$$s_\tau(t) = e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau) - c_{1,\tau}(t)\Phi(t-\tau) \tag{D.2}$$

To verify the claim, we derive the budget constraint

$$\frac{ds_\tau(t)}{dt} = \dot{s}_\tau(t) = \hat{r}_\tau(t)e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau) - \left[c_{1,\tau}(t)\phi(t-\tau) + \frac{dc_{1,\tau}(t)}{dt}\Phi(t-\tau)\right]$$

From (D.2) $e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau) = s_\tau(t) + c_{1,\tau}(t)\Phi(t-\tau)$, then

$$\dot{s}_\tau(t) = \hat{r}_\tau(t)\left[s_\tau(t) + c_{1,\tau}(t)\Phi(t-\tau)\right] - \left[c_{1,\tau}(t)\phi(t-\tau) + \frac{dc_{1,\tau}(t)}{dt}\Phi(t-\tau)\right]$$

from which it follows that

$$\dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau) + \hat{r}_\tau(t)c_{1,\tau}(t)\Phi(t-\tau) - \frac{dc_{1,\tau}(t)}{dt}\Phi(t-\tau) \quad \text{(D.3)}$$

If we derive (D.1), we obtain

$$\frac{dc_{1,\tau}(t)}{dt} = \hat{r}_\tau(t)\frac{\rho e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon - \int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}}{1-e^{-\rho}}s_\tau(\tau+1) = \hat{r}_\tau(t)c_{1,\tau}(t)$$

Therefore, (D.3) simplifies in

$$\dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau)$$

confirming our initial guess.

Now, we can use (D.2) to compute $s_\tau(\tau+1)$. We value (D.2) at $\tau+1$ and obtain

$$s_\tau(\tau+1) = e^{\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau+\pi_\tau) - c_{1,\tau}(\tau+1)\Phi(1) = e^{\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau+\pi_\tau) - c_{1,\tau}(\tau+1)$$

from which

$$c_{1,\tau}(\tau+1) = e^{\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau) - s_\tau(\tau+1)$$

Valuing (D.1) at $t = \tau+1$ leads to

$$c_{1,\tau}(\tau+1) = \frac{\rho}{1-e^{-\rho}}s_\tau(\tau+1)$$

By using the last two equations derived, we obtain

$$e^{\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau) - s_\tau(\tau+1) = \frac{\rho}{1-e^{-\rho}}s_\tau(\tau+1)$$

from which we derive the optimal savings when a consumer retires

$$s_\tau(\tau+1) = \frac{1-e^{-\rho}}{1-e^{-\rho}+\rho} e^{\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau+\pi_\tau) \tag{D.4}$$

so the explicit solution for consumption in both periods

$$c_{1,\tau}(t) = \frac{\rho e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}}{1-e^{-\rho}+\rho}(\omega_\tau+\pi_\tau) \qquad c_{2,\tau}(t) = \frac{\rho e^{\int_{\tau+1}^t \hat{r}_{\tau+1}(\epsilon)d\epsilon-\rho(t-(\tau+1))+\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}}{1-e^{-\rho}+\rho}(\omega_\tau+\pi_\tau)$$

D.2 Feasibility constraint

We assume that production takes places at each period, and not at each intra-period t. Then, the optimal savings stock when consumers of generation τ retire constitutes the physical capital for the next period, $K_{\tau+1} = L_\tau s_\tau(\tau+1)$. In per-capita terms

$$k_{\tau+1} = e^{-n} s_\tau(\tau+1) \tag{D.5}$$

Firms invest the resources in the intermediate good production and the R&D sector. Then, the discounted value of the resources available for investments equals the sum of resources invested in the intermediate goods production and R&D

$$s_\tau(\tau+1)e^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon} = m_\tau + y_\tau \tag{D.6}$$

where $m_\tau = \frac{\int_0^1 M_{i,\tau} di}{L_\tau}$ is the per-capita resources invested in R&D; $y_\tau = \int_0^1 y_{i,\tau} di$ is the per-capita resources invested in intermediate goods production.

The budget constraints in the consumer optimization problem are given in terms of flows. However, the feasibility constraint is usually represented in terms of stocks. Then, we need to integrate the flow of resources in the budget constraints to transform them into stocks.

The first-period consumer budget constraint is

$$\dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau)$$

Multiply each side with $e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}$ and integrate over τ and $\tau+1$

$$\int_\tau^{\tau+1} \dot{s}_\tau(t)e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt = \int_\tau^{\tau+1} \hat{r}_\tau(t)s_\tau(t)e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt - \int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t-\tau)e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt$$

The integration by parts of $\int_\tau^{\tau+1} \dot{s}_\tau(t)e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt$ gives

$$\int_{\tau}^{\tau+1}\dot{s}_\tau(t)e^{-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt = s_\tau(\tau+1)e^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon} - s_\tau(\tau) + \int_\tau^{\tau+1}\hat{r}_\tau s_\tau(t)e^{-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt$$

If we substitute this last expression in the first integration, using also the initial condition $s_\tau(\tau) = \omega_\tau + \pi_\tau$, we obtain

$$w_\tau + \pi_\tau = s_\tau(\tau+1)e^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon} + \int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t-\tau)e^{-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt \qquad \text{(D.7)}$$

Therefore, consumer's income equals the discounted optimal stock of savings plus the expected present value of the first-period consumption.

The second-period budget constraint is

$$\dot{s}_\tau(t) = \hat{r}_{\tau+1}(t)s_\tau(t) - c_{2,\tau}(t)$$

Multiply each side with $e^{-\int_{\tau+1}^t\hat{r}_{\tau+1}(\epsilon)d\epsilon}$ and integrate it over $\tau+1$ and $\tau+2$

$$\int_{\tau+1}^{\tau+2}\dot{s}_\tau(t)e^{-\int_{\tau+1}^t\hat{r}_{\tau+1}(\epsilon)d\epsilon}dt = \int_{\tau+1}^{\tau+2}\hat{r}_{\tau+1}(t)s_\tau(t)e^{-\int_{\tau+1}^t\hat{r}_{\tau+1}(\epsilon)d\epsilon}dt - \int_{\tau+1}^{\tau+2}c_{2,\tau}(t)e^{-\int_{\tau+1}^t\hat{r}_{\tau+1}(\epsilon)d\epsilon}dt$$

The integration by parts of the left-hand side term gives

$$\int_{\tau+1}^{\tau+2}\dot{s}_\tau(t)e^{-\int_{\tau+1}^t\hat{r}_{\tau+1}(\epsilon)d\epsilon}dt = s_\tau(\tau+2)e^{-\int_{\tau+1}^{\tau+2}\hat{r}_{\tau+1}(\epsilon)d\epsilon} - s_\tau(\tau+1) + \int_{\tau+1}^{\tau+2}\hat{r}_{\tau+1}(t)s_\tau(t)e^{-\int_{\tau+1}^t\hat{r}_{\tau+1}(\epsilon)d\epsilon}dt$$

If we substitute this last expression in the first integration, using also the no-waste condition $s_\tau(\tau+2) = 0$, we obtain

$$\int_{\tau+1}^{\tau+2} c_{2,\tau}(t)e^{-\int_{\tau+1}^t\hat{r}_{\tau+1}(\epsilon)d\epsilon}dt = s_\tau(\tau+1)$$

Therefore, the present value of the second-period consumption equals the optimal savings when the consumer retires.

However, in the feasibility constraint, we are interested in the consumption of generation $\tau-1$, and we need to take into account the population growth. If we introduce a lag and multiply both sides with e^{-n}, it follows that

$$e^{-n}s_{\tau-1}(\tau) = e^{-n}\int_\tau^{\tau+1} c_{2,\tau-1}(t)e^{-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt$$

Finally, we multiply each side by e^{r_τ} to simplify the construction of the feasibility constraint once we combine the stock versions of the budget constraints. Therefore, we obtain the second constraint

$$e^{-n} \int_{\tau}^{\tau+1} c_{2,\tau-1}(t) e^{r_\tau - \int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt = e^{r_\tau - n} s_{\tau-1}(\tau) \tag{D.8}$$

Combining (D.7) and (D.8), we obtain

$$w_\tau + \pi_\tau + e^{r_\tau - n} s_{\tau-1}(\tau) =$$
$$\int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t-\tau) e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt + s_\tau(\tau+1) e^{-\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon} + e^{-n} \int_\tau^{\tau+1} c_{2,\tau-1}(t) e^{r_\tau - \int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt$$

Using the monopolist profit maximization $\pi_\tau = p_\tau y_\tau - e^{r_\tau} k_\tau$ and the credit market conditions (D.5) and (D.6), it follows that

$$w_\tau + p_\tau y_\tau = \int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t-\tau) e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt + m_\tau + y_\tau + e^{-n} \int_\tau^{\tau+1} c_{2,\tau-1}(t) e^{r_\tau - \int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt$$

Using the the demand curve of intermediate goods (2.9) and the labor retribution (2.13), the (per-capita) feasibility constraint in Equation (2.18) obtains.

D.3 Proof of Proposition 5

The introduction of a pension scheme changes two elements of the previous computations: it reduces the income by the tax ψ_τ; and it increases the savings when an individual retire by the pension benefit ζ_τ. Instead of w_τ we now have $w_\tau - \psi_\tau$, and instead of $s_\tau(\tau+1)$, we now have $s_\tau(\tau+1) + \zeta_\tau$.

Solve $t \in [\tau+1, \tau+2]$

The problem becomes

$$\begin{aligned} \max_{c_{2,\tau}(t), s_\tau(t)} \quad & \int_{\tau+1}^{\tau+2} e^{-\rho(t-(\tau+1))} \ln\left(c_{2,\tau}(t)\right) dt \\ \text{subject to} \quad & \dot{s}_\tau(t) = \hat{r}_{\tau+1}(t) s_\tau(t) - c_{2,\tau}(t) \\ & s_\tau(\tau+1) + \zeta_\tau \text{ given}, s_\tau(\tau+2) = 0 \end{aligned}$$

The corresponding Hamiltonian is

$$\mathcal{H} : e^{-\rho(t-(\tau+1))} \ln\left(c_{2,\tau}(t)\right) + \lambda_2(t)\left(\hat{r}_{\tau+1}(t) s_\tau(t) - c_{2,\tau}(t)\right)$$

from which the first order necessary (and sufficient) conditions are

$$\begin{aligned}
&\partial\mathcal{H}/\partial c = 0: \quad e^{-\rho(t-(\tau+1))}/c_{2,\tau}(t) - \lambda_2(t) = 0\\
&\partial\mathcal{H}/\partial s = -\dot{\lambda}_2(t): \quad \lambda_2(t)\hat{r}_{\tau+1}(t) = -\dot{\lambda}_2(t)\\
&\lambda_2(\tau+2) \geq 0, \qquad s_\tau(\tau+2)\lambda_2(\tau+2) = 0
\end{aligned}$$

Integrating the second condition we obtain

$$\lambda_2(t) = \frac{1}{a}e^{\int_{\tau+1}^{t}\hat{r}_{\tau+1}(\epsilon)d\epsilon}$$

which together with the first condition gives

$$c_{2,\tau}(t) = ae^{\int_{\tau+1}^{t}\hat{r}_{\tau+1}(\epsilon)d\epsilon-\rho(t-(\tau+1))}$$

At $\tau+1$, we obtain $c_{2,\tau}(\tau+1) = a$. Then, the terminal condition and the budget constraint imply that the present value consumption is equal to the consumer wealth (the resources saved plus the benefit)

$$\begin{aligned}
s_\tau(\tau+1) + \zeta_\tau &= \int_{\tau+1}^{\tau+2}\frac{c_{2,\tau}(t)}{e^{\int_{\tau+1}^{t}\hat{r}_{\tau+1}(\epsilon)d\epsilon}}dt\\
&= \int_{\tau+1}^{\tau+2}\frac{ae^{\int_{\tau+1}^{t}\hat{r}_{\tau+1}(\epsilon)d\epsilon-\rho(t-(\tau+1))}}{e^{\int_{\tau+1}^{t}\hat{r}_{\tau+1}(\epsilon)d\epsilon}}dt\\
&= a\int_{\tau+1}^{\tau+2}e^{-\rho(t-(\tau+1))}dt\\
&= \frac{a}{\rho}(1-e^{-\rho})
\end{aligned}$$

and hence

$$a = \frac{\rho[s_\tau(\tau+1)+\zeta_\tau]}{1-e^{-\rho}}$$

Using $c_{2,\tau}(t) = ae^{\int_{\tau+1}^{t}\hat{r}_{\tau+1}(\epsilon)d\epsilon-\rho(t-(\tau+1)}$, we obtain the optimal consumption value as a function of the savings stock when the individual retires

$$U_2^*(s_\tau(\tau+1)) \equiv \int_{\tau+1}^{\tau+2}e^{-\rho(t-(\tau+1))}\ln\left(\frac{\rho[s_\tau(\tau+1)+\zeta_\tau]}{1-e^{-\rho}}e^{\int_{\tau+1}^{t}\hat{r}_{\tau+1}(\epsilon)d\epsilon-\rho(t-(\tau+1))}\right)dt$$

Solve $t \in [\tau, \tau+1]$

The problem reduces to

$$\max_{c_{1,\tau}(t), s_\tau(t)} \quad \int_\tau^{\tau+1} \ln\left(c_{1,\tau}(t)\right)\phi(t-\tau)dt + U_2^*(s_\tau(\tau+1))$$
$$\text{subject to} \quad \dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau)$$
$$s_\tau(\tau) = \omega_\tau + \pi_\tau - \psi_\tau$$

The Hamiltonian of the problem is

$$\mathcal{H} : \ln\left(c_{1,\tau}(t)\right)\phi(t-\tau) + U_2^*(s_\tau(\tau+1)) + \lambda_1(t)(\hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau))$$

from which, the first order conditions are

$$\partial\mathcal{H}/\partial c = 0 : \quad \phi(t-\tau)/c_{1,\tau}(t) - \lambda_1(t)\phi(t-\tau) = 0$$
$$\partial\mathcal{H}/\partial s = -\dot{\lambda}_1(t) : \quad \lambda_1(t)\hat{r}_\tau(t) = -\dot{\lambda}_1(t)$$
$$\lambda_1(\tau+1) \geq 0$$

Integrating the second expression we obtain

$$\lambda_1(t) = \frac{1}{b}e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}$$

which together with the first expression gives

$$c_{1,\tau}(t) = be^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}$$

By continuity $\lambda_1(\tau+1) = \lambda_2(\tau+1)$, from which we have

$$b = ae^{-\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon} = \frac{\rho}{1-e^{-\rho}}e^{-\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}[s_\tau(\tau+1) + \zeta_\tau]$$

