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CONDITIONAL MEAN RISK SHARING OF INDEPENDENT DISCRETE LOSSES IN LARGE POOLS

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Abstract

This paper considers a risk sharing scheme of independent discrete losses that combines risk retention at individual level, risk transfer for too expensive losses and risk pooling for the middle layer. This ensures that pooled losses can be considered as being uniformly bounded. We study the no-sabotage requirement and diversification effects when the conditional mean risk-sharing rule is applied to allocate pooled losses. The no-sabotage requirement is equivalent to Efron's monotonicity property for conditional expectations, which is known to hold under log-concavity. Elementary proofs of this result for discrete losses are provided for finite population pools. The no-sabotage requirement and diversification effects are then examined within large pools. It is shown that Efron's monotonicity property holds asymptotically and that risk can be eliminated under fairly general conditions which are fulfilled in applications.

Keywords: Risk pooling, Peer-to-Peer (P2P) insurance, Conditional mean risk-sharing, Likelihood ratio order, Log-concavity.

1 Introduction and motivation

Peer-to-Peer (P2P) insurance, or crowdsurance refers to emerging, technology-based risk sharing networks where participants pool their resources to mutually protect against the financial consequences of a given peril. Rooted in the sharing economy, it revives early forms of mutual insurance. P2P insurance aims to mitigate the conflict inherent to commercial insurance (where insurers keep the part of premiums that is not paid out for claims) and to lower prices or extend coverability.

Actuaries started to investigate the mathematics supporting P2P insurance quite recently. See e.g. Abdikerimova and Feng (2022) and Denuit (2019). The main challenge is to fairly distribute total losses among participants within P2P insurance funds aiming at standardization (that is, using general behavioral traits like risk aversion but not individual preferences). Appropriate risk-sharing rules are needed to that end. Inspired by properties that premium principles or risk measures may or may not satisfy, Denuit et al. (2022a) systematically studied the properties that risk-sharing rules could/should satisfy. This resulted in a list of reasonable (not necessarily independent) requirements that a risk-sharing scheme may fulfill. In particular, the no-sabotage (as termed by Carlier and Dana, 2003) or comonotonicity property (as termed in Denuit et al., 2022a) appears to be a desirable property since it ensures that the interests of all participants are aligned, in the sense that they all have an interest in keeping their losses as small as possible. Throughout this paper, we consider the conditional mean risk-sharing rule proposed by Denuit and Dhaene (2012), where the contribution paid in arrear by each participant is the conditional expectation of his or her loss, given the observed aggregate loss of the pool. This rule has been axiomatized by Jiao et al. (2022).

Under pure P2P insurance, participants' contributions are theoretically unlimited. A more effective solution can be obtained by combining self-insurance, or retention by individual participants, P2P pooling of the middle risk layer at the level of a community and risk transfer of the upper layer to an insurance company. By partnering with an established insurance company, the community can benefit not only from its risk-bearing capacity, but also from its experience in claim settlement and the related administration. Denuit (2020) designed such a scheme under the conditional mean risk-sharing rule and Denuit et al. (2023) extended it to any aggregate rule satisfying the no-sabotage or comonotonicity property.

In the present paper, we study the no-sabotage requirement and diversification effects obtained when the conditional mean risk-sharing rule is applied to allocate discrete independent losses among participants. This is in line with practice where insurance losses are generally discretized before performing actuarial calculations.

Under the conditional mean risk-sharing rule, the no-sabotage requirement is equivalent to Efron's monotonicity for conditional expectations given sums. The latter is known to hold true under log-concavity. We provide an elementary proof of this classical result for discrete distributions (since existing proofs rely on the basic composition formula and deal with probability density functions). The no-sabotage property is also carefully studied in large pools where log-concavity is recovered thanks to local limit theorems. We thus provide results comparable to those obtained in Denuit and Robert (2021b) where the loss distributions have been assumed to be absolutely continuous.

We also study diversification effects obtained with the conditional mean risk-sharing rule. Diversification effects of this risk-sharing rule when the size of the pool becomes larger and larger have already been documented in Denuit and Robert (2021a) for absolutely continuous losses or when the losses have compound Poisson distributions for which the claim severities are positive and absolutely continuous. Results have also been provided when losses are given by independent Bernoulli random variables multiplied by a capital belonging to a predefined menu in Denuit et al. (2022b). But no results seem to be available for general discrete distributions. We show that insurance at fair price can be recovered within large pools, under fairly general conditions which are fulfilled in applications to P2P insurance.

The remainder of this paper is organized as follows. Section 2 presents the 3-layer scheme studied in this paper where risk pooling operates in the middle layer. This ensures that insurance losses to be pooled are uniformly bounded from above. It also recalls the definition of the conditional mean risk-sharing rule. Section 3 presents the no-sabotage property and then discusses the likelihood ratio order and the log-concavity concept. This section also provides the reader with an elementary proof of Efron's monotonicity property for counting random variables. Log-concavity appears to be a strong requirement since it constrains in particular the loss distribution to be light-tailed. Section 4 is dedicated to large pools. It is proved that the no-sabotage property is generally recovered within large pools thanks to local limit theorems. The latter are also exploited to obtain risk elimination under fairly general conditions.

2 Risk retention, transfer and sharing

2.1 Collaborative system and pooled losses

As explained in the introduction, this paper considers a collaborative system built on self-insurance, or retention by individual participants, P2P pooling of the middle risk layer at the level of a community and risk transfer of larger losses to an insurance company. Let us briefly explain how this 3-layer system operates.

Consider n individuals, numbered $i = 1, 2, \ldots, n$. Each of them is exposed to some peril causing a random non-negative monetary loss at the end of a given observation period, taken to be the time interval $(0, 1)$. P2P insurance is not costless to operate so that running fees supplement participants' contributions. Therefore, some participants may consider retaining some risk, to save money. This may also be imposed by the sponsor, to guarantee that participants have interest to keep their losses under control. In practice, this is often achieved by applying a deductible or a quota-share arrangement. Let $Y_i \geq 0$ be the loss for participant i, in excess of individual retention.

In liability insurance for instance, some large losses may happen which exceed the risk-sharing capacities of the P2P insurance community. It is therefore useful to limit the maximum loss per participant, to some value m. This means that $\min\{Y_i, m\}$ is pooled and that $(Y_i - m)_+$ is transferred to the partnering insurer through an excess-of-loss cover and does not enter risk pooling. The corresponding premium must be supported directly by participant i, in addition to his or her own contributions to the pool.

