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Optimal Boundary Output Feedback Control by Triangularization of the Counterflow Heat Exchanger Model

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Abstract— **In this paper we are interested in the LQ-optimal boundary control of counterflow heat exchanger. The dynamics of this system is described (under some assumptions) by hyperbolic partial differential equations (PDEs) and contains singularities which do not guarantee in some cases the uniqueness of solution of the operator Riccati equation. To address this issue, we first propose a state transformation that involves solving a Riccati differential equation, and that allows to put the system in a lower triangular form. Next, for the reachability analysis, the model has been rewritten as an abstract boundary control system with bounded control and observation operators. Finally, the design of an optimal control law with integral action is considered. The results are illustrated by means of numerical simulations for the set point tracking, and show the interest of the control approach proposed in this paper.**

I. Introduction

We consider the hyperbolic linear system that characterizes the dynamics of countercurrent heat exchangers [1]:

$$\partial_t x(z,t) = \Lambda \partial_z x(z,t) + Mx(z,t), \tag{1}$$

where $\Lambda = \begin{bmatrix} -v_1 & 0 \\ 0 & v_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $M = \begin{bmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $x_1(z,t),\ x_2(z,t) \in \mathbb{R},\ z \in [0,1],\ t \geq 0$, denote the distributed state variables of each component typically temperatures, v_i and α_i represent the transport velocity and the heat transfer coefficient component of i, respectively. We associate to this equation the following boundary conditions:

$$x_1(0,t) = x_{1,in}(t),\ x_2(1,t) = x_{2,in}(t), \tag{2}$$

where $x_{1,in}$ and $x_{2,in}$ are the inlet temperatures of the heat exchanger, and we consider that $x_{1,in}$ is constant, and $x_{2,in}(t) = u(t) \in U$ as a control variable at time $t > 0$. Heat exchangers have a fundamental role in most industrial processes (chemical, biochemical, food processing industry, etc.), since heat is essential there. In industry they are generally used in a countercurrent configuration, because it offers the possibility of having a high efficiency, unlike the parallel-flow configuration.

Regarding the control of this system, we can distinguish two types of approaches in the literature: those that consider only the measurement of the values of the boundary states without needing to know their values inside the domain (see [2], [3]), and those that take into account the distributed nature of the system, and require the complete knowledge of the state for their implementation [4], [5]. In [6] a comparative analysis of the performances of these two control strategies is carried out on the axial dispersion model of a pulp bleaching tubular reactor. It emerges from this study that the synthesis of the control laws from the distributed parameter model makes it possible to increase the performance of the system.

In this paper we are interested in the LQ-optimal state-feedback control of the hyperbolic heat exchanger model (1)-(2). The choice of this approach is based on the fact that it offers better performance than many others (see for example [7] where a comparison is made with the backstepping approach). In general, whether bounded or unbounded (observation and control) operators, the design of an optimal feedback operator is obtained by solving a Operator Riccati Equation (ORE)[8],[9, Chapter 5]. Results specific to hyperbolic PDEs of the form (1)-(2) exist in the literature. In [10],[11],[12] it is shown that when the transport velocities of the fluids are different (as in the case of model (1)-(2)), there is a diagonal operator, unique solution of the ORE. It is easy to verify that in the case of the model (1)-(2), the use of such operator does not guarantee the uniqueness of solution of the ORE (there are two ordinary differential equations, and two coupled compatibility equations impossible to satisfy). This problem can be circumvented by putting the model (1)-(2) in a lower triangular form. Although that is impossible by classical approaches (the matrix M is singular, the system generates a non-normal semigroup [13]), we show the possibility of doing it by solving a certain Riccati differential equation. To guarantee zero set point tracking error, this control law is coupled to an integrator.

The paper is organized as follows: In Section II we define first the abstract Cauchy system, and by a state transformation based on the resolution of a Riccati differential equation we put the model of the system in the lower triangular form. Next, the model is rewritten as an abstract boundary control system with bounded control and observation operators, and we prove the stabilizability of the system which is a sufficient condition for the design of the optimal control law. Section III studies the design of the optimal boundary control law with integral action, and Section IV illustrates this control approach by some numerical simulations.

Notation

Throughout this paper, I_n denotes, the identity matrix of order n, $\mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) and $\mathbb{R}^n$ (or $\mathbb{C}^n$) denote the set real (or complex) of $n-$order matrices, respectively.

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$L^2(0,1) = L^2((0,1);\mathbb{C})$ denotes the Hilbert space of measurable square-integrable function with values in $\mathbb{C}$. $H^1(0,1) = H^1((0,1);\mathbb{C})$ is the Sobolev space of absolutely continuous $\mathbb{C}-$valued functions whose derivatives are in $L^2(0,1)$. $Z = H^1(0,1) \oplus H^1(0,1)$ and $X = L^2(0,1) \oplus L^2(0,1)$ are Hilbert spaces such that the injection $Z \subset X$ is continuous. The spaces U and Y are Hilbert spaces of control and observation values, respectively.