Therefore,

$$c_{1,\tau}(t) = \frac{\rho e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon - \int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}}{1-e^{-\rho}}[s_\tau(\tau+1) + \zeta_\tau] \tag{D.9}$$

Compute $s_\tau(\tau+1)$

We start from the first-period budget constraint

$$\dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau)$$

We guess that its integration leads to

$$s_\tau(t) = e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau - \psi_\tau) - c_{1,\tau}(t)\Phi(t-\tau) \tag{D.10}$$

To verify the claim, we derive the first-period budget constraint

$$\frac{ds_\tau(t)}{dt} = \dot{s}_\tau(t) = \hat{r}_\tau(t)e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau - \psi_\tau) - \left[c_{1,\tau}(t)\phi(t-\tau) + \frac{dc_{1,\tau}(t)}{dt}\Phi(t-\tau)\right]$$

From (D.10), we obtain $e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau - \psi_\tau) = s_\tau(t) + c_{1,\tau}(t)\Phi(t-\tau)$. Then,

$$\dot{s}_\tau(t) = \hat{r}_\tau(t)\left[s_\tau(t) + c_{1,\tau}(t)\Phi(t-\tau)\right] - \left[c_{1,\tau}(t)\phi(t-\tau) + \frac{dc_{1,\tau}(t)}{dt}\Phi(t-\tau)\right]$$

from which

$$\dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau) + \hat{r}_\tau(t)c_{1,\tau}(t)\Phi(t-\tau) - \frac{dc_{1,\tau}(t)}{dt}\Phi(t-\tau) \tag{D.11}$$

If we derive (D.9), we obtain

$$\frac{dc_{1,\tau}(t)}{dt} = \hat{r}_\tau(t)\frac{\rho e^{\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon - \int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}}{1-e^{-\rho}}[s_\tau(\tau+1) + \zeta_\tau] = \hat{r}_\tau(t)c_{1,\tau}(t)$$

Therefore, (D.11) simplifies in

$$\dot{s}_\tau(t) = \hat{r}_\tau(t)s_\tau(t) - c_{1,\tau}(t)\phi(t-\tau)$$

confirming our initial guess.

Now, we can use (D.10) to compute $s_\tau(\tau+1)$. If we value (D.10) at $\tau+1$, it follows that

$$s_\tau(\tau+1) = e^{\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau+\pi_\tau-\psi_\tau) - c_{1,\tau}(\tau+1)\Phi(1) = e^{\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau+\pi_\tau-\psi_\tau) - c_{1,\tau}(\tau+1)$$

from which

$$c_{1,\tau}(\tau+1) = e^{\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau - \psi_\tau) - s_\tau(\tau+1)$$

Valuing (D.9) at $t = \tau+1$, we obtain

$$c_{1,\tau}(\tau+1) = \frac{\rho}{1-e^{-\rho}}[s_\tau(\tau+1) + \zeta_\tau]$$

We can combine these the last two equations found and obtain

$$e^{\int_{\tau}^{\tau+1}\hat{r}_{\tau}(\epsilon)d\epsilon}(\omega_{\tau}+\pi_{\tau}-\psi_{\tau})-s_{\tau}(\tau+1)=\frac{\rho}{1-e^{-\rho}}[s_{\tau}(\tau+1)+\zeta_{\tau}]$$

from which

$$s_{\tau}(\tau+1)=\frac{1-e^{-\rho}}{1-e^{-\rho}+\rho}e^{\int_{\tau}^{\tau+1}\hat{r}_{\tau}(\epsilon)d\epsilon}(\omega_{\tau}+\pi_{\tau}-\psi_{\tau})-\frac{\rho}{1-e^{-\rho}+\rho}\zeta_{\tau} \tag{D.12}$$

Funded

In a funded pension scheme $\zeta_{\tau}=e^{r_{\tau}}\psi_{\tau}$ and the capital market clearing conditions are

$$k_{\tau+1}^{ff}=e^{-n}\left(s_{\tau}(\tau+1)+e^{r_{\tau}}\psi_{\tau}\right) \tag{D.13}$$

To verify that the condition is correct, we first check the feasibility constraint. Due to the pension scheme, in the first period a consumer pays ψ_{τ} to finance the pension, $s_{\tau}(\tau)=\omega_{\tau}+\pi_{\tau}-\psi_{\tau}$; and in the second period he receives the benefit $e^{r_{\tau}}\psi_{\tau}$. Then, Equations (D.7) and (D.8) update in

$$\omega_{\tau}+\pi_{\tau}-\psi_{\tau}=s_{\tau}(\tau+1)e^{-\int_{\tau}^{\tau+1}\hat{r}_{\tau}(\epsilon)d\epsilon}+\int_{\tau}^{\tau+1}c_{1,\tau}(t)\phi(t-\tau)e^{-\int_{\tau}^{t}\hat{r}_{\tau}(\epsilon)d\epsilon}dt$$

$$e^{-n}\int_{\tau}^{\tau+1}c_{2,\tau-1}(t)e^{r_{\tau}-\int_{\tau}^{t}\hat{r}_{\tau}(\epsilon)d\epsilon}dt=e^{r_{\tau}-n}\left[s_{\tau-1}(\tau)+e^{r_{\tau-1}}\psi_{\tau-1}\right]$$

Combining the two equations, we obtain

$$\begin{aligned}&\omega_{\tau}+\pi_{\tau}-\psi_{\tau}+e^{r_{\tau}-n}\left[s_{\tau-1}(\tau)+e^{r_{\tau-1}}\psi_{\tau-1}\right]=\\&\quad s_{\tau}(\tau+1)e^{-\int_{\tau}^{\tau+1}\hat{r}_{\tau}(\epsilon)d\epsilon}+\int_{\tau}^{\tau+1}c_{1,\tau}(t)\phi(t-\tau)e^{-\int_{\tau}^{t}\hat{r}_{\tau}(\epsilon)d\epsilon}dt+e^{-n}\int_{\tau}^{\tau+1}c_{2,\tau-1}(t)e^{r_{\tau}-\int_{\tau}^{t}\hat{r}_{\tau}(\epsilon)d\epsilon}dt\end{aligned}$$

Using (D.13) and the equilibrium factor prices, the feasibility constraint follows

$$f(k_{\tau})=s_{\tau}(\tau+1)e^{-\int_{\tau}^{\tau+1}\hat{r}_{\tau}(\epsilon)d\epsilon}+\int_{\tau}^{\tau+1}c_{1,\tau}(t)\phi(t-\tau)e^{-\int_{\tau}^{t}\hat{r}_{\tau}(\epsilon)d\epsilon}dt+e^{-n}\int_{\tau}^{\tau+1}c_{2,\tau-1}(t)e^{r_{\tau}-\int_{\tau}^{t}\hat{r}_{\tau}(\epsilon)d\epsilon}dt+\psi_{\tau}$$

Now, we can compute the capital level using (D.13)

$$k_{\tau+1}^{ff}=e^{-n}\left[\frac{1-e^{-\rho}}{1-e^{-\rho}+\rho}e^{\int_{\tau}^{\tau+1}\hat{r}_{\tau}(\epsilon)d\epsilon}(\omega_{\tau}+\pi_{\tau}-\psi_{\tau})+\frac{1-e^{-\rho}}{1-e^{-\rho}+\rho}e^{r_{\tau}}\psi_{\tau}\right]$$

from which

$$k^{ff}_{\tau+1} = e^{-n}\left[\frac{1-e^{-\rho}}{1-e^{-\rho}+\rho}e^{\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau+\pi_\tau)+\frac{1-e^{-\rho}}{1-e^{-\rho}+\rho}\left(e^{r_\tau}-e^{\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}\right)\psi_\tau\right] \tag{D.14}$$

The policy increases physical capital if the difference in the right-hand side is positive, i.e. $dk^{ff}_{\tau+1}/d\psi_\tau > 0$, if $e^{r_\tau} - e^{\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon} > 0$. Also, we have already shown that the difference is positive and equal to z. From (D.9), if savings increases, consumption in the first period also increases. A similar reasoning applies to the second-period consumption.

Pay-as-you-go

In a pay-as-you-go pension scheme $\zeta_\tau = e^n\psi_\tau$ and the capital market clearing conditions are

$$k^{payg}_{\tau+1} = e^{-n}s_\tau(\tau+1) \tag{D.15}$$

To verify that the condition is correct, we first check the feasibility constraint. Due to the pension scheme, in the first period a consumer pays ψ_τ to finance the pension, $s_\tau(\tau) = \omega_\tau + \pi_\tau - \psi_\tau$; and in the second period he receives the benefit $e^n\psi_{\tau+1}$. Then, Equations (D.7) and (D.8) update in

$$\omega_\tau + \pi_\tau - \psi_\tau = s_\tau(\tau+1)e^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon} + \int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t-\tau)e^{-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt$$

$$e^{-n}\int_\tau^{\tau+1} c_{2,\tau-1}(t)e^{r_\tau-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt = e^{-n}\left(e^{r_\tau}s_{\tau-1}(\tau)+e^n\psi_\tau\right)$$

Combining the two equations we obtain

$$\begin{aligned}&\omega_\tau+\pi_\tau-\psi_\tau+e^{r_\tau-n}s_{\tau-1}(\tau)+\psi_\tau = \\ &\quad s_\tau(\tau+1)e^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}+\int_\tau^{\tau+1}c_{1,\tau}(t)\phi(t-\tau)e^{-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt+e^{-n}\int_\tau^{\tau+1}c_{2,\tau-1}(t)e^{r_\tau-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt\end{aligned}$$

Using D.15) and the equilibrium factor prices, the feasibility constraint follows

$$f(k_\tau) = s_\tau(\tau+1)e^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}+\int_\tau^{\tau+1}c_{1,\tau}(t)\phi(t-\tau)e^{-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt+e^{-n}\int_\tau^{\tau+1}c_{2,\tau-1}(t)e^{r_\tau-\int_\tau^t\hat{r}_\tau(\epsilon)d\epsilon}dt$$

Now, we can compute the capital level using (D.15)

$$k_{\tau+1}^{payg} = e^{-n}\left(\frac{1-e^{-\rho}}{1-e^{-\rho}+\rho}e^{\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau+\pi_\tau-\psi_\tau)-\frac{\rho}{1-e^{-\rho}+\rho}e^n\psi_\tau\right) \quad \text{(D.16)}$$

The policy reduces savings and physical capital since $dk_{\tau+1}^{payg}/d\psi_\tau < 0$, if $\psi > 0$. From (D.12), we have

$$s_\tau(\tau+1)+e^n\psi_{\tau+1} = \frac{e^{\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}(1-e^{-\rho})}{1-e^{-\rho}+\rho}(\omega_\tau+\pi_\tau)+\frac{1-e^{-\rho}}{1-e^{-\rho}+\rho}(e^n-e^{\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon})\psi_\tau$$

Therefore, from (D.9), pay-as-you-go pensions have a negative effect on consumption if $(e^n - e^{\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon})(1-e^{-\rho}) < 0$. Since $1-e^{-\rho} > 0$, consumption decreases when $n < \int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon$.

D.4 Proof of Proposition 7

Compared to the proof of Proposition 5, only the first part of the derivation changes.

Solve $t \in [\tau+1, \infty]$

If we introduce a mortality rate, the problem becomes

$$\begin{aligned} \max_{c_{2,\tau}(t), s_\tau(t)} \quad & \int_{\tau+1}^{\infty} e^{-(\rho+d)(t-(\tau+1))}\ln(c_{2,\tau}(t))\,dt \\ \text{subject to} \quad & \dot{s}_\tau(t) = (\hat{r}_{\tau+1}(t)+d)s_\tau(t) - c_{2,\tau}(t) \\ & s_\tau(\tau+1)+\zeta_\tau \text{ given} \end{aligned}$$

The corresponding Hamiltonian is

$$\mathcal{H}: e^{-(\rho+d)(t-(\tau+1))}\ln(c_{2,\tau}(t)) + \lambda_2(t)\left((\hat{r}_{\tau+1}(t)+d)s_\tau(t) - c_{2,\tau}(t)\right)$$

From which, the first order necessary (and sufficient) conditions are

$$\begin{aligned} \partial\mathcal{H}/\partial c = 0: \quad & e^{-(\rho+d)(t-(\tau+1))}/c_{2,\tau}(t) - \lambda_2(t) = 0 \\ \partial\mathcal{H}/\partial s = -\dot{\lambda}_2(t): \quad & \lambda_2(t)(\hat{r}_{\tau+1}(t)+d) = -\dot{\lambda}_2(t) \end{aligned}$$

Integrating the second equation we obtain $\lambda_2(t) = \frac{1}{a}e^{-\int_{\tau+1}^{t}\hat{r}_{\tau+1}(\epsilon)d\epsilon - d(t-(\tau+1))}$, which together with the other conditions and the law of motion of savings give

$$c_{2,\tau}(t) = ae^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon - \rho(t-(\tau+1))}$$

Evaluating this last equation at $\tau+1$, we obtain $c_{2,\tau}(\tau+1) = a$. Now, the terminal condition and the budget constraint imply that the present value consumption equals consumer's wealth

$$\begin{aligned} s_\tau(\tau+1) + \zeta_\tau &= \int_{\tau+1}^{\infty} \frac{c_{2,\tau}(t)}{e^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon + d(t-(\tau+1))}} \\ &= \int_{\tau+1}^{\infty} \frac{ae^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon - \rho(t-(\tau+1))}}{e^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon + d(t-(\tau+1))}} \\ &= a\int_{\tau+1}^{\infty} e^{-(\rho+d)(t-(\tau+1))} \\ &= \frac{a}{\rho+d} \end{aligned}$$

from which

$$a = (\rho+d)[s_\tau(\tau+1) + \zeta_\tau]$$

Then, following the proof of Proposition 5, we obtain consumption in both periods

$$c_{1,\tau}(t) = (\rho+d)e^{\int_{\tau}^{t} \hat{r}_\tau(\epsilon)d\epsilon - \int_{\tau}^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}[s_\tau(\tau+1) + \zeta_\tau] \tag{D.17}$$

$$c_{2,\tau}(t) = (\rho+d)e^{\int_{\tau+1}^{t} \hat{r}_{\tau+1}(\epsilon)d\epsilon - \rho(t-(\tau+1))}[s_\tau(\tau+1) + \zeta_\tau]$$

