CYCLICALLY REDUCED ELEMENTS IN COXETER GROUPS

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ABSTRACT. Let W be a Coxeter group. We provide a precise description of the conjugacy classes in W, in the spirit of Matsumoto's theorem. This extends to all Coxeter groups an important result on finite Coxeter groups by M. Geck and G. Pfeiffer from 1993. In particular, we describe the cyclically reduced elements of W, thereby proving a conjecture of A. Cohen from 1994.

1. INTRODUCTION

Let (W, S) be a Coxeter system. By a classical result of J. Tits ([Tit69]), also known as Matsumoto's theorem (see [Mat64]), any given reduced expression of an element $w \in W$ can be obtained from any other expression of w by performing a finite sequence of braid relations and ss-cancellations (i.e. replacing a subword (s, s) for some $s \in S$ by the empty word). In particular, this yields a very simple and elegant solution to the word problem in Coxeter groups.

The conjugacy problem for Coxeter groups was solved about 30 years later, by D. Krammer in his thesis from 1994 (published in [Kra09]): there exists a cubic algorithm deciding whether two words on the alphabet S determine conjugate elements of W. However, Krammer's solution does not provide a sequence of "elementary operations" to pass from one word to the other, as do the braid relations and ss-cancellations in Matsumoto's theorem.

In this paper, we address the following long-standing open question on Coxeter groups: *Is there an analogue of Matsumoto's theorem for the conjugacy problem in Coxeter groups?*

A very natural elementary operation on words to consider for the conjugacy problem is that of cyclic shift: by extension, we say that an element $w' \in W$ is a **cyclic shift** of some $w \in W$ if there is some reduced decomposition $w = s_1 \dots s_d$ ($s_i \in S$) of w and some $r \in \{1, \dots, d\}$ such that $w' = s_{r+1} \dots s_d s_1 \dots s_r$. Such operations are, however, not sufficient to describe conjugacy classes in general, as for instance illustrated by the Coxeter group $W = \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle$ of type A_2, in which the simple reflections s and t are conjugate, but cannot be obtained from one another through a sequence of cyclic shifts. Nonetheless, in the terminology of [GP00, Chapter 3], the elements $w := s$ and $w' := t$ are **elementarily strongly conjugate**, meaning that $\ell_S(w) = \ell_S(w')$ and that there exists some $x \in W$ with $w' = x^{-1}wx$ such that either $\ell_S(x^{-1}w) = \ell_S(x) + \ell_S(w)$ or $\ell_S(wx) = \ell_S(w) + \ell_S(x)$.

Motivated by the representation theory of Hecke algebras, M. Geck and G. Pfeiffer proved in [GP93] that if W is finite, then for any conjugacy class $\mathcal{O}$ in W,

(1) any $w \in \mathcal{O}$ can be transformed by cyclic shifts into an element w' of minimal length in $\mathcal{O}$, and
(2) any two elements w, w' of minimal length in $\mathcal{O}$ are **strongly conjugate**, i.e. there exists a sequence $w = w_0, \dots, w_n = w'$ of elements of W such that w_{i-1} is elementarily strongly conjugate to w_i for each $i = 1, \dots, n$.

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Together with S. Kim, they later generalised this theorem (see [GKP00]) to the case of δ-twisted conjugacy classes for some automorphism δ of (W,S), that is, when $\mathcal{O}$ is replaced by $\mathcal{O}_\delta = \{\delta(v)^{-1}wv \mid v \in W\}$ for some $w \in W$. The proofs in [GP93] and [GKP00] involve a case-by-case analysis, with the help of a computer for the exceptional types. In [HN12], X. He and S. Nie gave a uniform (and computer-free) geometric proof of that theorem, which they later generalised, in [HN14], to the case of an affine Coxeter group W. In addition, they showed (for W affine) that

(3) if $\mathcal{O}$ is straight, then any two elements w, w' of minimal length in $\mathcal{O}$ are conjugate by a sequence of cyclic shifts,

where $\mathcal{O}$ is **straight** if it contains a straight element $w \in W$, that is, such that $\ell_S(w^n) = n\ell_S(w)$ for all $n \in \mathbb{N}$ (equivalently, every minimal length element of $\mathcal{O}$ is straight, see Lemma 2.4). Note that the straight elements in an arbitrary Coxeter group were characterised in [Mar14b, Theorem D]; these elements play an important role in the study of affine Deligne-Lusztig varieties (see [He14]), and also exhibit very useful dynamical properties (see e.g. [Mar14a] or [CH15]). Similar statements to (1) and (2) above were further obtained for an arbitrary Coxeter group W, but when $\mathcal{O}$ is replaced by some "partial" conjugacy class $\mathcal{O} = \{v^{-1}wv \mid v \in W_I\}$, for some finite standard parabolic subgroup $W_I \subseteq W$ (see [He07] and [Nie13]). Finally, we showed in [Mar14b, Theorem A] that for a certain class of Coxeter groups that includes the right-angled ones, (1) and (2) hold using only cyclic shifts.

In this paper, we prove the statements (1), (2) and (3) in full generality, namely, for an arbitrary Coxeter group W. Moreover, we actually prove a much more precise version of (2) by introducing a refined notion of "strong conjugation", which we call "tight conjugation" (see Definition 3.4) — in particular, if two elements are tightly conjugate, then they are strongly conjugate; when W is finite, the two notions coincide. Here is our main theorem.

Theorem A. *Let (W,S) be a Coxeter system. Let $\mathcal{O}$ be a conjugacy class of W, and let $\mathcal{O}_{\min}$ be the set of minimal length elements of $\mathcal{O}$. Then the following assertions hold:*

(1) *For any $w \in \mathcal{O}$, there exists an element $w' \in \mathcal{O}_{\min}$ that can be obtained from w by a sequence of cyclic shifts.*
(2) *If $w, w' \in \mathcal{O}_{\min}$, then w and w' are tightly conjugate.*
(3) *If $\mathcal{O}$ is straight, then any two elements $w, w' \in \mathcal{O}_{\min}$ are conjugate by a sequence of cyclic shifts.*

Note that the proof of Theorem A uses the results of [GKP00] (or [HN12]), but does not rely on [HN14]. In particular, we give an alternative, shorter proof that affine Coxeter groups satisfy Theorem A.

Recall that an element $w \in W$ is **cyclically reduced** if $\ell_S(w') = \ell_S(w)$ for every $w' \in W$ obtained from w by a sequence of cyclic shifts. Often, this terminology is used instead for elements of minimal length in their conjugacy class. An important reformulation of Theorem A(1) is that these two notions in fact coincide.

Corollary B. *An element $w \in W$ is cyclically reduced if and only if it is of minimal length in its conjugacy class.*

This proves a conjecture of A. Cohen (see [Coh94, Conjecture 2.18]).

The proof of Theorem A is of geometric nature, and uses the Davis complex X of (W,S) — here, we assume that S is finite, a safe assumption for the study of Theorem A (see Remark 6.1). This is a CAT(0) cellular complex on which W acts by cellular isometries. For instance, if W is affine, then X is just the standard geometric realisation of the Coxeter complex Σ of (W,S), and the CAT(0) metric $\mathrm{d}\colon X \times X \to \mathbb{R}_+$ is the usual Euclidean metric (see Example 2.7). For an element $w \in W$, the subset $\mathrm{Min}(w) \subseteq X$

of all $x \in X$ such that $\mathrm{d}(x, wx)$ is minimal will play an important role; it will also be crucial to investigate its combinatorial analogue $\mathrm{CombiMin}(w)$ (see §4), as highlighted in Remark 5.6. As a byproduct of our proofs, we are able to relate these two notions of "minimal displacement set" for w (see §7).

Corollary C. *Let $w \in W$. Then $\mathrm{Min}(w) \subseteq \mathrm{CombiMin}(w)$, and $\mathrm{CombiMin}(w)$ is at bounded Hausdorff distance from $\mathrm{Min}(w)$.*

Note that, while $\mathrm{Min}(w)$ is always connected (in the CAT(0) sense), its combinatorial analogue $\mathrm{CombiMin}(w)$ need not be (gallery-)connected, and this is precisely the reason why cyclic shifts are not sufficient to describe the conjugacy classes in W, and why one needs to also consider "tight conjugations" (see Remark 4.6).

We conclude the introduction with a short roadmap to the proof of Theorem A. Let $w \in W$, and let $\mathcal{O}$ (resp. $\mathcal{O}_{\min}$) denote its conjugacy class (resp. the set of elements of minimal length in $\mathcal{O}$). Let $\mathrm{Ch}(\Sigma)$ be the set of chambers of the Coxeter complex Σ ($\mathrm{Ch}(\Sigma)$ can be W-equivariantly identified with the set of vertices $\{v \mid v \in W\}$ of the Cayley graph of (W, S)).

