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Control design for thermodynamic systems on contact manifolds

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Abstract: We consider the problem of feedback control design for thermodynamic systems. Following past contributions, we consider the expression of closed-loop control in the Thermodynamic Phase Space (TPS) using contact geometry. Some of these past contributions identified restricted classes of admissible controllers by restricting the dynamics to a Legendre submanifold in the TPS. In this paper, we consider the problem of controlling a system through damping feedback control. By allowing this particular class of feedback design technique, we characterize the impact of control on the thermodynamic structure of the closed-loop system in the TPS. An example is considered to illustrate the proposed construction.

Keywords: Thermodynamic, Nonlinear damping feedback, Feedback control design, Contact geometry

1. INTRODUCTION

An approach to study thermodynamic systems is to use contact geometry, as an analogue of symplectic geometry for classical mechanics (Arnold, 1989). This approach, developed extensively to describe equilibrium thermodynamics (Mrugala et al., 1991). In the context of control systems analysis and feedback design for thermodynamic systems, contact geometry was considered, through a lift of a known control system, in (Eberard et al., 2007), (Favache et al., 2009), (Favache et al., 2010), (Gromov and Caines, 2015), (Ramirez et al., 2013b), and more recently in (Wang et al., 2015). Stability analysis problems were successfully addressed for control systems using the contact geometry approach, see for example the contribution by Maschke (2016). As presented in (Favache et al., 2010), both the energy and entropy functions can serve as the generating potential for the representation of thermodynamic systems using contact geometry. The Thermodynamic Phase Space (TPS) approach has the advantage of encoding both laws of thermodynamics in the expression of a system dynamics in an extended phase space (Grmela, 2002). The construction proposed in (Grmela, 2002), built on material from (Arnold, 1989), shows that the thermodynamic reciprocity relations are encoded within this framework. Contact geometry also serves as the basis for the geometrothermodynamics approach to nonequilibrium thermodynamics, see for example the original contribution (Quevedo, 2007) and applications presented in (Quevedo et al., 2011) and (Quevedo and Tapias, 2014), where the TPS is endowed with a metric, in the spirit of Weinhold and Ruppeiner (Quevedo, 2007), *i.e.*, by using the Hessian of the thermodynamic potential as a metric. An indefinite Riemannian metric was also introduced on the TPS in (Mrugala, 1996), a construction later used in (Preston and Vargo, 2008) to study geometric properties of constitutive surfaces defined for different thermodynamic potentials.

The results developed in (Eberard et al., 2007; Favache et al., 2009, 2010; Ramirez et al., 2013b; Wang et al., 2015; Gromov and Caines, 2015) were key to understand stability and stabilization problems for thermodynamic systems: By lifting a n-dimensional controlled dynamics to a $(2n+1)$-dimensional dynamical systems endowed with a contact structure, *i.e.*, a differential one-form encoding thermodynamics evolution constraints, it is possible to restrict stability and stabilization problems to admissible evolutions in an extended vector field. A further contribution to this understanding about the interplay between thermodynamic and feedback control is given in (Ramirez et al., 2013a) where two results pertinent to the present study were presented. First, it was demonstrated that the only state feedback preserving a given contact structure of a conservative control contact system is the constant one. Second, a feedback design approach as solution to matching equations, as used for passivity-based damping assignment for port-controlled Hamiltonian (Astolfi and Ortega, 2009), was considered in the TPS. The feedback design approaches from the contribution (Ramirez et al., 2013a) were characterized as restrictive, see for example the review given in (Maschke, 2016) where a characterization of the desired properties of controlled system in the TPS was proposed. However, it is not clear how far from the working assumptions from (Ramirez et al., 2013a) it is possible to go such that the resulting closed-loop system preserves the thermodynamic structure of the original system. The contribution(Maschke, 2016) pointed out the importance of the lift to the TPS as a key factor to assess stability (and indirectly feedback stabilization), as the choice of a lift is linked to the definition of the Reeb

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vector field, which is central to the analysis in the TPS provided in (Ramirez et al., 2013a).