Using savings (D.10) and consumption (D.17) both valued at $\tau+1$, it follows that

$$e^{\int_{\tau}^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau - \psi_\tau) - s_\tau(\tau+1) = (\rho+d)[s_\tau(\tau+1) + \zeta_\tau]$$

from which

$$s_\tau(\tau+1) = \frac{e^{\int_{\tau}^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau - \psi_\tau)}{1+\rho+d} - \frac{(\rho+d)\zeta_\tau}{1+\rho+d}$$

In a funded pension scheme $\zeta_\tau = e^{r_\tau}\psi_\tau$ and $k_\tau = e^{-n}(s_\tau(\tau+1) + e^{r_\tau}\psi_\tau)$, then

$$k_{\tau+1}^{ff} = \left[\frac{e^{\int_{\tau}^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon}(\omega_\tau + \pi_\tau - \psi_\tau)}{1+\rho+d} + \frac{e^{r_\tau}\psi_\tau}{1+\rho+d} \right] e^{-n}$$

In a pay-as-you-go pension scheme $\zeta_\tau = e^{n}\psi_\tau$ and $k_\tau = e^{-n}(s_\tau(\tau+1))$, then

$$k_{\tau+1}^{payg} = \left[\frac{e^{\int_{\tau}^{\tau+1} \hat{r}_{\tau}(\epsilon) d\epsilon}(\omega_{\tau} + \pi_{\tau} - \psi_{\tau})}{1+\rho+d} - \frac{(\rho+d)e^{n}\psi_{\tau}}{1+\rho+d} \right] e^{-n}$$

The two pension schemes have similar economic effects when the same amount of capital obtains,

$$k_{\tau+1}^{ff} = k_{\tau+1}^{payg}$$

from which

$$\left[\frac{e^{\int_{\tau}^{\tau+1} \hat{r}_{\tau}(\epsilon) d\epsilon}(\omega_{\tau}+\pi_{\tau}-\psi_{\tau})}{1+\rho+d} + \frac{e^{r_{\tau}}\psi_{\tau}}{1+\rho+d} \right] e^{-n} = \left[\frac{e^{\int_{\tau}^{\tau+1} \hat{r}_{\tau}(\epsilon) d\epsilon}(\omega_{\tau}+\pi_{\tau}-\psi_{\tau})}{1+\rho+d} - \frac{(\rho+d)e^{n}\psi_{\tau}}{1+\rho+d} \right] e^{-n}$$

which simplifies in

$$e^{r_{\tau}}\psi_{\tau} = -(\rho+d)e^{n}\psi_{\tau}$$

Therefore,

$$d = -(\rho + e^{r_{\tau}-n})$$

However, d cannot be negative. Therefore, a mortality rate such that the two schemes equal does not exist.

Appendix E

Additional results

E.1 The Cobb-Douglas production function case

We here solve the overlapping generation model studied in this chapter but without endogenous growth. We replace the Schumpeterian production function with a Cobb-Douglas one.

Firms

The production process is not instantaneous and lasts an entire period. Then, the production function is the following Cobb-Douglas

$$f(k_\tau) = Ak_\tau^\alpha$$

Markets are still competitive. Then, also the factor prices are defined as in the canonical case

$$R_\tau = f'(k_\tau) \quad \text{and} \quad \omega_\tau = f(k_\tau) - k_\tau f'(k_\tau) \tag{E.1}$$

Consumers

The consumption-savings problem remains unchanged with the only exception that $\pi_\tau = 0$, since there are no profits to redistribute.

Feasibility conditions and equilibrium definition

We assume that production takes place at each period, and not at each intra-period t. Then, the optimal savings stock when consumers of generation τ retire constitutes the physical capital for the next period, $K_{\tau+1} = L_\tau s_\tau(\tau + 1)$. In per-capita terms

$$k_{\tau+1} = e^{-n} s_\tau(\tau + 1) \tag{E.2}$$

The budget constraints in the consumer optimization problem are given in terms of flows. Instead, the feasibility constraint is usually represented in terms of stocks. Then, we need to integrate the flow of resources in the budget constraints to transform them into stocks.

The first-period consumer budget constraint is

$$\dot{s}_\tau(t) = \hat{r}_\tau(t) s_\tau(t) - c_{1,\tau}(t)\phi(t - \tau)$$

We multiply each side with $e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}$ and integrate it over τ and $\tau + 1$

$$\int_\tau^{\tau+1} \dot{s}_\tau(t) e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt = \int_\tau^{\tau+1} \hat{r}_\tau(t) s_\tau(t) e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt - \int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t - \tau) e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt$$

The integration by parts of $\int_\tau^{\tau+1} \dot{s}_\tau(t) e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt$ gives

$$\int_\tau^{\tau+1} \dot{s}_\tau(t) e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt = s_\tau(\tau + 1) e^{-\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon} - s_\tau(\tau) + \int_\tau^{\tau+1} \hat{r}_\tau s_\tau(t) e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt$$

By substituting this last expression in the first integration, using also the initial condition $s_\tau(\tau) = \omega_\tau$, we obtain

$$w_\tau = s_\tau(\tau + 1) e^{-\int_\tau^{\tau+1} \hat{r}_\tau(\epsilon)d\epsilon} + \int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t - \tau) e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon} dt$$

Therefore, consumer's income equals the discounted optimal stock of savings plus the expected present value of the first-period consumption.

The second-period budget constraint is

$$\dot{s}_\tau(t) = \hat{r}_{\tau+1}(t) s_\tau(t) - c_{2,\tau}(t)$$

We multiply each side with $e^{-\int_{\tau+1}^t \hat{r}_{\tau+1}(\epsilon)d\epsilon}$ and integrate it over $\tau + 1$ and $\tau + 2$

$$\int_{\tau+1}^{\tau+2} \dot{s}_\tau(t) e^{-\int_{\tau+1}^t \hat{r}_{\tau+1}(\epsilon)d\epsilon} dt = \int_{\tau+1}^{\tau+2} \hat{r}_{\tau+1}(t) s_\tau(t) e^{-\int_{\tau+1}^t \hat{r}_{\tau+1}(\epsilon)d\epsilon} dt - \int_{\tau+1}^{\tau+2} c_{2,\tau}(t) e^{-\int_{\tau+1}^t \hat{r}_{\tau+1}(\epsilon)d\epsilon} dt$$

The integration by parts of the left side term gives

$$\int_{\tau+1}^{\tau+2} \dot{s}_\tau(t) e^{-\int_{\tau+1}^t \hat{r}_{\tau+1}(\epsilon)d\epsilon} dt = s_\tau(\tau + 2) e^{-\int_{\tau+1}^{\tau+2} \hat{r}_{\tau+1}(\epsilon)d\epsilon} - s_\tau(\tau + 1) + \int_{\tau+1}^{\tau+2} \hat{r}_{\tau+1}(t) s_\tau(t) e^{-\int_{\tau+1}^t \hat{r}_{\tau+1}(\epsilon)d\epsilon} dt$$

By substituting this last expression in the first integration, using also the final condition $s_\tau(\tau+2)=0$, we obtain

$$\int_{\tau+1}^{\tau+2} c_{2,\tau}(t) e^{-\int_{\tau+1}^{t}\hat{r}_{\tau+1}(\epsilon)d\epsilon} dt = s_\tau(\tau+1)$$

Therefore, the present value of the second-period consumption equals the optimal savings when the consumer retires.

However, in the feasibility constraint, we are interested in the consumption of generation $\tau-1$, and we need to take into account the population growth. Therefore, if we introduce a lag, multiply both sides by $e^{r_\tau - n}$, the second constraint follows

$$e^{-n}\int_{\tau}^{\tau+1} c_{2,\tau-1}(t) e^{r_\tau-\int_{\tau}^{t}\hat{r}_{\tau}(\epsilon)d\epsilon} dt = e^{r_\tau-n} s_{\tau-1}(\tau)$$

We can now combine the stock version of the budget constraints

$$\begin{aligned} & w_\tau + e^{r_\tau-n}s_{\tau-1}(\tau) = \\ & \int_{\tau}^{\tau+1} c_{1,\tau}(t)\phi(t-\tau)e^{-\int_{\tau}^{t}\hat{r}_\tau(\epsilon)d\epsilon}dt + s_\tau(\tau+1)e^{-\int_{\tau}^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon} + e^{-n}\int_{\tau}^{\tau+1} c_{2,\tau-1}(t)e^{r_\tau-\int_{\tau}^{t}\hat{r}_\tau(\epsilon)d\epsilon}dt \end{aligned}$$

By using the capital market clearing condition (E.2) and the properties of the production function, the feasibility constraint obtains

$$f(k_\tau) = \int_{\tau}^{\tau+1} c_{1,\tau}(t)\phi(t-\tau)e^{-\int_{\tau}^{t}\hat{r}_\tau(\epsilon)d\epsilon}dt + s_\tau(\tau+1)e^{-\int_{\tau}^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon} + e^{-n}\int_{\tau}^{\tau+1} c_{2,\tau-1}(t)e^{r_\tau-\int_{\tau}^{t}\hat{r}_\tau(\epsilon)d\epsilon}dt \tag{E.3}$$

An equilibrium for the overlapping generation economy with consumption shocks is defined as follows

Definition. *A competitive equilibrium is a sequence of individual consumption and savings choices,* $\{c_{1,\tau}(t), c_{2,\tau}(t), s_\tau(\tau+1)\}_{\tau=0}^{\infty}$, *and factor prices,* $\{R_\tau, \omega_\tau\}_{\tau=0}^{\infty}$, *such that: (i) Consumers of each generation solve (2.3); (ii) Firms optimize according to (E.1); (iii) The feasibility condition (E.3) is satisfied.*

Social planner

A social planner with a discount rate β maximizes a weighted average of the utilities

$$\max_{c_{1,\tau}(t),c_{2,\tau}(t),k_\tau} \quad \sum_{\tau\in\mathbb{N}} \beta^{\tau-1}\left(\int_{\tau}^{\tau+1} \ln\left(c_{1,\tau}(t)\right)\phi(t-\tau)dt + \int_{\tau+1}^{\tau+2} e^{-\rho(t-(\tau+1))}\ln\left(c_{2,\tau}(t)\right)dt\right)$$

$$\text{subject to} \quad f(k_\tau) = \int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t-\tau)e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt + s_\tau(\tau+1)e^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon} + e^{-n}\int_\tau^{\tau+1} c_{2,\tau-1}(t)e^{r_\tau-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt$$

$$L_{\tau+1} = e^n L_\tau$$

$$L_0, K_0 \quad given \tag{E.4}$$

Define the total amount of resources consumed at τ as $c_\tau \equiv \int_\tau^{\tau+1} c_{1,\tau}(t)\phi(t-\tau)e^{-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt + e^{-n}\int_\tau^{\tau+1} c_{2,\tau-1}(t)e^{r_\tau-\int_\tau^t \hat{r}_\tau(\epsilon)d\epsilon}dt$. The budget constraint can be rewritten as

$$f(e^{-n}s_{\tau-1}(\tau)) = c_\tau + s_\tau(\tau+1)e^{-\int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon}$$

In a steady state $s_{\tau-1}(\tau) \equiv s_\tau(\tau+1) \equiv s^*$. It follows that

$$f(e^{-n}s^*) = c + s^* e^{-\int_\tau^{\tau+1}\hat{r}(\epsilon)d\epsilon}$$

A social planner maximizes overall consumption choosing a s^*, leading to

$$e^{r+\int_\tau^{\tau+1}\hat{r}(\epsilon)d\epsilon} = e^n$$

Consumption is maximized when $r + \int_\tau^{\tau+1}\hat{r}(\epsilon)d\epsilon = n$. (In Samuelson (1975) consumption is maximized when $r = n$.) The condition states that consumption is maximized when the population growth rate equals the rate of return of capital r plus the intra-period rate, $\int_\tau^{\tau+1}\hat{r}(\epsilon)d\epsilon$. The condition does not necessarily have to hold: population growth might be particularly low (as in the developed countries), or capital remuneration can be particularly low (as in the presence of low productivity); which leaves room for policy intervention.

Pensions

Let ψ_τ be the taxes paid by consumers of generation τ. The new initial condition is $s_\tau(\tau) = \omega_\tau - \psi_\tau$. In a funded case the return of the pension benefit corresponds to the market capital return, $e^{r_\tau}\psi_\tau$; in a pay-as-you go case to the population growth $e^n\psi_\tau$.

Proposition 8. *A funded social security increases both capital stock and consumption; while a pay-as-you-go scheme decreases consumption when $n < \int_\tau^{\tau+1}\hat{r}_\tau(\epsilon)d\epsilon$, and always decreases capital stock.*

Proof. From the proof of Proposition 5, but with $\pi_\tau = 0$. □

Chapter 3

Public bequests as a pension scheme

Abstract Governments have recently phased out pay-as-you-go pension schemes in favor of funded ones. Instead of dropping the intergenerational transfers of the pay-you-go pensions, in this chapter we propose to invert them. Resources would flow from the old generations to the young ones as if it were a system of public bequests. We show that the public bequests scheme outperforms the canonical pension schemes.

Keywords: Overlapping Generations - Pensions - Bequests.

JEL Classification: D90, E21, H55, O40[1]

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3.1 Introduction

An aging population may compromise the financial sustainability of a pay-as-you-go pension system. For this reason, some countries have started moving from a pay-as-you-go pension scheme to a funded one. However, this shift drops a source of intergenerational transfers, switching off an intergenerational risk-sharing mechanism.

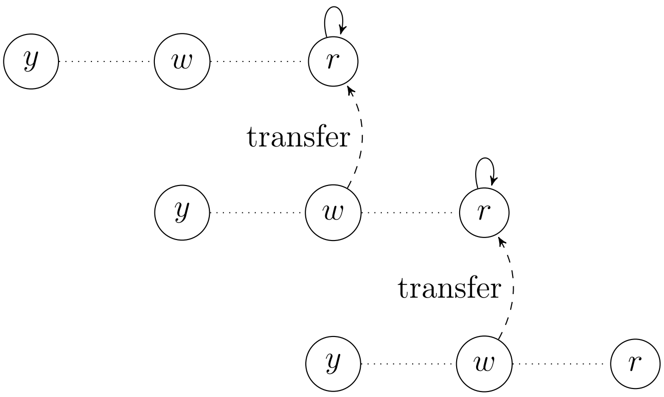
In this chapter, we address the problem of reforming a pension system by proposing to invert the intergenerational transfers. Instead of paying contributions for those already retired, workers would pay the future pensions of those still young, as if the pension scheme were a system of public bequests. We show that such a pension scheme leads to the optimal allocation of resources preserving intergenerational risk-sharing.