II. Dynamical analysis

A. Existence of equilibrium profiles

In the following we derive a condition on the model parameters (1)-(2) which guarantees the existence of the equilibrium profiles. For that, for all $z \in [0,1]$ the stationary solution of the PDEs (1), i.e the solution of the ordinary differential equation $\Lambda \frac{d\bar{x}(z)}{dz} + M\bar{x}(z) = 0$ is given by $\bar{x}(z) = \exp(-\Lambda^{-1}Mz)x(0)$. Taking into account the configuration of the system, the boundary condition of (1)-(2) is expressed by

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (0 \quad 1)\cdot \exp(-\Lambda^{-1}M) \end{pmatrix}^{-1} \begin{pmatrix} x_{1,in} \\ x_{2,in} \end{pmatrix}. \quad (3)$$

Thus, by [14, Lemma1], it is easy to verify that the determinant of the above matrix given by $\det := \beta_1 - \beta_2 e^{\beta_2 - \beta_1}$ is non-zero if and only if $\beta_1 \neq \beta_2$, with $\beta_1 = \frac{\alpha_1}{v_1}$ and $\beta_2 = \frac{\alpha_2}{v_2}$. Under this condition, a straightforward computation yield to the following equilibrium profiles:

$$\begin{aligned} \bar{x}_1(z) &= \frac{1}{\beta_1 - \beta_2 e^{(\beta_2-\beta_1)}} \left(\beta_1 e^{(\beta_2-\beta_1)z} - \beta_2 e^{(\beta_2-\beta_1)}\right) x_{1,in} \\ &\quad + \frac{\beta_1}{\beta_1 - \beta_2 e^{(\beta_2-\beta_1)}} \left(1 - e^{(\beta_2-\beta_1)z}\right) x_{2,in}, \\ \bar{x}_2(z) &= \frac{\beta_2}{\beta_1 - \beta_2 e^{(\beta_2-\beta_1)}} \left(e^{(\beta_2-\beta_1)z} - e^{(\beta_2-\beta_1)}\right) x_{1,in} \\ &\quad + \frac{1}{\beta_1 - \beta_2 e^{(\beta_2-\beta_1)}} \left(\beta_1 - \beta_2 e^{(\beta_2-\beta_1)z}\right) x_{2,in}. \end{aligned} \quad (4)$$

Computational details are given in [15, Chapter 2].

B. Abstract formulation

The linear system (1)-(2) can be formulated as an abstract boundary control problem on the Hilbert space X,

$$\begin{cases} \dot{\xi}(t) = \mathfrak{A}\xi(t), & \xi(0) = \xi_0, \\ \mathscr{B}\xi(t) = u(t), \\ y(t) = \mathscr{C}\xi(t), \end{cases} \quad (5)$$

where the linear operator $\mathfrak{A}$ is defined by

$$\mathfrak{A} = \begin{pmatrix} -v_1 \frac{d}{dz} - \alpha_1 \cdot I & \alpha_1 \cdot I \\ \alpha_2 \cdot I & v_2 \frac{d}{dz} - \alpha_2 \cdot I \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

on its domain of definition given by

$$\mathscr{D}(\mathfrak{A}) = \Big\{ \xi = (\xi_1 \ \xi_2)^\top \in X : \ \xi \text{ absolutely continuous (a.c)}, \ \frac{d\xi}{dz} \in X \text{ and } \xi_1(0) = 0 = \xi_2(1) \Big\}.$$

The input operator $\mathscr{B} : X \to Y$ is given by

$$\mathscr{B}\xi = \xi_2(1).$$

For all $\xi = (\xi_1 \ \xi_2)^\top \in X$, the output operator $\mathscr{C} : X \to Y$ is given by

$$\mathscr{C}\xi = \langle c, \xi_1 \rangle = \int_0^1 c(z)\xi_1(z)dz, \quad (6)$$

where $c(z)$ is a space-varying continuous function on the interval $[0,1]$.

The following result is stated in [16].

Corollary 2.1 (Well-posedness and stability): If $\beta_1 \neq \beta_2$ and for $\xi(0) = \xi_0 \in \mathscr{D}(\mathfrak{A})$, the abstract system (5) admits a unique solution $\xi \in C^0([0,+\infty), \mathscr{D}(\mathfrak{A})) \cap C^1([0,+\infty), X)$. The mild solution of (5) is given by:

$$\xi(t) = e^{\mathfrak{A}t}\xi_0, \qquad \forall\, t \in [0,+\infty).$$

Furthermore, the operator $\mathfrak{A}$ generates an exponentially stable C_0-semigroup denoted by $(\mathbb{T}(t))_{t\geq 0} = e^{\mathfrak{A}t}$ on X, i.e. there $N > 0$ and $\mu > 0$ such that

$$\|\mathbb{T}(t)\| \leq N e^{-\mu t}, \qquad t \geq 0.$$

C. Triangularized model

Let us perform a similarity transformation (i.e. Hilbert-space isomorphism) in order to get an equivalent state-space description whose generator is triangular.

