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Design and operations of gas transmission networks

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Abstract

Problems dealing with the design and the operations of gas transmission networks are challenging. The difficulty mainly arises from the simultaneous modeling of gas transmission laws and of the investment costs. The combination of the two yields a non-linear non-convex optimization problem. To obviate this shortcoming, we propose a new formulation as a multi-objective problem, with two objectives. The first one is the investment cost function or a suitable approximation of it; the second is the cost of energy that is required to transmit the gas. This energy cost is approximated by the total energy dissipated into the network. This bi-criterion problem turns out to be convex and easily solvable by convex optimization solvers. Our continuous optimization formulation can be used as an efficient continuous relaxation for problems with non-divisible restrictions such as a limited number of available commercial pipe dimensions.

Keywords: gas transmission networks, reinforcement, convex optimization.

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1 Introduction

Problems dealing with the design and the operations of gas transmission networks are challenging. The difficulty mainly arises from the simultaneous modeling of gas transmission laws and of the investment costs. The combination of the two yields a non-linear non-convex optimization optimization problem. To obviate this shortcoming, we propose a new formulation as a multi-objective problem, with two objectives. The first one is the investment cost function or a suitable approximation of it; the second is the cost of energy that is required to transmit the gas. This energy cost is approximated by the total energy dissipated into the network. This bi-criterion problem turns out to be convex and easily solvable by convex optimization solvers[1]. Our continuous optimization formulation can be used as an efficient continuous relaxation for problems with non-divisible restrictions such as a limited number of available commercial pipe dimensions.

A gas transmission network is represented by a graph with special characteristics. The arcs of the graph consist of pipes through which the gas can flow in one direction or another. The main attribute of the pipe is the internal diameter. The nodes can be input node, output node or pure transmission nodes. At an input node, the gas is injected at a certain pressure by means of a compressor. At an output node, the gas is released at a pressure controlled by a regulator.

The standard transmission problem is a *management problem*. Given the equipments (compressor, regulator and pipes), is it possible to transmit gas from supply nodes to demand nodes to meet constraints on the supplies and the deliveries with pressures at the nodes of the network that are compatible with the existing compressors and regulators. In the literature this problem is modeled with nonlinear relationships between flow and pressure. It consists in finding a feasible solution for this inequality system that meets demand and supply requirements. This problem is known to be very difficult in practice due to the nonlinearity and the non convexity of the feasible set of flow-pressure solutions. The authors in [6] propose a successive piecewise linear approximation method to solve the minimum supply cost problem.

The other transmission problem that has been widely studied in the literature is the problem of reinforcement, or *investment problem*. If the existing equipment is a bottleneck for the management and/or the operations problem, one may want to purchase additional equipment to expand

[1] A complete demonstration with on-line solutions can be found on the web at `http:www.ordecsys.com/gasdecsys`. In this demo, the solution engine is entirely based on open source tools

the transmission capabilities. In this paper, we consider purchase of new transmission lines, or the reinforcement of existing ones, but not purchase of new compressors. The state of the art approach for the continuous investment problem is to minimize a cost investment objective function on the set of feasible flow-pressure solutions [5]. The investment cost is often approximated by a quadratic function of the pipe diameters to be installed [5]. In practice, the final objective is to choose among a finite set of commercial pipes the optimal pipe reinforcement. Thus the problem belongs to the realm of mixed integer programming (MIP). In [3] the authors develop a Branch-and-Bound scheme. A trust-region successive linear programming method is proposed in [8]. The authors in [14] add binary variables per diameter on each arc and use the continuous relaxation in their MIP techniques. In those approaches the computation of the continuous relaxation problem is crucial because it has a direct impact on the overall efficiency algorithm. In this study we propose a convex formulation for the continuous relaxation that makes its computation easier.

The *operations problem* is another problem in the dealing with a gas transmission network. If the supplies and/or the demands are markets with prices, there is an issue about the amounts to be purchased and to be delivered on each market. The operations problem deals with the optimal choice of supply and delivery given the market prices and the network technical resources. A detailed infrastructure model with a complex contractual system which belongs to MIP programming is presented in [12]. More recent studies focus on new formulation to reflect the liberalization process in the natural gas industry in Europe. In this context the objective of the gas transportation companies is to minimize the energy cost in the compressors, i.e., the fuel consumption. In general this cost is a nonlinear function of the inlet and outlet pressures [1, 2]. In [4] the authors propose a two stage procedure for finding a local optimum solution. In the first stage, they compute a starting flow-pressure solution for their nonlinear programming second stage algorithm by minimizing the energy losses [9] in the network without the use of compressors. The authors in [11] have implemented a preprocessing reduction technique based on graph theory and nonlinear functional analysis in order to consider larger instances. In [7], a customized direct solution algorithm based on the study of the structure of the KKT systems arising in interior methods is developpped. For a more detailed description of the above mentioned problems dealing with gas transmission networks, we refer the reader to two recent thesis on the subject [13, 10].

In this paper, we propose a new approach to the classical operations and investment problem in gas transmission network. As stated in the literature, the energy that is injected in the network though the compressors has a cost, namely the energy cost. This cost is considered important enough to have generated much researches. Nevertheless we observe that when minimizing the supply or investment costs, this energy cost is always left out in traditional analyses. The main idea of the paper is that the managerial costs must be balanced with the energy cost. The proposed formulation is based on the minimum energy principle [9] which has the main advantage to leads to an unique solution of a convex optimization problem. The managerial costs are then handled through a weighted mutli-objective functions and making an appropriate assumption on the investment cost the overall objective function remains convex and easily solvable. The proposed approach is peculiar in that it does not provide only one solution but possibly a battery of solutions depending on the weight the user want to put on the energy cost versus the supply/investment costs. Given those possible solutions, the user must arbitrage between them. The problem being robust and easy computable makes it an appropriate solution method for the continuous relaxation in MIP approaches such that Branch-and-Bound algorithms.

The paper is organized as following. In the first section, we recall the different problems of gas transmission networks. In Section 3 we introduce a new formulation for the investment and operations problems. In Sections 4 and 5 we formulate the management, operations and investment problem as a single joint optimization problem and we propose a nex convex version. The next section is devoted to the characterizations of the solution. Then Section 7 provides an application to the Belgian gas network, and finally, a conclusion ends the paper.

2 Design and operations of gas transmission networks

We consider first an existing gas transmission network. It is modeled as a graph $G = (V, E)$, with certain arc and node characteristics. The arcs support flows in either direction, so it is convenient to put an arbitrary orientation on the graph. We denote $V_s \subset V$, $V_d \subset V$ and $V_t \subset V$, the subsets of nodes at which there is an inflow (supply nodes), an outflow (demand nodes), and no inflow and outflow (transit nodes), respectively. The subsets form a partition of V. Let $A \in \mathbb{R}^{|V|\times|E|}$ be the incidence matrix of the graph and $B \in \mathbb{R}^{|V|\times|V|}$ a diagonal matrix with diagonal element

$$B_{ii} = \begin{cases} 1 & \text{if} \quad i \in V_s \cup V_d \\ 0 & \text{if} \quad i \in V_t \end{cases}$$

Flows through the network are represented by a vector $x \in \mathbb{R}^{|E|}$ of flows on the arcs and a vector $y \in \mathbb{R}^{|V|}$ of inflows (if $i \in V_s$) or outflows (if $i \in V_d$).

Flows on the network must satisfy the mass balance equation

$$Ax - By = 0.$$

We also impose

$$y_i \geq 0, \; i \in V_s,$$

$$y_i \leq 0, \; i \in V_d.$$

Note that by definition of B, $(Ax)_i = 0$ if $i \in V_t$. Such node is a transmission node. The value of y_i at this node is then irrelevant for the problem. The default value is $y_i = 0$.