Figure 3.1 gives the intuition. Suppose a consumer who lives three periods: a young stage y; a working stage w; and a retired stage r. The dashed arrow represents the transfer of the pension scheme, while the second arrow points to the period where the resources transferred are consumed. A pay-as-you-go scheme links different generations, but the old generation immediately consumes the resources transferred; while a funded pension links different periods but within a single generation. Therefore, the two canonical pension schemes either link different generations or different periods. Instead, a public bequests scheme offers both links.

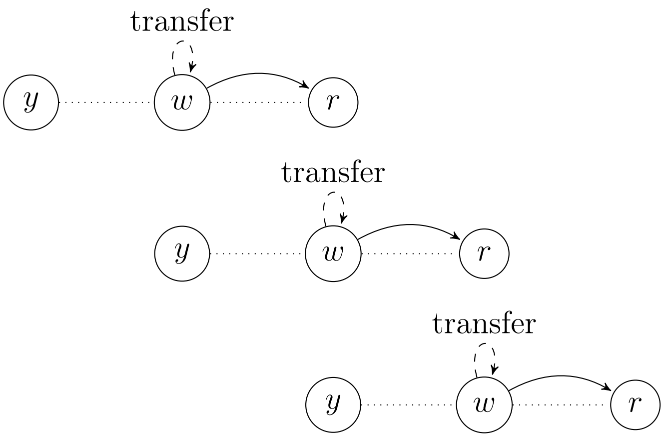
We formalize these intuitions in a three-period overlapping generations model, as in Boldrin and Montes (2005): the young generation invests in human capital; the middle-aged group works; the old one is retired. We show that the public bequests scheme maintains both capital accumulation and intergenerational risk-sharing. We first study the public bequests scheme in an economy without income shocks to clarify that dynamic efficiency drives the results, and then we introduce income shocks to study intergenerational risk-sharing.

This chapter is part of a growing literature studying the connections between public education and pensions schemes, e.g. Glomm and Kaganovich (2008) and Del Rey and Lopez-Garcia (2016).

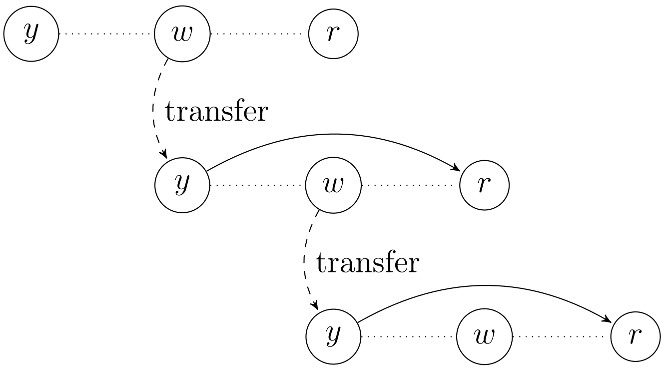
Kaganovich and Zilcha (1999) shows that if there exists a high degree of parents' altruism, education subsidies do not crowd out private investments and a pay-as-you-go system can improve growth and welfare. However, this result is susceptible to the choice of the parameters. Also, a combination of education



Pay-as-you-go scheme intergenerational links



Fully funded scheme intra-generational links



Public Bequests scheme intra- and inter-generational links.

Figure 3.1: Pension Schemes intra- and inter- generational links.

subsidies, fertility, and pensions produces mixed results depending on parameters' choice - see Rojas (2004), Cremer et al. (2011) and Yew and Zhang (2013). Given the severe sensitivity of the results on the parameters, we abstract away from fertility and altruism choices[2].

Boldrin and Montes (2005) shows how a system of public education and pay-as-you-go pensions restores the competitive market allocation when credit markets for financing education are absent. Docquier et al. (2007) points out that the allocation restored in Boldrin and Montes (2005) is still not optimal. (We report Docquier et al. (2007) social planner problem in the appendix and use it for our definition of optimality.) While Del Rey and Lopez-Garcia (2013) finds the optimal policy in a setting similar to Boldrin and Montes (2005), but assuming the existence of credit markets for financing education. We also assume perfect credit markets and discuss the assumption in the final remarks.

The next section introduces the three-period overlapping generations model without income shocks. Sections 3.3 analyzes the economic effects of the three - pay-as-you-go, funded and public bequests - pension schemes. We introduce the income shocks in Section 3.4 and conclude with some final remarks on the role of intergenerational transfers in human capital investments in Section 3.5.

3.2 An economy without uncertainty

We now modify Boldrin and Montes (2005) by introducing population growth. We assume that L_τ individuals form generation τ, and that population grows at a constant rate $n \in [-1, \infty)$, $L_\tau = (1+n)^\tau L_0$, with $L_0 = 1$ for simplicity.

The economy is composed of competitive firms producing a homogeneous good and consumers living three periods. We will now formalize production, agents' consumption-savings problem, the balanced growth path and the social planner solution.

3.2.1 Firms

Firms are competitive and produce a homogeneous good, employing physical and human capital. The production of human capital is a function of the stock of hu-

[2] We also abstract away from elastic labor supply as results would not qualitatively change. Moreover, we use a unitary model as collective models satisfying Slutsky conditions would lead to similar results, see Browning et al. (2006).

man capital inherited from the previous generation and investments in it[3]. Therefore, the production functions for the commodity and the human capital are

$$f(k_\tau, h_\tau) = k_\tau^\alpha h_\tau^{1-\alpha} \tag{3.1}$$

$$h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta} \tag{3.2}$$

where $k_\tau = K_\tau/L_\tau$ is the per-capita physical capital; $h_\tau = H_\tau/L_\tau$ is the per-capita human capital; $h_{\tau-1}$ is the human capital inherited; e_τ represents the investments in education; α and β are the Cobb-Douglas elasticities.

Firms remunerate human and physical capital respectively paying a wage w_τ and an interest rate $1 + r_\tau$. Since markets are competitive, factors' remuneration follows from (3.1)

$$\frac{df(k_\tau, h_\tau)}{dh_\tau} = w_\tau = (1-\alpha)(k_\tau/h_\tau)^\alpha \tag{3.3}$$

$$\frac{df(k_\tau, h_\tau)}{dk_\tau} = 1 + r_\tau = \alpha(k_\tau/h_\tau)^{\alpha-1} \tag{3.4}$$

3.2.2 Consumers

Each consumer of generation $\tau \in \mathbb{N}$ lives for three periods: in the first period he invests in human capital; in the second he works, receives a wage and consumes part of his wealth; in the last period he retires and consumes his savings.

Then, given an initial stock of human capital, the consumption-savings problem for each generation τ is

$$\begin{array}{ll} \max\limits_{s_\tau, e_\tau} & \ln(c_{1,\tau}) + \gamma \ln(c_{2,\tau}) \\ \text{subject to} & c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau \leq w_\tau h_\tau \\ & c_{2,\tau} \leq (1 + r_{\tau+1})s_\tau \\ & h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta} \end{array} \tag{3.5}$$

where $\gamma > 0$ is the discounting factor; s_τ are the saving; e_τ are the investments in human capital; $c_{1,\tau}$ and $c_{2,\tau}$ are respectively middle-aged and old consumption.

From the first order conditions, savings and human capital investments follow

[3] As in Uzawa (1965) and Lucas (1988) the production of human capital contributes to the stock of knowledge of all generations, making future generations more productive. Each generation may discard some obsolete knowledge or physical input but, for simplicity, we omit the depreciation rate.

$$s_\tau = \frac{\gamma}{1+\gamma}\left[w_\tau h_\tau - (1+r_\tau)e_\tau\right] \tag{3.6}$$

$$e_\tau = \frac{1-\alpha}{\alpha}\beta k_\tau \tag{3.7}$$

3.2.3 Feasibility constraint and equilibrium

Contrary to Boldrin and Montes (2005), we assume that a credit market for financing investments in human capital exists[4], and comment this point in the final remarks. Then, savings can be used as an input in both commodity and human capital production

$$s_\tau = (1+n)(k_{\tau+1} + e_{\tau+1}) \tag{3.8}$$

from which, together with the individual budget constraints, the feasibility constraint follows

$$c_{1,\tau} + s_\tau + \frac{c_{2,\tau-1}}{1+n} \leq f(k_\tau, h_\tau) \tag{3.9}$$

An equilibrium for this economy is defined as follows

Definition. *A competitive equilibrium is a sequence of individual savings and education choices, $\{s_\tau, e_\tau\}_{\tau=0}^{\infty}$, and factor prices, $\{r_\tau, w_\tau\}_{\tau=0}^{\infty}$, such that: (i) Consumers of each generation solve (3.5); (ii) Firms optimize according to (3.3) and (3.4); (iii) The feasibility condition (3.9) is satisfied.*

3.2.4 Growth and Social Planner

The Balanced Growth Path (BGP) of this economy is a constant (the derivation is in Appendix F.1)

$$\frac{k_\tau}{h_\tau} = \left[\frac{\eta}{\left(\beta\frac{1-\alpha}{\alpha}\eta\right)^\beta}\right]^{\frac{1}{1-\alpha(1-\beta)}} \tag{3.10}$$

where $\eta \equiv \frac{\gamma}{1+\gamma}\frac{(1-\alpha)(1-\beta)}{(1+n)\left(1+\frac{1-\alpha}{\alpha}\beta\right)}$ is the factor at which previous physical and human capital is converted into physical capital for the future generation, i.e. $k_{\tau+1} = \eta k_\tau^\alpha h_\tau^{1-\alpha}$.

[4] If credit markets for education do not exist, then consumers cannot borrow resources when young - and without an income - and $e_\tau = 0$.

The growth rate of both capital stocks along the balanced growth path is

$$g = \frac{k_{\tau+1} - k_\tau}{k_\tau} = \frac{\eta k_\tau^\alpha h_\tau^{1-\alpha}}{k_\tau} - 1$$

from which, using (3.10), we obtain

$$1 + g = \eta \left[\frac{\left(\beta \frac{1-\alpha}{\alpha} \eta\right)^\beta}{\eta} \right]^{\frac{1-\alpha}{1-\alpha(1-\beta)}} \tag{3.11}$$

We achieve equilibrium dynamic efficiency[5] if there is no capital over-accumulation, i.e. if

$$\frac{1 + r_\tau}{1 + n} = 1 + g \tag{3.12}$$

The social planner problem is as in Docquier et al. (2007) and, for completeness, we report the problem in Appendix F.2, where we also derive (3.12). Docquier et al. (2007) also emphasizes that consumers under-invests in human capital since the optimal solution is

$$e_\tau = \beta \frac{1-\alpha}{\alpha} k_\tau + (1 - \beta) \frac{1 + n}{1 + r_\tau} e_{\tau+1} \tag{3.13}$$

Therefore, consumers do not consider the positive externalities of their education on future generations, leaving room for policy interventions. Now that we have a stylized economy, we can show the economic effects of the pension schemes.

3.3 Pensions

In this section, we analyze the economic effects of the different pension schemes in the overlapping generations economy. We find that income taxation exasperates underinvestment in human capital and that a public bequests scheme does not alter capital accumulation.

Any pension scheme introduces an income tax, $0 < \psi < 1$, and offers a benefit ζ when a consumer retires. Then, the new consumer maximization problem becomes

[5] From (3.4) and (3.11), dynamic efficiency is reached, if $\alpha = \eta(1 + n)$; which means that, given how η is defined, $1 + n$ simplifies and efficiency only depends on α, β, and γ.

$$
\begin{aligned}
\max_{s_\tau, e_\tau} \quad & \ln(c_{1,\tau}) + \gamma \ln(c_{2,\tau}) \\
\text{subject to} \quad & c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau \leq (1-\psi)(w_\tau h_\tau) \\
& c_{2,\tau} \leq (1+r_{\tau+1})s_\tau + \zeta \\
& h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta}
\end{aligned}
$$

Each pension scheme offers a different benefit: the return of a pay-as-you-go scheme equals the population growth rate; the return of a funded scheme equals capital remuneration; the return of a public bequest equals capital remuneration compounded on two periods (remember Figure 3.1) and adjusted by the population growth. Therefore, for each pension scheme the benefit ζ is

$$
\zeta = \begin{cases} (1+n)\psi(w_{\tau+1}h_{\tau+1}) & \text{if pay-as-you-go} \\ (1+r_{\tau+1})\psi(w_\tau h_\tau) & \text{if funded} \\ \frac{(1+r_{\tau+1})(1+r_\tau)}{1+n}\psi(w_{\tau-1}h_{\tau-1}) & \text{if public bequests} \end{cases} \tag{3.14}
$$

By solving the consumer problem for each of the pension schemes, the following results follow

Proposition 9. *With respect to the market allocation:*

- *A pay-as-you-go scheme decreases the resources available for investments, and also human capital investments;*
- *A funded scheme affects neither the resources available nor human capital investments;*
- *A public bequests scheme does not affect the resources available for investments but decreases investments in human capital.*

Proof. In Appendix F.3. □

The effects of the canonical pension schemes are as expected. A pay-as-you-go scheme crowds out savings displacing capital, as the benefit depends on the income of another generation; which does not happen in the funded scheme where the benefit depends on the income of the same consumer.

Instead, the effects of the public bequest scheme on physical capital follow from the equilibrium dynamic efficiency condition (1.8), from which we derive

$$
\begin{aligned}
\frac{1+r_\tau}{1+n}\psi w_{\tau-1}h_{\tau-1} &= \frac{\alpha k_\tau^{\alpha-1}h_\tau^{1-\alpha}}{1+n}\psi(1-\alpha)k_{\tau-1}^{\alpha}h_{\tau-1}^{1-\alpha} \\
&= \frac{1-\alpha}{1+n}\psi\frac{(1+r_{\tau-1})k_{\tau-1}}{k_\tau}k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \frac{1-\alpha}{1+n}\psi\frac{(1+r_{\tau-1})}{1+g}k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \frac{1-\alpha}{1+n}\psi(1+n)k_\tau^\alpha h_\tau^{1-\alpha} \\
&= (1-\alpha)\psi k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \psi w_\tau h_\tau
\end{aligned}
$$

Therefore, the equilibrium dynamic efficiency condition allows us to transform the public bequest benefit into the funded one, where the longer compounding of the public bequest adds $(1+r_{\tau+1})$ to the derivation above, completing the equivalence of the benefits. Thus, the effect of a public bequest scheme on capital accumulation is neutral as in a funded pension scheme.