To clarify our understanding of the relations between control design and the thermodynamic representation approach based on contact geometry, we consider a particular class of physics-based feedback control design *prior* to the performing a lift of the dynamics to the TPS. Following previous investigations on damping feedback control (Hudon and Guay, 2013), we study systems in closed-loop in the TPS for which the control action has some physical sense, as for example damping feedback for port-controlled Hamiltonian systems (Ortega et al., 2002). With this study, we hope to shed a light on the admissible class of controllers, *i.e.*, feedback controllers preserving the thermodynamic structure of a given open-loop system.

This paper is organized as follows. Necessary background on feedback control in the TPS is given in Section 2. In Section 3, the lift of systems in closed-loop with damping feedback controllers is studied. An example is given in Section 4. Conclusions and future areas for investigation are discussed in Section 5.

2. BACKGROUND

In this section, we summarize the theory of thermodynamic systems expressed in the TPS, that is, using contact geometry, following the exposition proposed originally in (Eberard et al., 2007) and developed in (Ramirez et al., 2013a; Maschke, 2016) for control design and feedback stabilization.

We denote the n extensive variables by x^i, $i = 1, \ldots, n$, and the thermodynamic potential by x^0, for example the energy $x^0 = E(x)$ or the Entropy $x^0 = S(x)$. The n intensive variables are denoted by p_i and are dual to the extensive variables by the relations $p_i = \frac{\partial E}{\partial x^i}$ or $p_i = \frac{\partial S}{\partial x^i}$, depending on the choice of thermodynamic potential[1]. The thermodynamic phase space (TPS) is the $(2n+1)$-dimensional vector space endowed with the canonical contact structure

$$\theta = dx^0 + \sum_{i=1}^{n} p_i dx^i.$$

Definition 1. A one-form θ on a $2n+1$-dimensional manifold $\mathcal{M}$ is a contact form if $\theta \wedge (d\theta)^n \neq 0$ is a volume form. Then the pair $(\mathcal{M}, \theta)$ is called a contact manifold.

For a given set of canonical coordinates and any partition I and J of the set of indices $\{1, \ldots, n\}$, for any differentiable function $\phi(x^I, p_J)$ of n variables, $i \in I$, $j \in J$, the formulas

$$\begin{aligned} x^0 &= \phi - \sum_{i \in I} p_i \frac{\partial \phi}{\partial p_i} \\ x^i &= -\frac{\partial \phi}{\partial p_i}, i \in I, \\ p_j &= \frac{\partial \phi}{\partial x^j}, j \in J, \end{aligned} \tag{1}$$

define a Legendre submanifold $\mathcal{L}_\phi$ of $\mathbb{R}^{2n+1}$.

An important object to be studied in our context is the vector field governing the dynamics of a system in the TPS.

Definition 2. A (smooth) vector field X on the contact manifold $\mathcal{M}$ is a contact vector field with respect to a contact form θ if and only if there exists a smooth function $\rho \in C^\infty(\mathcal{M})$ such that

$$L_X \theta = \rho \theta,$$

where $L_X \cdot$ denotes the Lie derivative with respect to the vector field X. It is called a strict contact vector field if $\rho = 0$.

The problem considered in (Ramirez et al., 2013a; Maschke, 2016) is to study controlled balance equation system of the form

$$\frac{dx}{dt} = f(x, \phi_x) + \sum_{j=1}^{p} g^j(x, \phi_x) u_j, \tag{2}$$

where the gradient of the thermodynamic potential $\phi(x)$, $\frac{\partial \phi}{\partial x}$, is denoted by ϕ_x. The approach proposed in (Ramirez et al., 2013a; Maschke, 2016) first considers the lift of the controlled balance system (2) to the complete Thermodynamic Phase Space by defining the following contact Hamiltonian functions:

$$K_0 = (\phi_x - p)^T f(x, \phi_x) \tag{3}$$

$$K_C^j = (\phi_x - p)^T g^j(x, \phi_x). \tag{4}$$

By construction, the contact vector field X_K, generated by the function $K = K_0 + \sum_{j=1}^{p} K_C^j$, leaves invariant the Legendre submanifold $\mathcal{L}_\phi$ generated by the thermodynamic potential $\phi(x)$, defined as

$$\mathcal{L}_\phi = \{x_0 = \phi(x), x = x, p = \phi_x, x \in \mathbb{R}^n\}. \tag{5}$$