From now on, let $X_i = \min\{Y_i, m\}$ be the amount if losses pooled by participant i. Thus, $X_i \leq m$ holds for all i, where m corresponds to the priority of the yearly excess-of-loss cover limiting the amount entering pooling. Moreover, we assume that $X_1, X_2, \ldots$ are independent and valued in $\{0, 1, 2, \ldots, m\}$.

2.2 Conditional mean risk-sharing rule

Henceforth, we use the notation $S_n = \sum_{i=1}^n X_i$ for the total risk of the pool. In a risk pooling scheme, each participant in a pool of size n contributes an amount $h_{i,n}(s)$ where $s = \sum_{i=1}^n x_i$ is the sum of the observed realizations of $X_1, X_2, \ldots, X_n$.

Denuit and Dhaene (2012) studied the conditional mean risk allocation $h_{i,n}^{\text{cmrs}}$ defined as

$$h_{i,n}^{\text{cmrs}}(S_n) = \mathrm{E}[X_i | S_n], \;\; i = 1, 2, \ldots, n. \tag{2.1}$$

Clearly, the conditional mean risk sharing (2.1) allocates the full risk S_n as we obviously have

$$\sum_{i=1}^n h_{i,n}^{\text{cmrs}}(S_n) = \sum_{i=1}^n \mathrm{E}[X_i | S_n] = S_n.$$

In words, participant i must contribute the expected value of the risk X_i brought to the pool, given the total loss S_n. This rule is obviously fair in the actuarial sense as the expected contribution is equal to the expected loss, that is, $\mathrm{E}[h_{i,n}^{\text{cmrs}}(S_n)] = \mathrm{E}[X_i]$ for all $i = 1, 2, \ldots, n$.

3 No-sabotage condition

In the context of risk sharing, it is important that the interests of all participants are aligned, in the sense that they all have an interest in keeping their losses as small as possible. Under the conditional mean risk-sharing rule, this means that all functions $s \mapsto h_{i,n}^{\text{cmrs}}(s)$ are non-decreasing. This is referred to as the no-sabotage condition, after Carlier and Dana (2003) and renders individual contributions $h_{i,n}^{\text{cmrs}}(S_n)$ perfectly positively dependent (or comonotonic) since they are all non-decreasing functions of the random variable S_n. It implies that no individual contribution $h_{i,n}^{\text{cmrs}}(S_n)$ is allowed to increase more than the total losses S_n.

3.1 Likelihood ratio order

The likelihood ratio order plays a central role in the study of the no-sabotage condition. Recall that given two counting random variables V and W, V is said to be smaller than W in the likelihood ratio order, which is henceforth denoted as $V \preceq_{\text{LR}} W$, when the ratio of their respective probability mass functions $\text{P}[W = k]/\text{P}[V = k]$ is non-decreasing in $k \in \{0, 1, \ldots\}$. Equivalently,

$$V \preceq_{\text{LR}} W \Leftrightarrow \text{P}[V = k]\text{P}[W = k'] \geq \text{P}[V = k']\text{P}[W = k] \text{ for all } k \leq k' \in \{0, 1, \ldots\}.$$

When V and W are independent, this means that it is more likely that W assumes the larger value k' and V the smaller k, expressing the fact that W tends to be larger than V. We refer the reader to Denuit et al. (2005, Chapter 3) for more details concerning this stochastic order relation expressing the idea of "being larger than" for random variables.

3.2 Log-concavity

Recall that Z is said to be log-concave when the probability ratio $\text{P}[Z = k + 1]/\text{P}[Z = k]$ is non-increasing in $k \in \{0, 1, \ldots\}$. Clearly,

$$\begin{aligned} Z \text{ log-concave} \quad &\Leftrightarrow \quad \frac{\text{P}[Z = k - 1]}{\text{P}[Z = k]} \text{ non-decreasing in } k \in \{0, 1, \ldots\} \\ &\Leftrightarrow \quad \frac{\text{P}[Z + 1 = k]}{\text{P}[Z = k]} \text{ non-decreasing in } k \in \{0, 1, \ldots\} \Leftrightarrow Z \preceq_{\text{LR}} Z + 1. \end{aligned}$$

More generally,

$$Z \text{ log-concave} \Leftrightarrow Z \preceq_{\text{LR}} Z + j \text{ for any } j \in \{0, 1, \ldots\}. \tag{3.1}$$

Log-concavity implies that the distribution is light-tailed, as explained next. Define the discrete failure rate as

$$r_Z(k) = \frac{\mathrm{P}[Z = k]}{\mathrm{P}[Z \geq k]} \text{ provided } \mathrm{P}[Z \geq k] > 0.$$

Chapter 6 in Lai and Xie (2006) offers an introduction to discrete failure time models. In particular, if Z is log-concave then it has an increasing failure rate, that is, $r_Z(0) \leq r_Z(1) \leq r_Z(2) \leq \ldots$ holds true. Now,

$$\begin{aligned} 1 - r_Z(0) &= 1 - \mathrm{P}[Z = 0] = \mathrm{P}[Z > 0] \\ 1 - r_Z(1) &= \frac{\mathrm{P}[Z \geq 1] - \mathrm{P}[Z = 1]}{\mathrm{P}[Z \geq 1]} = \frac{\mathrm{P}[Z > 1]}{\mathrm{P}[Z > 0]} \end{aligned}$$

and in general

$$1 - r_Z(k) = \frac{\mathrm{P}[Z > k]}{\mathrm{P}[Z > k-1]}$$

so that

$$Z \text{ log-concave} \Rightarrow \mathrm{P}[Z > k] = \prod_{j=0}^{k} \big(1 - r_Z(j)\big) \leq \big(1 - r_Z(0)\big)^{k+1} = \big(\mathrm{P}[Z > 0]\big)^{k+1}$$

which expresses the light-tailed behavior of Z.

Notice that when the probability ratio $\mathrm{P}[Z = k+1]/\mathrm{P}[Z = k]$ is constantly equal to $c > 0$, only three distributions are possible: the Geometric distribution for which $\mathrm{P}[Z = k] = (1-p)^k p$ for some $p \in (0,1)$, the Uniform distribution and the distribution with probability mass function $\mathrm{P}[Z = k] = \frac{c^k}{1+c+c^2+\ldots+c^m}$ for $k \in \{0, 1, \ldots, m\}$. The discrete failure rate is constantly equal to p for the Geometric distribution.