Pay-as-you-go and public bequests schemes worsen the externalities on human capital investments since their benefits depend on the investment in human capital of another generation. A flat tax would not exasperate the human capital investment distortions, but it would also not allow us to address intergenerational risk-sharing, as we will show in the next section.

There are many ways to fix education externalities and, henceforth, we simply assume that education is mandatory and that e_τ satisfies (3.13). However, in Appendix G we show how to reach an equivalent result by making pension benefits contingent on human capital investments.

By assuming mandatory education which fixes the human capital externalities, the following results follow

Proposition 10. *After fixing education externalities:*

- *A pay-as-you-go scheme leads to a BGP lower than the optimal one;*
- *A funded scheme leads to the optimal BGP;*
- *A public bequests scheme leads to the optimal BGP.*

Proof. In Appendix F.4. □

The proof follows from the different effects of the pension schemes on savings and capital accumulation. Since the balanced growth path is the ratio between physical and human capital, by fixing the education externalities we keep the denominator constant and change the numerator with the pension scheme.

The results of this section emphasize the role of equilibrium dynamic efficiency and extended compounding on capital accumulation under a public bequest pension scheme. Once we fix education externalities, an optimal balanced growth path follows. However, in the next section, we show how the public bequests scheme outperforms the funded one thanks to intergenerational transfers.

3.4 An economy with uncertainty

Until now we have not defined a taxation-benefit value for the pension scheme as ψ has been taken as given. However, the government can use ψ to minimize income volatility between different generations. In this section, we show how the public bequests scheme outperforms the canonical ones in the presence of income shocks.

3.4.1 The model with uncertainty

A pension scheme can act as a social insurance against an income shock each generation may face. We model the risk by introducing income uncertainty in the second period of life. We also assume that the expected value of the income shock is 0, avoiding effects on human capital investments.

The setup of Section 3.2 remains unchanged except for labor income, which is now defined as $w_\tau h_\tau + \epsilon_\tau$, where $\epsilon_\tau \sim N(0, \sigma^2)$ represents the shock each consumer (representing his own generation) may face. The consumption-savings problem becomes

$$\begin{array}{ll} \max\limits_{s_\tau, e_\tau} & \mathbf{E}_{\epsilon_\tau}\left(\ln(c_{1,\tau})\right) + \gamma \ln(c_{2,\tau}) \\ \text{subject to} & c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau \leq w_\tau h_\tau + \epsilon_\tau \\ & c_{2,\tau} \leq (1+r_{\tau+1})s_\tau \\ & \epsilon_\tau \sim N\left(0, \sigma^2\right) \\ & h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta} \end{array} \tag{3.15}$$

Since the expected value of the shock is 0, the first order conditions generate the same results of the model without shocks. On the other hand, we can now compute the lifespan consumption volatility

$$\text{Var}\left(c_{1,\tau} + \frac{c_{2,\tau}}{1+r_{\tau+1}}\right) = \text{Var}\left(w_\tau h_\tau - (1+r_\tau)e_\tau + \epsilon_\tau\right) = \sigma^2 \tag{3.16}$$

Firms are still in a competitive setting and solve

$$\max_{k_\tau, h_\tau} f(k_\tau, h_\tau) - w_\tau h_\tau - (1+r_\tau)k_\tau \tag{3.17}$$

Since the expected value of the shock is 0, firm's optimization does not change, and factor prices are still as in (3.3) and (3.4).

From the budget constraint of the consumer and the capital market condition (3.8), the feasibility constraint follows

$$c_{1,\tau} + s_\tau + \frac{c_{2,\tau-1}}{1+n} \leq f(k_\tau, h_\tau) + \epsilon_\tau \tag{3.18}$$

which shows how the income shocks are equivalent to production shocks.

An equilibrium for this economy is defined as follows

Definition. *A competitive equilibrium with uncertainty is a sequence of individual savings and education choices, $\{s_\tau, e_\tau\}_{\tau=0}^{\infty}$, factor prices, $\{r_\tau, w_\tau\}_{\tau=0}^{\infty}$, and exogenous shocks $\{\epsilon_\tau\}_{\tau=0}^{\infty}$ such that: (i) Consumers of each generation solve (3.15); (ii) Consumption volatility satisfies (3.16); (iii) Firms solve (3.17); (iv) The feasibility condition (3.18) is satisfied.*

3.4.2 Social Planner

The shock ϵ hits every single individual (representing his generation), and its expected value is 0. By the law of large numbers, a social planner can solve the allocation problem as if there are no shocks, transferring resources from the generations who experienced a positive shock to those who had a negative one. Formally

$$\int_0^\infty c_{1,\tau} d\tau + \int_0^\infty s_\tau d\tau + \int_0^\infty \frac{c_{2,\tau-1}}{1+n} d\tau \leq \int_0^\infty f(k_\tau, h_\tau) d\tau + \int_0^\infty \epsilon_\tau d\tau$$

By construction, $\int_0^\infty \epsilon_\tau d\tau = 0$. Therefore, we can simplify the social planner feasibility constraint in

$$f(k_\tau, h_\tau) = c_{1,\tau} + \frac{c_{2,\tau-1}}{1+n} - (1+n)(e_{\tau+1} + k_{\tau+1}) \tag{3.19}$$

Therefore, the social planner problem is still as in Docquier et al. (2007). The difference between the competitive and the optimal allocation leaves room for policy intervention.

3.4.3 Pensions

As before, any pension scheme introduces an income tax, $0 < \psi < 1$, and offers a benefit ζ depending on the pension scheme. Then, the new consumer maximization problem becomes

$$\begin{aligned}
\max_{s_\tau} \quad & \ln(c_{1,\tau}) + \gamma \ln(c_{2,\tau}) \\
\text{subject to} \quad & c_{1,\tau} + s_\tau + (1 + r_\tau)e_\tau \leq (1 - \psi)(w_\tau h_\tau + \epsilon_\tau) \\
& c_{2,\tau} \leq (1 + r_{\tau+1})s_\tau + \zeta \\
& h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta} \\
& \epsilon_\tau, \epsilon_{\tau+1} \sim i.i.d.N\left(0, \sigma^2\right)
\end{aligned}$$

where the pension benefits are

$$\zeta = \begin{cases} (1+n)\psi(w_{\tau+1}h_{\tau+1} + \epsilon_{\tau+1}) & \text{if pay-as-you-go} \\ (1 + r_{\tau+1})\psi(w_\tau h_\tau + \epsilon_\tau) & \text{if funded} \\ \frac{(1+r_{\tau+1})(1+r_\tau)}{1+n}\psi(w_{\tau-1}h_{\tau-1} + \epsilon_{\tau-1}) & \text{if public bequests} \end{cases}$$

from which also the pension benefit is affected by the shock. The introduction of uncertainty allows us to use the pension schemes as an intergenerational risk-sharing mechanism. In Boldrin and Montes (2005) the pension scheme already defines a system of loans-repayments, while our approach leaves the policy instrument free to address uncertainty. Therefore,

Proposition 11. *After fixing education externalities:*

- *A pay-as-you-go scheme reduces income volatility;*
- *A funded scheme does not reduce income volatility;*
- *A public bequests scheme reduces income volatility.*

Proof. In Appendix F.5. □

As expected the funded scheme cannot be used to address the intergenerational risk since it offers only intragenerational transfers. Instead, the pay-as-you-go and the public bequests scheme both reduce the intergenerational risk by offering intergenerational transfers.

Propositions 10 and 11 together show that a public bequests scheme does not alter capital accumulation and constitutes a risk-sharing mechanism. Therefore, it combines the two main elements of the canonical schemes by offering both inter- and intra-generational links. However, as shown in the proof of Proposition 11, in the presence of uncertainty the public bequests system can only restore the second-best allocation as it links the generations pairwise, while the social planner can transfer the shocks across all generations.

3.5 Final Remarks

Boldrin and Montes (2005) formalizes how education subsidies are analogous to a loan granted by the middle-aged generation to the young one. When the young "borrowing" generation starts working, the middle-aged "lender" generation retires. Thus, a pay-as-you-go pension scheme becomes a system to repay the loan to the previous generation. In our framework, this system of loans-repayments fails. Public pensions are designed to reduce income uncertainty, instead of repaying the "loan" used to finance education.

Also, Boldrin and Montes (2005) justifies the analysis arguing that credit markets for financing investments in human capital do not exist. In a public bequests pension scheme, funds are set up at consumer "birth" and they are available when consumers invest in human capital. Therefore, consumers could use the fund to finance investments in human capital in two ways: they can borrow resources from it and they can use it as a collateral.

In the U.S. workers can borrow some resources from their pension fund set under the retirement savings plan 401(k), but they cannot use the fund as collateral. In the public bequest pension scheme, it is possible to imagine a similar mechanism, solving the absence of a credit market for education.

Therefore, since consumers could borrow from the fund, the public bequests pension scheme can be broadly interpreted as an education subsidy when the credit markets for financing human capital are imperfect.

To conclude, whether the transition to a public bequests scheme is politically sustainable in an aging population remains an open question. However, the transition would probably rely on the degree of altruism, as the share of young voters (who would immediately benefit from the transition) would be a minority.

Appendix F

Derivations and proofs

F.1 Balanced Growth Path derivation

Starting from physical capital, if we combine factors remuneration, (3.3) and (3.4), with savings and education conditions, (3.6) and (3.8), we obtain

$$\begin{aligned}(1+n)k_{\tau+1} &= s_\tau - (1+n)e_{\tau+1} \\ &= \frac{\gamma}{1+\gamma}\left[(1-\alpha)(k_\tau/h_\tau)^\alpha h_\tau - \alpha(k_\tau/h_\tau)^{\alpha-1}\frac{1-\alpha}{\alpha}\beta k_\tau\right] - (1+n)\frac{1-\alpha}{\alpha}\beta k_{\tau+1}\end{aligned}$$

from which

$$k_{\tau+1} = \frac{\gamma}{1+\gamma}\frac{(1-\alpha)(1-\beta)}{(1+n)\left(1+\frac{1-\alpha}{\alpha}\beta\right)}k_\tau^\alpha h_\tau^{1-\alpha} \tag{F.1}$$

We define η the rate converting previous physical and human capital into physical capital for the future generation

$$\eta \equiv \frac{\gamma}{1+\gamma}\frac{(1-\alpha)(1-\beta)}{(1+n)\left(1+\frac{1-\alpha}{\alpha}\beta\right)}$$

Using the production function of human capital (3.2), optimal investments in it (3.7), and the level of physical capital (F.1), we obtain the level of human capital

$$h_{\tau+1} = \left(\beta\frac{1-\alpha}{\alpha}\eta\right)^\beta k_\tau^{\alpha\beta}h_\tau^{1-\alpha\beta} \tag{F.2}$$

The balanced growth path equation follows from (F.1) and (F.2)

$$\frac{k_\tau}{h_\tau} = \left[\frac{\eta}{\left(\beta\frac{1-\alpha}{\alpha}\eta\right)^\beta}\right]^{\frac{1}{1-\alpha(1-\beta)}}$$

F.2 Social Planner problem

The social planner feasibility constraint in per-capita terms in each period is

$$f(k_\tau, h_\tau) = c_{1,\tau} + \frac{c_{2,\tau-1}}{1+n} - (1+n)(e_{\tau+1} + k_{\tau+1}) \tag{F.3}$$

The Lagrangian of the social planner is

$$\begin{aligned}\sum_{\tau=0}^{\infty} \hat{\gamma}^\tau \Bigg\{ & \ln(c_{1,\tau}) + \gamma \ln(c_{2,\tau}) \\ & + \lambda_{1,\tau} \left[f(k_\tau, h_\tau) - c_{1,\tau} - \frac{c_{2,\tau-1}}{1+n} - (1+n)(e_{\tau+1} + k_{\tau+1}) \right] \\ & + \lambda_{2,\tau} \left(e_\tau^\beta h_{\tau-1}^{1-\beta} - h_\tau \right) \Bigg\}\end{aligned} \tag{F.4}$$

Where $\lambda_{1,\tau}$ and $\lambda_{2,\tau}$ are the Lagrangian operators, and $\hat{\gamma}$ is the intergenerational discount rate of the social planner. From the first order conditions we obtain

$$\partial c_{1,\tau} = 0 : \frac{1}{c_{1,\tau}} = \lambda_{1,\tau} \tag{F.5a}$$

$$\partial c_{2,\tau-1} = 0 : \frac{\gamma}{c_{2,\tau-1}} = \frac{\hat{\gamma}\lambda_{1,\tau}}{1+n} \tag{F.5b}$$

$$\partial e_\tau = 0 : \lambda_{1,\tau}(1+n) = \lambda_{2,\tau}\beta e_\tau^{\beta-1} h_{\tau-1}^{1-\beta} \tag{F.5c}$$

$$\partial k_{\tau+1} = 0 : \lambda_{1,\tau}(1+n) = \hat{\gamma}\lambda_{1,\tau+1}(1+r_\tau) \tag{F.5d}$$

$$\partial h_{\tau+1} = 0 : \hat{\gamma}\lambda_{1,\tau+1} w_\tau + \hat{\gamma}\lambda_{2,\tau+1}(1-\beta) e_{\tau+1}^\beta h_\tau^{-\beta} = \lambda_{2,\tau} \tag{F.5e}$$

From condition (F.5b) we also obtain

$$\partial c_{2,\tau} = 0 : \frac{\gamma}{c_{2,\tau}} = \frac{\hat{\gamma}\lambda_{1,\tau+1}}{1+n}$$

Combining this with (F.5a) and (F.5d), we obtain

$$\frac{c_{2,\tau}}{c_{1,\tau}} = \gamma(1+r_\tau)$$

which gives the optimal savings stock and is satisfied in the competitive case. Then, savings satisfy (3.6) also in the social planner allocation.

From the derivative with respect to capital, (F.5d), we obtain the optimal accumulation of physical capital, from which we derive that there is no capital over-accumulation along the balanced growth path if

$$\frac{\hat{\gamma}(1+r_\tau)}{1+n} = \frac{\lambda_{1,\tau}}{\lambda_{1,\tau+1}} = \frac{c_{1,\tau+1}}{c_{1,\tau}} = 1+g \tag{F.6}$$

This is the condition reported in the Section 3.2 but with $\hat{\gamma} = 1$.