To every function K, there corresponds the contact vector field $\mathcal{X}_K$ given as

$$\begin{aligned} \mathcal{X}_K = &\left(K - \sum_{i=1}^{n} p_i \frac{\partial K}{\partial p_i}\right) \frac{\partial}{\partial x^0} + \frac{\partial K}{\partial x^0} \left(\sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i}\right) \\ &+ \sum_{j=1}^{n} \left(\frac{\partial K}{\partial x^j} \frac{\partial}{\partial p_j} - \frac{\partial K}{\partial p_j} \frac{\partial}{\partial x^j}\right). \end{aligned} \tag{6}$$

The corresponding dynamical system in the contact phase space is given as

[1] Generally speaking, any thermodynamic potential could be used, internal energy, entropy, Helmholtz free energy, or the Gibbs free energy. Those representations are related by Legendre transformations (?). The proper choice of a potential depends on the particular problem at hand. We do not make a particular choice here and in the sequel, and the thermodynamic potential is denoted by $\phi(x)$.

$$\dot{x}^0 = K - \sum_{i=1}^{n} p_i \frac{\partial K}{\partial p_i}$$
$$\dot{x}^i = -\frac{\partial K}{\partial p_i}$$
$$\dot{p}_i = p_i \frac{\partial K}{\partial x_0} + \frac{\partial K}{\partial x^i}. \quad (7)$$

Moreover, the restriction of $\mathcal{X}_K$ to $\mathcal{L}_\phi$ coincides with the balance equation system.

Conversely, for a given contact vector field, there exists a unique real function K, called a generating function, associated with the contact vector field X, given by

$$K = i_X \theta$$

and called the contact Hamiltonian. The function ρ in $L_X\theta = \rho\theta$ is given by $\rho = i_E dK$ where E is the Reeb vector field associated with the contact form θ and uniquely defined by $i_E\theta = 1$ and $i_E d\theta = 0$ where i_E denotes the contraction of differential forms by the vector field E.

To understand previous control design results in the TPS based on the above construction, we first consider the following definition, introduced in (Eberard et al., 2007), and used in more recent work (Ramirez et al., 2013a; Maschke, 2016).

Definition 3. A controlled contact system on the contact manifold $(\mathcal{M}, \theta)$, affine in the scalar input $u \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ is defined by the two functions $K_0 \in C^\infty(\mathcal{M})$, called the internal contact Hamiltonian and $K_C \in C^\infty(\mathcal{M})$ called the interaction (or control) contact Hamiltonian, and the state and output equations

$$\frac{d\tilde{x}}{dt} = X_{K_0} + \sum_{j=1}^{p} X_{K_C^j} u_j,$$

where X_{K_0} and $X_{K_C^j}$ are the contact vector fields of $(\mathcal{M}, \theta)$ generated by the contact Hamiltonians K_0 and K_C^j, respectively.

The central result, on which the theory of control in the TPS is established, is based on the following proposition, originally from (Mrugala et al., 1991) and reported in (Maschke, 2016).

Proposition 4. Let $(\mathcal{M}, \theta)$ be a contact manifold and denote $\mathcal{L}$ a Legendre submanifold. Then, the contact vector field X_K generated by the contact Hamiltonian $K \in C^\infty(M)$ is tangent to $\mathcal{L}$ if and only if it satisfies the invariance condition $K^{-1}(0) \supset \mathcal{L}$.

This motivated the following definition in (Eberard et al., 2007).

Definition 5. A conservative control contact system with respect to the Legendre submanifold $\mathcal{L}$ is a control contact system with the internal (resp. control) contact Hamiltonian K_0 (resp. K_C^j) satisfying the invariance condition $K^{-1}(0) \supset \mathcal{L}$.

In (Ramirez et al., 2013a), the above definition of conservative control contact systems was further restricted by constraining the contact Hamiltonian to first integrals of the Reeb vector field E, *i.e.*, by imposing

$$i_E dK_0 = i_E dK_C^j = 0.$$

Recently, extensions of the above feedback control construction approaches were reported in (Maschke, 2016), where non-strict equivalent conservative contact vector fields were generated. Those results were based, in part, on the following proposition.

