Chapter 9

Optimal design of profit sharing rates by FFT.

This chapter addresses the calculation of a fair profit sharing rate for participating policies with a minimum interest rate guaranteed. The bonus credited to policies depends on the guarantee, and on the performance of a non-rebalanced basket of two assets: a stock and a zero coupon bond. The dynamics of the instantaneous short rates is driven by a Hull and White model. The participation level is determined such that the return retained by the insurer is sufficient to hedge the interest rate guarantee. Given that the return of the total asset is not lognormal, we rely on a Fast Fourier Transform to compute the fair value of bonus and guarantee options. In numerical applications, we emphasize the relation existing between the insurer's asset, the guarantee and the fair profit sharing rate.

9.1 Introduction.

Most of life insurance products offer a minimal rate of return: guaranteed interests are credited periodically (usually, once a year) to policies. This guarantee being relatively low compared to the market performance, the insurer grants an extra bonus (a profit share) depending on the return of his assets portfolio. Our purpose is to value a fair profit sharing rate and to show how it is linked to the guarantee and to the insurer's investment strategy. Our contribution with respect to the existing literature is to consider this issue under stochastic interest rates and for an insurer having in portfolio a non-rebalanced basket of stocks and zero coupon bonds.

The recent studies on participating policies rely on the Briys and de Varenne (1997a, 1997b) model. Their aim was the valuation of the market price of insurance liabilities in a single period model. The asset of the insurance company is ruled by a geometric Brownian motion and the costs of guarantee and bonus are calculated by the Black and Scholes formula (1973). A multi-period extension has been proposed by Miltersen and Persson (2000). Bacinello (2001) has analysed in a contingent claims framework the pricing of participating policies and takes explicitly into account the mortality risk.

Grosen & Jørgensen (2000) have proposed a dynamic model to value participating contracts and the properties of this model were explored numerically by Monte Carlo simulations. Jørgensen (2001) and Grosen & Jørgensen (2002) have showed that a guaranteed participating policy may be split into four terms: a zero coupon bond, a bonus option and if any, a put option linked to the default risk and finally a rebate given to the policyholders in case of default prior to the maturity date. In papers of Bernard et al. (2005), as in Grosen & Jørgensen (2002), the possibility of an early payment is envisaged. The default mechanism is of structural type and the default barrier is exponential. Jensen et al. (2001) have used a finite difference approach to study a similar issue.

In the existing literature on fair valuation of insurance liabilities, the insurer's asset is usually modelled by a single geometric Brownian motion, which is correlated to stochastic interest rates. In this chapter, we price the bonus and guarantee options when the insurer's portfolio contains stocks and zero coupons. One also assumes that the short term interest rates are ruled by a Hull and White model. Considering a non-rebalanced basket of two assets allows us to emphasize the dependence between static investment strategies, the cost of the guarantee and the fair profit sharing rate. A similar issue was treated by Barbarin & Devolder (2005) but when assets are continuously rebalanced (constant proportion strategies).

The main drawback of our model is that the distribution of the whole portfolio is a sum of two lognormals and has no analytical expression. One rely then on a numerical method to price the options embedded in participating policies. We have opted for the Fast Fourier Transform approach (denoted FFT in the sequel) and in particular for the multi-factor method of Dempster and Hong (2000), initially developed to price spread options, in a faster way than Monte Carlo methods. For an introduction over FFT, we refer the interest reader to papers of Carr and Madan (1999) and Cerny (2004).

The outline of the chapter is as follows. Section 2 presents the insurer's balance sheet and defines what we call a fair participating rate. Section 3 details the assets dynamics and the interest rates modelling. So as to calculate the Fourier transform of options linked to the guarantee and bonus, we determine in section 4, the characteristic functions of assets under real and forward measures. Section 5 develops the multi period setting of Dempster and Hong (2000) and section 6 illustrates numerically our results.

9.2 Product's balance sheet.

We consider an insurance company having a time horizon T, selling a participating policy and guaranteeing a fixed interest rate r_g. At time $t = 0$, the premium paid in by the insured is denoted K. The insurer invests a fraction ρ of the received premium in stocks, S_t, and the rest in zero coupon bonds of maturity $T_p \geq T$, whose prices are denoted $P(t, T_p)$. The insurer 's initial balance sheet is then:

Assets	Liabilities
$P(0, T_p) = (1 - \rho).K$ $S_0 = \rho.K$	K

The investment strategy is assumed to be static: the asset manager doesn't reallocate his assets portfolio till maturity. For a short term planning horizon, this assumption fits relatively well the reality due to transaction costs. If the investments perform well, at maturity T , the total asset is higher than the liabilities and a positive surplus appears:

Assets	Liabilities
$P(T,T_p)$	$Surplus$
S_T	$K.e^{r_g.T}$

This surplus is partly redistributed to the policyholder as a profit share and to the shareholder as a dividend. If the return of asset is insufficient, the mathematical reserve is equal to the guaranteed amount, $K.e^{r_g.T}$, and the terminal surplus is negative. We don't insert the equity of the insurance company in our model and accordingly, we ignore the possible bankruptcy of the insurer. The main reason justifying this choice is that the most of the insurance products are managed in segregated accounts. The profit sharing rules must then be independent from the equity to avoid dumping.