The gas transmission is a pure feasibility problem. It outputs flows in the network that meet the supply and demand constraints. The laws of physics determine the pressures at the end nodes that are required to sustain the flow. We shall name this problem the *management problem*. In this paper we consider two other problems that are at a higher hierarchical level. The *operations problem* concerns purchase and delivery policies to optimize profits at the current market prices. The second problem is the *investment problem*. It deals with the reinforcement of existing arcs, or the creation of new arcs, so as to accommodate higher flows through the network. Both problems compute solutions which should ultimately be compatible with the management problem.

The gas flows into a transmission line $a = (i, j) \in E$ because of a pressure differential between the two end nodes of the line (arc) according to the formula (see, e.g., [5])

$$l_a \beta \frac{x_a |x_a|}{D_a^5} = p_i^2 - p_j^2. \tag{1}$$

In this formula, l_a is the length of the transmission line, β is a technical coefficient (usually the same for all pipes), D_a is the internal diameter of the pipe, and p_i is the pressure at node i. Note that the direction of the flow is determined by the sign of the pressure differential. The term on the left is associated with friction losses along the pipe. Formula (1) is a simplification of a more accurate expression, but it is commonly used in problems of the type considered in this paper.

The gas transmission problem can be worded as: is it possible to find the necessary pressure at the supply nodes (provided by the compressors) and the release pressure at the demand nodes (ensured by regulators) that enable flows satisfying the physical law (1), the mass conservation principle and the supply and demand requirements? In the literature [5, 6] the mathematical formulation is:

Find a flow (x, y) and a system of pressures p such that

$$\begin{aligned}
l_a \beta \frac{x_a |x_a|}{D_a^5} &= p_i^2 - p_j^2, \ \forall a \in E, \text{ such that } a = (i,j), \ i,j \in V && \text{(2a)} \\
\underline{p}_i \leq p_i &\leq \bar{p}_i, \ \forall i \in V_s \cup V_d && \text{(2b)} \\
Ax - By &= 0 && \text{(2c)} \\
\underline{\phi}_i \leq y_i &\leq \bar{\phi}_i, \ \forall i \in V_s \cup V_d. && \text{(2d)}
\end{aligned}$$

In that formulation, $\bar{\phi}_i \leq 0$ for $i \in V_d$ and $\underline{\phi}_i \geq 0$ for $i \in V_s$.

The operations problem focuses on the difference between the revenue of deliveries minus the total supply costs. The operations cost is associated with the input and output flows. We denote it $\mathcal{O}(y)$. It is a linear function of the vector of supply and demand y. Defining c as the price vector ($c_i \geq 0$, if $i \in V_s$, $c_i = 0$, if $i \in V_t$, and $c_i \leq 0$, if $i \in V_d$). Thus

$$\mathcal{O}(y) = \langle c, y \rangle. \tag{3}$$

The minimization of the operations must take into account physical constraints. Because the purchased gas has to flow through the network, the operations policy is strongly dependent on the physical characteristics of the network. Suppose for instance, that the supply is much cheaper at some supply nodes than at any other one. One would certainly like to purchase as much as possible from that node. But if some demand nodes are very remote from this unique supply node, the gas should be injected at a considerable pressure to have a chance to reach the remote demand nodes. The compressor and regulator equipments limit the pressure adjustments, and operations must account for this constraint.

The investment problem focuses on reinforcing existing connections or building new ones to accommodate new supplies and demands. A short qualitative discussion is useful to clarify the stakes. Consider a single connection line from a supply node to a demand one. Suppose that the current pipe carries a certain flow with a differential of the squared pressures given by equation (2a). To meet a higher demand with the existing pipe, one must increase the pressure differential, but the increase may quickly hit the technical limits of the compressor. The alternative is either to change the pipe for one with a larger internal diameter, or to reinforce the pipe with a new one. The issue on larger and more complex networks is to determined the location and the dimension of new pipes so as to meet given supply and demand. The investment problem is the one of finding the least cost investment. To formulate this cost we introduce the set of new arcs E_n that supplement the existing arcs E_e. The investment cost is separable and can be written

$$\mathcal{I}(D) = \sum_{a \in E_n} \mathcal{I}_a(D_a). \tag{4}$$

We shall provide an analytic expression later.

The problem of optimal design and operations can be formulated as

$$\min_{\substack{x,y,p \\ D_a,\, a \in E_n}} \quad \mathcal{I}(D) + \mathcal{O}(y) \tag{5a}$$

$$l_a \beta \frac{x_a |x_a|}{D_a^5} = p_i^2 - p_j^2, \ \forall a \in E \cup E_n, \quad \text{such that } a = (i,j), \ i,j \in V \tag{5b}$$

$$\underline{p}_i \leq p_i \leq \bar{p}_i, \ \forall i \in V_s \cup V_d \tag{5c}$$

$$Ax - By = 0 \tag{5d}$$

$$\underline{\phi}_i \leq y_i \leq \bar{\phi}_i, \ \forall i \in V_s \cup V_d. \tag{5e}$$

In that formulation, the vector x, y and p are decision variables. Constraints (5b)–(5e) guarantee that there exists a feasible stationary flow for the set of diameters D. This problem is known to be non-convex, even if the objective is convex. The thrust of this paper is to provide an alternative convex formulation of the design and operations problem.

3 A new formulation of the investment and operations problems

The first step in our search for a convex formulation is to provide a new characterization of stationary flows. Essentially, we prove that conditions (2a)–(2d) are the variational inequality formulation of a convex minimization problem, whose objective can be interpreted as the total energy in the gas transmission network. Hence the search for a feasible flow in (2) boils down to solving a convex minimization problem, a considerable simplification. In particular, the minimization problem automatically handles the flow orientation, even on a network with loops.

Consider the following optimization problem

$$\min_{x,y} \quad \mathcal{E}(x) - \langle d, y \rangle \tag{6a}$$

$$Ax - By = 0, \tag{6b}$$

where $\mathcal{E}(x)$ is the separable function

$$\mathcal{E}(x) = \sum_{a \in E} l_a \frac{\beta}{D_a^5} \frac{|x_a|^3}{3}, \tag{7}$$

and d is a fixed vector in $\mathbb{R}^{|V|}$ satisfying $d_i = 0$, $\forall i \in V_t$. The condition on the parameter d can be written in matrix form as $Bd = d$. Note that $\mathcal{E}$ is continuously differentiable except at $x_a = 0$. In the next section we will interpret problem (6) as the minimization of total energy. In the sequel we shall refer to "energy" to designate the two terms $\mathcal{E}(x)$ and $\langle d, y \rangle$.

Problem (6) can be formulated as a problem in x only. In view of $Bd = d$ one has

$$\langle d, y \rangle = \langle Bd, y \rangle = \langle d, By \rangle = \langle d, Ax \rangle = \langle A^T d, x \rangle.$$

These relations make it possible to eliminate all rows $i \in V_s \cup V_d$ in the constraint $Ax - By = 0$. We are left with the constraints $(Ax)_i = 0$, for $i \in V_t$. Let I be the $|V| \times |V|$ identity matrix; then, $I - B$ is a diagonal matrix whose only unit entries are on the diagonal elements $i \in V_t$. The condition $(Ax)_i = 0$, for $i \in V_t$ is conveniently represented by

$$(I - B)Ax = 0.$$

Problem (6) can now be reformulated as

$$\min \quad \mathcal{E}(x) - \langle A^T d, x \rangle \tag{8a}$$

$$(I - B)Ax = 0. \tag{8b}$$

This problem in the x variable alone always has a solution and this solution is unique since the objective is strictly convex. The vector y of inflows and outflows can be retrieved by the simple relation $y = Ax$. Indeed, the constraint $(I - B)Ax = 0$ implies $y_i = 0$, $i \in V_t$.