The first step is to view the elements of $\mathcal{O}$ and $\mathcal{O}_{\min}$ geometrically, as chambers of Σ: we consider the map $\pi_w\colon \mathrm{Ch}(\Sigma) \to W : v \mapsto v^{-1}wv$, which satisfies $\pi_w(\mathrm{Ch}(\Sigma)) = \mathcal{O}$ and $\pi_w^{-1}(\mathcal{O}_{\min}) = \mathrm{CombiMin}(w)$. The second step is to interpret the operations of cyclic shifts and tight conjugations geometrically, at the level of $\mathrm{Ch}(\Sigma)$, by defining two "elementary geometric operations", say of type (I) and (II), allowing to pass from one chamber $C \in \mathrm{Ch}(\Sigma)$ to another chamber $D \in \mathrm{Ch}(\Sigma)$, in such a way that passing from C to D with an operation of type (I) implies that one can pass from $\pi_w(C)$ to $\pi_w(D)$ using cyclic shifts, and passing from C to D with an operation of type (II) implies that one can pass from $\pi_w(C)$ to $\pi_w(D)$ using tight conjugations; this strategy is implemented in Sections 4 and 5, and makes use of the analogue of Theorem A for twisted conjugacy classes in finite Coxeter groups established in [GKP00] and [HN12]. Theorem A then amounts to showing that one can pass from the chamber $C_0 := \{1_W\}$ (representing $\pi_w(C_0) = w$) to any other chamber C by a sequence of geometric operations of type (I) and (II) (see Section 6). This geometric formulation of the problem allows one to take advantage of the tools provided by CAT(0) geometry, and of the specific properties of Davis complexes described in §2.4–2.6. Finally, note that the analogue of Theorem A for untwisted conjugacy classes in finite Coxeter groups, first established in [GP93], is also used to prove Theorem A when w has finite order.

2. Preliminaries

2.1. **Basic definitions.** Basics on Coxeter groups and complexes can be found in [AB08, Chapters 1–3]. The notions introduced below are illustrated on Figure 1 (see Example 2.2).

Throughout this paper, (W, S) denotes a Coxeter system of finite rank (see Remark 6.1). We let $\Sigma = \Sigma(W, S)$ be the associated Coxeter complex, with set of roots (or half-spaces) Φ. Let also $C_0 := \{1_W\}$ be the fundamental chamber of Σ, and $\Pi := \{\alpha_s \mid s \in S\}$ be the corresponding set of simple roots (i.e. the roots containing C_0 and whose wall is a wall of C_0). Write $\mathrm{Ch}(\Sigma) := \{wC_0 \mid w \in W\}$ for the set of chambers of Σ. We will often identify a chamber subcomplex A of Σ with its underlying set $\mathrm{Ch}(A) \subseteq \mathrm{Ch}(\Sigma)$ of chambers.

Two chambers $D, E \in \mathrm{Ch}(\Sigma)$ are ***s*-adjacent** for some $s \in S$ if they are s-adjacent in the Cayley graph $\mathrm{Cay}(W, S)$. A **gallery** Γ between two chambers $D, E \in \mathrm{Ch}(\Sigma)$ is a sequence of chambers $D = D_0, D_1, \dots, D_r = E$ such that, for each $i \in \{1, \dots, r\}$, the chamber D_{i-1} is (distinct from and) s_i-adjacent to D_i for some $s_i \in S$. The sequence $(s_1, \dots, s_r) \in S^r$ is the **type** of Γ, and $\ell(\Gamma) := r$ the **length** of Γ. The gallery Γ is

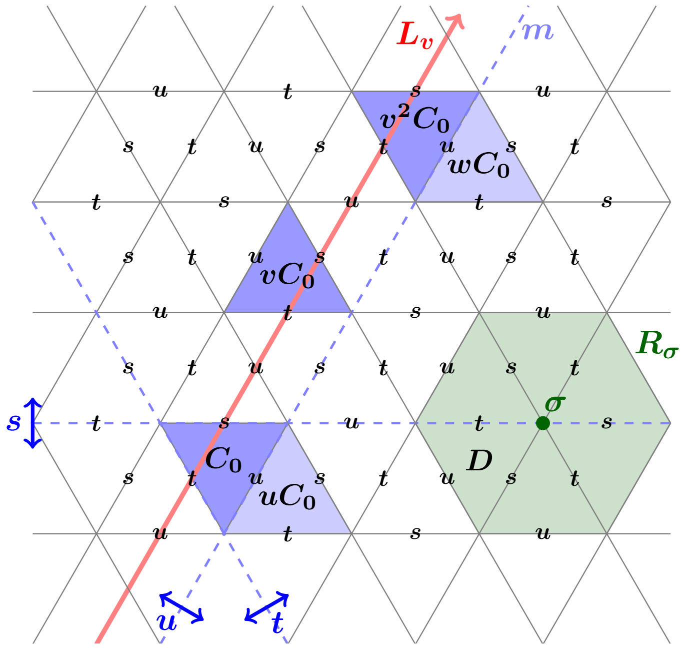

FIG. 1. Coxeter complex of type $\widetilde{A}_2$

minimal if it is a gallery of minimal length between C and D; in this case, we set $\mathrm{d_{Ch}}(C,D) := \ell(\Gamma)$ and call it the **chamber distance** between C and D. We let

$$\Gamma(C,D) := \{E \in \mathrm{Ch}(\Sigma) \mid \mathrm{d_{Ch}}(C,D) = \mathrm{d_{Ch}}(C,E) + \mathrm{d_{Ch}}(E,D)\}$$

denote the set of chambers on a minimal gallery from C to D. If $C = vC_0$ and $D = wvC_0$ for some $v, w \in W$, there is a bijective correspondence between minimal galleries Γ from C to D and reduced expressions (on the alphabet S) for $v^{-1}wv$, mapping the gallery Γ to its type. In particular, if we again denote by $\ell\colon W \to \mathbb{N}$ the word length on W with respect to S, then $\ell(v^{-1}wv)$ coincides with $\ell(\Gamma)$, or else with the number of walls crossed by Γ (i.e. the number of walls separating C from D).

To each simplex σ of Σ, one associates its corresponding **residue** R_σ, which is the set of chambers of Σ containing σ. A **wall of** R_σ is a wall of Σ containing σ. For a subset $I \subseteq S$, we let $W_I := \langle I \rangle \subseteq W$ denote the **standard parabolic subgroup** of type I. The **parabolic subgroups** of W are then the conjugates of the standard parabolic subgroups, or equivalently, the stabilisers in W of some simplex (resp. residue) of Σ. The simplex σ (resp. the residue R_σ) is **spherical** if its stabiliser P_σ in W is finite; it is **standard** if P_σ is a standard parabolic subgroup (equivalently, if σ is a face of C_0, resp. if $C_0 \in R_\sigma$). Thus, if σ is a face of vC_0 for some $v \in W$, then $P_\sigma = vW_Iv^{-1}$ for some subset $I \subseteq S$. For any $w \in W$, there is a smallest parabolic subgroup $\mathrm{Pc}(w)$ containing w, called the **parabolic closure** of w.

For each $I \subseteq S$, we set $\Pi_I := \{\alpha_s \mid s \in I\}$. Let N_I be the stabiliser in W of Π_I. Note that the conjugation action of any $n_I \in N_I$ on W_I induces an automorphism of W_I preserving I (called a **diagram automorphism**). We write $N_W(W_I)$ for the normaliser of W_I in W, and we call I **spherical** if W_I is finite. The following lemma follows from [Lus77, Lemma 5.2].

Lemma 2.1. *Let $I \subseteq S$. Then $N_W(W_I) = W_I \rtimes N_I$. Moreover,*

$$\ell(w_In_I) = \ell(w_I) + \ell(n_I) \quad \textit{for all } w_I \in W_I \textit{ and } n_I \in N_I. \tag{2.1}$$

Example 2.2. For the benefit of the reader unfamiliar with Coxeter groups and complexes, we illustrate the above notions on an example. Consider the (affine) Coxeter group $W = W(\widetilde{A}_2)$ of type $\widetilde{A}_2$, with standard generating set $S = \{s, t, u\}$ and presentation $W = \langle s, t, u \mid s^2 = t^2 = u^2 = 1 = (st)^3 = (su)^3 = (tu)^3 \rangle$.