Proposition 6. Consider a contact Hamiltonian function $K \in C^\infty(\mathcal{M})$ satisfying the invariance relation $K^{-1}(0) \supset \mathcal{L}$ with respect to a Legendre submanifold $\mathcal{L}$, and consider a real function $\epsilon \in C^\infty(\mathcal{M})$, defining a contact Hamiltonian $\bar{K} = \epsilon K$. Then, the contact vector field $X_{\bar{K}}$ leaves invariant the Legendre submanifold $\mathcal{L}$ and a sufficient condition for the two contact vector fields X_K and $X_{\bar{K}}$ to be equal on their restriction to the Legendre submanifold $\mathcal{L}$ is $(\epsilon - 1)|_{\mathcal{L}} = 0$.

To summarize, the approach advocated in the contributions (Eberard et al., 2007), (Ramirez et al., 2013b), (Ramirez et al., 2013a), and (Maschke, 2016) for the design of admissible feedback controllers in the TPS first considered a lift of the control affine system (2) to the extended space, and then considered the class of controllers that preserved the invariance of the Legendre submanifold. In the present contribution, we reverse the approach and first design a stabilizing controller for a given control affine system and then, after a lift to the TPS, study the impact of the control action on the contact structure.

3. FEEDBACK CONTROL DESIGN

We now present the main developments of this paper. We study balance equation system of the form

$$\frac{dx}{dt} = f(x, \phi_x) + \sum_{j=1}^{p} g^j(x, \phi_x) u_j,$$

for which we consider damping feedback control, also referred to as Jurdjevic–Quinn control (Malisoff and Mazenc, 2009). First, we recall the main result of the Jurdjevic–Quinn damping control approach (Malisoff and Mazenc, 2009).

Theorem 7. Given a smooth control affine system

$$\dot{x} = f(x) + g(x)u$$

and a function $\psi(x)$ such that $\mathcal{L}_f\psi < 0$ for every $x \in \mathbb{R} \setminus \{0\}$. Suppose moreover that

$$\mathrm{span}\{f(x), \mathrm{ad}_f^k g_i(x) : i = 1, \ldots, p, k \in \mathbb{N}\} = \mathbb{R}^n$$

on $\mathbb{R}^n \setminus \{0\}$. Then the feedback laws

$$u_i(x) = -(\mathcal{L}_{g_i}\psi)(x), \forall i \in 1, \ldots, p$$

globally asymptotically stabilizes the control affine system to the origin.

In the present note, we apply the above approach to damping feedback stabilization for systems of the form (2). To simplify the exposition, we consider single input systems, *i.e.*, we assume that $p = 1$ in (2) and that $u \in \mathcal{U} \subset \mathbb{R}$, where $\mathcal{U}$ is the set of admissible control in one dimension. We further assume the following.

Assumption 8. The thermodynamic potential $\phi_x(x)$ is an admissible Jurdjevic–Quinn function centered at the equilibrium of the open-loop system which is located at the origin, *i.e.*, the potential $\phi(x)$ is such that $\mathcal{L}_{f(x,\phi_x)}\psi(x) < 0$ for every $x \in \mathbb{R} \setminus \{0\}$.

Assumption 9. We assume that the reachability condition holds, *i.e.*, we assume that

$$\mathrm{span}\{f(x), \mathrm{ad}_f^k g(x) : k \in \mathbb{N}\} = \mathbb{R}^n$$

on $\mathbb{R}^n \setminus \{0\}$.

Under the above assumptions, the closed-loop system given by

$$\dot{x} = f(x, \phi_x) - g(x, \phi_x) g^T(x, \phi_x)\phi_x \tag{8}$$

is globally asymptotically stable at the origin.