Theorem 1 *Assume $(x^*, y^*; p^*)$ solves problem (2) with $\mathcal{E}(x)$ defined by (7). Then problem (8) with $d_i = (p_i^*)^2$, $i \in V_s \cup V_d$ and $d_i = 0$, $i \in V_t$, has solution x^*.*

Proof: From the definition of (7),

$$\mathcal{E}'_a(x_a^*) = l_a \beta \frac{x_a |x_a|}{D_a^5}.$$

So by (5b), one has

$$\mathcal{E}'_a(x_a^*) = (p_{i_a}^*)^2 - (p_{j_a}^*)^2.$$

Let $\mu \in \mathbb{R}^{|V|}$ be defined by $\mu_i = (p_i^*)^2$, $i \in V$. Define $d = B\mu$ and $\lambda = -\mu$. We may write

$$\mathcal{E}'(x^*) = A^T\mu = A^T(B\mu + (I-B)\mu) = A^T d - A^T(I-B)\lambda.$$

Therefore, the following conditions hold

$$\begin{aligned} \mathcal{E}'(x^*) - A^T d + A^T(I-B)\lambda &= 0 \qquad (9a) \\ (I-B)Ax^* &= 0. \qquad (9b) \end{aligned}$$

Equations (9a) and (9b) are the necessary and sufficient optimality conditions of the strictly convex problem (8). ■

We now discuss whether the solution of (8), or a variant of it, provides an answer to the feasibility problem (2)? Consider problem (8) with additional bounds constraints on the flows

$$\begin{aligned} \min \quad & \mathcal{E}(x) - \langle A^T d, x\rangle \qquad (10a) \\ & \underline{\phi} \le Ax \le \bar{\phi}. \qquad (10b) \end{aligned}$$

where $\underline{\phi}$ and $\bar{\phi}$ are vectors in $\mathbb{R}^{|V|}$, that extend the vectors defined in problem (2) to $\underline{\phi}_i = 0 = \bar{\phi}_i$, for $i \in V_t$. Let $d \in \mathbb{R}^{|V|}$ be an arbitrary vector satisfying $Bd = d$. Assuming that the problem with this d has a solution, we can write the first order optimality conditions as

$$\begin{aligned} \mathcal{E}'(x) - A^T d + A^T(\pi^+ - \pi^-) &= 0 \qquad (11a) \\ \langle \pi^+, Ax - \bar{\phi}\rangle &= 0 \qquad (11b) \\ \langle \pi^-, \underline{\phi} - Ax\rangle &= 0 \qquad (11c) \\ \underline{\phi} \le Ax \le \bar{\phi} & \qquad (11d) \\ \pi^+ \ge 0,\ \pi^- \ge 0. & \qquad (11e) \end{aligned}$$

Theorem 2 *Assume the vectors $(x; \pi^+, \pi^-)$ satisfy the necessary and sufficient optimality conditions (11) for problem (10) with d defined by (12). Let*

$$\gamma = -\min\{d_i - \underline{p}_i^2 - \pi_i^+ + \pi_i^-\}.$$

Then $\mu = d - \pi^+ + \pi^- + \gamma e$, where $e \in \mathbb{R}^{|V|}$ is the vector of all ones, defines a set of admissible pressure $p_i = \sqrt{\mu_i}$ for problem (2) with no upper bound in (2b) ($\bar{p} = +\infty$).

Proof: By construction, $\mu \geq d \geq 0$. Therefore, $p_i = \sqrt{\mu_i}$, $\forall\ i \in V$, is well-defined and $\underline{p}_i \leq p_i$, $i \in V_s \cup V_d$. So (2b), (2c) and (2d) are satisfied. To prove (2a), we note that

$$A^T \mu = A^T(d - \pi^+ + \pi^- + \gamma e) = A^T d + A^T(\pi^- - \pi^+)$$

because A is an incidence matrix and $A^T e = 0$. Hence, $\mathcal{E}'(x) = A^T \mu$, which is the same as $l_a \beta \frac{x_a |x_a|}{D_a^5} = p_i^2 - p_j^2$, $\forall a \in E$, such that $a = (i,j)$, $i,j \in V$. This concludes the proof. ∎

Remark 1 *Theorem 2 does not guarantee that the computed pressures meet the upper bound constraints $p_i \leq \bar{p}_i$, $i \in V_s \cup V_d$. If it turns out that the solution satisfy this extra condition, then this solution solves problem (2) in its original formulation.*

Remark 2 *A sensible choice for d could be*

$$d_i = \begin{cases} \underline{p}_i^2, \ i \in V_s \cup V_d \\ d_i = 0, \ i \in V_t. \end{cases} \tag{12}$$

Let us establish the relationship between the two formulations of the minimum energy problem, namely problem (8) and problem (10).

Theorem 3 *Let*

$$z^* = \min\{\mathcal{E}(x) - \langle A^T d, x\rangle \mid \underline{\phi} \leq Ax \leq \bar{\phi}\}$$

be the optimal value of (10) with optimal primal and dual solutions $(x^*; (\pi^+)^*, (\pi^-)^*)$. Let also

$$t^* = \min\{\mathcal{E}(x) - \langle A^T \delta, x\rangle \mid (I - B)Ax = 0\}$$

be the optimal value of problem (8) with $\delta = B(d + (\pi^-)^* - (\pi^+)^*) = d + B((\pi^-)^* - (\pi^+)^*)$. Then x^* solves problem (8) and

$$t^* = z^* + \langle (\pi^+)^*, \bar{\phi}\rangle - \langle (\pi^-)^*, \underline{\phi}\rangle.$$

Proof: By duality

$$\begin{aligned} z^* &= \min_x \max_{\pi^+ \geq 0, \pi^- \geq 0} \{\mathcal{E}(x) - \langle A^T d, x\rangle + \langle \pi^+, Ax - \bar{\phi}\rangle + \langle \pi^-, -Ax + \underline{\phi}\rangle\} \\ &= \max_{\pi^+ \geq 0, \pi^- \geq 0} \left\{ -\langle \pi^+, \bar{\phi}\rangle + \langle \pi^+, \underline{\phi}\rangle + \min_x \{\mathcal{E}(x) - \langle A^T d - (\pi^+ - \pi^-), x\rangle\}\right\} \\ &= -\langle (\pi^+)^*, \bar{\phi}\rangle + \langle (\pi^+)^*, \underline{\phi}\rangle + \mathcal{E}(x^*) - \langle A^T(d - (\pi^+)^* + (\pi^-)^*), x^*\rangle. \end{aligned}$$

The necessary and sufficient optimality condition for the inner minimization problem is

$$\mathcal{E}'(x^*) = A^T(d - (\pi^+)^* + (\pi^-)^*).$$

On the other hand, the necessary and sufficient optimality condition for problem (8) is

$$\mathcal{E}'(x) - A^T\delta + A^T(I - B)\lambda = 0$$

for some $\lambda \in \mathbb{R}^{|V|}$. If we take $\lambda = -(I - B)(d - (\pi^+)^* + (\pi^-)^*)$ and $x = x^*$, then

$$\begin{aligned}
&\mathcal{E}'(x^*) - A^T\delta + A^T(I - B)\lambda = \\
&\quad = \mathcal{E}'(x^*) - A^T\delta + A^T\lambda \quad \text{(because } (I - B)\lambda = \lambda\text{)} \\
&\quad = \mathcal{E}'(x^*) - A^T\left(B(d - (\pi^+)^* + (\pi^-)^*) + (I - B)(d - (\pi^+)^* + (\pi^-)^*)\right) \\
&\quad = \mathcal{E}'(x^*) - A^T(d - (\pi^+)^* + (\pi^-)^*) = 0.
\end{aligned}$$

This proves that x^* solves problem (8). We have

$$z^* - t^* = \langle A^T(\delta - d), x^*\rangle = \langle B((\pi^-)^* - (\pi^+)^*), Ax^*\rangle.$$

Recall that $(I - B)Ax^* = 0$. Thus

$$\begin{aligned}
z^* - t^* &= \langle (\pi^-)^* - (\pi^+)^*, BAx^*\rangle \\
&= \langle (\pi^-)^* - (\pi^+)^*, BAx^*\rangle + \langle (\pi^-)^* - (\pi^+)^*, (I - B)Ax^*\rangle \\
&= \langle (\pi^-)^* - (\pi^+)^*, Ax^*\rangle \\
&= \langle (\pi^-)^*, \underline{\phi}\rangle - \langle (\pi^+)^*, \bar{\phi}\rangle.
\end{aligned}$$

The last equation stems from the complementarity conditions

$$\langle (\pi^+)^*, Ax^* - \bar{\phi}\rangle = 0 = \langle (\pi^-)^*, -Ax^* + \underline{\phi}\rangle.$$

■

It seems reasonable to argue that the energy that is necessary to move the flows in the system is given by the optimal value t^* of problem (8) with no bound on the inflows and outflows. The difference $\langle (\pi^+)^*, \bar{\phi}\rangle - \langle (\pi^-)^*, \underline{\phi}\rangle$ between t^* and z^* is a simple function of optimal dual variables for problem (10). It is thus a simple by-product of the solving of (10).