The ratio $\mathrm{P}[Z = k+1]/\mathrm{P}[Z = k]$ can be used to order discrete distributions. Considering Proposition 3.4.10 in Denuit et al. (2005), if Z is log-concave then there is a unique crossing between $\mathrm{P}[Z = k+1]/\mathrm{P}[Z = k]$ and the horizontal line at any height $1 - p \in (0,1)$ corresponding to the Geometric distribution, so that the latter dominates Z in the convex order when p is selected so that expected values coincide. Hence, Z has uniformly smaller stop-loss premiums compared to those of the Geometric distribution with the same mean, as well as uniformly smaller moments of any order. This confirms that log-concavity implies that the distribution is light-tailed.

Let us show that $(j,k) \mapsto \mathrm{P}[Z = j - k]$ is a TP2 function when Z is log-concave. This result is well known but generally stated for random variables with a probability density function. Here, we provide a direct proof of it for counting random variables, deduced from the close relationship between log-concavity and likelihood ratio order recalled earlier.

Property 3.1. *If the counting random variable Z is log-concave then*

$$\mathrm{P}[Z=j'-k']\mathrm{P}[Z=j-k]\geq \mathrm{P}[Z=j'-k]\mathrm{P}[Z=j-k'] \text{ for any } j\leq j' \text{ and } k\leq k'.$$

Proof. Clearly,

$$Z \text{ log-concave} \Rightarrow \frac{\mathrm{P}[Z=l-k+1]}{\mathrm{P}[Z=l-k]} \text{ non-increasing in } l\geq k$$

so that if Z is log-concave then $Z+k$ also is for any $k\in\{0,1,\ldots\}$. Considering (3.1), we then see that

$$Z \text{ log-concave} \Rightarrow Z+k \preceq_{\mathrm{LR}} Z+k' \text{ for any } k\leq k',$$

which is equivalent to the stated inequality. This ends the proof. □

It is well documented that the likelihood ratio order is not preserved by convolution, in general. However, log-concavity ensures its stability under convolution. This result is generally established by applying the so-called basic composition formula to probability density functions. We propose here a direct proof of this result for integer-valued random variables, inspired from Hu and Zhu (2001). The direct reasoning given below will be adapted in Section 4 to deal with large pools.

Property 3.2. *Let U, V and W be independent random variables valued in $\{0,1,2\ldots\}$. If W is log-concave then $U\preceq_{LR}V\Rightarrow U+W\preceq_{LR}V+W$.*

Proof. We have to show that

$$\mathrm{P}[U+W=l]\mathrm{P}[V+W=l']\geq \mathrm{P}[U+W=l']\mathrm{P}[V+W=l] \text{ for all } l\leq l'\in\{0,1,\ldots\}.$$

This is equivalent to show that $\Delta\geq 0$ for all $l\leq l'\in\{0,1,\ldots\}$ where

$$\Delta=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\mathrm{P}[U=j]\mathrm{P}[V=k]\Big(\mathrm{P}[W=l-j]\mathrm{P}[W=l'-k]-\mathrm{P}[W=l'-j]\mathrm{P}[W=l-k]\Big).$$

Let us rewrite Δ as

$$\begin{aligned}
\Delta &= \sum_{j\neq k} \mathrm{P}[U=j]\mathrm{P}[V=k]\Big(\mathrm{P}[W=l-j]\mathrm{P}[W=l'-k]-\mathrm{P}[W=l'-j]\mathrm{P}[W=l-k]\Big)\\
&= \sum_{j>k} \mathrm{P}[U=j]\mathrm{P}[V=k]\Big(\mathrm{P}[W=l-j]\mathrm{P}[W=l'-k]-\mathrm{P}[W=l'-j]\mathrm{P}[W=l-k]\Big)\\
&\quad +\sum_{j>k} \mathrm{P}[U=k]\mathrm{P}[V=j]\Big(\mathrm{P}[W=l-k]\mathrm{P}[W=l'-j]-\mathrm{P}[W=l'-k]\mathrm{P}[W=l-j]\Big).
\end{aligned}$$

Now, $U \preceq_{\mathrm{LR}} V$ implies that $\mathrm{P}[U=k]\mathrm{P}[V=j] \geq \mathrm{P}[U=j]\mathrm{P}[V=k]$ for all $j>k$ and Property 3.1 shows that $\mathrm{P}[W=l-k]\mathrm{P}[W=l'-j] \geq \mathrm{P}[W=l'-k]\mathrm{P}[W=l-j]$ when $j>k$. Hence,

$$\begin{aligned}
\Delta &\geq \sum_{j>k} \mathrm{P}[U=j]\mathrm{P}[V=k]\Big(\mathrm{P}[W=l-j]\mathrm{P}[W=l'-k]-\mathrm{P}[W=l'-j]\mathrm{P}[W=l-k]\\
&\qquad\qquad +\mathrm{P}[W=l-k]\mathrm{P}[W=l'-j]-\mathrm{P}[W=l'-k]\mathrm{P}[W=l-j]\Big)\\
&= 0,
\end{aligned}$$

which ends the proof. $\square$

3.3 Efron's monotonicity property

Consider two independent, integer-valued random variables Y and Z. This sections studies the monotonicity of the conditional expectation $\mathrm{E}[Y|Y+Z=k]$ as a function of $k \in \{0,1,\ldots\}$. This property is referred to as Efron's monotonicity, after Efron (1965).

The next result shows that log-concavity of Z ensures that Efron's monotonicity property holds true. The proof proposed in this paper uses the size-biased transform, whose definition is recalled next. Given a random variable X valued in $\{0,1,\ldots\}$ with finite expectation, define $\widetilde{X}$ valued in $\{1,2,\ldots\}$ with probability mass function function

$$\mathrm{P}[\widetilde{X}=k]=\frac{k\mathrm{P}[X=k]}{\mathrm{E}[X]},\ \ k\in\{1,2,\ldots\};$$

the random variable $\widetilde{X}$ is said to be a size-biased version of X. We refer the reader to Arratia et al. (2019) for a detailed account of the size-biased transform.