Consider the state transformation $\zeta = S\xi$ with $S \in \mathscr{L}(Z)$ a linear bounded operator given by:

$$S := \begin{pmatrix} I & \vartheta \cdot I \\ 0 & I \end{pmatrix}, \quad (7)$$

with $\vartheta \in [0,1]$ a C^1-bounded function which satisfies certain conditions. Applying the state transformation defined by S, we find the operator $\tilde{\mathfrak{A}} := S\mathfrak{A}S^{-1}$ given by:

$$\tilde{\mathfrak{A}} = \begin{pmatrix} -v_1 \frac{d}{dz} - (\alpha_1 + \alpha_2\vartheta)\cdot I & (*) \\ \alpha_2 \cdot I & v_2 \frac{d}{dz} - \alpha_2(1-\vartheta)\cdot I \end{pmatrix},$$

on its domain $\mathscr{D}(\tilde{\mathfrak{A}}) = \mathscr{D}(\mathfrak{A})$, with $(*)$ equal to:

$$(*) := \left(v_1 \frac{d\vartheta}{dz} - \alpha_2 \vartheta^2(z) - (\alpha_1 + \alpha_2)\vartheta(z) + \alpha_1 \right) \cdot I = 0. \quad (8)$$

For the operator $\tilde{\mathfrak{A}}$ to be lower triangular, it suffices that ϑ satisfies the following Riccati differential equation (RDE):

$$v_1 \frac{d\vartheta}{dz} - \alpha_2 \vartheta^2(z) - (\alpha_1 + \alpha_2)\vartheta(z) + \alpha_1 = 0, \quad \vartheta(0) = 0, \quad (9)$$

to which we associate the following linear Hamiltonian system (see [17, Chapter 3] for more information):

$$\frac{d}{dz}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha_1+\alpha_2}{v_1} & -\frac{\alpha_2}{v_1} \\ \beta_1 & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad \begin{pmatrix} \theta_1(0) = 1 \\ \theta_2(0) = 0 \end{pmatrix}. \quad (10)$$

Lemma 2.1: If $\theta(z) = (\theta_1(z) \ \theta_2(z))^\top$ is solution of system (10), then $\vartheta(z) = \theta_2(z)(\theta_1(z))^{-1}$ is solution of the RDE (9) for all $z \in [0,1]$ such that $\vartheta_1(z) \neq 0$.

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Theorem 2.1: The RDE (9) admits a unique bounded C^1-solution on $[0,1]$ given by:

$$\vartheta(z) = \alpha_2 \frac{e^{-\frac{\alpha_2}{v_1}z} - e^{-\beta_1 z}}{\alpha_2 e^{-\frac{\alpha_2}{v_1}z} + \alpha_1 e^{\beta_1 z}}. \quad (11)$$

Consequently, the operator $S \in \mathscr{L}(Z)$ given by (7) is a similarity transformation which triangularizes the operator $\mathfrak{A}$.
Likewise, we define the input operator $\tilde{\mathscr{B}}$ by

$$\tilde{\mathscr{B}}\zeta(t) = \mathscr{B}S^{-1}\zeta(t) = \mathscr{B}\begin{pmatrix}\zeta_1(t) - \vartheta(\cdot)\zeta_2(t)\\ \zeta_2(t)\end{pmatrix} = \zeta_2(1,t), \quad (12)$$

and $\forall \zeta = (\zeta_1, \zeta_2)^\top \in X$ the observation operator $\tilde{\mathscr{C}} = \mathscr{C}S^{-1} \in \mathscr{L}(X,Y)$ by

$$\tilde{\mathscr{C}}\zeta(t) = \int_0^1 c(z)\zeta_1(z,t)dz - \int_0^1 c(z)\vartheta(z)\zeta_2(z,t)dz. \quad (13)$$

Since S is a regular and differentiable operator, then for all $\zeta \in X$ the dynamic abstract system (5) is equivalent to:

$$\begin{cases} \dot{\zeta}(t) = \tilde{\mathfrak{A}}\zeta(t), \qquad \zeta(0) = \zeta_0,\\ \tilde{\mathscr{B}}\zeta(t) = u(t),\\ y(t) = \tilde{\mathscr{C}}\zeta(t). \end{cases} \quad (14)$$

D. Boundary control system

For reachability analysis, it is necessary to return to a "state representation" form: $\dot{\zeta}(t) = \tilde{\mathfrak{A}}\zeta(t) + \tilde{B}u(t)$, with homogeneous boundary conditions. This is possible by considering some transformations which consist in extracting the boundary control part of the dynamic model (1)-(2) and rewriting it as a boundary control Fattorini model [18]. In that case, the goal is to find a bounded operator $\tilde{B} \in \mathscr{L}(U,X)$ such that

a) for all $u(t) \in U$, $\tilde{B}u \in \mathscr{D}(\tilde{\mathfrak{A}})$,
b) the operator $\tilde{\mathfrak{A}}\tilde{B} \in \mathscr{L}(U,X)$,
c) $\tilde{\mathscr{B}}\tilde{B}u = u$ for all $u \in U$.