We know investigate both open-loop and closed-loop vector fields in the Thermodynamic Phase Space (TPS). Following the approach proposed originally in (Eberard et al., 2007), we first define two contact Hamiltonian functions K_{OL} and K_{CL} to lift the original dynamics $f(x, \phi_x)$ and the closed-loop system (8) with respect to the thermodynamic potential $\phi(x)$. The contact Hamiltonian functions are hence given by

$$K_{\mathrm{OL}} = (\phi_x - p)^T f(x, \phi_x) \tag{9}$$

$$K_{\mathrm{CL}} = (\phi_x - p)^T \Big(f(x, \phi_x) - g(x, \phi_x) g^T(x, \phi_x)\phi_x \Big). \tag{10}$$

We then construct the contact vector fields. For the open-loop system, we have:

$$\begin{aligned} \dot{x}^0 &= \phi_x^T f(x, \phi_x) \\ \dot{x} &= f(x, \phi_x) \\ \dot{p} &= \frac{\partial}{\partial x} \left((\phi_x - p)^T f(x, \phi_x) \right). \end{aligned} \tag{11}$$

For the closed-loop system, we have the following vector field:

$$\begin{aligned} \dot{x}^0 &= \phi_x^T \tilde{f}(x, \phi_x) \\ \dot{x} &= \tilde{f}(x, \phi_x) \\ \dot{p} &= \frac{\partial}{\partial x} \left((\phi_x - p)^T \tilde{f}(x, \phi_x) \right), \end{aligned} \tag{12}$$

with $\tilde{f}(x, \phi_x) = f(x, \phi_x) - g(x, \phi_x) g^T(x, \phi_x)\phi_x$.

By construction the $(2n+1)$ vector field (11) leaves the contact one-form

$$\theta = d\phi + \sum_{i=1}^{n} p_i dx^i$$

invariant. Moreover, following (Favache et al., 2009), we have the following result characterizing an isolated equilibrium in the extended phase space.

Proposition 10. An isolated equilibrium of the dynamical system (2) $f(\bar{x}) \equiv 0$ coincides with an isolated equilibrium $(\bar{x}^0, \bar{x}, \bar{p})$ on $\mathcal{L}_\phi$ of the extended system (11).

The damping feedback control proposed above preserves the equilibrium of the dynamical system, and hence, the equilibrium of system (12) matched the equilibrium of system (11), which by assumption is the origin.

The key questions to be investigated are the following:

(1) Does the system (12) is a contact vector field? and,
(2) If so, does the contact vector field $X_{K_{\mathrm{CL}}}$ leaves the Legendre submanifold invariant.

To answer the first question, we set $\theta = dx^0 - p_i dx^i$, and use the following result from (McDuff and Salamon, 1998, Chap. 3).

Lemma 11. Let E be the Reeb vector field of the contact form θ. Let X be a vector field. Then X is a contact vector field if and only if there exists a function H such that

$$\begin{aligned} i_X \theta &= -H \\ i_X d\theta &= dH - (i_E dH)\theta. \end{aligned}$$

Using this result, we can prove the following theorem.

Theorem 12. The $2n+1$ dimensional vector field generated by the lift of the Hamiltonian function K_{CL} as given in (12) is a contact vector for the contact one-form $\theta = dx^0 - p_i dx^i$.

Proof: By definition of a Reeb vector field E, *i.e.*, for a given θ,

$$\begin{aligned} i_E \theta &= 1 \\ i_E d\theta &= 0, \end{aligned}$$

we have for $\theta = dx^0 - p_i dx^i$ that $E = \frac{\partial}{\partial x^0}$. Following Lemma 11, computing $i_{X_{\mathrm{CL}}}\theta$, we have

$$H = -i_{X_{\mathrm{CL}}}\theta = -\phi_x \tilde{f}(x, \phi_x) + p^T \tilde{f}.$$

As dH is not an explicit function of x^0, second condition amounts to ensuring that $i_{X_{\mathrm{CL}}} d\theta = dH$.