Finally, from the last optimal condition (F.6), and the derivative with respect to the investments in human capital (F.5e), we can derive the optimal accumulation of human capital

$$\hat{\gamma} w_\tau + \hat{\gamma}\frac{\lambda_{2,\tau+1}}{\lambda_{1,\tau+1}}(1-\beta)e_{\tau+1}^{\beta}h_\tau^{-\beta} = \frac{\lambda_{2,\tau}}{\lambda_{1,\tau+1}}$$

Updating the derivative with respect to e_τ (F.5c), we obtain $\frac{\lambda_{2,\tau+1}}{\lambda_{1,\tau+1}} = \frac{1+n}{\beta e_{\tau+1}^{\beta-1}h_\tau^{1-\beta}}$; and combining (F.5c) with (F.5d), we have $\frac{\lambda_{2,\tau}}{\lambda_{1,\tau+1}} = \frac{\hat{\gamma}(1+r_\tau)}{\beta e_\tau^{\beta-1}h_{\tau-1}^{1-\beta}} = \frac{\hat{\gamma}(1+r_\tau)e_\tau}{\beta h_\tau}$. We can use these quotients in the previous equation to obtain

$$w_\tau + \frac{1+n}{\beta e_{\tau+1}^{\beta-1}h_\tau^{1-\beta}}(1-\beta)e_{\tau+1}^{\beta}h_\tau^{-\beta} = \frac{(1+r_\tau)e_\tau}{\beta h_\tau}$$

from which

$$\beta h_\tau w_\tau + (1-\beta)(1+n)e_{\tau+1} = (1+r_\tau)e_\tau$$

using factors remuneration the optimal level of investments in human capital obtains

$$e_\tau = \beta\frac{1-\alpha}{\alpha}k_\tau + (1-\beta)\frac{1+n}{1+r_\tau}e_{\tau+1} \tag{F.7}$$

F.3 Proof of Proposition 9

Pay-as-you-go

A pay-as-you-go pension scheme introduces a tax on individual income, $0 < \psi < 1$, in the working period of life, while offering a benefit $(1+n)\psi$ in the last period of retirement. Then, the new consumer maximization problem becomes

$$\begin{aligned}
\max_{s_\tau, e_\tau} \quad & \ln(c_{1,\tau}) + \gamma\ln(c_{2,\tau}) \\
\text{subject to} \quad & c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau \leq (1-\psi)w_\tau h_\tau \\
& c_{2,\tau} \leq (1+r_{\tau+1})s_\tau + (1+n)\psi w_{\tau+1}h_{\tau+1} \\
& h_\tau = e_\tau^{\beta}h_{\tau-1}^{1-\beta}
\end{aligned} \tag{F.8}$$

From the first order conditions, we obtain

$$s_\tau = \frac{\gamma}{1+\gamma}\left[\omega_\tau h_\tau(1-\psi) - (1+r_\tau)e_\tau\right] - \frac{1+n}{1+\gamma}\frac{w_{\tau+1}}{1+r_{\tau+1}}\psi h_{\tau+1} \tag{F.9}$$

$$e_\tau = \frac{1-\alpha}{\alpha}\beta(1-\psi)k_\tau \tag{F.10}$$

Taxes reduce wage income, reducing at the same time the expected return of human capital investment; which decreases the investments compared to the competitive case without social security (which, as a reminder, is already lower than socially optimum). We now show that the feasibility constraint is satisfied.

The capital market clearing condition is still

$$s_\tau = (1+n)(k_{\tau+1} + e_{\tau+1}) \tag{F.11}$$

From the budget constraint

$$c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau + \frac{c_{2,\tau-1}}{1+n} \leq (1-\psi)(w_\tau h_\tau + \epsilon_\tau) + (1+r_\tau)\frac{s_{\tau-1}}{1+n} + \psi w_\tau h_\tau$$

By using the capital market equilibrium condition (F.11), the feasibility constraint is satisfied

$$c_{1,\tau} + s_\tau + \frac{c_{2,\tau-1}}{1+n} \leq f(k_\tau, h_\tau)$$

Funded

A funded pension scheme introduces a tax on wages for each generation τ, $0 < \psi < 1$, offering a benefit $(1+r_{\tau+1})\psi$ when an individual of generation τ retires. The consumer problem becomes

$$\begin{array}{ll} \max\limits_{s_\tau, e_\tau} & \ln(c_{1,\tau}) + \gamma\ln(c_{2,\tau}) \\ \text{subject to} & c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau \leq (1-\psi)w_\tau h_\tau \\ & c_{2,\tau} \leq (1+r_{\tau+1})\left(s_\tau + \psi w_\tau h_\tau\right) \\ & h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta} \end{array} \tag{F.12}$$

From the first order conditions with respect to s_τ, we obtain the optimal amount of savings

$$s_\tau = \frac{\gamma}{1+\gamma}\left[\omega_\tau h_\tau(1-\psi) - (1+r_\tau)e_\tau\right] - \frac{\psi}{1+\gamma}w_\tau h_\tau \tag{F.13}$$

The first order condition with respect to e_τ is

$$\frac{\beta(1-\psi)w_\tau h_\tau/e_\tau - (1+r_\tau)}{(1-\psi)w_\tau h_\tau - (1+r_\tau)e_\tau - s_\tau} + \frac{\beta\gamma\psi w_\tau h_\tau/e_\tau}{s_\tau + \psi w_\tau h_\tau} = 0$$

using the savings condition (F.13), we obtain

$$\frac{\beta(1-\psi)w_\tau h_\tau/e_\tau - (1+r_\tau)}{w_\tau h_\tau - (1+r_\tau)e_\tau} + \frac{\beta\gamma\psi w_\tau h_\tau/e_\tau}{\gamma w_\tau h_\tau - \gamma(1+r_\tau)e_\tau} = 0$$

In the second term γ simplifies, which leads to the same denominator of the first term. Then,

$$\beta(1-\psi)w_\tau h_\tau/e_\tau - (1+r_\tau) + \beta\psi w_\tau h_\tau/e_\tau = 0$$

Combining the market remuneration of labor (3.3) and capital (3.4), we have $\frac{w_\tau}{1+r_\tau} = \frac{1-\alpha}{\alpha}\frac{k_\tau}{h_\tau}$, leading to the optimal level of investments in education for a consumer under a funded pension scheme

$$e_\tau = \frac{1-\alpha}{\alpha}\beta k_\tau \tag{F.14}$$

which is as in the case without social security.

The optimal savings level is different from the case without pension schemes, but in a funded system the resources collected through taxation are available for investments

$$s_\tau + \psi w_\tau h_\tau = (1+n)(k_{\tau+1} + e_{\tau+1}) \tag{F.15}$$

Combining the optimal savings (F.13) and (F.15) gives back (F.1).

From the budget constraint

$$c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau + \frac{c_{2,\tau-1}}{1+n} \leq (1-\psi)w_\tau h_\tau + (1+r_\tau)\left[\frac{s_{\tau-1} + \psi w_{\tau-1}h_{\tau-1}}{1+n}\right]$$

By using the capital market equilibrium condition (F.15), the feasibility constraint is satisfied

$$c_{1,\tau} + s_\tau + \frac{c_{2,\tau-1}}{1+n} + \psi w_\tau h_\tau \leq f(k_\tau, h_\tau)$$

Public bequests

A public bequests scheme introduces a tax ψ and offers a benefit $\frac{(1+r_{\tau+1})(1+r_\tau)}{1+n}\psi$. The consumer problem becomes

$$
\begin{aligned}
\max_{s_\tau, e_\tau} \quad & \ln(c_{1,\tau}) + \gamma \ln(c_{2,\tau}) \\
\text{subject to} \quad & c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau \leq (1-\psi)w_\tau h_\tau \\
& c_{2,\tau} \leq (1+r_{\tau+1}) \left[s_\tau + \frac{1+r_\tau}{1+n}\psi w_{\tau-1} h_{\tau-1} \right] \\
& h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta}
\end{aligned}
\tag{F.16}
$$

From the first order conditions, we obtain

$$
s_\tau = \frac{\gamma}{1+\gamma}\left[\omega_\tau h_\tau (1-\psi) - (1+r_\tau)e_\tau\right] - \frac{1+r_\tau}{1+\gamma}\frac{\psi}{1+n} w_{\tau-1}h_{\tau-1} \tag{F.17}
$$

$$
e_\tau = \frac{1-\alpha}{\alpha}\beta(1-\psi)k_\tau \tag{F.18}
$$

As in the pay-as-you-go case, each consumer has an extra source of income which does not depend on his salary. At the same time, wage income is taxed, so consumers invest less in human capital compared to the case without social security.

As in a funded scheme, resources are available for investments. The capital market clearing condition is

$$
s_\tau + \frac{\psi}{1+n}(1+r_\tau)w_{\tau-1}h_{\tau-1} = (1+n)(k_{\tau+1} + e_{\tau+1}) \tag{F.19}
$$

Using (F.17) and (F.19) we obtain

$$
(1+n)(k_{\tau+1}+e_{\tau+1}) = \frac{\gamma}{1+\gamma}\left[\omega_\tau h_\tau(1-\psi) - (1+r_\tau)e_\tau\right] + \frac{\gamma}{1+\gamma}\left[\frac{(1+r_\tau)\psi w_{\tau-1}h_{\tau-1}}{1+n}\right]
$$

From the equilibrium dynamic efficiency condition (1.8), we can derive

$$
\begin{aligned}
\frac{\gamma(1+r_\tau)}{(1+\gamma)(1+n)}\psi w_{\tau-1}h_{\tau-1} &= \frac{\gamma(\alpha k_\tau^{\alpha-1}h_\tau^{1-\alpha})}{(1+\gamma)(1+n)}\psi(1-\alpha)k_{\tau-1}^{\alpha}h_{\tau-1}^{1-\alpha} \\
&= \frac{\gamma}{1+\gamma}\frac{1-\alpha}{1+n}\psi\frac{(1+r_{\tau-1})k_{\tau-1}}{k_\tau}k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \frac{\gamma}{1+\gamma}\frac{1-\alpha}{1+n}\psi\frac{(1+r_{\tau-1})}{1+g}k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \frac{\gamma}{1+\gamma}\frac{1-\alpha}{1+n}\psi(1+n)k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \frac{\gamma}{1+\gamma}(1-\alpha)\psi k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \frac{\gamma}{1+\gamma}\psi w_\tau h_\tau
\end{aligned}
$$

Therefore, on the BGP the resources available for investments equal the ones available in a funded scheme.

From the consumer budget constraint

$$
c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau + \frac{c_{2,\tau-1}}{1+n} \leq (1-\psi)w_\tau h_\tau + (1+r_\tau)\left[\frac{s_{\tau-1} + \frac{1+r_{\tau-1}}{1+n}\psi w_{\tau-2}h_{\tau-2}}{1+n}\right]
$$

By using the capital market equilibrium condition (F.19), the feasibility constraint is satisfied

$$
c_{1,\tau} + s_\tau + \frac{c_{2,\tau-1}}{1+n} + \psi w_\tau h_\tau \leq f(k_\tau, h_\tau)
$$

F.4 Proof of Proposition 10

Social Planner

The capital market clearing condition (3.8) is

$$
k_{\tau+1} = \frac{s_\tau}{1+n} - e_{\tau+1}
$$

Substituting the savings condition in the previous equation, we obtain

$$
k_{\tau+1} = \frac{\gamma}{1+\gamma}\frac{(1-\alpha)k_\tau^\alpha h_\tau^{1-\alpha}}{1+n} - \frac{\gamma}{1+\gamma}\frac{(1+r_\tau)e_\tau}{1+n} - e_{\tau+1}
$$

Using the optimal investments in education (F.7) to substitute e_τ, it follows that

$$k_{\tau+1} = \frac{\gamma}{1+\gamma}\frac{(1-\alpha)k_\tau^\alpha h_\tau^{1-\alpha}}{1+n} - \frac{\gamma}{1+\gamma}\frac{(1+r_\tau)}{1+n}\left[\beta\frac{1-\alpha}{\alpha}k_\tau + (1-\beta)\frac{1+n}{1+r_\tau}e_{\tau+1}\right] - e_{\tau+1}$$

Using the capital remuneration equilibrium $1+r_\tau = \alpha(k_\tau/h_\tau)^{\alpha-1}$, we obtain

$$k_{\tau+1} = \frac{\gamma}{1+\gamma}\frac{(1-\alpha)(1-\beta)}{1+n}k_\tau^\alpha h_\tau^{1-\alpha} - \left[\frac{\gamma}{1+\gamma}(1-\beta)+1\right]e_{\tau+1}$$

And using again the optimal investments in education (F.7) to substitute $e_{\tau+1}$, it follows that

$$k_{\tau+1} = \tfrac{\gamma}{1+\gamma}\tfrac{(1-\alpha)(1-\beta)}{1+n}k_\tau^\alpha h_\tau^{1-\alpha} - \left[\tfrac{\gamma}{1+\gamma}(1-\beta)+1\right]\left[\beta\tfrac{1-\alpha}{\alpha}k_{\tau+1} + (1-\beta)\tfrac{1+n}{1+r_{\tau+1}}e_{\tau+2}\right]$$

from which

$$k_{\tau+1} = \tfrac{\gamma}{1+\gamma}\tfrac{(1-\alpha)(1-\beta)}{(1+n)\left[1+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}\right]}k_\tau^\alpha h_\tau^{1-\alpha} - \tfrac{\frac{\gamma}{1+\gamma}(1-\beta)+1}{1+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}}\tfrac{(1-\beta)(1+n)}{1+r_{\tau+1}}e_{\tau+2}$$

Therefore, the social planner balanced growth path is

$$\begin{aligned} k_{\tau+1} &= \frac{s_\tau}{1+n} - e_{\tau+1} \\ &= \eta^{sp}k_\tau^\alpha h_\tau^{1-\alpha} - \mathbb{C} \end{aligned} \tag{F.20}$$

where

$$\eta^{sp} \equiv \frac{\gamma}{1+\gamma}\frac{(1-\alpha)(1-\beta)}{(1+n)\left[1+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}\right]}$$

is the factor at which the economy transforms previous physical and human capital into the new physical capital in the social planner allocation; and

$$\mathbb{C} \equiv \frac{\frac{\gamma}{1+\gamma}(1-\beta)+1}{1+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}}\frac{(1-\beta)(1+n)}{1+r_{\tau+1}}e_{\tau+2}$$

is the social planner adjustment to take into account education externalities.

Since the ratio $\frac{1+n}{1+r_\tau} = \frac{\hat{\gamma}}{1+g}$ equals the economic growth rate, $\mathbb{C}$is a constant on the BGP. $\mathbb{C}$ can be viewed as the social planner adjustment to take into account the externalities of human capital on future generations.