If the operator $\tilde{B}$ is chosen such that for all $u \in U$

$$\tilde{B}u(t) = \begin{pmatrix} b_1(z)\\ b_2(z)\end{pmatrix} u(t), \quad (15)$$

where $b_i(z)$ $(i = 1,2)$ are continuous functions such that $b_i(1) = 1$, then it can be easily observed that a) and b) hold. Moreover, c) is also satisfied since

$$\tilde{\mathscr{B}}\tilde{B}u = b_2(z=1)u = u.$$

Now we are in a position to define the new operator

$$\tilde{A} : \mathscr{D}(\tilde{A}) \to X \quad \text{by} \quad \tilde{A}\zeta = \tilde{\mathfrak{A}}\zeta,$$

on its domain $\mathscr{D}(\tilde{A}) = \mathscr{D}(\tilde{\mathfrak{A}}) \cap \ker(\tilde{\mathscr{B}})$. Let us consider the new state $v(t) = \zeta(t) - \tilde{B}u(t)$ and the new input $\tilde{u}(t) = \dot{u}(t)$. Then, by using the augmented state $\zeta^e(t) = \begin{pmatrix}u(t) & v(t)\end{pmatrix}^\top \in X^e := U \oplus X$, the system can be written as follows

$$\begin{cases} \dot{\zeta}^e(t) = A^e\zeta^e(t) + B^e\tilde{u}(t), \qquad \zeta(0) = \zeta_0\\ y(t) = C^e\zeta^e(t), \end{cases} \quad (16)$$

where the operator $A^e \in U \oplus \mathscr{D}(\tilde{A})$, together with the operators $B^e \in \mathscr{L}(U, X^e)$ and $C^e \in \mathscr{L}(X^e, Y)$ by

$$A^e = \begin{pmatrix} 0 & 0\\ \tilde{\mathfrak{A}}\tilde{B} & \tilde{A}\end{pmatrix}, \quad B^e = \begin{pmatrix} I\\ -\tilde{B}\end{pmatrix}, \quad C^e = \tilde{\mathscr{C}}\begin{pmatrix}\tilde{B} & I\end{pmatrix}. \quad (17)$$

The operator $\tilde{\mathfrak{A}}\tilde{B}$ is given explicitly by

$$\tilde{\mathfrak{A}}\tilde{B} = \begin{pmatrix} -v_1\dfrac{db_1}{dz} - (\alpha_1 + \alpha_2\vartheta(z))b_1\\ \alpha_2 v_1 + v_2\dfrac{db_2}{dz} - \alpha_2(1-\vartheta(z))b_2 \end{pmatrix} \cdot I := \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} \cdot I.$$

It should be noted that the spectrum of the operator A^e is such that: $\sigma(A^e) = \sigma(\mathfrak{A}) \cup \{0\}$. Despite this, it is always possible to find an operator from Lyapunov who guarantees the exponential stability of the semigroup generated by A^e.

Theorem 2.2 (Stabilizability): Let us consider the operators A^e and B^e given by (17). Then $\Sigma(A^e, B^e, -)$ is exponentially stabilizable.

Proof: Let $K^e = (k_1 \;\; K) \in \mathscr{L}(X^e, \mathbb{R})$ with $k_1 \in \mathbb{R}$ and $K \in \mathscr{L}(X, \mathbb{R})$ a bounded operator such that $(A^e - B^eK^e)$ generates an exponentially stable C_0-semigroup $(\mathbb{T}_{(A^e-B^eK^e)}(t))_{\geq 0}$ on X^e. If we choose $k_1 > 0$ and $K = 0$, the resulting closed-loop operator become:

$$A^e - B^eK^e = \begin{pmatrix} -k_1 & 0\\ \tilde{\mathfrak{A}}\tilde{B} - \tilde{B}k_1 & \tilde{A}\end{pmatrix} = \begin{pmatrix} A^e_{11} & 0\\ A^e_{21} & A^e_{22}\end{pmatrix}.$$

This operator is a triangular operator whose diagonal operators A^e_{11} and A^e_{22} generate exponentially C_0-semigroup on $\mathbb{R}$ and X (according to the Corollary 2.1), respectively. Thus, by [9, Lemma 3.2.2] and for $t \geq 0$, $\mathbb{T}_{(A^e-B^eK^e)}(t)$ is exponentially stable on X^e. ■

III. LQ-OPTIMAL CONTROL

Having defined the system in the form of a boundary control observation system, we now turn to the calculation of the optimal state feedback operator.
The design of LQ-optimal control law amounts to find a control that minimizes the cost functional

$$J(\zeta_0^e, \tilde{u}^e) = \int_0^\infty \left(\langle B^{e*}\zeta^e(t), B^{e*}\zeta^e(t)\rangle + \langle \tilde{u}^e(t), r\tilde{u}^e(t)\rangle\right)dt. \quad (18)$$