Proposition 3.3. *Consider independent random variables Y and Z valued in $\{0,1,\ldots\}$. If Z is log-concave then the conditional expectation $\mathrm{E}[Y|Y+Z=k]$ is non-decreasing in $k\in\{0,1,\ldots\}$.*

Proof. Using Proposition 2.2(ii) in Denuit (2019), we have

$$\mathrm{E}[Y|Y+Z=k] = \mathrm{E}[Y]\frac{\mathrm{P}[\widetilde{Y}+Z=k]}{\mathrm{P}[Y+Z=k]}$$

where the size-biased version $\widetilde{Y}$ is assumed to be independent of Y and Z. Now,

$$\frac{\mathrm{P}[\widetilde{Y}=k]}{\mathrm{P}[Y=k]} = \frac{k}{\mathrm{E}[X]} \text{ increases in } k \in \{0,1,\ldots\} \Rightarrow Y \preceq_{\mathrm{LR}} \widetilde{Y}.$$

By Property 3.2, the log-concavity of Z ensures that $Y+Z \preceq_{\mathrm{LR}} \widetilde{Y}+Z$ so that $\frac{\mathrm{P}[\widetilde{Y}+Z=k]}{\mathrm{P}[Y+Z=k]}$ is non-decreasing in k. This ends the proof. □

Compared to Efron (1965), Property 3.3 does not impose any log-concavity condition on Y. Only Z is assumed to be log-concave there. The result is known but generally stated for probability density functions. See e.g. Denuit and Robert (2021b) or Pellerey and Navarro (2022).

As a consequence of Property 3.3, we can conclude that $s \mapsto h_{i,n}^{\mathrm{cmrs}}(s)$ is a non-decreasing function if $\sum_{j\neq i} X_j$ is log-concave. Log-concavity of losses is therefore a sufficient condition for no sabotage but it appears to be restrictive for applications to insurance. For a large pool however, log-concavity does not appear to be such a restrictive assumption as explained in the following section.

4 Large pools

In this section, we let the size of the pool increase to infinity. The no-sabotage property has already been studied in large pools by Denuit and Robert (2021b) when the loss distributions are absolutely continuous. This assumption however appears as too strong for insurance applications since the probability a participant has no loss is in general high. In our framework this probability will be assumed to be larger than the probabilities to have a loss with size $k \in \{1,2,\ldots\}$ for all participants. This is in line with empirical data where the no-loss probability largely exceeds 0.5 for individual insurance contracts.

Diversification effects of the conditional mean risk-sharing rule for large pools has been studied in Denuit and Robert (2021a) for absolutely continuous losses or when the losses have compound Poisson distributions for which the claim severities are positive and absolutely continuous. We show further in this section that insurance at fair price can also be recovered within large pools for discrete losses, under fairly general conditions.

4.1 Local limit theorem

Local limit theorems (LLTs) describe how the probability mass function of a sum of random variables follows the Normal curve. In probability theory, these theorems have often been considered as being of secondary importance when compared with Central limit theorems, as explained in the recent survey by Szewczak and Weber (2022). But they are essential for our results. Let $X_1, X_2, X_3, \ldots$ be independent integer-valued random variables with

$$\mu_k = \mathrm{E}[X_k], \;\; a_n = \mathrm{E}[S_n] = \sum_{k=1}^{n} \mu_k \text{ and } b_n^2 = \mathrm{Var}[S_n] = \sum_{k=1}^{n} \mathrm{E}[(X_k - \mu_k)^2].$$

LLTs provide conditions such that

$$\sup_k \left| b_n \mathrm{P}[S_n = k] - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(k-a_n)^2}{2b_n^2} \right) \right| \to 0 \tag{4.1}$$

as $n \to \infty$.

Let us assume that $\mathrm{P}[X_i = 0] \geq \mathrm{P}[X_i = j]$ for every participant i and all $j \in \{1, 2, \ldots\}$. Such an assumption is generally satisfied in insurance applications, where the probability mass at zero dominates all other probabilities. As proved in Theorem 3 of Petrov (1975), a necessary and sufficient condition for the convergence in Eq. (4.1) to hold when $X_1, X_2, X_3, \ldots$ are uniformly bounded integer-valued random variables is that

$$\text{g.c.d} \left\{ l : \sum_{i=1}^{\infty} \mathrm{P}[X_i = l] = \infty \right\} = 1,$$

where g.c.d is the greatest common divisor of the terms enclosed in parentheses. The previous condition means that the g.c.d. of the positive integers such that there is an infinite number of X_i that can take as values these positive integers must be equal to 1.

In this paper, we need stronger conditions to get asymptotic expansions and uniform estimates for the remainder term. We will use the following result given in Petrov (1975).

Proposition 4.1. *(Theorem 12 in Petrov (1975)) Assume that:*

(I) $\mathrm{P}[X_k = 0] \geq \mathrm{P}[X_k = j]$ *for all* k *and* j*;*

(II) there exists a constant $c > 0$ *such that* $\liminf_{n\to\infty} \frac{1}{n} b_n^2 \geq c$*;*

(III) $\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{E}[|X_k - \mu_k|^p] < \infty$ *and* $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{|j-\mu_k|>n^\tau} |j - \mu_k|^p = 0$*;*

(IV)

$$g.c.d\left\{l : \frac{1}{\log n}\sum_{k=1}^{n} \mathrm{P}[X_k = 0]\mathrm{P}[X_k = l] \to \infty\right\} = 1,$$

for some integer $p \geq 3$ and some positive $\tau < 1/2$. Then

$$b_n \mathrm{P}[S_n = k] = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)\left(1 + \sum_{\nu=1}^{p-2} \frac{p_{\nu n}(x)}{n^{\nu/2}} + o\left(\frac{1}{n^{(p-2)/2}}\right)\right)$$

uniformly in k ($-\infty < k < \infty$), where $x = (k - a_n)/b_n$ and $p_{\nu n}(\cdot)$ is a polynomial of degree 3ν with coefficients depending only on the cumulants of the random variables $X_1, \ldots, X_n$ up to and including order $\nu + 2$.

LLTs are often studied by using specific indicators. For an integer-valued random variable X, define

$$\vartheta_X = \sum_{j=-\infty}^{\infty} \min\big\{\mathrm{P}[X = j], \mathrm{P}[X = j + 1]\big\}$$

and

$$\delta_X = \sum_{j=-\infty}^{\infty} \big|\mathrm{P}[X = j] - \mathrm{P}[X = j - 1]\big|.$$

The latter series is often used to measure smoothness. Let us recall some useful properties of these indicators, that can be found in Szewczak and Weber (2022). For any positive real values a and b, one has $a + b - |a - b| = 2\min\{a, b\}$ so that

$$\begin{aligned} \vartheta_X &= \frac{1}{2}\sum_{j=-\infty}^{\infty} \Big(\mathrm{P}[X = j] + \mathrm{P}[X = j + 1] - \big|\mathrm{P}[X = j] - \mathrm{P}[X = j + 1]\big|\Big) \\ &= 1 - \frac{\delta_X}{2} \end{aligned}$$

or $\delta_X = 2(1 - \vartheta_X)$.