The social planner allocation of human capital is

$$\begin{aligned} h_{\tau+1} &= e_{\tau+1}^{\beta} h_{\tau}^{1-\beta} \\ &= \left[\beta \frac{1-\alpha}{\alpha} k_{\tau+1} + (1-\beta) \frac{1+n}{1+r_{\tau+1}} e_{\tau+2}\right]^{\beta} h_{\tau}^{1-\beta} \\ &= \left[\beta \frac{1-\alpha}{\alpha} \eta^{sp} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha} - \beta \frac{1-\alpha}{\alpha} \mathbb{C} + (1-\beta) \frac{1+n}{1+r_{\tau+1}} e_{\tau+2}\right]^{\beta} h_{\tau}^{1-\beta} \end{aligned} \tag{F.21}$$

The BGP is linked to how the economy converts previous capital stocks into new ones. The ratio between the two capital stocks is

$$\frac{k_{\tau+1}}{h_{\tau+1}} = \frac{\eta^{sp} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha} - \mathbb{C}}{\left[\beta \frac{1-\alpha}{\alpha} \eta^{sp} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha} - \beta \frac{1-\alpha}{\alpha} \mathbb{C} + (1-\beta) \frac{1+n}{1+r_{\tau+1}} e_{\tau+2}\right]^{\beta} h_{\tau}^{1-\beta}}$$

The derivative $\partial \frac{k_{\tau+1}}{h_{\tau+1}} / \partial \eta^{sp}$ is

$$\frac{k_{\tau}^{\alpha} h_{\tau}^{1-\alpha}\left[\beta \frac{1-\alpha}{\alpha} \eta^{sp} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha} - \beta \frac{1-\alpha}{\alpha} \mathbb{C} + (1-\beta) \frac{1+n}{1+r_{\tau+1}} e_{\tau+2}\right] - \beta(\eta^{sp} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha} - \mathbb{C})(\beta \frac{1-\alpha}{\alpha} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha})}{\left[\beta \frac{1-\alpha}{\alpha} \eta^{sp} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha} - \beta \frac{1-\alpha}{\alpha} \mathbb{C} + (1-\beta) \frac{1+n}{1+r_{\tau+1}} e_{\tau+2}\right]^{\beta+1} h_{\tau}^{1-\beta}} \tag{F.22}$$

The denominator is undoubtedly positive. Then, the numerator gives the sign of the derivative.

$$(1-\beta) k_{\tau}^{\alpha} h_{\tau}^{1-\alpha} \left[\eta^{sp} \beta \tfrac{1-\alpha}{\alpha} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha}\right] - (1-\beta) \left[\beta \tfrac{1-\alpha}{\alpha} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha} \mathbb{C}\right] + (1-\beta) \tfrac{1+n}{1+r_{\tau+1}} e_{\tau+2} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha}$$

The first term is always positive. Moreover, using the definition of $\mathbb{C}$, we can write the last two terms of the previous expression as

$$\left[1 - \frac{(1-\beta)\beta \frac{1-\alpha}{\alpha}\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)}{1+\beta \frac{1-\alpha}{\alpha}\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)}\right] \frac{(1-\beta)(1+n)}{1+r_{\tau+1}} e_{\tau+2} k_{\tau}^{\alpha} h_{\tau}^{1-\alpha}$$

which is certainly positive. The sign of the derivative is therefore positive

$$\frac{\partial \frac{k_{\tau+1}}{h_{\tau+1}}}{\partial \eta^{sp}} > 0 \tag{F.23}$$

The ratio between stocks is positively correlated to the rate at which the economy transforms previous resources into new ones. The result is not obvious since this rate also affects the accumulation of human capital.

Pay-as-you-go

The capital market clearing conditions are as in the social planner and competitive case (3.8)

$$k_{\tau+1} = \frac{s_\tau}{1+n} - e_{\tau+1}$$

Substituting the savings condition under a pay-as-you-go scheme (F.9) in the previous equation, we obtain

$$k_{\tau+1} = \frac{\gamma}{1+\gamma}\frac{(1-\alpha)k_\tau^\alpha h_\tau^{1-\alpha}}{1+n}(1-\psi) - \frac{\gamma}{1+\gamma}\frac{(1+r_\tau)e_\tau}{1+n} - \frac{\psi}{1+\gamma}\frac{1-\alpha}{\alpha}k_{\tau+1} - e_{\tau+1}$$

We are assuming mandatory education, therefore, by substituting e_τ with (F.7) it follows that

$$k_{\tau+1} = \tfrac{\gamma}{1+\gamma}\tfrac{(1-\alpha)(1-\psi)}{1+n}k_\tau^\alpha h_\tau^{1-\alpha} - \tfrac{\gamma}{1+\gamma}\tfrac{\beta(1-\alpha)}{1+n}k_\tau^\alpha h_\tau^{1-\alpha} - \tfrac{\psi}{1+\gamma}\tfrac{1-\alpha}{\alpha}k_{\tau+1} - \left[\tfrac{\gamma}{1+\gamma}(1-\beta)+1\right]e_{\tau+1}$$

Repeating the same process for $e_{\tau+1}$ leads to

$$\begin{aligned} k_{\tau+1} = \frac{\gamma}{1+\gamma}\frac{(1-\alpha)(1-\psi-\beta)}{1+n}k_\tau^\alpha h_\tau^{1-\alpha} \\ - \frac{\psi}{1+\gamma}\frac{1-\alpha}{\alpha}k_{\tau+1} - \left[\frac{\gamma}{1+\gamma}(1-\beta)+1\right]\left[\beta\frac{1-\alpha}{\alpha}k_{\tau+1} + (1-\beta)\frac{1+n}{1+r_{\tau+1}}e_{\tau+2}\right] \end{aligned}$$

from which

$$\begin{aligned} k_{\tau+1} = \frac{\gamma}{1+\gamma}\frac{(1-\alpha)(1-\beta-\psi)}{(1+n)\left[1+\frac{\psi}{1+\gamma}\frac{1-\alpha}{\alpha}+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}\right]}k_\tau^\alpha h_\tau^{1-\alpha} \\ - \frac{\frac{\gamma}{1+\gamma}(1-\beta)+1}{1+\frac{\psi}{1+\gamma}\frac{1-\alpha}{\alpha}+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}}\frac{(1-\beta)(1+n)}{1+r_{\tau+1}}e_{\tau+2} \end{aligned}$$

Therefore, the pay-as-you-go balanced growth path is

$$\begin{aligned} k_{\tau+1} &= \frac{s_\tau}{1+n} - e_{\tau+1} \\ &= \eta^{payg}k_\tau^\alpha h_\tau^{1-\alpha} - \mathbb{C}^{payg} \end{aligned} \tag{F.24}$$

where

$$\eta^{payg} \equiv \frac{\gamma}{1+\gamma} \frac{(1-\alpha)(1-\beta-\psi)}{(1+n)\left[1+\frac{\psi}{1+\gamma}\frac{1-\alpha}{\alpha}+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}\right]}$$

is the rate at which the economy transforms previous physical and human capital into new physical capital, and

$$\mathbb{C}^{payg} \equiv \frac{\frac{\gamma}{1+\gamma}(1-\beta)+1}{1+\frac{\psi}{1+\gamma}\frac{1-\alpha}{\alpha}+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}} \frac{(1-\beta)(1+n)}{1+r_{\tau+1}} e_{\tau+2}$$

is the adjustment due to human capital externalities, both defined analogously to the previous case.

From ψ, it immediately follows that

$$\eta^{payg} < \eta^{sp} \tag{F.25}$$

Under a pay-as-you-go scheme, the economy converts previous capital stock into new one at a lower rate, since the pension scheme crowds out savings.

The human capital level is

$$\begin{aligned} h_{\tau+1} &= e_{\tau+1}^{\beta} h_{\tau}^{1-\beta} \\ &= \left[\beta\frac{1-\alpha}{\alpha}k_{\tau+1} + (1-\beta)\frac{1+n}{1+r_{\tau+1}}e_{\tau+2}\right]^{\beta} h_{\tau}^{1-\beta} \\ &= \left[\beta\frac{1-\alpha}{\alpha}\eta^{payg}k_{\tau}^{\alpha}h_{\tau}^{1-\alpha} - \beta\frac{1-\alpha}{\alpha}\mathbb{C}^{payg} + (1-\beta)\frac{1+n}{1+r_{\tau+1}}e_{\tau+2}\right]^{\beta} h_{\tau}^{1-\beta} \end{aligned} \tag{F.26}$$

The pay-as-you-go stocks - (F.24) and (F.26) - and the social planner ones - (F.20) and (F.21) - have a similar structure. The only difference is that we have replaced η^{sp} with η^{payg}, and $\mathbb{C}$ with $\mathbb{C}^{payg}$.

From (F.23), if the rate at which the economy transforms old resources into new ones decreases, then the BGP decreases too. From (F.25) a pay-as-you-go pension scheme decreases the rate of conversion. Therefore, a pay-as-you-go scheme leads the economy to a rate of growth lower than the optimal one[1].

[1] $\mathbb{C}^{payg}$ is lower than $\mathbb{C}$, which might lead to an increase of the BGP since the adjustment has a negative effect on physical capital accumulation. This effect is negligible. A tax ψ alters $k_{\tau+1}$ as follows: if $\psi = 0$ we have the social planner problem; while if $\psi > 0$ we reduce the only positive term in the expression, $\frac{\gamma}{1+\gamma}\frac{(1-\alpha)k_{\tau}^{\alpha}h_{\tau}^{1-\alpha}}{1+n}$. In the extreme, inadmissible, case of $\psi = 1$, η^{payg} becomes negative while $\mathbb{C}^{payg}$ does not change sign.

Funded

Remind the capital market clearing condition in case of a funded pension scheme (F.15)

$$k_{\tau+1} = \frac{s_\tau + \psi w_\tau h_\tau}{1+n} - e_{\tau+1}$$

Using the savings condition (F.13)

$$k_{\tau+1} = \frac{\gamma}{1+\gamma}\frac{(1-\alpha)k_\tau^\alpha h_\tau^{1-\alpha}}{1+n} - \frac{\gamma}{1+\gamma}\frac{(1+r_\tau)e_\tau}{1+n} - e_{\tau+1}$$

Unsurprisingly we obtain the social planner solution. A funded scheme does not alter capital accumulation, while at the same time we corrected the human capital externality assuming mandatory education. Since also $h_{\tau+1}$ equals the social planner case (F.21), the BGP under a funded scheme and mandatory education lead to the optimal growth.

Public bequests

The capital market clearing condition in case of a public bequests pension scheme (F.19) is

$$k_{\tau+1} = \frac{s_\tau}{1+n} + \frac{(1+r_\tau)}{(1+n)^2}\psi w_{\tau-1}h_{\tau-1} - e_{\tau+1}$$

Using the savings condition (F.17), we obtain

$$k_{\tau+1} = \frac{\gamma}{(1+\gamma)(1+n)}\left[\omega_\tau h_\tau(1-\psi) - (1+r_\tau)e_\tau\right] - \frac{1+r_\tau}{(1+\gamma)(1+n)^2}\psi w_{\tau-1}h_{\tau-1} + \frac{1+r_\tau}{(1+n)^2}\psi w_{\tau-1}h_{\tau-1} - e_{\tau+1}$$

From the equilibrium dynamic efficiency condition, we can derive that

$$\begin{aligned}
\frac{\gamma(1+r_\tau)}{(1+\gamma)(1+n)^2}\psi w_{\tau-1}h_{\tau-1} &= \frac{\gamma(\alpha k_\tau^{\alpha-1}h_\tau^{1-\alpha})}{(1+\gamma)(1+n)^2}\psi(1-\alpha)k_{\tau-1}^\alpha h_{\tau-1}^{1-\alpha} \\
&= \frac{\gamma}{1+\gamma}\frac{1-\alpha}{(1+n)^2}\psi\frac{(1+r_{\tau-1})k_{\tau-1}}{k_\tau}k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \frac{\gamma}{1+\gamma}\frac{1-\alpha}{(1+n)^2}\psi\frac{(1+r_{\tau-1})}{1+g}k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \frac{\gamma}{1+\gamma}\frac{1-\alpha}{(1+n)^2}\psi(1+n)k_\tau^\alpha h_\tau^{1-\alpha} \\
&= \frac{\gamma}{1+\gamma}\frac{1-\alpha}{1+n}\psi k_\tau^\alpha h_\tau^{1-\alpha}
\end{aligned}$$

from which

$$k_{\tau+1} = \frac{\gamma}{(1+\gamma)(1+n)}(1-\alpha)(1-\psi)k_\tau^\alpha h_\tau^{1-\alpha} - \frac{\gamma}{(1+\gamma)(1+n)}(1+r_\tau)e_\tau + \frac{\gamma}{(1+\gamma)}\frac{1-\alpha}{1+n}\psi k_\tau^\alpha h_\tau^{1-\alpha} - e_{\tau+1}$$

We are assuming mandatory education, therefore, using the optimal condition (F.7) to substitute e_τ, we obtain

$$k_{\tau+1} = \frac{\gamma}{(1+\gamma)(1+n)}(1-\alpha)(1-\beta)k_\tau^\alpha h_\tau^{1-\alpha} - \left[\frac{\gamma}{1+\gamma}(1-\beta)+1\right]e_{\tau+1}$$

Doing the same for $e_{\tau+1}$, it follows that

$$k_{\tau+1} = \frac{\gamma}{(1+\gamma)(1+n)}(1-\alpha)(1-\beta)k_\tau^\alpha h_\tau^{1-\alpha} - \left[\frac{\gamma}{1+\gamma}(1-\beta)+1\right]\left[\beta\frac{1-\alpha}{\alpha}k_{\tau+1} + (1-\beta)\frac{1+n}{1+r_{\tau+1}}e_{\tau+2}\right]$$

from which

$$k_{\tau+1} = \frac{\gamma}{1+\gamma}\frac{(1-\alpha)(1-\beta)}{(1+n)\left[1+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}\right]}k_\tau^\alpha h_\tau^{1-\alpha} - \frac{\frac{\gamma}{1+\gamma}(1-\beta)+1}{1+\left(\frac{\gamma}{1+\gamma}(1-\beta)+1\right)\beta\frac{1-\alpha}{\alpha}}\frac{(1-\beta)(1+n)}{1+r_{\tau+1}}e_{\tau+2}$$

Therefore, we obtain the social planner condition

$$k_{\tau+1} = \eta^{sp}k_\tau^\alpha h_\tau^{1-\alpha} - \mathbb{C}$$

from which the optimal balanced growth path follows (as human capital stock is also optimal due to mandatory education).