For all $\tilde{\zeta}^e \in \mathscr{D}(A^e)$, the solution to this optimization problem can be obtained by finding a non-negative self-adjoint $\Pi(\mathscr{D}(A^e)) \subset \mathscr{D}(A^{e*})$ operator, solution of the ORE (see [19, Chapter 4]):

$$\left[A^{e*}\Pi + \Pi A^e + C^{e*}C^e - \Pi B^e r^{-1} B^{e*}\Pi\right]\zeta^e = 0, \quad (19)$$

and where r is a positive function. In this case, the quadratic cost functional (18) is minimized and the closed-loop system is stabilized by the unique control $\tilde{u}(t)$ given for any $t \geq 0$ by

$$\tilde{u} = -K^e\zeta^e(t) = -\frac{1}{r}B^{e*}\Pi\zeta^e(t), \quad (20)$$

with $\zeta^e(t) = e^{(A^e - B^eK^e)t}\zeta_0^e$.
Solving the operator Riccati equation (19) could be a very challenging problem. For linear hyperbolic systems of the

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form (1), it is shown in [12, Theorem 5] and [11] that the operator Riccati equation (19) admits a diagonal solution $\Pi = \mathrm{diag}(\pi_1, \Pi_r) \in \mathscr{L}(X^e)$, with $\Pi_r = \mathrm{diag}(\pi_2, \pi_3) \in \mathscr{L}(X)$. Thus, for all $\xi^e \in \mathscr{D}(A^e)$ the ORE (19) can be decoupled and converted to the following operator equations:

$$(\tilde{\mathscr{C}}\tilde{B})^*(\tilde{\mathscr{C}}\tilde{B}) - \pi_1 r^{-1}\pi_1 = 0, \quad (21)$$
$$\tilde{A}^*\Pi_r + \Pi_r\tilde{A} + \tilde{\mathscr{C}}^*\tilde{\mathscr{C}} - \Pi_r\tilde{B}r^{-1}\tilde{B}^*\Pi_r = 0, \quad (22)$$
$$(\tilde{\mathfrak{A}}\tilde{B})^*\Pi_r + (\tilde{\mathscr{C}}\tilde{B})^*\tilde{\mathscr{C}} - \pi_1 r^{-1}\tilde{B}^*\Pi_r = 0, \quad (23)$$
$$\Pi_r(\tilde{\mathfrak{A}}\tilde{B}) + \tilde{\mathscr{C}}^*(\tilde{\mathscr{C}}\tilde{B}) - \Pi_r r^{-1}\tilde{B}\pi_1 = 0. \quad (24)$$

Now, we can solve these four equations separately. The equation (21) can be easily solved through straightforward calculations and gives the expression of π_1:

$$\pi_1 = \sqrt{r\left(\langle c, b_1\rangle - \langle c, \vartheta b_2\rangle\right)^2} = \sqrt{r}\,|\langle c, b_1 - \vartheta b_2\rangle|. \quad (25)$$

To solve the operator Riccati equation (22) we need the adjoint operator $\tilde{A}$. Its expression is stated in the following lemma.

Lemma 3.1: Let $\tilde{A}$, the operator defined by

$$\tilde{A}\varphi = \Lambda\frac{d\varphi}{dz} + \underbrace{\begin{pmatrix} -(\alpha_1 + \alpha_2\vartheta)\cdot I & 0 \\ \alpha_2 \cdot I & -\alpha_2(1-\vartheta)\cdot I \end{pmatrix}}_{M_1}\varphi.$$

For all $\zeta \in X$ the adjoint of $\tilde{A}$ is given by $\tilde{A}^*\zeta = -\Lambda\frac{d\zeta}{dz} + M_1^\top\zeta$, and its domain $\mathscr{D}(\tilde{A}^*) = \left\{\zeta = (\zeta_1, \zeta_2) \in X : \zeta \text{ a.c. }, \frac{d\zeta}{dz} \in X, \zeta_1(1) = 0, \zeta_2(0) = 0\right\}$.

Proof: Indeed, it suffices to observe that the usual pairing identity $\langle \tilde{A}^*\zeta, \varphi\rangle = \langle \zeta, \tilde{A}\varphi\rangle$ holds $\forall \varphi \in \mathscr{D}(\tilde{A})$ and $\forall \zeta \in \mathscr{D}(\tilde{A}^*)$. To check this, let us consider the operator $\tilde{A}^*$ as given in Lemma 3.1 on its domain $\mathscr{D}(\tilde{A}^*)$ and observe that, for all $\varphi \in \mathscr{D}(\tilde{A})$ and for all $\zeta \in \mathscr{D}(\tilde{A}^*)$ we have that,

$$\begin{aligned}\langle \tilde{A}^*\zeta, \varphi\rangle &= \int_0^1 \left(-\Lambda\frac{d\zeta}{dz} + M_1^\top\zeta(z)\right)^\top \varphi(z)dz \\ &= \int_0^1 \left(M_1^\top\zeta(z)\right)^\top\varphi(z)dz - \int_0^1\left(\Lambda\frac{d\zeta}{dz}\right)^\top\varphi(z)dz \\ &= -\int_0^1 \frac{d\zeta^\top}{dz}\Lambda\varphi(z)dz + \int_0^1 \zeta^\top(z)M_1\varphi(z)dz.\end{aligned}$$