If the distribution of X is unimodal, in the sense that there exists an integer j_0 such that $\mathrm{P}[X = j - 1] \leq \mathrm{P}[X = j]$ for all $j \leq j_0$ and $\mathrm{P}[X = j + 1] \leq \mathrm{P}[X = j]$ for all $j \geq j_0$ then

$$\begin{aligned} \delta_X &= \sum_{j \leq j_0} \big(\mathrm{P}[X = j] - \mathrm{P}[X = j - 1]\big) + \sum_{j > j_0} \big(\mathrm{P}[X = j - 1] - \mathrm{P}[X = j]\big) \\ &= \sum_{j \leq j_0} \mathrm{P}[X = j] + \sum_{j \geq j_0} \mathrm{P}[X = j] - \left(\sum_{j < j_0} \mathrm{P}[X = j] + \sum_{j > j_0} \mathrm{P}[X = j]\right) \\ &= 2\mathrm{P}[X = j_0] = 2\max_j \mathrm{P}[X = j]. \end{aligned}$$

This is for instance the case when X is log-concave since every log-concave distribution is unimodal.

With individual yearly insurance losses, it is common to have the largest probability mass at zero and then a unique mode on strictly positive values, that is,

$$
\begin{aligned}
&\mathrm{P}[X=0] \geq \mathrm{P}[X=j] \text{ for all } j=1,2,\dots \\
&\mathrm{P}[X=j-1] \leq \mathrm{P}[X=j] \text{ for all } j=2,\dots,j_0 \\
&\mathrm{P}[X=j+1] \leq \mathrm{P}[X=j] \text{ for all } j \geq j_0.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\delta_X &= \mathrm{P}[X=0]+\mathrm{P}[X=0]-\mathrm{P}[X=1]+\sum_{j=2}^{j_0}\big(\mathrm{P}[X=j]-\mathrm{P}[X=j-1]\big) \\
&\quad +\sum_{j>j_0}\big(\mathrm{P}[X=j-1]-\mathrm{P}[X=j]\big) \\
&= \mathrm{P}[X=0]-2\mathrm{P}[X=1]+\sum_{j\leq j_0}\mathrm{P}[X=j]+\sum_{j\geq j_0}\mathrm{P}[X=j]-\left(\sum_{j=1}^{j_0-1}\mathrm{P}[X=j]+\sum_{j>j_0}\mathrm{P}[X=j]\right) \\
&= 2\big(\mathrm{P}[X=0]-\mathrm{P}[X=1]+\mathrm{P}[X=j_0]\big).
\end{aligned}
$$

Sometimes, there is a local mode at the maximum value m, because of truncation, that is, $\mathrm{P}[X=m-1]<\mathrm{P}[X=m]$. We then get

$$
\begin{aligned}
\delta_X &= \mathrm{P}[X=0]+\mathrm{P}[X=0]-\mathrm{P}[X=1]+\sum_{j=2}^{j_0}\big(\mathrm{P}[X=j]-\mathrm{P}[X=j-1]\big) \\
&\quad +\sum_{j=j_0+1}^{m-1}\big(\mathrm{P}[X=j-1]-\mathrm{P}[X=j]\big)+\mathrm{P}[X=m]-\mathrm{P}[X=m-1] \\
&= 2\big(\mathrm{P}[X=0]-\mathrm{P}[X=1]+\mathrm{P}[X=j_0]-\mathrm{P}[X=m-1]+\mathrm{P}[X=m]\big).
\end{aligned}
$$

We will use the following LLT given in McDonald (1979) for establishing risk diversification of the conditional mean risk sharing rule.

Proposition 4.2. *(Theorem 1 in McDonald (1979)). Assume that:*

(I) $\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{E}[\exp(a|X_k|)]<\infty$ *for some positive constant* a*;*

(II) there exists a constant $c>0$ *such that* $\liminf_{n\to\infty}\frac{1}{n}b_n^2\geq c$*;*

(III) $\liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\vartheta_{X_k}>0$.

Let $\omega(n)$ be a sequence such that $\lim_{n\to\infty}\omega(n)=\infty$. Then

$$\mathrm{P}\left[S_n=k\right]=\frac{1}{b_n\sqrt{2\pi}}\exp\left(-\frac{(k-a_n)^2}{2b_n^2}+\frac{(k-a_n)^3}{n^2}\lambda_n\left(\frac{k-a_n}{n}\right)\right)\left(1+O\left(\frac{k-a_n}{n}\right)\right)$$

uniformly for k in

$$1\leq|k-a_n|\leq\frac{n}{\omega(n)}$$

where $\lambda_n(\tau)$ is a special power series converging uniformly with respect to n for sufficiently small τ.

Condition (III) of Proposition 4.2 requires that two consecutive integer values must be taken by a large number of random variables X_k. This excludes the classical counter-example with $X_1\in\{0,1\}$ and all other $X_k\in\{0,2\}$ where $\mathrm{E}[X_1|S_n]=X_1$ and the conditional mean risk-sharing rule fails to diversify risk. Condition (III) of Proposition 4.2 generally holds true in the applications to risk sharing as it can be seen from the values of ϑ_X computed in the cases commonly encountered with insurance losses.

Condition (III) of Proposition 4.2 can be replaced with

(III') for $h=2,3,\ldots$

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\max_{1\leq r\leq h}\mathrm{P}[X_k=r\bmod h]<1.$$

Condition (III') is formulated by Moskvin (1972) in the following way: the maximal step of the variables X_k, $1\leq k\leq n$, is "on the average" equal to unity.

Conditions (I)-(II)-(III) of Proposition 4.2 generally hold true when individual losses X_i have been obtained by discretizing a continuous severity model over $(0,\infty)$, like Gamma or Inverse Gaussian for instance, supplemented with a probability mass at zero.

4.2 No-sabotage condition for large pools

LLTs lead to show that the assumption that log-concavity for sums approximately holds true in a sufficiently large pool and for a sufficiently large interval around the mean of the sum, so that the no-sabotage requirement is almost valid for the conditional mean risk-sharing rule operated among many participants. The next result shows that the no-sabotage condition asymptotically holds for intervals larger than those where the Central limit theorem holds.