The Coxeter complex $\Sigma = \Sigma(W, S)$ is then the simplicial complex associated to the tesselation of the Euclidean plane by congruent equilateral triangles (see Figure 1). Fixing a fundamental chamber (i.e. a triangle) C_0, the generators s, t, u act as orthogonal reflections across the walls (i.e. lines) delimiting C_0 (see the dashed lines on Figure 1). The W-action on Σ is then simply transitive of the set $\mathrm{Ch}(\Sigma)$ of chambers. The roots $\alpha_s, \alpha_t, \alpha_u$ are the half-spaces respectively delimited by the walls of s, t, u (i.e. the lines fixed by s, t, u) and containing C_0.

To each codimension 1 face of C_0 (i.e. edge of C_0, say contained in the wall of $x \in S$), one attributes its type $x \in S$. One then extends this labelling to all edges by requiring the W-action to be type-preserving. Two distinct chambers are then x-adjacent if they share a common edge of type x. Together with these x-adjacency relations, $\mathrm{Ch}(\Sigma)$ then coincides with the Cayley graph of (W, S).

One can reconstruct Σ group-theoretically by attaching to each face of C_0 its stabiliser in W: the chamber C_0 (together with its faces) is isomorphic to the poset of standard parabolic subgroups, ordered by the opposite of the inclusion relation (indeed, $\mathrm{Stab}_W(C_0) = W_\varnothing = \{1\}$, the stabiliser of the edge of C_0 labelled $x \in S$ is $W_{\{x\}}$, the stabiliser of the vertex of C_0 at the intersection of the edges labelled x and y is $W_{\{x,y\}}$, and the stabiliser of the empty simplex is W). Using the W-action, Σ can thus be defined as the poset $\{wW_I \mid w \in W,\ I \subseteq S\}$, ordered by the opposite of the inclusion relation (the W-action being by left translation).

An example of 0-dimensional simplex (i.e. vertex) σ, as well as the corresponding residue R_σ, are pictured on Figure 1. Since $D = ustuC_0 \in R_\sigma$, the stabiliser of R_σ is the parabolic subgroup $P_\sigma = (ustu)W_{\{s,t\}}(ustu)^{-1}$.

Finally, note that the element $w := sutsutu$ commutes with u, and hence $w \in N_W(W_I)$ for $I = \{u\}$. The decomposition $w = w_I n_I$ with $w_I \in W_I$ and $n_I \in N_I$ provided by Lemma 2.1 is then given by $w_I = u$ and $n_I = sutsut$.

2.2. **Straight elements.** An element $w \in W$ is called **straight** if $\ell(w^n) = n\ell(w)$ for all $n \in \mathbb{N}$. We record for future reference the following basic properties of straight elements.

Lemma 2.3 ([Mar14b, Lemma 4.1])**.** *Let $w \in W$ be straight. Then w is of minimal length in its conjugacy class. Moreover, if $w \in N_W(W_I)$ for some spherical subset $I \subseteq S$, then $w \in N_I$.*

Lemma 2.4 ([Mar14b, Lemma 4.2])**.** *Let $v, w \in W$ be such that $\ell(v^{-1}wv) = \ell(w)$. Then w is straight if and only if $v^{-1}wv$ is straight.*

2.3. **Projections.** The general reference for this section is [MPW15, Chapter 21]. Given a chamber $D \in \mathrm{Ch}(\Sigma)$ and a residue R, there is a unique chamber $E \in R$ at minimal distance from D, called the **projection** of D on R, and denoted $\mathrm{proj}_R(D)$. Alternatively, $\mathrm{proj}_R(D)$ is the unique chamber E of R such that D and E lie on the same side of every wall of R. In particular, one has the following **gate property**:

$$\mathrm{d}_{\mathrm{Ch}}(D, E) = \mathrm{d}_{\mathrm{Ch}}(D, \mathrm{proj}_R(D)) + \mathrm{d}_{\mathrm{Ch}}(\mathrm{proj}_R(D), E) \quad \text{for any chamber } E \in R.$$

As $\mathrm{proj}_R \colon \mathrm{Ch}(\Sigma) \to R$ maps galleries to galleries, it does not increase the chamber distance. Two residues R, R' are **parallel** if the projection map $\mathrm{proj}_R|_{R'} \colon R' \to R$ is bijective (in which case $\mathrm{proj}_{R'}|_R \colon R \to R'$ is its inverse). Equivalently, R and R' are parallel if and only if they have the same walls if and only if they have the same stabiliser in W. In that case, $\mathrm{d}_{\mathrm{Ch}}(D, \mathrm{proj}_R(D))$ is independent of the choice of chamber $D \in R'$. Finally, if $R \subseteq R'$, then $\mathrm{proj}_R(D) = \mathrm{proj}_R(\mathrm{proj}_{R'}(D))$ for all $D \in \mathrm{Ch}(\Sigma)$.

Example 2.5. Keeping the notations of Example 2.2, the projection of C_0 on the residue R_σ is the chamber D. Let now τ_1 be the common edge shared by C_0 and uC_0, and let τ_2 be the common edge shared by wC_0 and uwC_0 (also denoted v^2C_0 on Figure 1, where $v := sut$). Then the residues $R_{\tau_1} = \{C_0, uC_0\}$ and $R_{\tau_2} = \{wC_0, uwC_0\}$ have the same walls (i.e. the wall m of u), and are therefore parallel. The bijection between R_{τ_1} and R_{τ_2} induced by the projections identifies C_0 with uwC_0 and uC_0 with wC_0.

Example 2.6. Let R be a residue, and assume that R and wR are parallel for some $w \in W$. Then w normalises $\mathrm{Stab}_W(R)$. If, moreover, R is spherical and standard, so that $\mathrm{Stab}_W(R) = W_I$ for some spherical subset $I \subseteq S$, then $w = w_I n_I$ for some $w_I \in W_I$ and $n_I \in N_I$ by Lemma 2.1. By definition of N_I, we then have $\mathrm{proj}_{wR}(C_0) = n_I C_0$.

2.4. **Davis complex.** The general reference for this section is [Dav98]. We briefly recall the construction of the **Davis complex** X of (W, S), which is a complete, uniquely geodesic metric realisation of Σ. Let $\Sigma_{(1)}$ be the flag complex of Σ, that is, $\Sigma_{(1)}$ is the simplicial complex with vertices the simplices of Σ and simplices the flags of simplices of Σ. Let also $\Sigma^s_{(1)}$ denote the subcomplex of $\Sigma_{(1)}$ with vertices the spherical simplices of Σ. Then X is the geometric realisation of $\Sigma^s_{(1)}$ (hence a cellular subcomplex of the barycentric subdivision of the geometric realisation of Σ), together with a suitably defined CAT(0) metric $\mathrm{d}\colon X \times X \to \mathbb{R}_+$ extending the canonical Euclidean metrics on its cells. Each (open) cell σ of X corresponds to a unique spherical simplex wW_I of Σ — namely, σ is (the realisation of) the union of all flags of spherical simplices whose upper bound is wW_I — and the W-action on the spherical simplices of Σ induces a cellular isometric W-action on X.

For each $x \in X$, there is a unique (open) cell $\mathrm{supp}(x)$ containing x, called the **support** of x. In particular, $\mathrm{Stab}_W(x) = \mathrm{Stab}_W(\mathrm{supp}(x))$ is a spherical (i.e. finite) parabolic subgroup of W. In this paper, we shall identify the roots, walls and chambers of Σ with the corresponding *closed* subsets of X. In particular, a chamber $D \in \mathrm{Ch}(\Sigma) \approx \mathrm{Ch}(X)$ will be identified with the set of $x \in X$ whose support corresponds either to D *or to a (spherical) face of* D.

Example 2.7. Keeping the notations of Example 2.2, the Davis complex X of (W, S) coincides with the geometric realisation of Σ (i.e. the tesselated Euclidean plane pictured on Figure 1), together with the Euclidean metric d (note that, in this example, all nonempty simplices of Σ are spherical). The open cells of dimension $0, 1, 2$ are, respectively, the vertices, the edges without endpoints, and the open triangles, and these form a partition of X.

2.5. **Actions on CAT(0)-spaces.** Basics on CAT(0) spaces can be found in [BH99]. Consider the W-action on X. For an element $w \in W$, we let

$$|w| := \inf\{\mathrm{d}(x, wx) \mid x \in X\} \in [0, +\infty)$$

denote its **translation length**, and we set

$$\mathrm{Min}(w) := \{x \in X \mid \mathrm{d}(x, wx) = |w|\} \subseteq X.$$

By a classical result of M. Bridson ([Bri99]), $\mathrm{Min}(w)$ is a nonempty closed convex subset of X for all $w \in W$ (in particular, the infimum defining $|w|$ is always attained). More precisely, if w has finite order, then $|w| = 0$ and $\mathrm{Min}(w)$ is the fixed-point set of w (see e.g. [AB08, Theorem 11.23]). If w has infinite order, then $|w| > 0$ (otherwise, w would fix a point x, and hence would belong to the finite parabolic subgroup $\mathrm{Stab}_W(x)$) and $\mathrm{Min}(w)$ is the union of all w-axes, where a **w-axis** is a geodesic line stabilised by w (on which w then acts as a translation).