To conclude this section, we briefly discuss an extension of the base problem (6). In practical cases, it is sometimes desirable to introduce a mechanism to compensate for excessive pressure drop along a chain of pipes.

The mechanism is a compressor, and an arc endowed with a compressor is usually called *active*. The compressor injects potential energy in the system, but its action is directional. The action of the device is independent of its location on the arc. Therefore, to differentiate the two energy modifications along the arc, gain in potential energy and loss in friction, we choose to replace an active arc (i, j) by two adjacent arcs (i, j') and (j', j). The first arc has the same length as (i, j) and is a *passive* arc. The second, with length 0 (no friction loss), is where the potential energy is injected. The change in the mathematical model boils down to replacing the variable $x_{i,j}$ by $x_{i,j'}$ and $x_{j',j}$ with the constraints $x_{i,j'} = x_{j',j}$ and $x_{j',j} \geq 0$. In the objective, the friction term (7) in the objective is associated with $x_{i,j'}$, while the compressor potential energy is given by the term $-d_{j',j}x_{j',j}$, where $d_{j',j} \geq 0$ is the potential gain per unit of flow.

4 The joint optimization problem

Theorem 2 motivates the search for an interpretation of the quantity $\mathcal{E}(x) - \langle d, y\rangle$ that defines the objectives of problems (8) and (10). The flows in the gas transmission network constitutes a physical system linked to the external world by input and output flows. We showed that the stationary of this system is the (unique) minimizer of a certain function, which, according to the physical *minimum energy principle*, should be interpreted as a total energy. We shall link the first term $\mathcal{E}(x)$ to the energy dissipated within the system, while the second one $-\langle d, y\rangle$ connects the system with the external world.

In the literature, the quantity

$$\mathcal{E}'_a(x_a) = l_a\beta\frac{x_a|x_a|}{D_a^5}$$

is interpreted as the friction loss along the arc a per unit of flow and per unit of time. Its integral with respect to x_a can thus be viewed as the *energy* that is dissipated per unit of time. We shall call this energy[2] *friction*. Let us propose an interpretation of the linear term $-\langle d, y\rangle$. For a supply node $i \in V_s$, the quantity $-d_iy_i \leq 0$ is the energy that the external world puts into the system by pushing a flow y_i at a certain pressure $p_i = \sqrt{d_i}$. For a demand node $i \in V_d$, the quantity $-d_iy_i \geq 0$ is the energy returned by

[2]As far as physics is concerned, the present view is simplistic and approximative. The actual laws of compressible flows is much more complicated, but the simplified form is commonly accepted for global design and analyzes of transmission networks.

the system to the outside world. It is delivered by the release of a flow $-y_i \geq 0$ at the pressure $p_i = \sqrt{d_i}$. The net balance $-\langle d, y \rangle$ takes the form of a *potential energy.* The sum of the friction and potential energies is the total energy of the system. The stationary, or equilibrium, state is one that minimizes the total energy, as expressed in problem (10).

The formulation of the management problem as the minimization of total energy has the further advantage of making apparent a third category of costs that are not explicitly taken into account in traditional analyzes. Namely, the energy dissipated in the gas transmission has to be supplied by some source external to the network, essentially via the compressors. This supplied energy has a cost, which depends on the cost of power supply but also on the equipment efficiencies, which are known to be often nonlinear functions of the flows. Nevertheless, the dissipated energy is a reasonable surrogate for the *management costs* associated with the previously described management problem. The central idea of this paper is that the other managerial costs, those associated with the operations policy of purchases and sales, and those associated with investments, should be balanced with the energy cost. This gives a new approach to the classical operations and investment problem in gas transmission network.

We are now in position to formalize the joint investment-operations-management problem as joint minimization problem. The objective combines the management cost $\mathcal{E}(x; D) - \langle d, y \rangle$ with the operations cost (3) and the investment cost (4) by means of two positive coefficients $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. The problem is now

$$\min_{(x,y)} \min_{(D_a \leq \bar{D},\, a \in E_n)} \quad \mathcal{E}(x; D) - \langle d, y \rangle + \alpha_1 \mathcal{I}(D) + \alpha_2 \langle c, y \rangle \tag{13a}$$

$$\underline{\phi} \leq y = Ax \leq \bar{\phi}. \tag{13b}$$

An important issue is whether this problem is convex and easily solvable. We shall show in the next section that the answer is positive, if we make an appropriate assumption on the investment cost function.

In the above discussion, we did not include the mechanical devices that allow insertion and release of flows at given pressures. Those devices are compressors and regulators. They are located[3] at supply nodes (compressors) and at demand node (regulator). The compressor is powered by some external energy, say electric power. The regulator is a passive device, like

[3]As mentioned in the end of the previous section, compressors may also be inserted at intermediary nodes to compensate for excessive pressure drop in some parts of the network. We do not consider this case here.

a faucet, which causes energy dissipating turbulence. One could imagine a more comprehensive energy model that would explicitly include the energy at the regulators and compressors. This has been done water distribution systems[4] equipped with two-mode faucet devices (fully open or fully closed). The energy dissipated in the regulator is proportional to the outflow at the power 3. The analysis of a compressor is much more complex; we completely leave it out of the present paper.

5 A convex version of the joint optimization problem

In the literature, the investment cost is often approximated by a quadratic function. In this paper, we use another approximation. Namely,

Assumption 1

$$\mathcal{I}_2(D) = l(k_1 D^{2.5} + k_2), \tag{14}$$

where l is the length of the arc.

It turns out that this cost function fits reasonably well data that can be found on pipes, for gas, water or other fluid transmission. For instance, [5] proposes the equation

$$\mathcal{I}(D) = l(2.5180\ 10^{-5} D^2 + 7.4782\ 10^{-3} D + 7.7476). \tag{15}$$

It was obtained by least fitting a quadratic polynomial to real data with D in the range 180 to 900. (See Figure 8 in [5].) The reported correlation coefficient is $r^2 = 0.998$. We have computed the best approximation of (15) by a function of type (14) (in the sense of the L_2 norm) and obtained

$$\mathcal{I}_2(D) = l(1.0408\ 10^{-6} D^{2.5} + 11.2155). \tag{16}$$

To check the validity of this approximation, we computed the correlation coefficient on the range of D and obtained $r^2 = 0.994$. This value hides non-negligible discrepancies, which can be visualized on the following graph. The relative approximation error is relatively high for small diameters (up to 17%, but less than 4% for diameters in the range [300,900]).

[4]See the software NeatWork and its user guide.