The last equality comes from the fact that Λ is a real-valued diagonal matrix, and therefore $\Lambda^\top = \Lambda$. Let us integrate by part the first term on $[0,1]$. Thus, we get:

$$\begin{aligned}\langle \tilde{A}^*\zeta, \varphi\rangle &= \left[-\zeta^\top(z)\Lambda\varphi(z)\right]_0^1 + \int_0^1 \zeta^\top(z)\Lambda\varphi(z)dz \\ &\quad + \int_0^1 \zeta^\top(z)M_1\varphi(z)dz \\ &= -\zeta^\top(1)\Lambda\varphi(1) + \zeta^\top(0)\Lambda\varphi(0) \\ &\quad + \int_0^1 \zeta^\top(z)\left(\Lambda\frac{d}{dz} + M_1\right)\varphi(z)dz.\end{aligned}$$

Now, using the definition of $\mathscr{D}(\tilde{A}^*)$, we get, for all $\zeta \in \mathscr{D}(\tilde{A}^*)$,

$$\begin{aligned}\zeta^\top(0)\Lambda\varphi(0) - \zeta^\top(1)\Lambda\varphi(1) &= v_1\underbrace{\zeta_1(1)}_{=0}\varphi_1(1) - v_2\zeta_2(1)\varphi_2(1) \\ &\quad - v_1\zeta_1(0)\varphi_1(0) + v_2\underbrace{\zeta_2(0)}_{=0}\varphi_2(0).\end{aligned}$$

Combining the above expression and the definitions of $\tilde{A}$ and its domain $\mathscr{D}(\tilde{A})$ yields the following identity, for all $\varphi \in \mathscr{D}(\tilde{A})$ and for all $\zeta \in \mathscr{D}(\tilde{A}^*)$:

$$\begin{aligned}\langle \tilde{A}^*\zeta, \varphi\rangle &= -v_1\zeta_1(0)\underbrace{\varphi_1(0)}_{=0} - v_2\zeta_2(1)\underbrace{\varphi_2(1)}_{=0} \\ &\quad + \int_0^1 \zeta^\top(z)\left(\Lambda\frac{d}{dz} + M_1\right)\varphi(z)dz \\ &= \int_0^1 \zeta^\top(z)\left(\Lambda\frac{d}{dz} + M_1\right)\varphi(z)dz \\ &= \langle \zeta, \tilde{A}\varphi\rangle.\end{aligned}$$

Thus we have that the operator $\tilde{A}^*$ defined in Lemma 3.1 and its domain $\mathscr{D}(\tilde{A}^*)$ characterize the adjoint operator of $\tilde{A}$. ■

By using the expressions of $\tilde{A}$ and its adjoint $\tilde{A}^*$, and since $\frac{d\Pi_r}{dz}\cdot I = -\frac{d\cdot}{dz}\cdot\Pi_r\cdot I + \Pi\cdot\frac{d\cdot}{dz}$, it can be easily observed that the following identity holds

$$\tilde{A}^*\Pi_r + \Pi_r\tilde{A} = \Lambda\frac{d\Pi_r}{dz} + M_1^\top\Pi_r + \Pi_r M_1. \quad (26)$$

Then, for all $\zeta \in \mathscr{D}(\tilde{A})$ and $\Pi_r(\mathscr{D}(\tilde{A})) \subset \mathscr{D}(\tilde{A}^*)$ equation (22) can be converted into the following matrix differential equation:

$$\begin{cases} \Lambda\dfrac{d\Pi_r}{dz}\zeta = -M_1^\top\Pi_r\zeta - \Pi_r M_1\zeta - \tilde{\mathscr{C}}\langle\tilde{\mathscr{C}}, \zeta\rangle \\ \qquad + r^{-1}\langle\tilde{B}, \Pi_r\zeta\rangle\Pi_r\tilde{B}, \\ \pi_2(1) = 0, \quad \pi_3(0) = 0. \end{cases} \quad (27)$$

Note that the operator Riccati equation (27) is an integro-differential equation that is difficult to solve. Indeed, since $U = Y = \mathbb{R}$, then

$$\tilde{B}^*\zeta := \int_0^1 \left(b_1(z)\zeta_1(z) + b_2(z)\zeta_2(z)\right)dz,$$

and equation $\tilde{\mathscr{C}}\zeta$ given by (13), which represent average value of $b_1\zeta_1 + b_2\zeta_2$ and $c\zeta_1 - c\vartheta\zeta_2$ on $[0,1]$. To simplify the problem, we have approximated the average values by distributed functions; this makes it possible to see $\tilde{B}\tilde{B}^*$ and $\tilde{\mathscr{C}}^*\tilde{\mathscr{C}}$ as matrices instead of operators. In other words:

$$\tilde{B}\tilde{B}^*\zeta :\approx \begin{pmatrix} b_1^2 & b_1b_2 \\ b_1b_2 & b_2^2 \end{pmatrix}\begin{pmatrix}\zeta_1 \\ \zeta_2\end{pmatrix},$$

and

$$\tilde{\mathscr{C}}^*\tilde{\mathscr{C}}\zeta :\approx \begin{pmatrix} c^2 & -\vartheta c^2 \\ -\vartheta c^2 & \vartheta^2c^2 \end{pmatrix}\begin{pmatrix}\zeta_1 \\ \zeta_2\end{pmatrix}.$$