Example 2.8. Keeping the notations of Examples 2.2 and 2.7, the element $v := sut$ is a glide reflection, i.e. the composition of a translation with axis L_v (depicted on Figure 1) with a flip around that axis. In particular, L_v is the unique v-invariant line, and hence $\mathrm{Min}(v) = L_v$. On the other hand, $\mathrm{Min}(v^2) = X$, that is, v^2 is a translation across the whole plane.

2.6. **Walls.** Given $x, y \in X$, we let $[x, y]$ denote the unique geodesic segment between x and y. If $[x, y]$ intersects a wall m of X in at least two points, then it is entirely contained in m (see [Nos11, Lemma 2.2.6]). Let $w \in W$ be of infinite order. A w-axis L is **transverse** to a wall m if it intersects m in a single point (in which case the two components of $L \setminus m$ lie on different sides of m, see [Nos11, Lemma 2.3.1]); in that case, any w-axis is transverse to m, and m is called w**-essential**. Note that, given any two points $x, y \in L$, there are only finitely many w-essential walls intersecting $[x, y] \subseteq L$. In particular, for any $x \in L$, there exists a (nonempty) open geodesic segment $\sigma \subseteq L$ containing x in its closure and contained in some (open) cell $\mathrm{supp}(\sigma)$ (i.e. σ does not intersect any w-essential wall).

3. Tight conjugation

We start by recalling the conjugation operations introduced in [GP93] (see Definitions 3.1 and 3.3), and then introduce a refinement of these notions, which we call "tight conjugation" (see Definition 3.4). Since we will use some results from [Mar14b], in which yet another conjugation operation is introduced, we also recall that notion (see Definition 3.2), and then relate all the above notions in Lemma 3.5.

Definition 3.1 ([GP93])**.** Let $w, w' \in W$ and $s \in S$. We write $w \xrightarrow{s} w'$ if $w' = sws$ and $\ell(w') \leq \ell(w)$. We write $w \to w'$ if there is a sequence $w = w_0, w_1, \ldots, w_n = w'$ of elements of W such that, for each i, $w_{i-1} \xrightarrow{s_i} w_i$ for some $s_i \in S$.

Definition 3.2 ([Mar14b])**.** Let $w, w' \in W$. We say that w is **elementarily related** to w' if w' admits a decomposition that is a cyclic shift of some reduced decomposition of w, that is, there is some reduced decomposition $w = s_1 \ldots s_n$ ($s_i \in S$) of w and some $k \in \{1, \ldots, n\}$ such that $w' = s_k \ldots s_n s_1 \ldots s_{k-1}$. We say that w is κ**-related** to w', which we denote by $w \sim_\kappa w'$, if there is a sequence $w = w_0, \ldots, w_n = w'$ of elements of W such that w_{i-1} is elementarily related to w_i for each i.

Definition 3.3 ([GP93])**.** Two elements $w, w' \in W$ are called **elementarily strongly conjugate** if $\ell(w') = \ell(w)$, and there exists some $x \in W$ with $w' = x^{-1}wx$ such that either $\ell(x^{-1}w) = \ell(x) + \ell(w)$ or $\ell(wx) = \ell(w) + \ell(x)$; we then write $w \overset{x}{\sim} w'$. We further call $w, w' \in W$ **strongly conjugate** if there is a sequence $w = w_0, \ldots, w_n = w'$ of elements of W such that w_{i-1} is elementarily strongly conjugate to w_i for each i; we then write $w \sim w'$.

Definition 3.4. Two elements $w, w' \in W$ are called **elementarily tightly conjugate** if $\ell(w) = \ell(w')$ and one of the following holds:

(1) there exists some $s \in S$ such that $w \xrightarrow{s} w'$.
(2) there exists some spherical subset $I \subseteq S$ such that $w \in N_W(W_I)$, and some $x \in W_I$ such that $w \overset{x}{\sim} w'$.

We further call $w, w' \in W$ **tightly conjugate** if there is a sequence $w = w_0, \ldots, w_n = w'$ of elements of W such that w_{i-1} is elementarily tightly conjugate to w_i for each i; we then write $w \approx_t w'$.

We now show that Definitions 3.1 and 3.2 are equivalent, and that "tight conjugation" is indeed a refinement of "strong conjugation" (but of course the two notions coincide if W is finite).

Lemma 3.5. *Let $w, w' \in W$. Then the following assertions hold.*

(1) $w \to w' \iff w \sim_\kappa w'$.
(2) $w \approx_t w' \implies w \sim w'$.
(3) *If $\ell(w) = \ell(w')$, then* $w \to w' \implies w \approx_t w'$.

Proof. (1) For the forward implication, it is sufficient to show that if $w \xrightarrow{s} sws$ for some $s \in S$, then $w \sim_\kappa sws$. If $sws = w$, this is clear. We may thus assume that $sws \neq w$ and $\ell(sws) \leq \ell(w)$. Then either $\ell(sw) < \ell(w)$ or $\ell(ws) < \ell(w)$ (see the condition (F) in [AB08, page 79]), and hence w is elementarily related to sws by the exchange condition ([AB08, Condition (E) page 79]). The converse is clear.

(2) It is sufficient to show that if $w \xrightarrow{s} sws$ with $\ell(w) = \ell(sws)$ for some $s \in S$, then either $w = sws$ (so that $w \overset{1}{\sim} sws$) or $w \overset{s}{\sim} sws$. Assume that $\ell(sws) = \ell(w)$, and that $\ell(sw) < \ell(w)$ and $\ell(ws) < \ell(w)$ for some $s \in S$, and let us show that $sws = w$. As $\ell(sw) < \ell(w)$, the exchange condition implies that w has a reduced decomposition $w = s_1 \dots s_n$ with $s_1 = s$. Similarly, as $\ell(ws) < \ell(w)$, the exchange condition implies that there is some $i \in \{1, \dots, n\}$ such that $w = s_1 \dots \widehat{s_i} \dots s_n s$, where $\widehat{s_i}$ indicates the omission of s_i. If $i \neq 1$, then $\ell(sws) = \ell(s_2 \dots \widehat{s_i} \dots s_n) = \ell(w) - 2$, a contradiction. Thus $i = 1$ and $sws = ss_2 \dots s_n = w$, as desired.

(3) This holds by definition of tight conjugation. $\square$

Definition 3.6. For $w, w' \in W$, we write $w \to\approx_t w'$ if there is some $w'' \in W$ such that $w \to w''$ and $w'' \approx_t w'$.

4. The complex $\mathrm{CombiMin}(w)$

In this section, we establish some basic properties of the combinatorial analogue $\mathrm{CombiMin}(w)$ of $\mathrm{Min}(w)$ for an element $w \in W$, and show how it is related to the conjugation operation $\to$ from Definition 3.1.

Definition 4.1. For $w \in W$, set

$$\mathrm{CombiMin}(w) := \{D \in \mathrm{Ch}(X) \mid \mathrm{d_{Ch}}(D, wD) \text{ is minimal}\}.$$

Alternatively, $\mathrm{CombiMin}(w)$ is the set of chambers $D = vC_0$ ($v \in W$) such that $v^{-1}wv$ is of minimal length in the conjugacy class of w. In other words, $\mathrm{CombiMin}(w)$ coincides with the inverse image under the map

$$\pi_w\colon\ \mathrm{Ch}(X) \to W : vC_0 \mapsto v^{-1}wv$$

of the set of conjugates of w of minimal length.

Definition 4.2. Let $w \in W$. A chamber subcomplex A of X is called w**-convex** if $\Gamma(D, w^\varepsilon D) \subseteq A$ for any chamber D of A and any $\varepsilon \in \{\pm 1\}$.