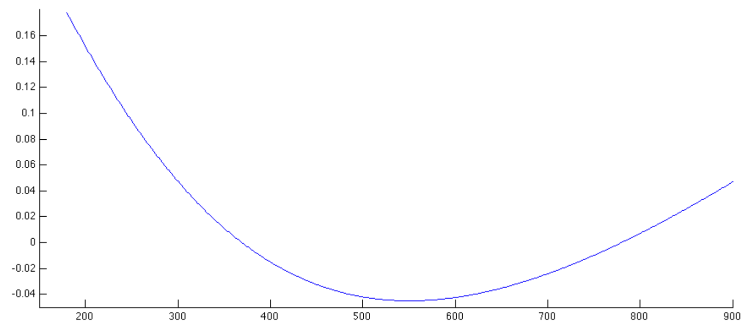

Figure 1: Relative error on the cost approximation,

It is reasonable to assume that there is a maximal admissible diameter $\bar{D}$. The main problem (13) can be formulated as

$$\begin{aligned} \min_{(x,y)} \quad & \min_{(D_a \le \bar{D},\, a \in E_n)} \{\mathcal{E}(x;D) + \alpha_1 \mathcal{I}(D)\} - \langle d - \alpha_2 c, y \rangle \\ & (I - B)Ax = 0 \\ & \underline{\chi} \le y = BAx \le \bar{\chi}. \end{aligned}$$

In this formulation, one first minimizes in D with fixed x, and then minimizes in x and y. The minimization problem on D concerns the new arcs only and is separable. Assuming that $a \in E_n$ is an arc on which an investment takes place, we want to solve

$$\mathcal{C}_a(x_a) = \min_{D_a} \left\{ l_a \frac{\beta}{3} \frac{|x_a|^3}{D_a^5} + \alpha_1 l_a k_1 D_a^{2.5} \mid D_a \le \bar{D}_a \} \right\}. \tag{17}$$

In this formulation, we dropped the constant term k_2 of the investment cost function, because it plays no role in the optimal sizing of D. In the sequel, we drop the subscript in k_1 and use k instead.

Theorem 4 *$\mathcal{C}_a(x_a)$ is convex. Moreover*

$$\mathcal{C}_a(x_a) = \begin{cases} l_a \beta^{1/3} \Big(\dfrac{3\alpha_1 k}{2}\Big)^{2/3} |x_a| & \textit{if} \quad |x_a| < \bar{x} \\ l_a \dfrac{\beta}{3} \dfrac{|x_a|^3}{\bar{D}_a^5} + \alpha_1 l_a k \bar{D}_a^{5/2} & \textit{if} \quad |x_a| \ge \bar{x}, \end{cases}$$

with $\bar{x} = \left(\frac{3\alpha_1 k}{2\beta}\right)^{1/3} \bar{D}^{5/2}$.

Proof: For the sake of a simpler presentation, we shall drop the index a in (17) and take $\alpha_1 = \alpha$. The problem to be solved is

$$\frac{\mathcal{C}(x)}{l} = \min_{D \leq \bar{D}} \left\{ \frac{\beta}{3} \frac{|x|^3}{D^5} + \alpha k D^{2.5} \right\}.$$

Assume first that the optimal diameter satisfies $D < \bar{D}$. The function is clearly convex in D. Its minimum is achieved at the unique zero of its derivative

$$-\frac{5\beta}{3} \frac{|x|^3}{D^6} + \frac{5}{2} \alpha k D^{1.5} = 0.$$

Namely,

$$D^{7.5} = \frac{2\beta}{3\alpha k} |x|^3.$$

Substituting the optimal value in the cost function per unit of length, we obtain

$$\begin{aligned}
\mathcal{C}(x)/l &= \left(\frac{\beta (\alpha k)^2}{3} \right)^{1/3} \left(\frac{1}{2} \right)^{2/3} |x| + \left(\frac{\beta (\alpha k)^2}{3} \right)^{1/3} 2^{1/3} |x| \\
&= \left(\frac{\beta (\alpha k)^2}{3} \right)^{1/3} \left[2^{-2/3} + 2^{1/3} \right] |x| \\
&= \beta^{1/3} \left(\frac{3\alpha k}{2} \right)^{2/3} |x|.
\end{aligned}$$

This solution prevails as long as the optimal diameter

$$D = \left(\frac{2\beta}{3\alpha k} \right)^{2/15} |x|^{2/5} < \bar{D},$$

equivalently as long as

$$|x| < \bar{x} = \left(\frac{3\alpha k}{2\beta} \right)^{1/3} \bar{D}^{5/2}.$$

If $|x| \geq \bar{x}$, the derivative at $\bar{D}$ is

$$-\frac{5\beta}{3} \frac{|x|^3}{\bar{D}^6} + \frac{5}{2} \alpha k \bar{D}^{1.5} \leq -\frac{5\beta}{3} \frac{\bar{x}^3}{\bar{D}^6} + \frac{5}{2} \alpha k \bar{D}^{1.5} = 0.$$

By convexity, the condition on the sign of the derivative implies that the minimum cost occurs at $D = \bar{D}$. ■

Remark 3 *The limit flow $\bar{x}$ in the theorem depends on the maximum diameter $\bar{D}$ but not on the length of the arc. It is thus the same for all arcs.*

The compound cost is thus a convex function. It is continuously differentiable at any $x_a \neq 0$. One easily checks that the left and right derivatives at x_a, with $|x_a| = \bar{x}$, are both equal to

$$\mathcal{C}_a'(x_a) = l_a \beta^{1/3} \left(\frac{3\alpha_1 k}{2}\right)^{2/3} \frac{x_a}{\bar{x}}.$$

On the other hand, the compound cost is non differentiable at $x_a = 0$. Its subgradient set at the origin is the interval

$$\partial_x C(x)_{x=0} = \left[-l_a \beta^{1/3} \left(\frac{3\alpha k}{2}\right)^{2/3}, \; l_a \beta^{1/3} \left(\frac{3\alpha k}{2}\right)^{2/3} \right].$$

Note that the bounds of the interval are simple functions of the ratio k/β and the tradeoff factor α. This coefficient is the same on all arcs subject to investment.

It is also interesting to compute the pure investment cost (per unit of length) when $|x_a| < \bar{x}$

$$cD_a^{2.5} = \frac{1}{\alpha} \left(\frac{2\beta(\alpha k)^2}{3}\right)^{1/3} |x_a| = \left(\frac{2\beta k^2}{3\alpha}\right)^{1/3} |x_a|.$$

6 Characterizations of the solution

The extension of the existing network consists in duplicating all the arcs. The problem is to size the pipes on the new arcs. Let us denote A_e and A_n the incidence matrices associated with graphs $G_e = (V, E_e)$ and $G_n = (V, E_n)$. Let also x_{E_e} and x_{E_n} be the vectors of flows on the arcs of E_e and E_n. The simplified formulation of the gas network design can be written as

$$\min\{\mathcal{C}_e(x_{E_e}) + \mathcal{C}_n(x_{E_n}) + \mathcal{C}_2(y) \mid \underline{\phi} \leq y = A_e x_{E_e} + A_n x_{E_n} \leq \bar{\phi}\}.$$

On the existing arcs E_e, the term $\mathcal{C}_e(x_{E_e})$ represents the energy component. It is a separable function written as

$$\mathcal{C}_e(x_{E_e}) = \sum_{a \in E_e} l_a \frac{\beta}{3D_a^5} |x_a|^3.$$

On the new arcs E_n, the term $\mathcal{C}_n(x_{E_n})$ represents the combination of the management cost (dissipated energy) and the investment cost for an optimally chosen diameter. The function is also separable

$$\mathcal{C}_n(x_{E_n}) = \sum_{a \in E_n} \mathcal{C}_a(x_a),$$

with

$$C_a(x_a) = \begin{cases} l_a \beta^{1/3} \left(\dfrac{3\alpha_1 k}{2} \right)^{2/3} |x_a| & \text{if} \quad |x_a| < \bar{x} \\ l_a \dfrac{\beta}{3} \dfrac{|x_a|^3}{\bar{D}_a^5} + \alpha_1 l_a k \bar{D}_a^{5/2} & \text{if} \quad |x_a| \geq \bar{x}. \end{cases}$$

Note that the breaking point value in the formula $\bar{x} = \left(\frac{3\alpha_1 k}{2\beta} \right)^{1/3} \bar{D}_a^{5/2}$ is the same for all arcs.