Our first purpose is to determine the bonus as a contractual fraction γ of terminal surplus such that the pricing is fair both for the policyholder and the shareholder. Mathematically, the contract is fair at time $t=0$ if the price of the guarantee is equal to the fair value of the surplus kept by the insurer:

$$\underbrace{E^Q\left(e^{-\int_0^T r_s.ds}.\left[K.e^{r_g.T}-(P(T,T_p)+S_T)\right]_+|\mathcal{F}_0\right)}_{Put}$$
$$=(1-\gamma).\underbrace{E^Q\left(e^{-\int_0^T r_s.ds}.\left[P(T,T_p)+S_T-K.e^{r_g.T}\right]_+|\mathcal{F}_0\right)}_{Call} \quad (9.2.1)$$

Where Q is the pricing measure and $\mathcal{F}_0$ is the initial filtration. The left and right sides of the equality (9.2.1) are respectively an European put option and an European call option, of strike price $K.e^{r_g.T}$, written on a basket of stocks and bonds. Our second purpose is to calculate the expected real bonus for a given participating rate:

$$Expected\ Bonus \quad = \quad \gamma.E^P\left(\left[P(T,T_p)+S_T-K.e^{r_g.T}\right]_+|\mathcal{F}_0\right) \quad (9.2.2)$$

Where P is the real measure. This measure is particularly interesting for the marketing of such policies. Before any further developments, we describe the dynamics of assets.

9.3 Assets dynamics and interest rate modelling.

As mentioned earlier, the insurer invests in two assets: stocks and zero coupon bonds. The real financial probability space is noted $(\Omega,\mathcal{F},P)$ on which is defined a 2-dimensional Brownian motion $W_t^P=\left(W_t^{r,P},W_t^{S,P}\right)$ generating the filtration

$$\mathcal{F}=(\mathcal{F}_t)_t=\sigma\left\{\left(W_u^{r,P},W_u^{S,P}\right)\,:\,u\leq t\right\}$$

The financial market is complete and then admits an unique equivalent measure Q (the risk neutral measure) under which the discounted asset prices are martingales. The instantaneous risk-free rate r_t is modelled by a Hull & White model which has the following dynamic under P :

$$dr_t = a.(b(t) - r_t).dt + \sigma_r.\underbrace{\left(dW_t^{r,P} + \lambda_r.dt\right)}_{dW_t^{r,Q}} \tag{9.3.1}$$

Under the risk neutral measure, the dynamics of interest rates becomes:

$$dr_t = a.(b(t) - r_t).dt + \sigma_r.dW_t^{r,Q} \tag{9.3.2}$$

Where $W_t^{r,Q}$ is a Brownian motion under Q. The constants a, σ_r and λ_r are respectively the speed of mean reversion, the volatility of r_t and the cost of the risk coupled to r_t. The level of mean reversion, $b(t)$, is chosen to fit the initial yield curve and depends on instantaneous forward rate $f(0,t)$[1]:

$$b(t) = \frac{1}{a}.\frac{\partial}{\partial t}f(0,t) + f(0,t) + \frac{\sigma^2}{2.a^2}.(1 - e^{-2.a.t})$$

where

$$f(0,t) = -\frac{\partial}{\partial t}\log P(0,t)$$

In appendix A, more details on the calibration of $f(0,t)$ are given. The market value $P(t,T_p)$ of a zero coupon bond of maturity T_p , obeys to the next SDE:

$$\begin{aligned}\frac{dP(t,T_p)}{P(t,T_p)} &= r_t.dt - \sigma_r.B(t,T_p).\left(dW_t^{r,P} + \lambda_r.dt\right)\\ &= r_t.dt - \sigma_r.B(t,T_p).dW_t^{r,Q}\end{aligned}$$

Where $B(t,T_p)$ is defined as follows:

$$B(t,T_p) = \frac{1}{a}.\left(1 - e^{-a.(T_p-t)}\right)$$

In appendix B, the analytical formula of $P(t,T_p)$ is remembered. The second asset available on markets are stocks S_t driven by a geometric motion, correlated to interest rates:

$$\begin{aligned}\frac{dS_t}{S_t} &= r_t.dt + \sigma_{Sr}.\left(dW_t^{r,P} + \lambda_r.dt\right) + \sigma_S.\left(dW_t^{S,P} + \lambda_S.dt\right)\\ &= r_t.dt + \sigma_{Sr}.dW_t^{r,Q} + \sigma_S.dW_t^{S,Q}\end{aligned}$$

where constants σ_{Sr}, σ_S and λ_S are respectively the correlation between stocks and interest rates, the intrinsic volatility of stocks and the market price of risk.

[1]If $R(t,T,T+\Delta)$ is the forward rate as seen at time t for the period between time T and time $T+\Delta$, the instantaneous forward rate $f(t,T)$ is equal to the following limit $f(t,T) = \lim_{\Delta\to 0} R(t,T,T+\Delta)$. If $P(0,t)$ is the price of a zero coupon bond maturing at time t, one has that $f(0,t) = -\frac{\partial}{\partial t}\log P(0,t)$ (see Hull (1997), for further details).