Lemma 4.3. *Let $w \in W$. Then* $\mathrm{CombiMin}(w)$ *is w-convex.*

Proof. Let $\varepsilon \in \{\pm 1\}$ and $D \in \mathrm{CombiMin}(w)$. Let $E \in \Gamma(D, w^\varepsilon D)$, and let Γ_1 (resp. Γ_2) be a minimal gallery from D to E (resp. from E to $w^\varepsilon D$), so that $\ell(\Gamma_1) + \ell(\Gamma_2) = \mathrm{d_{Ch}}(D, w^\varepsilon D)$. Then the concatenation of Γ_2 with $w^\varepsilon \Gamma_1$ is a gallery from E to $w^\varepsilon E$, and hence $\mathrm{d_{Ch}}(E, w^\varepsilon E) \leq \mathrm{d_{Ch}}(D, w^\varepsilon D)$, yielding the claim. $\square$

Lemma 4.4. *Let $w \in W$ be straight, and let R be a spherical residue with* $\mathrm{Stab}_W(R) = \mathrm{Stab}_W(wR)$. *Let $C, D \in R \cap$* $\mathrm{CombiMin}(w)$. *Then $\pi_w(C) = \pi_w(D)$.*

Proof. Let $u, v \in W$ be such that $C = uC_0$ and $D = vC_0$, and let us show that $u^{-1}wu = v^{-1}wv$. Note that w is of minimal length in its conjugacy class by Lemma 2.3. Hence $w_0 := u^{-1}wu$ is straight by Lemma 2.4 (because $uC_0 \in \mathrm{CombiMin}(w)$). On the other hand, writing $\mathrm{Stab}_W(R) = uW_Iu^{-1}$ for some spherical subset $I \subseteq S$, the hypotheses imply that $w_0 \in N_W(W_I)$. From Lemma 2.3, we then deduce that $w_0 \in N_I$. In particular, $w_0C_0 = \mathrm{proj}_{w_0R_0}(C_0)$ and hence $\mathrm{d_{Ch}}(C_0, w_0C_0) = \mathrm{d_{Ch}}(E, \mathrm{proj}_{w_0R_0}(E))$ for all $E \in R_0 := u^{-1}R$ (see Example 2.6). Finally, since $E := u^{-1}vC_0 \in R_0 \cap \mathrm{CombiMin}(w_0)$ by assumption, we have $\mathrm{d_{Ch}}(E, w_0E) = \mathrm{d_{Ch}}(C_0, w_0C_0)$ and hence $w_0E = \mathrm{proj}_{w_0R_0}(E)$.

Since $v^{-1}uw_0R_0 = w_0R_0$ (because $u^{-1}v \in W_I = \mathrm{Stab}_W(R_0)$ and $w_0 \in N_W(W_I)$), we conclude that

$$w_0E = \mathrm{proj}_{w_0R_0}(u^{-1}vC_0) = u^{-1}v\,\mathrm{proj}_{v^{-1}uw_0R_0}(C_0) = u^{-1}v\,\mathrm{proj}_{w_0R_0}(C_0) = u^{-1}vw_0C_0,$$

that is, $w_0u^{-1}v = u^{-1}vw_0$, or else $v^{-1}wv = v^{-1}uw_0u^{-1}v = w_0 = u^{-1}wu$, as desired. □

Lemma 4.5. *Let $w \in W$. Let $C, D \in \mathrm{Ch}(X)$ be two chambers connected by a gallery $\Gamma \subseteq \mathrm{CombiMin}(w)$. Then $\pi_w(C) \to \pi_w(D)$.*

Proof. Let $u, v \in W$ be such that $C = uC_0$ and $D = vC_0$, and let us show that $u^{-1}wu \to v^{-1}wv$. Reasoning inductively on $\ell(\Gamma)$, we may assume that uC_0, vC_0 are adjacent, say $v = us$ for some $s \in S$. As $uC_0, vC_0 \in \mathrm{CombiMin}(w)$, we have $\ell(u^{-1}wu) = \ell(v^{-1}wv)$, and hence $u^{-1}wu \xrightarrow{s} v^{-1}wv$, as desired. □

Remark 4.6. Let $w \in W$ be of minimal length in its conjugacy class (i.e. $C_0 \in \mathrm{CombiMin}(w)$). Lemma 4.5 implies that if $\mathrm{CombiMin}(w)$ is gallery-connected, then every conjugate w' of w with $\ell(w') = \ell(w)$ can be obtained from w by a sequence of cyclic shifts (in particular, Theorem A(2) holds for w). The necessity of introducing "tight conjugations" as well comes from the fact that $\mathrm{CombiMin}(w)$ need not be gallery-connected, as for instance illustrated by the Coxeter group $W = \langle s, t \mid s^2 = t^2 = (st)^3 = 1\rangle$ of type A_2, with $w = s$.

5. The complex $\mathscr{C}^w$

In this section, we define for each $w \in W$ a chamber subcomplex $\mathscr{C}^w$ of X such that for any chamber $D = vC_0$ of $\mathscr{C}^w$ ($v \in W$), the conjugate $\pi_w(D) = v^{-1}wv$ of w can be obtained from w through a sequence of cyclic shifts and tight conjugations.

Definition 5.1. Let $w \in W$. Consider the following conditions, which a chamber subcomplex A of X may or may not satisfy.

(CM0) $C_0 \in \mathrm{Ch}(A)$.
(CM1) A is w-convex.
(CM2) If R is a spherical residue such that $\mathrm{Stab}_W(R) = \mathrm{Stab}_W(wR)$ and $R \cap \mathrm{Ch}(A) \neq \varnothing$, then $R \cap \mathrm{CombiMin}(w) \subseteq A$.

The smallest chamber subcomplex of X satisfying (CM0) and (CM1), denoted $\mathscr{C}_w$, was introduced in [Mar14b, §3]. In this paper, we will instead consider the smallest chamber subcomplex of X satisfying (CM0), (CM1) and (CM2), which we denote by $\mathscr{C}^w$.

Lemma 5.2. *Let $w \in W$ be of minimal length in its conjugacy class. Then $\mathscr{C}^w \subseteq \mathrm{CombiMin}(w)$.*

Proof. We have to check that $\mathrm{CombiMin}(w)$ satisfies (CM0), (CM1) and (CM2). But (CM0) holds by assumption, (CM1) by Lemma 4.3, and (CM2) is clear. □

Lemma 5.3. *Let $w \in W$, and let $C, D \in \mathrm{Ch}(X)$ be such that $D \in \Gamma(C, w^\varepsilon C)$ for some $\varepsilon \in \{\pm 1\}$. Then $\pi_w(C) \to \pi_w(D)$. If, moreover, $C \in \mathrm{CombiMin}(w)$, then $\pi_w(C) \approx_t \pi_w(D)$.*

Proof. Let $u, v \in W$ be such that $C = uC_0$ and $D = vC_0$. Let Γ be a minimal gallery from uC_0 to $w^\varepsilon uC_0$ containing vC_0, and let $(s_1, \dots, s_d)$ be its type, so that $u^{-1}w^\varepsilon u = s_1 \dots s_d$. Then the gallery Γ' from vC_0 to $w^\varepsilon vC_0$ obtained by concatenating the sub-gallery of Γ from vC_0 to $w^\varepsilon uC_0$ with the sub-gallery of $w^\varepsilon\Gamma$ from $w^\varepsilon uC_0$ to $w^\varepsilon vC_0$ is of type $(s_{r+1}, \dots, s_d, s_1, \dots, s_r)$, where $r := \mathrm{d_{Ch}}(uC_0, vC_0)$. Hence $v^{-1}w^\varepsilon v = s_{r+1} \dots s_d s_1 \dots s_r$, yielding the first claim. The second claim then follows from Lemma 3.5(3). □

Lemma 5.4. *Let $w \in W$ and let $C \in \mathrm{Ch}(X)$ be such that $C \in R$ for some spherical residue R with $\mathrm{Stab}_W(R) = \mathrm{Stab}_W(wR)$. Let $D \in \mathrm{Ch}(X)$ be such that $D \in R \cap \mathrm{CombiMin}(w)$. Then $\pi_w(C) \to\approx_t \pi_w(D)$. If, moreover, $C \in \mathrm{CombiMin}(w)$, then $\pi_w(C) \approx_t \pi_w(D)$.*

Proof. Let $u, v \in W$ be such that $C = uC_0$ and $D = vC_0$. As $u^{-1}R$ is a standard spherical residue, there is some spherical subset $I \subseteq S$ such that $\mathrm{Stab}_W(u^{-1}R) = W_I$. By assumption, $u^{-1}wu$ normalises W_I, and hence there exist by Lemma 2.1 some $w_I \in W_I$ and $n_I \in N_I$ such that $u^{-1}wu = n_I w_I$. Moreover, as $vC_0 \in R$, there is some $x \in W_I$ such that $v = ux$. Let $\delta\colon W_I \to W_I$ denote the diagram automorphism of W_I defined by $\delta(z) := n_I^{-1} z n_I$. Then

$$v^{-1}wv = x^{-1}n_I w_I x = n_I \cdot \delta(x)^{-1} w_I x.$$

Note that the element $\delta(x)^{-1}w_I x$ is of minimal length in its δ-twisted conjugacy class $\mathcal{O}_\delta(w_I) := \{\delta(z)^{-1} w_I z \mid z \in W_I\}$: otherwise, we find some $z \in W_I$ such that $\ell(\delta(xz)^{-1}w_I xz) < \ell(\delta(x)^{-1}w_I x)$. We then deduce from (2.1) in Lemma 2.1 that

$$\ell((vz)^{-1}wvz) = \ell(n_I \cdot \delta(xz)^{-1}w_I xz) < \ell(n_I \cdot \delta(x)^{-1}w_I x) = \ell(v^{-1}wv),$$

contradicting our assumption that $v^{-1}wv$ is of minimal length in its conjugacy class (i.e. $vC_0 \in \mathrm{CombiMin}(w)$).