This argument is commonly used in optimal control of distributed parameter systems [11]. Consequently, equation

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(27) becomes:

$$\begin{cases} v_1 \dfrac{d\pi_2}{dz} - r^{-1} b_1^2 \pi_2^2(z) - 2(\alpha_1 + \alpha_2 \vartheta)\pi_2(z) + c^2 = 0, \\ v_2 \dfrac{d\pi_3}{dz} + b_2^2 r^{-1} \pi_3^2(z) + 2\alpha_2(1-\vartheta)\pi_3(z) - \vartheta^2 c^2 = 0, \\ \alpha_2 \pi_3 - r^{-1} b_1 b_2 \pi_2(z)\pi_3(z) - c^2\vartheta = 0, \\ \pi_2(1) = 0, \quad \pi_3(0) = 0. \end{cases} \tag{28}$$

On the other hand, the equations (23) and (24) are equivalent, and represent compatibility equations. They can be written explicitly and converted to the following algebraic equations:

$$\begin{cases} \gamma_1 \pi_2 - r^{-1} b_1 \pi_1 \pi_2 + c^2(b_1 - \vartheta b_2) = 0 \\ \gamma_2 \pi_3 - r^{-1} b_2 \pi_1 \pi_3 - c^2 \vartheta (b_1 - \vartheta b_2) = 0. \end{cases} \tag{29}$$

Now we are able to express the stabilizing compensator based on the state-feedback boundary control of the dynamical system (5).

Proposition 3.1: Let us consider the boundary control system (5). If π_1 and $\Pi_r = \text{diag}(\pi_2, \pi_3)$ are the solutions of (25) and (28), then the optimal state-feedback control law that minimizes the cost criterion (18) along the trajectories of the system (5) is given by

$$u(t) = e^{-\omega t} u(0) + \frac{1}{r}\int_0^t \int_0^1 e^{-\omega(t-\tau)} b_1 \pi_2(z) \xi_1(z,\tau) dz d\tau + \frac{1}{r}\int_0^t \int_0^1 e^{-\omega(t-\tau)} (b_1 \vartheta \pi_2(z) + b_2 \pi_3(z)) \xi_2(z,\tau) dz d\tau, \tag{30}$$

with $\omega = \frac{1}{r}\left(\sqrt{r}|\langle c, b_1 - \vartheta b_2\rangle| + \langle b_1, b_1\pi_2(z)\rangle + \langle b_2, b_2\pi_3(z)\rangle\right)$ a positive constant.

Proof: Let us consider the dynamic extended system (16). By using equation (19), it follows that

$$\dot{u}(t) = \tilde{u}(t) = -\frac{1}{r} B^{e*} \Pi \zeta^e(t) = -\frac{1}{r}\pi_1(z) u(t) + \frac{1}{r}\tilde{B}^* \Pi_r(z) v(t),$$

with $\zeta^e(t) = (u(t) \;\; v(t))^\top$. Considering the fact that $v(t) = \zeta(t) - \tilde{B}u(t)$ and expression of π_1 given by equation (25), the previously equation become:

$$\begin{aligned} \dot{u}(t) &= -\frac{1}{r}\left(\sqrt{r}|\langle c, b_1\rangle - \langle c, \vartheta b_2\rangle| + \langle b_1, b_1 \pi_2(z)\rangle\right) \\ &\quad + \langle b_2, b_2\pi_3(z)\rangle - \langle b_1, \pi_2(z)\zeta_1(t)\rangle - \langle b_2, \pi_3(z)\zeta_2(t)\rangle) \\ &= -\omega u(t) + \frac{1}{r}\int_0^1 (b_1 \pi_2(z)\zeta_1(z,t) + b_2\pi_3(z)\zeta_2(z,t)) dz \\ u(t) &= e^{-\omega t} u_0 + \frac{1}{r}\int_0^t \int_0^1 e^{-\omega(t-\tau)} b_1 \pi_2(z) \zeta_1(z,\tau) dz d\tau \\ &\quad + \frac{1}{r}\int_0^t \int_0^1 e^{-\omega(t-\tau)} b_2 \pi_3(z) \zeta_2(z,\tau) dz d\tau. \end{aligned}$$

By the state transformation (7) it follows that $\zeta_1(z,t) = \xi_1(z,t) + \vartheta \xi_2(z,t)$, $\zeta_2(z,t) = \xi_2(z,t)$, which conclude the proof. ■

Note that for a given set point $y_d(t) \in Y$, the control law obtained does not make it possible to cancel the static error of the closed-loop system (30). Indeed, this is due to the fact that the dynamic model of the heat exchanger does not have open-loop integration. Classical control theory therefore leads us to add an integrator to the control chain. Thus, the resulting control law is given by:

$$\begin{cases} u(t) = e^{-\omega t} u(0) + \dfrac{1}{r}\displaystyle\int_0^t \int_0^1 e^{-\omega(t-\tau)} (b_1 \pi_2(z) \xi_1(z,\tau) \\ \qquad + (b_1 \vartheta \pi_2(z) + b_2\pi_3(z))\xi_2(z,\tau)) dz d\tau + \chi(t), \\ \dfrac{d\chi(t)}{dt} = y_d(t) - y(t), \end{cases} \tag{31}$$

with $\chi(t) \in Y$ being the tracking error integral.