Proposition 4.3. *Let $X_1,X_2,X_3,\ldots$ be independent integer-valued random variables such that there is an integer m such that $X_i\leq m$ holds for all i. Suppose also conditions (I) and*

(II) of Proposition 4.1 are fulfilled. Let $\omega(n)$ be a sequence such that $\lim_{n\to\infty}\omega(n) = \infty$. There exists n_0 such that for $n \geq n_0$

$$\mathrm{E}[X_i|S_n = k] \textit{ is non-decreasing in } k \textit{ such that } 1 \leq |k - a_n| \leq n^{3/4}/\omega(n).$$

Proof. Without loss of generality we consider the case $i = 1$. Let us define $S_n^{(\backslash 1)} = \sum_{k=2}^n X_k$, $a_n^{(\backslash 1)} = \sum_{k=2}^n \mu_k$ and $(b_n^{(\backslash 1)})^2 = \sum_{k=2}^n \mathrm{E}[(X_k - \mu_k)^2]$. The proof then proceeds as follows. Using Proposition 2.2(ii) in Denuit (2019), we have

$$\mathrm{E}[X_1|X_1 + S_n^{(\backslash 1)} = k] = \mathrm{E}[X_1]\frac{\mathrm{P}[\widetilde{X_1} + S_n^{(\backslash 1)} = k]}{\mathrm{P}[X_1 + S_n^{(\backslash 1)} = k]}$$

where the size-biased version $\widetilde{X_1}$ is assumed to be independent of X_1 and $S_n^{(\backslash 1)}$ and satisfies

$$\frac{\mathrm{P}[\widetilde{X_1} = l]}{\mathrm{P}[X_1 = l]} = \frac{l}{\mathrm{E}[X_1]},\ l \in \{0, 1, \ldots\}.$$

Therefore $\mathrm{E}[X_i|S_n = l]$ is non-decreasing in l over the range of integer values $\mathcal{I}_n = \{l : 1 \leq |l - a_n| \leq n^{3/4}/\omega(n)\}$ if and only if

$$\frac{\mathrm{P}[X_1 + S_n^{(\backslash 1)} = l]}{\mathrm{P}[\widetilde{X_1} + S_n^{(\backslash 1)} = l]} \text{ is non-increasing in } l \in \mathcal{I}_n,$$

or equivalently

$$\mathrm{P}[X_1 + S_n^{(\backslash 1)} = l]\mathrm{P}[\widetilde{X_1} + S_n^{(\backslash 1)} = l+1] \geq \mathrm{P}[X_1 + S_n^{(\backslash 1)} = l+1]\mathrm{P}[\widetilde{X_1} + S_n^{(\backslash 1)} = l],\ \ l \in \mathcal{I}_n.$$

This is equivalent to show that $\Delta_l \geq 0$ for all $l \in \mathcal{I}_n$ where

$$\Delta_l = \sum_{j=0}^{l}\sum_{k=1}^{l}\mathrm{P}[X_1 = j]\mathrm{P}[\widetilde{X_1} = k]\Big(\mathrm{P}[S_n^{(\backslash 1)} = l-j]\mathrm{P}[S_n^{(\backslash 1)} = l+1-k] - \mathrm{P}[S_n^{(\backslash 1)} = l+1-j]\mathrm{P}[S_n^{(\backslash 1)} = l-k]\Big).$$

Let us assume that l is large enough such that $l > m$ and let us rewrite Δ_l as

$$\begin{aligned}
\Delta_l &= \frac{1}{\mathrm{E}[X_1]} \sum_{j=0}^{m} \sum_{k=1}^{m} k\mathrm{P}[X_1 = j]\mathrm{P}[X_1 = k]\Big(\mathrm{P}[S_n^{(\backslash 1)} = l - j]\mathrm{P}[S_n^{(\backslash 1)} = l + 1 - k] \\
&\qquad\qquad - \mathrm{P}[S_n^{(\backslash 1)} = l + 1 - j]\mathrm{P}[S_n^{(\backslash 1)} = l - k]\Big) \\
&= \frac{1}{\mathrm{E}[X_1]} \sum_{0 \le j < k \le m} k\mathrm{P}[X_1 = j]\mathrm{P}[X_1 = k]\Big(\mathrm{P}[S_n^{(\backslash 1)} = l - j]\mathrm{P}[S_n^{(\backslash 1)} = l + 1 - k] \\
&\qquad\qquad - \mathrm{P}[S_n^{(\backslash 1)} = l + 1 - j]\mathrm{P}[S_n^{(\backslash 1)} = l - k]\Big) \\
&\quad + \frac{1}{\mathrm{E}[X_1]} \sum_{1 \le k < j \le m} k\mathrm{P}[X_1 = j]\mathrm{P}[X_1 = k]\Big(\mathrm{P}[S_n^{(\backslash 1)} = l - j]\mathrm{P}[S_n^{(\backslash 1)} = l + 1 - k] \\
&\qquad\qquad - \mathrm{P}[S_n^{(\backslash 1)} = l + 1 - j]\mathrm{P}[S_n^{(\backslash 1)} = l - k]\Big) \\
&= \frac{1}{\mathrm{E}[X_1]} \sum_{k=1}^{m} k\mathrm{P}[X_1 = 0]\mathrm{P}[X_1 = k]\Big(\mathrm{P}[S_n^{(\backslash 1)} = l]\mathrm{P}[S_n^{(\backslash 1)} = l + 1 - k] \\
&\qquad\qquad - \mathrm{P}[S_n^{(\backslash 1)} = l + 1]\mathrm{P}[S_n^{(\backslash 1)} = l - k]\Big) \\
&\quad + \frac{1}{\mathrm{E}[X_1]} \sum_{1 \le j < k \le m} (k - j)\mathrm{P}[X_1 = j]\mathrm{P}[X_1 = k]\Big(\mathrm{P}[S_n^{(\backslash 1)} = l - j]\mathrm{P}[S_n^{(\backslash 1)} = l + 1 - k] \\
&\qquad\qquad - \mathrm{P}[S_n^{(\backslash 1)} = l + 1 - j]\mathrm{P}[S_n^{(\backslash 1)} = l - k]\Big) \\
&= \frac{1}{\mathrm{E}[X_1]} \sum_{0 \le j < k \le m} (k - j)\mathrm{P}[X_1 = j]\mathrm{P}[X_1 = k]\Big(\mathrm{P}[S_n^{(\backslash 1)} = l - j]\mathrm{P}[S_n^{(\backslash 1)} = l + 1 - k] \\
&\qquad\qquad - \mathrm{P}[S_n^{(\backslash 1)} = l + 1 - j]\mathrm{P}[S_n^{(\backslash 1)} = l - k]\Big).
\end{aligned}$$