By [HN12, Theorem 3.1] applied to the Coxeter system (W_I, I) and to the automorphism δ of W_I, we can find some $w_I' \in W_I$ of minimal length in $\mathcal{O}_\delta(w_I)$ such that $w_I \to_\delta w_I'$ and $w_I' \sim_\delta \delta(x)^{-1}w_I x$, where $\to_\delta$ and $\sim_\delta$ are the analogues of $\to$ and $\sim$ in W_I for δ-twisted conjugacy classes (i.e. one can transform w_I into w_I' by a sequence of elementary operations of the form $z \mapsto \delta(s)^{-1}zs$ ($z \in W_I$ and $s \in I$) where $\ell(\delta(s)^{-1}zs) \leq \ell(z)$, and one can transform w_I' into $\delta(x)^{-1}w_I x$ by a sequence of elementary operations of the form $z \mapsto \delta(y)^{-1}zy$ ($z, y \in W_I$) where $\ell(\delta(y)^{-1}zy) = \ell(z)$ and either $\ell(\delta(y)^{-1}z) = \ell(y) + \ell(z)$ or $\ell(zy) = \ell(z) + \ell(y)$).

Using again (2.1) and the fact that $n_I \cdot \delta(y)^{-1}zy = y^{-1}n_I zy$ for all $y, z \in W_I$, we deduce that

$$n_I \cdot w_I \to n_I \cdot w_I' \approx_t n_I \cdot \delta(x)^{-1}w_I x,$$

that is, $u^{-1}wu = n_I w_I \to\approx_t n_I \cdot \delta(x)^{-1}w_I x = x^{-1}n_I w_I x = v^{-1}wv$, proving the first claim. The second claim then follows from Lemma 3.5(3). ☐

Proposition 5.5. *Let $w \in W$. If $C \in \mathrm{Ch}(\mathscr{C}^w) \cap \mathrm{CombiMin}(w)$, then $w \to\approx_t \pi_w(C)$.*

Proof. Note that $w = \pi_w(C_0)$. By definition of $\mathscr{C}^w$, the chamber C can be obtained from the chamber C_0 after performing a finite sequence of steps of one of the following two types:

(I) going from a chamber $C \in \mathrm{Ch}(\mathscr{C}^w)$ to a chamber $D \in \Gamma(C, w^\varepsilon C)$ for some $\varepsilon \in \{\pm 1\}$.

(II) going from a chamber $C \in R \cap \mathrm{Ch}(\mathscr{C}^w)$ for some spherical residue R with $\mathrm{Stab}_W(R) = \mathrm{Stab}_W(wR)$ to a chamber $D \in R \cap \mathrm{CombiMin}(w)$.

Hence the proposition follows from a straightforward induction on the number of steps of type (I) and (II) needed to go from C_0 to C, by using Lemmas 5.3 and 5.4. ☐

Remark 5.6. Let $w \in W$. If $\mathrm{CombiMin}(w) \subseteq \mathscr{C}^w$, then Proposition 5.5 implies that $w \to\approx_t u$ for any u of minimal length in the conjugacy class of w, thus proving Theorem A(1,2) in that case. This idea will be implemented in the next section to complete the proof of Theorem A.

6. The conjugacy problem in (W,S)

This section is devoted to the proof of Theorem A.

Remark 6.1. Note that, in order to prove Theorem A, there is no loss of generality in assuming that (W,S) has finite rank (i.e. that S is finite), justifying our standing assumption from the beginning of §2.1. Indeed, if $w,w'\in W$ are conjugate, say $w'=v^{-1}wv$ for some $v\in W$, then there is some finite subset $J\subseteq S$ such that $w,w',v\in W_J$, and it is thus sufficient to show that w and w' are related by a suitable sequence of elementary operations inside the finite rank Coxeter system (W_J,J).

Proposition 6.2. *Let $w\in W$ be of infinite order. Let L be a w-axis, and let $\sigma\subseteq L$ be a nonempty open geodesic segment that is contained in some open cell* $\mathrm{supp}(\sigma)$*. Let R be the spherical residue corresponding to* $\mathrm{supp}(\sigma)$*. Let D be a chamber. Then $\mathrm{d_{Ch}}(C,wC)\leq \mathrm{d_{Ch}}(D,wD)$, where $C:=\mathrm{proj}_R(D)$.*

In particular, if $D\in\mathrm{CombiMin}(w)$, then $\mathrm{proj}_R(D)\in\mathrm{CombiMin}(w)$.

Proof. Without loss of generality, we may assume that $C=C_0$: indeed, write $C=vC_0$ for some $v\in W$. Then $L':=v^{-1}L$ is an axis for $w':=v^{-1}wv$ containing the nonempty open geodesic segment $\sigma':=v^{-1}\sigma$, and $R':=v^{-1}R$ is the spherical residue corresponding to the cell $\mathrm{supp}(\sigma'):=v^{-1}\mathrm{supp}(\sigma)$ supporting σ'. Moreover, setting $D':=v^{-1}D$, we have $\mathrm{proj}_{R'}(D')=v^{-1}\mathrm{proj}_R(D)=v^{-1}C=C_0$. As $\mathrm{d_{Ch}}(C,wC)=\mathrm{d_{Ch}}(C_0,w'C_0)$ and $\mathrm{d_{Ch}}(D,wD)=\mathrm{d_{Ch}}(D',w'D')$, the claim follows.

Assume thus that $C=C_0$; in particular, R is standard. Let $I\subseteq S$ be such that $\mathrm{Stab}_W(R)=W_I$. Let us show that

$$\ell(w)=\mathrm{d_{Ch}}(C_0,wC_0)\leq \mathrm{d_{Ch}}(D,wD). \tag{6.1}$$

Note that the walls of R coincide with the walls containing σ, or else with the walls containing L. In particular, w stabilises this set of walls, so that the residues R and wR are parallel, and $w\in N_W(W_I)$. Write $w=n_Iw_I$ for some $w_I\in W_I$ and $n_I\in N_I$, so that $\ell(w)=\ell(n_I)+\ell(w_I)$ (see Lemma 2.1). Thus the chambers C_0 and n_IC_0 lie on the same side of any wall of R.

Let $n\in\mathbb{N}^*$ be such that $w^n=n_I^n$ (i.e. for each $r\in\mathbb{N}$, there is some $w_r\in W_I$ such that $w^r=n_I^rw_r$; since W_I is finite, we find some $r,s\in\mathbb{N}^*$ with $r<s$ such that $w_r=w_s$, and one can take $n:=s-r$). Let Γ be a gallery from D to w^nD obtained by concatenating minimal galleries Γ_i from $w^{i-1}D$ to w^iD for $i=1,\dots,n$. Thus $\ell(\Gamma)=n\,\mathrm{d_{Ch}}(D,wD)$.

Since D,C_0,n_IC_0 lie on the same side of any wall of R (equivalently, of wR), we have $\mathrm{proj}_{wR}(D)=n_IC_0$, and hence

$$\mathrm{proj}_R(w^{-1}D)=w^{-1}\mathrm{proj}_{wR}(D)=w^{-1}n_IC_0=w_I^{-1}C_0.$$

In particular, for each $i\in\{1,\dots,n\}$, the number of walls of R (or equivalently, of w^iR) crossed by Γ_i is

$$\mathrm{d_{Ch}}(\mathrm{proj}_{w^iR}(w^{i-1}D),\mathrm{proj}_{w^iR}(w^iD))=\mathrm{d_{Ch}}(\mathrm{proj}_R(w^{-1}D),\mathrm{proj}_R(D))=\mathrm{d_{Ch}}(C_0,w_IC_0).$$

As $\ell(\Gamma)$ is also the number of times Γ crosses a wall, and as D and $w^nD=n_I^nD$ lie on the same side of any wall of R, we deduce that

$$n\,\mathrm{d_{Ch}}(D,wD)=\ell(\Gamma)\geq \mathrm{d_{Ch}}(D,w^nD)+n\,\mathrm{d_{Ch}}(C_0,w_IC_0).$$

