Weak action representability and categories of algebras

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Action representable categories

Definition (F. Borceux, G. Janelidze and G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X)$ is representable. This means that there exists an object $[X]$ of $\mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]).$$

- Prototype examples: **Grp**, **Lie** and any abelian category.
- In **Grp**, the actor of X is $\mathrm{Aut}(X)$.
- In **Lie**, the actor of X is $\mathrm{Der}(X)$.
- In any abelian category, the actor of X is 0.

Weakly action representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ admits a weak representation. This means there exist an object T of $\mathcal{C}$ and a monomorphism of functors

$$\tau \colon \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, T).$$

An object T as above is called *weak representing object* of X, the pair (T, τ) is called *weak representation* of $\mathrm{SplExt}(-, X)$ and $(\varphi \colon B \to T) \in \mathrm{Im}(\tau_B)$ is called *acting morphism*.

- Examples: **Assoc**, **Leib** and any action representable category.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden, C. Vienne, 2025)

Let $\mathcal{V}$ be an action accessible variety of non-associative algebras, let $\mathbf{PAlg}$ *be the category of partial algebras and let* $U\colon \mathcal{V} \to \mathbf{PAlg}$ *be the forgetful functor.*

- *Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors*

$$\tau\colon\ \mathrm{SplExt}(-,X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-),\mathcal{E}(X)).$$

- *If $\mathcal{E}(X)$ is an object of $\mathcal{V}$, then it is a weak representing object of X.*

Definition

The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X. When τ is a natural isomorphism, we say that $\mathcal{E}(X)$ is an *external actor* of X.

Question: does the converse of the implication

$$\textit{weakly action representable category} \Rightarrow \textit{action accessible category}$$

hold in the context of varieties of algebras over a field?

Proposition (J. R. A. Gray, 2025)

Let $\mathcal{C}$ be an action representable category and let $\mathcal{X}$ be a Birkhoff subcategory of $\mathcal{C}$. Suppose there exist two monomorphisms $m\colon S \rightarrowtail B$, $m'\colon S \rightarrowtail B'$ in $\mathcal{X}$, two monomorphisms $u\colon B \rightarrowtail D$, $u'\colon B' \rightarrowtail D$ in $\mathcal{C}$, an object X of $\mathcal{X}$ and a monomorphism $v\colon D \rightarrowtail [X]$ such that:

- *m and m' cannot be amalgamated in $\mathcal{X}$.*
- *The pair (u, u') is an amalgam of (m, m'), i.e., $um = u'm'$.*
- *The split extensions with kernel X corresponding to vu and vu' are in $\mathcal{X}$.*

Then, the category $\mathcal{X}$ is not weakly action representable.

Theorem (J. R. A. Gray, 2025)

The varieties $\mathbf{Sol}_t(\mathbf{Grp})$ *($t \geq 3$) are not weakly action representable.*

Theorem (X. García Martínez, M. M., 2025)

The varieties $\mathbf{Sol}_2(\mathbf{Grp})$ *and* $\mathbf{Nil}_k(\mathbf{Grp})$ *($k \geq 3$) are not weakly action representable.*

Theorem (X. García Martínez, M. M., 2025)

The varieties $\mathbf{Sol}_t(\mathbf{Lie})$ *($t \geq 2$) and* $\mathbf{Nil}_k(\mathbf{Lie})$ *($k \geq 3$) are not weakly action representable.*

- **Open question:** is the variety $\mathbf{Nil}_2(\mathbf{Grp})$ weakly action representable?

Unitary algebras

Definition

A variety of non-associative algebras $\mathcal{V}$ is *unit-closed* if for any algebra X of $\mathcal{V}$, the algebra $\tilde{X}$ obtained by adjoining to X the external unit 1, together with the identities $x \cdot 1 = 1 \cdot x = x$, is still an object of $\mathcal{V}$.

Examples

- **Assoc**, **CAssoc**, **Alt** are unit-closed.
- **Lie**, **Leib**, **JJord** are not unit-closed.

Remark

The subcategory $\mathcal{V}_1$ of unitary algebras of $\mathcal{V}$ is an *ideally exact category*.

Ideally exact categories

Definition (G. Janelidze, 2024)

Let $\mathcal{C}$ be a category with pullbacks, with initial object 0 and terminal object 1. $\mathcal{C}$ is *ideally exact* if it is Barr exact, protomodular, has finite coproducts and the unique morphism $0 \to 1$ is a regular epimorphism.

- **Examples: Ring**, **CRing**, **MVAlg**, **PrAlg**, **GAlg**, any cotopos...

Remark (G. Janelidze, 2024)

There is a semi-abelian category $\mathcal{V}$ and a monadic adjunction with cartesian unit

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{V}. \arrow[l, bend right, "F"'] \end{tikzcd}$$

Examples

- One may take $\mathcal{V} = (\mathcal{C} \downarrow 0)$ with U and F defined in the obvious way.
- If $\mathcal{V}$ is a unit-closed variety of non-associative algebras, then the monadic adjunction with cartesian unit is given by

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{V}, \arrow[l, bend right, "F"'] \end{tikzcd}$$

where U forgets the unit and $F(X) = \mathbb{F} \ltimes X$, for any X in $\mathcal{V}$.

Action representability of ideally exact categories

Definition (G. Janelidze, 2024)

An ideally exact category $\mathcal{C}$ is *action representable* if the functor

$$\mathrm{SplExt}_{\mathcal{V}}(-, U(X)) \colon \mathcal{V}^{\mathrm{op}} \to \mathbf{Set}$$

is representable for any object X of $\mathcal{C}$.

Theorem (G. Janelidze, 2024)

The ideally exact categories ***Ring*** *and* ***CRing*** *are action representable.*

Theorem (M. M., F. Piazza, 2025)

The ideally exact categories $\mathbf{Assoc}_1$, $\mathbf{CAssoc}_1$, $\mathbf{Alt}_1$, $\mathbf{Pois}_1$, $\mathbf{CPois}_1$ *are action representable.*

Thank you!

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