Computing dH, we obtain

$$dH = -\frac{\partial}{\partial x} \left(\phi_x \tilde{f}(x, \phi_x) - p^T \tilde{f} \right)^T dx + \tilde{f}^T dp.$$

Since $d\theta = -dp_i \wedge dx^i = dx^i \wedge dp_i$ by the property of the wedge product, we have

$$i_{X_{\mathrm{CL}}} d\theta = \tilde{f}^T dp - \frac{\partial}{\partial x} \Big(\Big((\phi_x - p)^T \tilde{f}(x, \phi_x) \Big) \Big).$$

Hence $i_{X_{\mathrm{CL}}} d\theta = dH$ and by virtue of Lemma 11, X_{CL} is a contact vector field. ■

To answer the second question, we use the first part of Proposition 6, and we work directly with the contact Hamiltonian functions K_{OL} and K_{CL}. The question translates to the existence of a function ϵ such that

$$K_{\mathrm{CL}} = \epsilon K_{\mathrm{OL}}.$$

We state the following theorem, based on Proposition 6.

Corollary 13. For $(\phi_x - p)^T f(x, \phi_x) \neq 0$, there exists a function ϵ such that

$$K_{\mathrm{CL}} = \epsilon K_{\mathrm{OL}},$$

and the contact vector field associated with the closed-loop dynamics leaves invariant the Legendre submanifold $\mathcal{L}$.

Proof: We begin with the desired equality

$$K_{\rm CL} = \epsilon K_{\rm OL},$$

and recall the definitions of the contact Hamiltonian functions (9) and (10):

$$\begin{aligned} K_{\rm OL} &= (\phi_x - p)^T f(x, \phi_x) \\ K_{\rm CL} &= (\phi_x - p)^T \Big(f(x, \phi_x) - g(x, \phi_x) g^T(x, \phi_x) \phi_x \Big). \end{aligned}$$

Distributing and re-organizing, we obtain

$$\begin{aligned} - (\phi_x - p)^T g(x, \phi_x) g^T(x, \phi_x) \phi_x = \\ (\epsilon - 1) (\phi_x - p)^T f(x, \phi_x). \end{aligned}$$

By assumption, if $(\phi_x - p)^T f(x, \phi_x) \neq 0$, we have directly

$$\epsilon = 1 - \frac{(\phi_x - p)^T g(x, \phi_x) g^T(x, \phi_x) \phi_x}{(\phi_x - p)^T f(x, \phi_x)}.$$

■

The key issue, illustrated in the example of the next Section, is to consider the points where the condition $(\phi_x - p)^T f(x, \phi_x) \neq 0$ fails.

4. EXAMPLE

To illustrate the above construction, we consider the lift to the TPS of a controlled system generated by thermodynamic potential. Consider the control affine system

$$\dot{x} = f(x) + g(x) u,$$

with $x \in \mathbb{R}^n$ and $u \in \mathcal{U} \subset \mathbb{R}$. We further consider a case where the drift dynamics is generated by a thermodynamic potential $F(x)$ with a constant structure matrix $\mathcal{F}$, *i.e.*,

$$f(x) = \mathcal{F} \cdot \nabla^T F(x).$$

Examples of such dynamical systems are metriplectic systems, derived originally in the work by Morrison (1986), but also gradient systems and Hamiltonian systems. In the present case, we assume that the generating potential is quadratic in its arguments, *i.e.*, we consider the case where $F(x)$ is given by

$$F(x) = \sum_{i=1}^{n} \frac{1}{2} (x_i)^2,$$

and as a result, the convex generating potential vanishes at the origin. We further assume that the constant structure matrix $\mathcal{F}$ is negative definite, and as a result, the quadratic potential function can be used as a Jurdjevic–Quinn function, *i.e.*,

$$\begin{aligned} \dot{F}(x) &= \nabla F(x) \cdot f(x) \\ &= \nabla F(x) \mathcal{F} \nabla^T F(x) < 0 \end{aligned}$$

on $\mathbb{R}^n \setminus \{0\}$. Following the approach proposed above, the damping controller is set to

$$u(x) = -\kappa L_{g(x)} F(x),$$

where κ is a positive constant. This leads to the closed-loop system

$$\dot{x} = \mathcal{F} \nabla^T F(x) - \kappa g(x) g^T(x) \nabla^T F(x).$$

Provided that the maximal invariant set is $\{0\}$, By Lasalle's invariance principle, the origin of the system is asymptotically stable in closed-loop

$$\begin{aligned} \dot{F}(x) &= \nabla F(x) \cdot \big(\mathcal{F} \nabla^T F(x) - \kappa g(x) g^T(x) \nabla^T F(x) \big) \\ &= \nabla F(x) \big(\mathcal{F} - \kappa g^T(x) g(x) \big) \nabla^T F(x) < 0 \end{aligned}$$

on $\mathbb{R}^n \setminus \{0\}$.