F.5 Proof of Proposition 11

Pay-as-you-go

A pay-as-you-go pension scheme introduces a tax on individual income, $0 < \psi < 1$, in the working period of life, while offering a benefit $(1+n)\psi$ in the last period of retirement. The consumer problem without uncertainty (F.8) updates in

$$\begin{aligned}
\max_{s_\tau} \quad & \mathbf{E}_{\epsilon_\tau}(\ln(c_{1,\tau})) + \gamma\mathbf{E}_{\epsilon_{\tau+1}}(\ln(c_{2,\tau})) \\
\text{subject to} \quad & c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau \leq (1-\psi)(w_\tau h_\tau + \epsilon_\tau) \\
& c_{2,\tau} \leq (1+r_{\tau+1})s_\tau + (1+n)\psi(w_{\tau+1}h_{\tau+1} + \epsilon_{\tau+1}) \\
& \epsilon_\tau, \epsilon_{\tau+1} \sim N\left(0,\sigma^2\right) i.i.d. \\
& h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta} \\
& e_\tau = \hat{e}_\tau
\end{aligned} \tag{F.27}$$

The first order condition remains unchanged compared to the case without uncertainty. After the introduction of a pay-as-you-go scheme, consumption volatility is given by

$$\begin{aligned}\mathrm{Var}\left(c_{1,\tau}+\frac{c_{2,\tau}}{1+r_{\tau+1}}\right)&=\mathrm{Var}\left((1-\psi)\left(w_\tau h_\tau+\epsilon_\tau\right)-(1+r_\tau)e_\tau+\frac{1+n}{1+r_{\tau+1}}\psi\left(w_{\tau+1}h_{\tau+1}+\epsilon_{\tau+1}\right)\right)\\&=(1-\psi)^2\sigma^2+\psi^2\left(\frac{1+n}{1+r_{\tau+1}}\right)^2\sigma^2\end{aligned}$$

The policy maker minimizes the volatility setting $\psi=\frac{1}{1+\left(\frac{1+n}{1+r_{\tau+1}}\right)^2}$. Therefore, volatility is lower than the market allocation, but still not to 0 as in the social planner solution. As a side note, the policy seems to depend on the future interest rates. However, from the market remuneration of capital, (3.4), the interest rate is a constant on the BGP, since it depends on the ratio between physical and human capital.

$$1+r_\tau=\alpha\left(\frac{k_\tau^{payg}}{h_\tau^{payg}}\right)^{\alpha-1}$$

Funded

A funded pension scheme introduces a tax on wages for each generation τ, $0<\psi<1$, offering a benefit $(1+r_{\tau+1})\psi$ when an individual of generation τ retires. The consumer problem without uncertainty (F.12) updates in

$$\begin{aligned}\max_{s_\tau}\quad & \mathbf{E}_{\epsilon_\tau}\left(\left(\ln(c_{1,\tau})\right)+\gamma\left(\ln(c_{2,\tau})\right)\right)\\ \text{subject to}\quad & c_{1,\tau}+s_\tau+(1+r_\tau)e_\tau\leq(1-\psi)(w_\tau h_\tau+\epsilon_\tau)\\ & c_{2,\tau}\leq(1+r_{\tau+1})\left[s_\tau+\psi(w_\tau h_\tau+\epsilon_\tau)\right]\\ & \epsilon_\tau\sim N\left(0,\sigma^2\right)\\ & h_\tau=e_\tau^\beta h_{\tau-1}^{1-\beta}\\ & e_\tau=\hat{e}_\tau\end{aligned}\tag{F.28}$$

Therefore, the first order condition remains unchanged. The funded pension scheme has no risk-sharing elements since it does not transfer resources between generations.

$$\mathrm{Var}\left(c_{1,\tau}+\frac{c_{2,\tau}}{1+r_{\tau+1}}\right)=\sigma^2$$

Public bequests

The public bequests scheme introduces a tax ψ on wages of generation τ; while it offers a benefit $\frac{(1+r_{\tau+1})(1+r_\tau)}{1+n}\psi$. The consumer problem without uncertainty (F.16) updates in

$$\begin{aligned}
\max_{s_\tau} \quad & \mathbf{E}_{\epsilon_\tau}\left(\ln(c_{1,\tau}) + \gamma \ln(c_{2,\tau})\right) \\
\text{subject to} \quad & c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau \leq (1-\psi)(w_\tau h_\tau + \epsilon_\tau) \\
& c_{2,\tau} \leq (1+r_{\tau+1})\left[s_\tau + \frac{1+r_\tau}{1+n}\psi(w_{\tau-1}h_{\tau-1} + \epsilon_{\tau-1})\right] \\
& \epsilon_\tau \sim N\left(0, \sigma^2\right) \\
& h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta} \\
& e_\tau = \hat{e}_\tau
\end{aligned} \tag{F.29}$$

From the budget constraints, consumption volatility obtains

$$\mathrm{Var}\left(c_{1,\tau} + \frac{c_{2,\tau}}{1+r_{\tau+1}}\right) = (1-\psi)^2\sigma^2 + \psi^2\left(\frac{1+r_\tau}{1+n}\right)^2\sigma^2$$

Analogously to the pay-as-you-go case, if $\psi = \frac{1}{1+\left(\frac{1+r_\tau}{1+n}\right)^2}$, then the pension scheme minimizes the volatility without bringing it to 0, and the policy is constant on the BGP.

Appendix G

Additional results

G.1 Optimal tax policy

We can design a tax policy offering two options:

- (I) The consumer invests $\hat{e}_\tau$ equal to the social optimum (F.7), and he is eligible to receive the full pension benefit once he retires;
- (II) The consumer freely chooses e_τ, but, if $e_\tau < \hat{e}_\tau$, he receives only a fraction $0 \leq \pi \leq 1$ of the pension benefit.

Therefore, a consumer is entitled to the full amount of the benefit only if he chooses the education level $\hat{e}$. The policy maker can choose a π such that consumers will prefer in expectations $\hat{e}$ rather than e.

Define $b_\tau(e)$ the pension benefit received by a consumer of generation τ under human capital investments e. Then, the policymaker can implement the optimal level of human capital investments, if the following individual incentive constraint is satisfied

$$(1-\psi)w_\tau h_\tau - (1+r_\tau)e_\tau + \pi b_\tau(e) \leq (1-\psi)w_\tau h_\tau - (1+r_\tau)\hat{e}_\tau + b_\tau(\hat{e}) \tag{G.1}$$

The policy maker solves the inequality by choosing the fraction of the benefit received, π. Before defining π, we reformulate the previous inequality.

Using the market remuneration of labor, (3.3), we can rewrite $w_\tau h_\tau$ as

$$w_\tau h_\tau = (1-\alpha)\left(\frac{k_\tau}{h_\tau}\right)^\alpha h_\tau = (1-\alpha)k_\tau^\alpha h_\tau^{1-\alpha} = (1-\alpha)f(k_\tau, h_\tau)$$

Using also the human capital production function (3.2), we can express the relation with respect to e_τ

$$w_\tau h_\tau = (1-\alpha)k_\tau^\alpha \left(e_\tau^\beta h_{\tau-1}^{1-\beta}\right)^{1-\alpha} = (1-\alpha)f(k_\tau(e_\tau), e_\tau, h_{\tau-1}(e_{\tau-1})) \equiv (1-a)f(e_\tau)$$

where the last equivalence is just to compact the notation, since investments in education already made, $e_{\tau-1}$, are taken as given.

Thus, we can write the individual incentive condition (G.1) as

$$(1-\psi)(1-\alpha)f(e_\tau) - (1+r_\tau)e_\tau + \pi b_\tau(e) \leq (1-\psi)(1-\alpha)f(\hat{e}_\tau) - (1+r_\tau)\hat{e}_\tau + b_\tau(\hat{e}) \tag{G.2}$$

Define π as the ratio of contributions to the pension scheme through taxation

$$\pi = \frac{f(e_\tau)}{f(\hat{e}_\tau)} \tag{G.3}$$

from which

$$\pi = \frac{f(e_\tau)}{f(\hat{e}_\tau)} = \frac{(1-\psi)(1-\alpha)f(e_\tau)}{(1-\psi)(1-\alpha)f(\hat{e}_\tau)} = \frac{(1-\psi)w_\tau(e_\tau)h_\tau(e_\tau)}{(1-\psi)w_\tau(\hat{e}_\tau)h_\tau(\hat{e}_\tau)}$$

Therefore, the fraction of the benefit received is proportional to the share of contribution to the pension scheme, compared to the contribution under optimal investment in human capital.

Substituting (G.3) into (G.2), we obtain

$$(1-\psi)(1-\alpha)f(e_\tau) - (1+r_\tau)e_\tau + \frac{f(e_\tau)}{f(\hat{e}_\tau)} b_\tau(e) \leq (1-\psi)(1-\alpha)f(\hat{e}_\tau) - (1+r_\tau)\hat{e}_\tau + b_\tau(\hat{e})$$

from which

$$\frac{f(e_\tau)}{f(\hat{e}_\tau)} b_\tau(e) \leq (1-\psi)(1-\alpha)(f(\hat{e}_\tau) - f(e_\tau)) - (1+r_\tau)(\hat{e}_\tau - e_\tau) + b_\tau(\hat{e})$$

$$\frac{f(e_\tau)}{f(\hat{e}_\tau)} b_\tau(e) - b_\tau(e) \leq (1-\psi)(1-\alpha)(f(\hat{e}_\tau) - f(e_\tau)) - (1+r_\tau)(\hat{e}_\tau - e_\tau) + b_\tau(\hat{e}) - b_\tau(e)$$

$$\frac{f(e_\tau)}{f(\hat{e}_\tau)} - 1 \leq \frac{(1-\psi)(1-\alpha)(f(\hat{e}_\tau) - f(e_\tau)) - (1+r_\tau)(\hat{e}_\tau - e_\tau) + b_\tau(\hat{e}) - b_\tau(e)}{b_\tau(e)}$$

Since $e_\tau < \hat{e}_\tau$, the term on the right hand side is always positive, while the one on the left hand side is always negative. This means that for any $b_\tau(e)$, i.e. any pension scheme, a tax scheme with $\pi = \frac{f(e_\tau)}{f(\hat{e}_\tau)}$ can offer enough incentives to the consumers to implement the optimal investments in human capital.

Proposition 12. *A tax policy implements the optimal investments in human capital for any possible pension scheme, if the benefit received is proportional to the share of contribution to the pension scheme, with respect to the contribution under optimal investments in human capital, i.e.* $\pi = \frac{f(e_\tau)}{f(\hat{e}_\tau)}$.

G.2 Robustness to a funded scheme with partial insurance

We introduce the intergenerational risk-sharing mechanism studied in Gordon and Varian (1988). Consumers share the risk creating a fund which redistributes the shocks. Suppose that the government taxes individuals of each generation τ when old of an amount $\psi\epsilon_\tau$, and that it transfers the resources to the middle-aged generation $\tau+1$ adjusting them by the population growth rate. The transfers constitute the intergenerational risk-sharing mechanism integrating a fully funded scheme. The consumer problem becomes

$$\begin{aligned} \max_{s_\tau, e_\tau} \quad & \mathbf{E}_{\epsilon_\tau}\left((\ln(c_{1,\tau})) + \gamma\left(\ln(c_{2,\tau})\right)\right) \\ \text{subject to} \quad & c_{1,\tau} + s_\tau + (1+r_\tau)e_\tau \leq (1-\psi)(w_\tau h_\tau + \epsilon_\tau) + \frac{\psi\epsilon_{\tau-1}}{1+n} \\ & c_{2,\tau} \leq (1+r_{\tau+1})\left[s_\tau + \psi(w_\tau h_\tau + \epsilon_\tau)\right] - \psi\epsilon_\tau \\ & \epsilon_\tau \sim N\left(0, \sigma^2\right) \\ & h_\tau = e_\tau^\beta h_{\tau-1}^{1-\beta} \end{aligned} \tag{G.4}$$

The expected value of the shocks is still 0. Solving the consumer problem we still obtain the first order conditions of the fully funded case already studied.

Also, the market clearing condition for the capital market is still as in the funded case, i.e. (F.15). Combining the consumer budget constraints we obtain

$$c_{1,\tau} + s_\tau + \frac{c_{2,\tau-1}}{1+n} + (1+r_\tau)e_\tau \leq$$
$$(1-\psi)(w_\tau h_\tau + \epsilon_\tau) + \frac{\psi\epsilon_{\tau-1}}{1+n} + \frac{1+r_\tau}{1+n}\left[s_{\tau-1} + \psi(w_{\tau-1}h_{\tau-1} + \epsilon_{\tau-1})\right] - \frac{\psi\epsilon_{\tau-1}}{1+n}$$

Using the capital market condition and the properties of the Cobb-Douglas production function, the feasibility constraint obtains

$$c_{1,\tau} + s_\tau + \frac{c_{2,\tau-1}}{1+n} + \psi(w_\tau h_\tau + \epsilon_\tau) \leq f(k_\tau)$$

Also, from the budget constraints the lifetime consumption volatility is

$$\mathrm{Var}\left(c_{1,\tau} + \frac{c_{2,\tau}}{1+r_{\tau+1}}\right) = \frac{\psi^2\sigma^2}{(1+n)^2} - \psi^2\sigma^2$$

whose condition for a minimum is $\frac{1}{(1+n)^2} = 1$, or

$$n = 0$$

It follows that

Proposition 13. *The adjusted funded pension scheme offers risk sharing if there is no population growth.*

Gordon and Varian (1988) already states a similar result when there is no population growth. However, Abel (1988) finds that the funded risk-sharing mechanism also works with a non-stationary population. We contradict Abel's result because we analyze risk sharing in a general equilibrium model, while he derives his results in a partial equilibrium setup.

Conclusion

The first part of this dissertation defines an equilibrium where players indirectly communicate through a signal aggregating their private information; while the second part focuses on the links between pensions and economic growth.

Each set of results came with many challenges, which in hindsight were necessary and fruitful moments of investigation. To name a few from each chapter: the indirect renegotiation-proof equilibrium required me to model information about the strategies in a static setting; the shocks in the second chapter create a hybrid structure of discrete periods and continuous time; the design of the public bequests scheme brings all the problems of designing a new pension system almost from scratch.

The theoretical models I developed in this dissertation can be further extended to include more features and answer more questions. Moreover, future empirical research may apply these models to real life problems. My only hope is that this dissertation will not be perceived as an isolated set of results but, that it will pave the way to future advances in our understanding of how economies work.

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