In particular,

$$\ell(w)=\ell(n_I)+\mathrm{d_{Ch}}(C_0,w_IC_0)\leq \ell(n_I)+\mathrm{d_{Ch}}(D,wD)-\frac{\mathrm{d_{Ch}}(D,w^nD)}{n}. \tag{6.2}$$

Finally, note that L is also an n_I-axis, as $n_Ix=n_Iw_Ix=wx\in L$ for any $x\in L$. As C_0 and n_IC_0 are not separated by any wall containing L (that is, by any wall of R), it follows from [Mar14b, Lemma 4.3] that n_I is straight. Hence

$$n\ell(n_I)=\mathrm{d_{Ch}}(C_0,n_I^nC_0)=\mathrm{d_{Ch}}(C_0,w^nC_0).$$

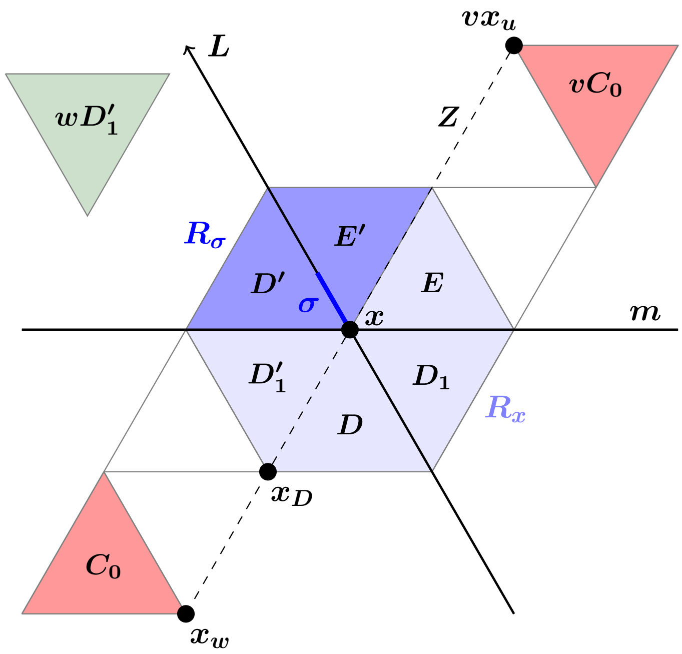

FIG. 2. Proof of Proposition 6.3

Moreover, $n_I^n = w^n$ is straight as well, hence of minimal length in its conjugacy class by Lemma 2.3. In particular, $\mathrm{d_{Ch}}(C_0, w^nC_0) \leq \mathrm{d_{Ch}}(D, w^nD)$. Therefore, $\mathrm{d_{Ch}}(D, w^nD) \geq n\ell(n_I)$, and (6.1) follows from (6.2). $\square$

Proposition 6.3. *Let $w \in W$ be of infinite order, and let u be of minimal length in the conjugacy class of w. Then $w \to\approx_t u$. If, moreover, w is straight, then $w \to u$.*

Proof. By [Mar14b, Corollary 3.5], we find some $w_1, u_1 \in W$ with $w \sim_\kappa w_1$ and $u \sim_\kappa u_1$, such that there exist some $x_w \in \mathrm{Min}(w_1) \cap C_0$ and some $x_u \in \mathrm{Min}(u_1) \cap C_0$. Note that u_1 is still of minimal length in its conjugacy class and $u_1 \sim_\kappa u$; similarly, if w is straight, then w_1 is still straight by Lemma 2.4. In view of Lemma 3.5(1,3), there is thus no loss of generality in assuming that $w = w_1$ and $u = u_1$.

Let $v \in W$ be such that $u = v^{-1}wv$. In particular, $vC_0 \in \mathrm{CombiMin}(w)$. Moreover, $Z := [x_w, vx_u] \subseteq \mathrm{Min}(w)$ as $\mathrm{Min}(w)$ is convex. Let $\Gamma_Z(C_0, vC_0)$ be the set of chambers of $\Gamma(C_0, vC_0)$ intersecting Z nontrivially (note that there always exists a minimal gallery from C_0 to vC_0 containing Z, see [Mar14b, Lemma 3.1]).

Claim: *Let $D \in \Gamma_Z(C_0, vC_0) \cap \mathrm{Ch}(\mathscr{C}^w)$ with $D \neq vC_0$. Then there exists some $E \in \Gamma_Z(C_0, vC_0) \cap \mathrm{Ch}(\mathscr{C}^w)$ with $\mathrm{d_{Ch}}(E, vC_0) < \mathrm{d_{Ch}}(D, vC_0)$. If, moreover, w is straight, then $\pi_w(D) \to \pi_w(E)$.*

Indeed, let $x_D \in D \cap Z$, and let $D = D_0, D_1, \dots, D_k = vC_0$ be a minimal gallery from D to vC_0 containing $[x_D, vx_u]$ ($k \geq 1$). Let $x \in (D_0 \cap D_1) \cap [x_D, vx_u]$. Let also L be the w-axis through x, and let $\sigma \subseteq L$ be a nonempty open geodesic segment containing x in its closure and contained in some (open) cell $\mathrm{supp}(\sigma)$. Let R_x (resp. R_σ) be the spherical residue consisting of all chambers containing x (resp. σ). In particular, $R_\sigma \subseteq R_x$ and $D \in R_x$. Let $D' := \mathrm{proj}_{R_\sigma}(D)$, $E = \mathrm{proj}_{R_x}(vC_0)$ and $E' := \mathrm{proj}_{R_\sigma}(vC_0) = \mathrm{proj}_{R_\sigma}(E)$ (see Figure 2). Note that $E \in \Gamma_Z(C_0, vC_0)$, because $\mathrm{d_{Ch}}(vC_0, D) = \mathrm{d_{Ch}}(vC_0, E) + \mathrm{d_{Ch}}(E, D)$ by the gate property. Moreover, $D \neq E$, for otherwise $\mathrm{d_{Ch}}(vC_0, D_1) = \mathrm{d_{Ch}}(vC_0, D) + 1$, a contradiction. In particular, $\mathrm{d_{Ch}}(E, vC_0) < \mathrm{d_{Ch}}(D, vC_0)$.

Let now $\Gamma_D = (D = D'_0, D'_1, \dots, D'_l = D')$ be a minimal gallery from D to D'. We claim that $\Gamma_D \subseteq \mathscr{C}^w$. Indeed, assume for a contradiction that there is some $i \in \{1, \dots, l\}$

such that $D'_{i-1} \in \mathrm{Ch}(\mathscr{C}^w)$ but $D'_i \notin \mathrm{Ch}(\mathscr{C}^w)$. Let m be the wall separating D'_{i-1} from D'_i. Note that $x \in D'_{i-1} \cap D'_i$. Hence if L is transverse to m, then we find some $\varepsilon \in \{\pm 1\}$ such that D'_{i-1} and $w^{\varepsilon} D'_{i-1}$ lie on different sides of m. Hence in that case, $D'_i \in \Gamma(D'_{i-1}, w^{\varepsilon} D'_{i-1})$, so that $D'_i \in \mathrm{Ch}(\mathscr{C}^w)$ by (CM1), a contradiction (this is illustrated for $i = 2$ on Figure 2). Thus m contains L. In particular, m contains σ, and hence m is a wall of R_σ. But by definition of D', the minimal gallery Γ_D does not cross any wall of R_σ, a contradiction. This proves that $\Gamma_D \subseteq \mathscr{C}^w$, and hence in particular that $D' \in \mathrm{Ch}(\mathscr{C}^w)$.

We next claim that $E' \in \mathrm{Ch}(\mathscr{C}^w)$. Indeed, note that the walls of R_σ coincide with the walls containing L. In particular, w stabilises this set of walls. In other words, $\mathrm{Stab}_W(R_\sigma) = \mathrm{Stab}_W(wR_\sigma)$. On the other hand, since $vC_0 \in \mathrm{CombiMin}(w)$ by assumption, Proposition 6.2 implies that $E' \in \mathrm{CombiMin}(w)$. Hence $E' \in \mathrm{Ch}(\mathscr{C}^w)$ by (CM2).

Finally, let Γ_E be a minimal gallery from E' to E. Then $\Gamma_E \subseteq \mathscr{C}^w$, for exactly the same reasons that $\Gamma_D \subseteq \mathscr{C}^w$. In particular, $E \in \mathrm{Ch}(\mathscr{C}^w)$.

If, moreover, w is straight (in particular, w is of minimal length in its conjugacy class by Lemma 2.3), then $\Gamma_D, \Gamma_E \subseteq \mathscr{C}^w \subseteq \mathrm{CombiMin}(w)$ by Lemma 5.2, and hence $\pi_w(D) \to \pi_w(D')$ and $\pi_w(E') \to \pi_w(E)$ by Lemma 4.5. As $\pi_w(D') = \pi_w(E')$ by Lemma 4.4, this proves the claim.