The objective here is to understand the effects of the damping control on the system in the TPS. Following the lifting approach described above and setting $\phi(x) = F(x)$, we define two contact Hamiltonian functions. Since we considered $F(x)$ to be quadratic, the gradient $\phi_x(x)$ is simply the vector of the extensive variables x. The open-loop contact Hamiltonian $K_{\rm OL}$ is thus given by

$$K_{\rm OL} = (x - p)^T \mathcal{F} x.$$

The closed-loop contact Hamiltonian $K_{\rm CL}$ is given by

$$K_{\rm CL} = (x - p)^T \Big(\mathcal{F} - \kappa g(x) g^T(x) \Big) x.$$

Obviously, if $\kappa \equiv 0$, both contact Hamiltonian functions agree. The first question however is to decide if the damping feedback control leaves invariant the Legendre submanifold. Following Proposition 6, it is the case if there exists a smooth function ϵ such that $K_{\rm CL} = \epsilon K_{\rm OL}$. Distributing and re-arranging the above expressions, we have

$$(\epsilon - 1)(x - p)^T \mathcal{F} x = -\kappa (x - p)^T g(x) g^T(x) x,$$

and hence we have

$$\epsilon = 1 - \kappa \frac{(x - p)^T g(x) g^T(x) x}{(x - p)^T \mathcal{F} x},$$

and hence, the damping feedback control for this particular case leaves invariant the Legendre submanifold. However, the last expression outlines two issues:

(1) The function ϵ is ill-defined at the origin (equilibrium point of both dynamical systems); and
(2) The value of $(\epsilon - 1)|$ to the Legendre submanifold might not be defined, depending on the structure of $g(x)$, and hence, following Proposition 6, the respective open-loop and closed-loop contact vector fields $X_{\rm OL}$ and $X_{\rm CL}$ might not be equal on their restriction to the Legendre submanifold.

The second point was expected from the development given in Section 4. Indeed, both contact vector fields different. The contact vector field corresponding to open-loop dynamics is:

$$\begin{aligned} \dot{x}^0 &= x^T \mathcal{F} x \\ \dot{x} &= \mathcal{F} x \\ \dot{p} &= (\mathcal{F}^T + \mathcal{F}) x - p^T \mathcal{F}. \end{aligned}$$

The contact vector field corresponding to closed-loop dynamics is:

$$
\begin{aligned}
\dot{x}^0 &= x^T \Theta(x) x \\
\dot{x} &= \Theta(x) x \\
\dot{p} &= (\Theta^T(x) + \Theta(x)) x - p^T \Theta(x),
\end{aligned}
$$

with $\Theta(x) = \mathcal{F} - \kappa g(x) g^T(x)$.

By construction, the equilibrium point of the open-loop and closed-loop dynamics coincide (at the origin). The origin is also the equilibrium for both contact vector fields.

The key issue, however, resides in the fact that the function ϵ is ill-defined at the origin, hence leaving opened the question of the invariance of the Legendre submanifold $\mathcal{L}$ at the origin under damping feedback. This issue is common in the solution of feedback equivalence problems, where feedback transformations are ill-defined at the equilibrium point, see for example (Gardner, 1989). Further investigations, building on the method described in (Gardner, 1989), are required to understand this issue of feedback equivalence between controlled systems in the TPS.

5. CONCLUSIONS

This paper considers the problem of feedback control design for thermodynamic systems. By considering a fixed form of damping control and then lifting the closed-loop system to the Thermodynamic Phase Space, it is shown that even if the contact vector fields obtained by lifting the open-loop system and the closed-loop system might differ when restricted to the Legendre submanifold, the closed-loop system might leave invariant the Legendre submanifold, except at the equilibrium point. This issue is well documented in the literature treating the problem of feedback equivalence. Further investigations will consider the problem from a feedback equivalence point of view following the classical approach given in (Gardner, 1989) and (Bryant et al., 2003).

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