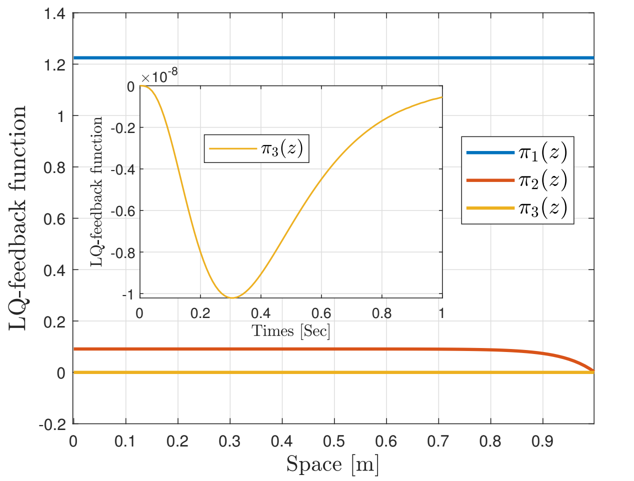

Fig. 1: LQ-feedback functions $\pi_1(z), \pi_2(z)$ and $\pi_3(z)$ for $r(z) = 1.5$.

IV. Numerical simulations

To illustrate the performance of the integral action LQ-controller, we have chosen the following operating conditions: $\xi_{1,int}(t) = 60°C$, $\xi_2(1,1) = 8°C$, and $\xi(z,0) = \tilde{\xi}(z,0)$. For all $z \in [0,1]$ the weighting function is chosen as $r(z) = 1.5$. Figure 1 shows the resulting LQ-optimal state feedback functions $\pi_1(z), \pi_2(z)$ and $\pi_3(z)$. Note that the function $\pi_1(z)$ is constant and positive as defined by (25). Observe that the function $\pi_2(z)$ induce a positive spatially varying feedback on the internal temperature $\xi_1(\cdot,t)$ (see equation (6)). Moroever, observe that the function $\pi_3(z)$ is almost identically zero, i.e., there is a very low gain feedback on the external temperature $\xi_2(\cdot,t)$; this result is not surprising in view of the definition of the output function (6). From the LQ optimal state feedback functions, we get the numerical value of the control law decay constant (31) as $\omega = 0.8735$. For the set point tracking test, a set point step of $15°C$ ($y_d(t) = 35°C$) and $-10°C$ ($y_d(t) = 25°C$) was imposed at time $t = 168s$ and $t = 335s$. The simulation results are summarized in Figures 2a-2d. We note good tracking performance (Figure 2a) and an acceptable variation of the manipulated variable. We end this simulation step by presenting the $\mathbb{X}-$norm of the closed-loop state vector $(\tilde{\xi}_1(\cdot,t), \tilde{\xi}_2(\cdot,t), \chi(t))$ as a function of time, see Figure 2d. One observes that the equilibrium corresponding to each reference value is reached exponentially fast on each subinterval.

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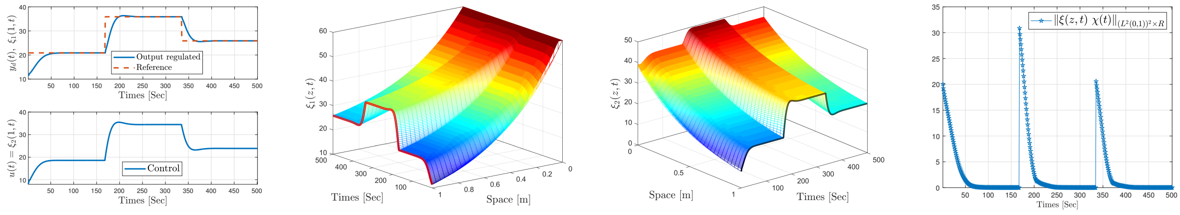

(a) Reference,output and control variables. (b) 3D state variable $\xi_1(z,t)$. (c) 3D state variable $\xi_2(z,t)$. (d) $(L^2(0,1),\mathbb{R})^2 \times \mathbb{R}$−norm.

Fig. 2: Closed-loop simulation

V. Conclusion

In this paper we have studied the LQ-optimal control problem of a counterflow heat exchanger. This system, governed by two coupled PDEs, is known to be poorly conditioned. Indeed, the operator which carries the dynamics of the system contains singularities which prevent the system to benefit from the theoretical results which exist in the literature. First, by a state transformation based on the resolution of a Riccati differential equation we were able to put the model of the system in the lower triangular form. Next, in order not to resort to additional computation steps, in particular by solving a spectral factorization problem, the model has been rewritten as an abstract boundary control problem with bounded control and observation operators, using the Fattorini approach, from which we developed an LQ-optimal control law with integral action. The simulation results of the control law have shown the relevance of the proposed approach.

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