Then we have

$$\begin{aligned}
& \mathrm{P}[S_n^{(\backslash 1)} = l - j]\mathrm{P}[S_n^{(\backslash 1)} = l + 1 - k] - \mathrm{P}[S_n^{(\backslash 1)} = l + 1 - j]\mathrm{P}[S_n^{(\backslash 1)} = l - k] \\
= \; & \mathrm{P}[S_n^{(\backslash 1)} = l + 1 - j]\mathrm{P}[S_n^{(\backslash 1)} = l - k]\left(\frac{\mathrm{P}[S_n^{(\backslash 1)} = l - j]}{\mathrm{P}[S_n^{(\backslash 1)} = l - (j-1)]}\frac{\mathrm{P}[S_n^{(\backslash 1)} = l - (k-1)]}{\mathrm{P}[S_n^{(\backslash 1)} = l - k]} - 1\right).
\end{aligned}$$

Let us study the sign of

$$\ln\left(\frac{\mathrm{P}[S_n^{(\backslash 1)} = l - j]}{\mathrm{P}[S_n^{(\backslash 1)} = l - (j-1)]}\frac{\mathrm{P}[S_n^{(\backslash 1)} = l - (k-1)]}{\mathrm{P}[S_n^{(\backslash 1)} = l - k]}\right)$$

for $l \in \mathcal{I}_n$. Let

$$x_{lj} = \frac{l - j - a_n^{(\backslash 1)}}{b_n^{(\backslash 1)}} \quad \text{and} \quad x_{lk} = \frac{l - k - a_n^{(\backslash 1)}}{b_n^{(\backslash 1)}}.$$

By Proposition 4.1 with $p = 4$

$$\begin{aligned}
&\ln\left(\frac{\mathrm{P}[S_n^{(\setminus 1)} = l-j]}{\mathrm{P}[S_n^{(\setminus 1)} = l-(j-1)]}\frac{\mathrm{P}[S_n^{(\setminus 1)} = l-(k-1)]}{\mathrm{P}[S_n^{(\setminus 1)} = l-k]}\right)\\
= \quad & -\frac{1}{2}\left((x_{lj}^2 - x_{lj-1}^2) - (x_{lk}^2 - x_{lk-1}^2)\right)\\
&+\sum_{\nu=1}^{2}\frac{1}{n^{\nu/2}}\Big(\big(p_{\nu n}\left(x_{lj}\right) - p_{\nu n}\left(x_{lj-1}\right)\big) - \big(p_{\nu n}\left(x_{lk}\right) - p_{\nu n}\left(x_{lk-1}\right)\big)\Big) + o\left(\frac{1}{n}\right).
\end{aligned}$$

We have

$$\frac{1}{2}\left((x_{lj}^2 - x_{lj-1}^2) - (x_{lk}^2 - x_{lk-1}^2)\right) = \frac{k-j}{(b_n^{(\setminus 1)})^2}$$

and

$$\big(p_{\nu n}\left(x_{lj}\right) - p_{\nu n}\left(x_{lj-1}\right)\big) - \big(p_{\nu n}\left(x_{lk}\right) - p_{\nu n}\left(x_{lk-1}\right)\big) = O\left(\frac{\left(x_{l0}\right)^{3\nu-2}}{(b_n^{(\setminus 1)})^2}\right).$$

We deduce that, if for $\nu = 1, 2$, $\left(x_{l0}\right)^{3\nu-2} = o\left(n^{\nu/2}\right)$, then the dominant term is $(k-j)/(b_n^{(\setminus 1)})^2$, which is equivalent to have

$$\frac{l - a_n^{(\setminus 1)}}{b_n^{(\setminus 1)}} = o\left(n^{1/4}\right).$$

This condition is satisfied for $l \in \mathcal{I}_n$. It follows that, for $l \in \mathcal{I}_n$,

$$\lim_{n\to\infty} b_n^2\left(\frac{\mathrm{P}[S_n^{(\setminus 1)} = l-j]}{\mathrm{P}[S_n^{(\setminus 1)} = l-(j-1)]}\frac{\mathrm{P}[S_n^{(\setminus 1)} = l-(k-1)]}{\mathrm{P}[S_n^{(\setminus 1)} = l-k]} - 1\right) = j-k < 0.$$

Finally, we can conclude that there exists n_0 such that for $n \geq n_0$,

$$\left(\frac{b_n}{\mathrm{P}[S_n^{(\setminus 1)} = l]}\right)^2 \Delta_l < 0$$

uniformly for $1 \leq |l - a_n| \leq n^{3/4}/\omega(n)$. $\square$

4.3 Risk-reducing effect, risk elimination and insurance at fair price

The conditional mean risk allocation is regarded as beneficial by all risk-averse economic agents, because

$$\mathrm{E}[u_i(w_i - X_i)] \leq \mathrm{E}[u_i(w_i - h_{i,n}^{\mathrm{cmrs}}(S_n))] \text{ for all } i \in \{1, 2, \ldots, n\}, \tag{4.2}$$

whatever the concave utility function u_i and the initial wealth level w_i. This is a direct application of Jensen inequality for conditional expectations. Inequality (4.2) shows that provided total losses are distributed among participants according to the conditional mean risk-sharing rule, every risk-averse agent prefers joining the pool and paying $h_{i,n}^{\text{cmrs}}(S_n)$ over the stand-alone position, keeping the individual loss X_i. Interestingly, this is true whatever the distribution of the other losses X_j, $j \neq i$, comprised in the pool.

Enlarging the pool is always beneficial, even with heterogeneous losses provided S_n is allocated among participants according to the conditional mean risk-sharing rule $h_{i,n}^{\text{cmrs}}$. Precisely, for any independent losses X_i, we have

$$\mathrm{E}[u_i(w_i - h_{i,n}^{\text{cmrs}}(S_n))] \leq \mathrm{E}[u_i(w_i - h_{i,n+1}^{\text{cmrs}}(S_{n+1}))] \text{ for all } n \text{ and } i \in \{1, 2, \ldots, n\}, \tag{4.3}$$

for any concave utility function u_i and initial wealth w_i. See e.g. Denuit and Robert (2023) and the references therein.

Risk elimination may be obtained at the limit, when it can be shown that $h_{i,n}^{\text{cmrs}}(S_n)$ converges to the pure premium $\mathrm{E}[X_i]$ as the number n of participants tends to infinity. Insurance at fair price is recovered at the limit when such a law of large numbers applies to the pool under consideration. We now provide sufficient conditions such that risk elimination becomes possible within infinitely large pools with independent integer-valued losses.

Proposition 4.4. *Let $X_1, X_2, X_3, \ldots$ be independent integer-valued random variables. Assume that:*

(I) there exists an integer m such that $|X_i| \leq m$, for all $i \geq 1$;

(II) there exists a constant $c > 0$ such that $\liminf_{n\to\infty} \frac{1}{n} b_n^2 \geq c$;

(III) $\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \vartheta_{X_k} > 0$.