The claim readily implies that $vC_0 \in \mathrm{Ch}(\mathscr{C}^w)$, so that $w \to\approx_t u$ by Proposition 5.5. If, moreover, w is straight, then the claim implies that $w = \pi_w(C_0) \to \pi_w(vC_0) = u$, as desired. ☐

Lemma 6.4. *Let $I, J \subseteq S$ be such that W_I and W_J are conjugate. Then there is some $x \in W$ with $x^{-1}Ix = J$ such that $w \approx_t x^{-1}wx$ for all $w \in W_I$.*

Proof. By [Kra09, Proposition 3.1.6], there exists some $x \in W$ such that $x^{-1}\Pi_I = \Pi_J$. Hence by [Kra09, Theorem 3.1.3] (see also [Deo82, Proposition 5.5]), we find a sequence $I = I_0, I_1, \dots, I_{k+1} = J$ of subsets of S and elements $s_i \in S \setminus I_i$ $(i = 0, \dots, k)$ such that the following hold for each $i \in \{0, \dots, k\}$:

(1) In the Coxeter diagram of (W, S), the connected component K_i of $I_i \cup \{s_i\}$ containing s_i is spherical. We set $x_i := w_{K_i \setminus \{s_i\}} w_{K_i}$, where for a spherical subset $T \subseteq S$ we denote by w_T the longest element of W_T.
(2) $x_i^{-1}\Pi_{I_i} = \Pi_{I_{i+1}}$.
(3) $I_{i+1} = (I_i \cup \{s_i\}) \setminus \{t_i\}$ for some $t_i \in K_i$.
(4) $x = x_0 \dots x_k$ and $\ell(x) = \sum_{j=0}^{k} \ell(x_j)$.

(Note that x_i is denoted $\nu(I_i, s_i)$ in [Kra09, Theorem 3.1.3].) Let $w \in W_I$. Reasoning inductively on k, it is sufficient to show that if $J = (I \cup \{s\}) \setminus \{t\}$ for some $s \in S \setminus I$ and some $t \in K$, where K is the connected component of $I \cup \{s\}$ containing s (K is spherical), then $w \overset{x}{\sim} x^{-1}wx$, where $x := w_{K \setminus \{s\}} w_K \in W_K$ (note that w normalises W_K).

Set $I' := I \setminus K$, so that I' and K are not connected in $I' \cup K = I \cup \{s\}$. Write $w = w(I') \cdot w(K^s)$ with $w(I') \in W_{I'}$ and $w(K^s) \in W_{K^s}$, where we set for short $K^s := K \setminus \{s\} = I \setminus I'$. We have to show that $\ell(x^{-1}w) = \ell(x) + \ell(w)$. Recall from [AB08, Proposition 1.77] that for any spherical subset $T \subseteq S$ and any $v \in W_T$, we have $w_T = w_T^{-1}$ and $\ell(w_T v) = \ell(v w_T) = \ell(w_T) - \ell(v)$. Hence,

$$
\begin{aligned}
\ell(x^{-1}w) &= \ell(w_K^{-1} \cdot w_{K^s}^{-1} w(K^s)) + \ell(w(I')) = \ell(w_K) - \ell(w_{K^s}^{-1} w(K^s)) + \ell(w(I')) \\
&= \ell(w_K) - \ell(w_{K^s}) + \ell(w(K^s)) + \ell(w(I')) = \ell(w_{K^s} w_K) + \ell(w(K^s) w(I')) \\
&= \ell(x) + \ell(w),
\end{aligned}
$$

as desired. ☐

Proposition 6.5. *Let $w \in W$ be of finite order, and let u be of minimal length in the conjugacy class of w. Then $w \to\approx_t u$.*

Proof. By [Mar14b, Corollary C], there is some $w_1 \in W$ of minimal length in its conjugacy class such that $w \to w_1$, and we may thus assume that w is of minimal length in its conjugacy class. In particular, by [CF10, Proposition 4.2], w has standard parabolic closure $\mathrm{Pc}(w) = W_I$ $(I \subseteq S)$, while u has standard parabolic closure $\mathrm{Pc}(u) = W_J$ $(J \subseteq S)$. As W_I and W_J are conjugate, we find some $w' \in W_J$ such that $w \approx_t w'$ by Lemma 6.4. On the other hand, by [HN12, Theorem 3.1(2)] applied in the finite Coxeter group W_J, we have $w' \sim u$ (and hence $w' \approx_t u$). Thus $w \approx_t w' \approx_t u$, as desired. □

Here is a reformulation of Theorem A.

Theorem 6.6. *Let $w \in W$, and let u be of minimal length in the conjugacy class of w. Then $w \to\approx_t u$. If, moreover, w is straight, then $w \to u$.*

Proof. This sums up Propositions 6.3 and 6.5. □

7. Comparison of $\mathrm{Min}(w)$ and $\mathrm{CombiMin}(w)$

This final section is devoted to the proof of Corollary C. We start with the analogue of Proposition 6.2 for elements $w \in W$ of finite order.

Lemma 7.1. *Let $w \in W$ be of finite order. Let $x \in \mathrm{Min}(w)$, and let R be the spherical residue corresponding to x. Let $D \in \mathrm{Ch}(X)$. Then $\mathrm{d_{Ch}}(C, wC) \leq \mathrm{d_{Ch}}(D, wD)$, where $C := \mathrm{proj}_R(D)$. In particular, if $D \in \mathrm{CombiMin}(w)$, then $\mathrm{proj}_R(D) \in \mathrm{CombiMin}(w)$.*

Proof. As $wR = R$, we have $wC = \mathrm{proj}_R(wD)$, so that the lemma follows from the fact that proj_R does not increase the chamber distance. □

Proposition 7.2. *Let $w \in W$. Then $\mathrm{Min}(w) \subseteq \mathrm{CombiMin}(w)$.*

Proof. Let $x \in \mathrm{Min}(w)$. If w has finite order, then by Lemma 7.1, the projection on the residue corresponding to x of any chamber $D \in \mathrm{CombiMin}(w)$ is a chamber $C \in \mathrm{CombiMin}(w)$ containing x. Assume now that w has infinite order, and let L be the w-axis through x. Let $\sigma \subseteq L$ be a nonempty open geodesic segment containing x in its closure and contained in some (open) cell. Then Proposition 6.2 yields some chamber $D \in \mathrm{CombiMin}(w)$ containing σ, and hence also x. □

Lemma 7.3. *Let $w \in W$. Then the centraliser $C_W(w)$ of w in W acts cocompactly on both $\mathrm{Min}(w)$ and $\mathrm{CombiMin}(w)$.*

Proof. The fact that $C(w) := C_W(w)$ acts cocompactly on $\mathrm{Min}(w)$ follows from [Rua01, Theorem 3.2], and a straightforward adaptation of the proof of [Rua01, Theorem 3.2], which we now provide, also shows that $C(w)$ acts cocompactly on $\mathrm{CombiMin}(w)$.

Indeed, fix some $D \in \mathrm{CombiMin}(w)$, and assume for a contradiction that there is a sequence of chambers $(D_n)_{n\in\mathbb{N}} \subseteq \mathrm{CombiMin}(w)$ such that $\mathrm{d_{Ch}}(D_n, C(w)D) \geq n$ for all $n \in \mathbb{N}$. Write $D_n = v_n D$ for some $v_n \in W$. By hypothesis, we have $\mathrm{d_{Ch}}(D, v_n^{-1}wv_nD) = \mathrm{d_{Ch}}(D_n, wD_n) = \mathrm{d_{Ch}}(D, wD)$ for all $n \in \mathbb{N}$. In particular, $\{v_n^{-1}wv_n \mid n \in \mathbb{N}\}$ is finite. Hence, up to extracting a subsequence, we may assume that $v_n^{-1}wv_n = v_0^{-1}wv_0$ for all $n \in \mathbb{N}$, that is, $v_nv_0^{-1} \in C(w)$ for all $n \in \mathbb{N}$. But then

$$n \leq \mathrm{d_{Ch}}(D_n, C(w)D) \leq \mathrm{d_{Ch}}(D_n, v_nv_0^{-1}D) = \mathrm{d_{Ch}}(D, v_0^{-1}D) = \mathrm{d_{Ch}}(D_0, D)$$

for all $n \in \mathbb{N}$, a contradiction. □

Proof of Corollary C: Let $w \in W$. Then $\mathrm{Min}(w) \subseteq \mathrm{CombiMin}(w)$ by Proposition 7.2, and $\mathrm{CombiMin}(w)$ is at bounded Hausdorff distance from $\mathrm{Min}(w)$ by Lemma 7.3. □

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