The last component $\mathcal{C}_2$ combines the management cost (energy) at the inflow and outflow nodes with the purchase and selling values of the in and out flows

$$\mathcal{C}_2(y) = -\langle d - \alpha_2 c, y \rangle.$$

If we associate the dual variables $\pi^+ \geq 0$ and $\pi^- \geq 0$ to the constraints $y \leq \bar{\chi}$ and $-y \leq -\underline{\chi}$, and the dual variable λ to $Ax - y = 0$, we have the first order optimality conditions

$$\begin{array}{rcl} (\pi^- - \pi^+) + \lambda & = & -d + \alpha_2 c \\ \langle \pi^+, \bar{\chi} - y \rangle & = & 0 \\ \langle \pi^-, y - \underline{\chi} \rangle & = & 0 \\ -A_e^T \lambda & = & \mathcal{C}_e'(x_{E_e}) \\ -A_n^T \lambda & \in & \partial_{x_{E_n}} \mathcal{C}_n(x_{E_n}). \end{array}$$

Denoting $\Delta_a = |(A_n^T \lambda)_a| / l_a$ the friction loss on arc a per unit of length, we conclude from the last inclusion the inequalities

$$\begin{array}{rcll} \Delta_a & \leq & \beta^{1/3} \left(\dfrac{3\alpha_1 k}{2} \right)^{2/3}, & \text{if} \;\; x_a = 0 \\ \Delta_a & = & \beta^{1/3} \left(\dfrac{3\alpha_1 k}{2} \right)^{2/3}, & \text{if} \;\; 0 < |x_a| \leq \bar{x} \\ \Delta_a = \beta \dfrac{|x_a|^2}{\bar{D}_a^5} & > & \beta^{1/3} \left(\dfrac{3\alpha k}{2} \right)^{2/3}, & \text{if} \;\; |x_a| > \bar{x}. \end{array}$$

From these formulas, we derive the following conclusion:

Theorem 5 *Let* $\nu = \beta^{1/3}\left(\frac{3\alpha_1 k}{2}\right)^{2/3}$. *Depending of the relative value of* Δ_a *with respect to the critical factor* ν_a, *we have*

- $\Delta_a < \nu \Rightarrow x_a = 0$ *and* $D_a = 0$
- $\Delta_a = \nu \Rightarrow 0 < |x_a| \leq \bar{x}$ *and* $0 < D_a \leq \bar{D}$
- $\Delta_a > \nu \Rightarrow |x_a| > \bar{x}$ *and* $D_a = \bar{D}$.

We can also derive an interesting consequence for any new arc $a \in E_n$ that reinforces an existing arc in E_e (i.e., an arc with same origin and destination, and same length). Let a_n and a_e be a pair of such arcs. The two arcs having the same extremities, we have $(A_n^T\lambda)_{a_n} = (A_e^T\lambda)_{a_e}$ and $l_{a_e} = l_a = l_{a_n}$. If furthermore $x_{a_n} \neq 0$, we have

$$\frac{l_a\beta}{D_{a_e}^5}x_{a_e}|x_{a_e}| = l_a\beta^{1/3}\left(\frac{3\alpha k}{2}\right)^{2/3}\frac{x_{a_n}}{|x_{a_n}|}$$

and thus

$$|x_{a_e}| = D_{a_e}^{5/2}\left(\frac{3\alpha k}{2\beta}\right)^{1/3}.$$

The sign of x_{a_e} is the same as the sign of $-(A_e^T\lambda)_{a_e}$. The above relation provides a handy way to compute exact solutions.

The pure investment problem is another interesting case. Given node points and potential connexions between those nodes, one seeks which and where pipes are to be installed to meet the demand and supply constraints with the usual pressure constraints. We can use our analysis, with the proviso that it it is based on the variable investment cost $lk_1D^{2.5}$ and does not consider the fixed part lk_2. If we assume $\bar{D} = \infty$, it is easily seen that neglecting the fixed cost leads to a linear programming problem, because each component in the objective is linear in the absolute value of the flow[5] on the arc. In practice, the constant term k_2 cannot be neglected. For diameters in the range $[200, 900]$ the variable part $k_1D^{2.5}$ and the fixed part k_2 in formula (16) have comparable magnitude. Therefore, one must resort to a mixed integer linear programming. We simply add for each potential arc a binary variable $\eta \in \{0, 1\}$ and write the combined objective

$$l\beta^{1/3}\left(\frac{3\alpha_1 k_1}{2}\right)^{2/3}|x| + l\alpha_1 k_2\eta, \text{ subject to } |x| \leq M\eta$$

[5] The flow on the arc can be expressed as the difference of two opposite positive flows; its absolute value is bounded above by the sum of the two flows. Hence, the transformation into a linear programming problem.

for some large M.

Among possible optimal solutions of the problem with variable investment cost only, there is always a tree. Thus, loops can be avoided in the optimal solution, but there is a situation where they may pop-up naturally in the computations. We show here that a single connexion can always be replaced by several pipes in parallel with the same performance relative to variable investment cost and to the energy.

Theorem 6 *A pipe with diameter D connecting two nodes has the same variable investment cost and induces the same friction as m parallel identical pipes with diameter $m^{-2/5}D$. If the fixed part of the investment cost is non-zero, a single pipe is always more efficient than m parallel pipes.*

Proof: Since the quantity of interest are all proportional to the length of the pipes, we assume without loss of generality that this length is one. The investment cost for the m pipes is $mk_1(m^{-2/5}D)^{2.5} = k_1D^{2.5}$. This proves the first statement in the theorem. To prove the second statement, we note that the flows among the identical pipes. Let x be the flow though the unique pipe with diameter D and assume that the flows on the m pipes with diameter $m^{-2/5}D$ is x/m. The energy term in the case with m pipes is

$$m\beta\frac{|x/m|^3}{(m^{-2/5}D)^5} = \beta\frac{|x|^3}{D^5}.$$

This proves the second claim and the theorem. ■

So far, we have developed an approach that computes a solution that minimizes combined cost with given parameters $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. Theorem 2 ensures that problem (13) has a primal solution that meets all the flow constraints, but the associated dual solution may well be off-track. The idea is to play with the parameters α_1 and α_2 until one gets a satisfactory solution. From a qualitative point of view, it is easy to get an idea of the impact of these two parameters.

Consider first the case $\alpha_1 > 0$ and $\alpha_2 = 0$. The higher α_1, the lesser the investment, and for α_1 large enough, no investment will take place. As shown in Theorem 2, solving the management problems yields a system of pressures that satisfy the lower bounds, but not necessarily the upper bounds. A violation of the upper bounds on the supply side indicates that more energy than what the compressors can provide is necessary to meet the flow constraints. To decrease the pressures, it would be necessary to dissipate less energy in the network, something that can be achieved by investing into new arcs. By lowering α_1, the cost of investment may become sufficiently low so as to trade energy for investment, and thus contribute to lower the pressure differential. A parametric analysis permits to find the more attractive trade-off.

A similar analysis can be performed with respect to α_2. Note that the operations cost $\langle c, y\rangle$ are similar to the input/output energy $\langle d, y\rangle$, but with opposite sign. In the previous qualitative analysis, we started with the assumption that the solution of the pure management problem (with no investment and/or operations) cost had no feasible solution with respect to the pressure requirements. Suppose now that it has a feasible solution. This solution may not be attractive from the view point of the operations cost. By increasing α_2, the (negative) operations cost are taken more intro consideration. Problem (13) produce more attractive solution from the view point of operations, but with higher management cost, which translates into higher input pressures.

7 An application to the Belgian gas network

To illustrate the application of our methodology we use data on the Belgian gas network, which can be found in [5, 6]. The network has a tree shape, with only 20 nodes. There are 24 arcs; five of them are just doubling existing arcs. This network is considered by practitioners as a good reference for benchmarking. To our knowledge, it is the only one available in the open literature with exhaustive information on the network and the associated parameters. We do not reproduce here the full information on the network; it can be found in the appendix of [6]. Table 1 displays the information relative to the arcs, which we find useful in the analysis of our results. In this table, $\underline{\phi}$ and $\bar{\phi}$ are the lower and upper bound on the inflows (negative values) and the outflows (positive values), and $\underline{p}$ and $\bar{p}$ are the lower and upper bounds on the pressures at the nodes. The last column c gives the unit supply cost.