Then, for all i, $\mathrm{E}[X_i|S_n] \xrightarrow{L_1} \mathrm{E}[X_i]$ as $n \to \infty$.

Proof. Without loss of generality we consider the case $i = 1$. Let us define $S_n^{(\backslash 1)} = \sum_{k=2}^{n} X_k$, $a_n^{(\backslash 1)} = \sum_{k=2}^{n} \mu_k$ and $(b_n^{(\backslash 1)})^2 = \sum_{k=2}^{n} \mathrm{E}[(X_k - \mu_k)^2]$. The proof then proceeds as follows. First, notice that Lyapunov's condition holds for partial sums S_n, $n \geq 1$, i.e. for any $\delta > 0$, we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathrm{E}[|X_k - \mu_k|^{2+\delta}]}{b_n^{2+\delta}} = 0.$$

Therefore, for any $t \in \mathbb{R}$,

$$\mathrm{P}\left[\frac{S_n - a_n}{b_n} \leq t\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-u^2/2\right) \mathrm{d}u \text{ as } n \to \infty.$$

Second, notice that

$$\begin{aligned}
\mathrm{E}\left[X_1|S_n=s\right] &= \sum_{l=0}^{m} l\mathrm{P}[X_1=l|S_n=s] = \sum_{l=0}^{m} l\frac{\mathrm{P}[X_1=l]\mathrm{P}[S_n=s|X_1=l]}{\mathrm{P}[S_n=s]} \\
&= \sum_{l=0}^{m} l\frac{\mathrm{P}[X_1=l]\mathrm{P}[S_n^{(\setminus 1)}=s-l]}{\sum_{j=0}^{m}\mathrm{P}[X_1=j]\mathrm{P}[S_n^{(\setminus 1)}=s-j]}
\end{aligned}$$

and

$$\mathrm{E}\left[X_1|S_n=s\right]-\mathrm{E}\left[X_1\right] = \sum_{l=0}^{m} l\mathrm{P}[X_1=l]\sum_{j=0}^{m} d_{l,j,n}(s)$$

with

$$d_{l,j,n}(s) = \mathrm{P}[X_1=j]\frac{\mathrm{P}[S_n^{(\setminus 1)}=s-l]-\mathrm{P}[S_n^{(\setminus 1)}=s-j]}{\sum_{r=0}^{m}\mathrm{P}[X_1=r]\mathrm{P}[S_n^{(\setminus 1)}=s-r]}.$$

Then, we have, for $t>0$,

$$\begin{aligned}
\mathrm{E}\left[\left|\mathrm{E}\left[X_1|S_n\right]-\mathrm{E}\left[X_1\right]\right|\right] &= \mathrm{E}\left[\left|\mathrm{E}\left[X_1|S_n\right]-\mathrm{E}\left[X_1\right]\right|\mathrm{I}\left[\left|S_n-a_n^{(\setminus 1)}\right|\le tb_n^{(\setminus 1)}\right]\right] \\
&\quad +\mathrm{E}\left[\left|\mathrm{E}\left[X_1|S_n\right]-\mathrm{E}\left[X_1\right]\right|\mathrm{I}\left[\left|S_n-a_n^{(\setminus 1)}\right|> tb_n^{(\setminus 1)}\right]\right] \\
&= I_n(t)+II_n(t).
\end{aligned}$$

The second part, $II_n(t)$, is bounded as follows

$$\begin{aligned}
II_n(t) &\le 2m\mathrm{P}\left[\left|\frac{S_n-a_n^{(\setminus 1)}}{b_n^{(\setminus 1)}}\right|>t\right] \\
&\le 2m\mathrm{P}\left[\left|\frac{S_n-a_n}{b_n}\right|>\sqrt{\left(1-\frac{\mathrm{E}[(X_1-\mu_1)^2]}{b_n^2}\right)}\left(t-\frac{\mu_1}{b_n^{(\setminus 1)}}\right)\right].
\end{aligned}$$

and therefore

$$\limsup_{n\to\infty} II_n(t) \le 2m\frac{1}{\sqrt{2\pi}}\int_t^{\infty}\exp\left(-u^2/2\right)\mathrm{d}u$$

so that

$$\lim_{t\to\infty}\limsup_{n\to\infty} II_n(t)=0.$$

Let us now consider the first part, $I_n(t)$. We have

$$I_n(t) \le \sum_{l=0}^{m} l\mathrm{P}[X_1=l]\sum_{j=0}^{m}\sum_{s:1\le|s-a_n^{(\setminus 1)}|\le tb_n^{(\setminus 1)}}|d_{l,j,n}(s)|\mathrm{P}[S_n=s].$$

By Proposition 4.2 for which all the conditions are satisfied, we have

$$\begin{aligned} \mathrm{P}\left[S_n^{(\backslash 1)} = s\right] &= \frac{1}{b_n^{(\backslash 1)}\sqrt{2\pi}} \exp\left(-\frac{(s-a_n^{(\backslash 1)})^2}{2(b_n^{(\backslash 1)})^2} + \frac{(s-a_n^{(\backslash 1)})^3}{n^2}\lambda_n\left(\frac{s-a_n^{(\backslash 1)}}{n}\right)\right) \\ &\quad \times \left(1 + O\left(\frac{s-a_n^{(\backslash 1)}}{n}\right)\right) \end{aligned}$$

uniformly for s in

$$1 \leq \left|s - a_n^{(\backslash 1)}\right| \leq \frac{n}{\omega(n)}$$

where $\lambda_n(\tau)$ is a special power series converging uniformly with respect to n for sufficiently small τ, and $\lim_{n\to\infty} \omega(n) = \infty$. We easily deduce choosing $\omega(n) = n/(tb_n^{(\backslash 1)})$ that

$$\lim_{n\to\infty} \sup_{1\leq|s-a_n^{(\backslash 1)}|\leq tb_n^{(\backslash 1)}} \sup_{0\leq l,j\leq m} |d_{l,j,n}(s)| = 0$$

and therefore that, for all $t > 0$,

$$\lim_{n\to\infty} I_n(t) = 0.$$

This ends the proof. □

Proposition 4.4 shows that insurance at fair price can be recovered at the limit, by pooling among a large number of participants. The added value of the insurance company remains for covering the upper layer, since the assumption $X_i \leq m$ in Condition (I) is needed to establish Proposition 4.4. The middle risk layer can be effectively pooled among participants, offering them insurance at fair price for medium losses.

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