As mentioned earlier, the basic flow equation is an approximation. In particular, the coefficient β in (1) is not a universal constant independent of the diameter D_{ij}. The authors propose a more complex formula, where

$$\beta_{ij} = \frac{\bar{\beta}}{[2\log(74D_{ij})]^2}. \tag{18}$$

We performed three classes of experiments. The first one deals with the pure transmission problem. The primary goal is to check the consistency

Nodes	$\underline{\phi}$	$\bar{\phi}$	$\underline{p}$	$\bar{p}$	c
1	8.87	11.594	0.0	77.0	2.28
2	0	8.4	0.0	77.0	2.28
3	$-\infty$	-3.918	30.0	80.0	0
4	0	0	0.0	80.0	0
5	0	4.8	0.0	77.0	2.28
6	$-\infty$	-4.034	30.0	80.0	0
7	$-\infty$	-5.256	30.0	80.0	0
8	20.344	22.012	50.0	66.2	1.68
9	0	0	0.0	66.2	0
10	$-\infty$	-6.365	30.0	66.2	0
11	0	0	0.0	66.2	0
12	$-\infty$	-2.12	0.0	66.2	0
13	0	1.2	0.0	66.2	1.68
14	0	0.96	0.0	66.2	1.68
15	$-\infty$	-6.848	0.0	66.2	0
16	$-\infty$	-15.616	50.0	66.2	0
17	0	0	0.0	66.2	0
18	0	0	0.0	63.0	0
19	$-\infty$	-0.222	0.0	66.2	0
20	$-\infty$	-1.919	25.0	66.2	0

Table 1: Data for the Belgian network.

of our approach with the results in [6]. The second class of experiments deals with the supply cost problem. In these two categories of experiments, the diameters are fixed: therefore, we use the value β_{ij} given by (18). The last class deals with the design problem. Since diameters are unknown, we replace β_{ij} by an average value 1.72 10^{13} and compute "optimal diameters" with this average. However, once the diameter are fixed, we rerun the optimization to compute the flows and the pressure with β given by (18).

7.1 The gas transmission problem

In [6], the authors propose a solution that meets all the flow and bound constraints. This solution involves a compressor on the arc 22 leading to node 18; the impact per unit of flow is easily computed from the published figures and has value 1589. We first checked that our approach of the gas transmission problem computes the very same flows as in [6] when the pressures at the inflow and outflow nodes are those given in [6] and no constraint is set on the in- and out-flows. In Table 2 we report two other experiments. In each of them, the d vector was taken to be the lower bound $\underline{p}$ and the supply and demand flows are constrained to lie between their bounds. In a first case we set the compressor impact at 1589, and in the other at 400. We report these data in the second and third columns of Table 2. While in both cases, our solution satisfies the flow constraints, we observe that the flows are slightly different from [6]. (See the shift of supply from node 1 to node 5.) In [6], the authors were looking for a least supply cost. Interestingly enough, their solution and ours have the same supply cost. The non-uniqueness of a minimal supply cost solution was mentioned as a possibility in the quoted paper.

Node	Solution in [6]		Compressor at 1589		Compressor at 400	
#	Pressure	Demand/Supply	Pressure	Demand/Supply	Pressure	Demand/Supply
1	55.82	10.9145	55.42	8.9348	61.16	8.9348
2	55.79	8.4	55.40	8.4	61.14	8.4
3	55.66	-3.9212	55.29	-3.918	61.04	-3.918
4	54.11	0	54.11	0	59.97	0
5	53.03	2.8148	55.42	4.7912	61.16	4.7912
6	52.28	-4.034	53.31	-4.034	59.25	-4.034
7	52.37	-5.256	53.28	-5.256	59.22	-5.256
8	59.85	22.012	59.85	22.012	65.20	22.012
9	59.41	0	59.41	0	64.79	0
10	57.59	-6.365	57.59	-6.365	63.13	-6.365
11	56.42	0	56.42	0	62.06	0
12	54.52	-2.12	54.52	-2.12	60.34	-2.12
13	53.19	1.2	53.19	1.2	59.14	1.2
14	52.98	0.96	52.98	0.96	58.96	0.96
15	51.65	-6.848	51.65	-6.848	57.77	-6.848
16	50.00	-15.616	50.00	-15.616	56.29	-15.616
17	55.62	0	55.62	0	61.34	0
18	63.00	0	63.00	0	58.73	0
19	35.74	-0.222	35.74	-0.222	27.52	-0.222
20	33.84	-1.919	33.84	-1.919	25.00	-1.919

Table 2: Results for the gas transmission problem.

7.2 Optimizing the supply costs

As pointed out in the previous section, the goal in [6] is to compute a least cost supply. The optimal reported value is 91.0562. We already observed that we can produce alternative feasible solutions with the same cost. To make sure that this value could not be improved, we incorporated the supply cost in our model (13) with a very large coefficient α_2, to put a maximum emphasis on minimizing the cost. This did not change the result, probably because the given supply cost figures are neutral. To make the instance more relevant, we modified the supply costs at node 1 and 2, from 2.28 to 2. We used two values for α_2 0 and 10000, the first one to disregard the cost, and the second one to focus almost exclusively on the costs. Table 3 gives the optimal pressures, the demand/supply values and the computed supply cost for the two values of α_2.

Node	$\alpha_2 = 0$		$\alpha_2 = 10000$	
#	Pressure	Demand/Supply	Pressure	Demand/Supply
1	61.16	8.9348	61.66	11.594
2	61.14	8.4	61.63	8.4
3	61.04	-3.918	61.50	-3.918
4	59.97	0	59.97	0
5	61.16	4.7912	58.24	2.132
6	59.25	-4.034	57.85	-4.034
7	59.22	-5.256	58.06	-5.256
8	65.20	22.012	65.20	22.012
9	64.79	0	64.79	0
10	63.13	-6.365	63.13	-6.365
11	62.06	0	62.06	0
12	60.34	-2.12	60.34	-2.12
13	59.14	1.2	59.14	1.2
14	58.96	0.96	58.96	0.96
15	57.77	-6.848	57.77	-6.848
16	56.29	-15.616	56.29	-15.616
17	61.34	0	61.34	0
18	58.73	0	58.73	0
19	27.52	-0.222	27.52	-0.222
20	25.00	-1.919	25.00	-1.919
Supply cost	86.2025		85.4579	

Table 3: Pressures and in- and out-flows with different weights on the investment cost.

7.3 Investment problem

In this section we use model (13) to find optimal pipe diameters. In these experiments we made the simplifying assumptions:

1. There is no upper bound on the diameters ($\bar{D} = \infty$).
2. The search for an optimal investment is performed under the assumption that the fixed part of the investment cost is neglected ($k_2 = 0$ in formula (16)). However, we report the full investment cost with k_2 set at its correct value 11.2155.
3. There is no compressor at the intermediate node 18.

7.3.1 Optimization from scratch, with no pre-existing pipes

Our goal is to compare the network in [6] with one designed to meet the same constraints and to minimize the investment cost. The experiments were performed with increasing values of α_1. The larger values of α_1 lead to lesser investment cost as shown in Table 4. In that table, one also finds the diameters of the existing network [6] (column 3) and the proposed alternatives. While the optimization is performed with respect to the new formula (16), we also computed the cost with respect to (15) and checked that the relative difference is within $\pm 4\%$.

With respect to cost, the results with $\alpha_1 = 6$ are superior. However, larger values of α_1 have an impact on the pressures that may result in constraint violations. This is shown in Table 5. Actually, in all the reported cases, the pressures meet their bound constraints, but we notice that the pressure profile is flatter with low values of α_2. For $\alpha_1 = 6$, the pressures increase in most places and is nearly at its upper limit at node 8. Even though, the problem specification does not discard this situation, we suspect that a flatter profile may be preferred as more robust.

	Arc		Increasing weight on the cost function			
#	(O,D)	[6]	$\alpha_1 = 1$	$\alpha_1 = 1.6$	$\alpha_1 = 5$	$\alpha_1 = 6$
1	(1,2)	890	650.3	610.8	524.7	512.1
2	(1,2)	890	650.3	610.8	524.7	512.1
3	(2,3)	890	834.7	784	673.5	657.3
4	(2,3)	890	834.7	784	673.5	657.3
5	(3,4)	890	998.9	938.3	806	786.7
6	(5,6)	590.1	604.3	567.6	487.6	475.9
7	(6,7)	590.1	0	0	0	0
8	(7,4)	590.1	671.7	630.9	542	529
9	(4,14)	890	829.9	779.5	669.7	653.6
10	(8,9)	890	902.8	848	728.4	711
11	(8,9)	395.5	902.8	848	728.4	711
12	(9,10)	890	902.8	848	728.4	710.9
13	(9,10)	395.5	902.8	848	728.4	711
14	(10,11)	890	787.6	739.8	635.5	620.1
15	(10,11)	395.5	787.6	739.8	635.5	620.4
16	(11,12)	890	979.8	920.3	790.6	771.6
17	(12,13)	890	915.1	859.6	738.4	720.7
18	(13,14)	890	952.6	894.7	768.6	750.1
19	(14,15)	890	1201	1128	969	945.8
20	(15,16)	890	1038.4	975.3	837.9	817.7
21	(11,17)	395.5	469	440.5	378.4	369.3
22	(17,18)	315.5	469	440.5	378.4	369.3
23	(18,19)	315.5	469	440.5	378.4	369.3
24	(19,20)	315.5	448.9	421.7	362.2	353.5
Cost						
formula	(16)	14,170	15,669	14,252	11,611	11,274
formula	(15)	13,800	15,400	14,236	12,000	11,800

Table 4: Optimal diameters for different weights on the investment cost.

Node		Increasing weight on the cost function			
#	[6]	$\alpha_1 = 1$	$\alpha_1 = 1.6$	$\alpha_1 = 5$	$\alpha_1 = 6$
1	55.82	53.75	55.13	60.70	61.99
2	55.79	53.63	54.96	60.36	61.62
3	55.66	53.46	54.72	59.89	61.09
4	54.11	52.71	53.71	57.84	58.82
5	53.03	53.75	55.13	60.70	61.99
6	52.28	52.40	53.29	56.98	57.85
7	52.37	52.12	52.90	56.18	56.96
8	59.85	55.04	56.85	64.10	65.77
9	59.41	54.90	56.67	63.74	65.36
10	57.59	54.33	55.90	62.24	63.70
11	56.42	53.60	54.91	60.27	61.51
12	54.52	52.38	53.27	56.94	57.81
13	53.19	51.19	51.64	53.53	53.99
14	52.98	51.04	51.43	53.10	53.50
15	51.65	50.75	51.04	52.25	52.54
16	50.00	50.00	50.00	50.00	50.00
17	55.62	53.25	54.45	59.33	60.47
18	63.00	52.39	53.28	56.94	57.81
19	35.74	49.01	48.61	46.87	46.43
20	33.84	48.79	48.31	46.18	45.63

Table 5: Optimal pressures for different weights on the investment cost.

We note that the optimization with respect to the variable investment cost led to a solution with twin pipes on connexions $(1, 2), (2, 3), (8, 9), (9, 10), (10, 11)$. Each pair could be replaced by a single pipe with diameter $2^{2/5}$ larger with no effect on the flows and on the variable investment cost. This alternative solution would induce a diminution of the fixed investment cost of

$$k_2(l_{1,2} + l_{2,3} + l_{8,9} + l_{8,10} + l_{10,11}) = 672.93.$$

We did not propose this solution for two reasons. First, those arcs are all reinforced in the network in [6]. Second, the single pipe solution implies pipes of diameter $m^{0.4} \approx 1.32$ larger and in some cases, those diameters are out of the range of the pipes in the existing network.

7.3.2 Optimal reinforcement of an existing network

In this section we test our approach on the reinforcement of the existing network. We increase the bounds of the demands and the supplies with a factor 1.3 to make the existing design under-dimensioned, i.e., the pressures exceed the upper bounds to satisfy the demands. We start with the existing design but as previously we do not use a compressor at node 18. We compute for different values of α_1 the new diameters to be installed in all the network to make the design feasible with respect to the pressure constraints. We recall that $\alpha_1 \geq 0$ is a coefficient that balances in the objective the investment cost with the energy in the network. We report the optimal diameters and the two costs computed with (16) and (15) in Tables 6 and the associated pressures in Table 7.

Arc	Increasing weight on the cost function			
#	$\alpha_1 = 1$	$\alpha_1 = 5$	$\alpha_1 = 10$	$\alpha_1 = 15$
1	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0
3	92.2	0.0	0.0	0.0
4	92.2	0.0	0.0	0.0
5	717.8	0.0	0.0	0.0
6	488.5	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0
8	453.4	0.0	0.0	0.0
9	231.7	0.0	0.0	0.0
10	795.0	462.6	225.4	0.0
11	795.0	462.6	225.4	0.0
12	795.0	462.6	225.4	0.0
13	795.1	462.6	225.4	0.0
14	594.5	0.0	0.0	0.0
15	594.5	0.0	0.0	0.0
16	723.7	0.0	0.0	0.0
17	575.8	0.0	0.0	0.0
18	665.5	0.0	0.0	0.0
19	1098.6	701.2	485.0	284.7
20	835.7	290.3	0.0	0.0
21	397.2	201.1	0.0	0.0
22	458.4	326.6	268.4	231.7
23	458.4	326.6	268.4	231.7
24	431.0	299.5	238.9	198.7
Cost with (16)	10,511	3,206	2,382	1,693
Cost with (15)	10,557	3,163	2,209	1,524

Table 6: Optimal diameters for the reinforcement problem.

8 Conclusion

The gas transmission problem is definitely more complex than usually described, because it involves hidden elements: the true operating costs (what is the real energy cost in $ to activate a compressor) and the almost variability of the operating conditions (variations in demand and supply, variations in delivery prices). Moreover, the Belgian gas network involves storage points. In the data, the storage points were all supply nodes. Truly enough,

Node		Increasing weight on the cost function			
#	[6]	$\alpha_1 = 1$	$\alpha_1 = 5$	$\alpha_1 = 10$	$\alpha_1 = 15$
1	70.54	54.03	57.45	58.24	58.86
2	70.52	53.99	57.42	58.21	58.83
3	70.37	53.80	57.24	58.03	58.65
4	68.80	52.98	55.30	56.12	56.76
5	70.54	53.82	57.45	58.24	58.86
6	67.75	52.41	53.98	54.82	55.48
7	67.70	52.35	53.92	54.76	55.42
8	76.42	55.50	63.02	64.91	65.79
9	75.83	55.35	62.61	64.28	65.10
10	73.43	54.75	60.96	61.70	62.29
11	71.87	53.96	59.08	59.84	60.44
12	69.34	52.63	55.97	56.78	57.42
13	67.58	51.30	53.77	54.61	55.27
14	67.31	51.13	53.43	54.27	54.94
15	65.54	50.82	52.49	52.76	52.97
16	63.33	50.00	50.00	50.00	50.22
17	70.81	53.61	58.14	58.57	59.18
18	73.46	52.76	55.72	54.68	54.10
19	29.13	49.40	45.47	36.52	27.52
20	25.00	49.18	44.76	35.10	25.00

Table 7: Optimal pressures for the reinforcement problem.

the storage points must be refilled once in a while. Presumably, they become demand points on the network, a situation that dramatically changes the operating conditions. Clearly, reinforcement of an existing network should account for these various solution.

An important issue is thus to measure the ability of an existing network to cope with many different operating conditions. By formulating the gas transmission problem as a convex optimization problem with linear constraints, we have a tool to perform many simulations on very large networks, at low computational cost. Moreover, there is no need in this approach to be concerned with loops in the network. The optimization problem automatically handle them.

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