9.4 Change of measure and characteristic functions.

The options present in eq. (9.2.1) don't have any closed form expression given that the distribution of the total asset is unknown (the sum of two lognormals has no analytical expression). So as to simplify future calculation, we perform a change of measure toward the forward F_T measure. Given that the expectation of a discounted payoff under Q is equal to the price of a zero coupon bond times the expected payoff under F_T, the price of the call option present in relation (9.2.1) is rewritten as follows:

$$\begin{aligned} &E^Q\left(e^{-\int_0^T r_s.ds}.\left[P(T,T_p)+S_T-K.e^{r_g.T}\right]_+|\mathcal{F}_0\right)= \\ &\qquad P(0,T).\underbrace{E^{F_T}\left(\left[P(T,T_p)+S_T-K.e^{r_g.T}\right]_+|\mathcal{F}_0\right)}_{computed\,by\,FFT} \end{aligned}$$

Once that the market value of the call is valued by FFT, the price of the put option involved in equation (9.2.1) is directly inferred by the put call parity:

$$Call+P(0,T).K.e^{r_g.T} \;=\; Put+P(0,T_p)+S_0 \tag{9.4.1}$$

The next step required to execute the FFT algorithm is the calculation of the characteristic function of $(\ln S_T\,,\,\ln P(T,T_p))$ under the forward measure:

Proposition 9.4.1. *The characteristic function of* $(\ln S_T\,,\,\ln P(T,T_p))$ *under* F_T *is given by:*

$$\begin{aligned} \phi_T^{F_T}(u_1,u_2) &= E^{F_T}\left(\exp\left(i.u_1.\ln S_T+i.u_2.\ln P(T,T_p)\right)|\mathcal{F}_0\right) \\ &= \exp\left(i.u_1.\mu_{S_T}^{F_T}+i.u_2.\mu_{P(T,T_p)}^{F_T}+\frac{1}{2}.\Sigma(u_1,u_2)\right) \end{aligned} \tag{9.4.2}$$

Where

$$\begin{aligned} \mu_{S_T}^{F_T} &= \ln S_0-\left(\frac{\sigma_{Sr}^2}{2}+\frac{\sigma_S^2}{2}\right).T-\sigma_{Sr}.\sigma_r.\int_0^T B(s,T).ds \\ &\quad -\log\left(P(0,T)\right)-\frac{\sigma_r^2}{2}.\int_0^T B(s,T)^2.ds \end{aligned} \tag{9.4.3}$$

$$\begin{aligned} \mu_{P(T,T_p)}^{F_T} &= \ln P(0,T_p)+\sigma_r^2.\int_0^T B(s,T_p).B(s,T).ds-\frac{\sigma_r^2}{2}\int_0^T B(s,T_p)^2.ds \qquad (9.4.4) \\ &\quad -\log\left(P(0,T)\right)-\frac{\sigma_r^2}{2}.\int_0^T B(s,T)^2.ds \end{aligned}$$

$$\begin{aligned} \Sigma(u_1,u_2) &= -(u_1+u_2)^2.\sigma_r^2.\int_0^T B(s,T)^2ds-u_2.\sigma_r^2.\int_0^T B(s,T_p)^2ds-u_1^2.\sigma_{Sr}^2.T \\ &\quad +2.(u_1+u_2).\left(u_2.\sigma_r^2.\int_0^T B(s,T).B(s,T_p).ds-u_1.\sigma_r.\sigma_{Sr}.\int_0^T B(s,T).ds\right) \\ &\quad +2.u_1.u_2.\sigma_r.\sigma_{Sr}.\int_0^T B(s,T_p).ds-u_1^2.\sigma_S^2.T \end{aligned} \tag{9.4.5}$$

The proof of this proposition and the detail of integrals in $\mu_{S_T}^{F_T}$, $\mu_{P(T,T_p)}^{F_T}$, $\Sigma(u_1, u_2)$ depending on $B(s,T)$, $B(s,T_p)$, are provided in appendix C. The calculation of the expected real bonus, eq. (9.2.2), doesn't require any change of measure. The only information necessary to run the FFT algorithm is the characteristic function of $(\ln S_T\, , \ln P(T,T_p))$ under the measure P:

Proposition 9.4.2. *The characteristic function of* $(\ln S_T\, , \ln P(T,T_p))$ *under the real measure* P *is as follows:*

$$\begin{aligned}\phi_T^P(u_1,u_2) &= E^P\left(\exp\left(i.u_1.\ln S_T + i.u_2.\ln P(T,T_p)\right)|\mathcal{F}_0\right)\\ &= \exp\left(i.u_1.\mu_{S_T}^P + i.u_2.\mu_{P(T,T_p)}^P + \frac{1}{2}.\Sigma(u_1,u_2)\right)\end{aligned} \qquad (9.4.6)$$

Where

$$\begin{aligned}\mu_{S_T}^P &= \ln S_0 + \left(\sigma_{Sr}.\lambda_r + \sigma_S.\lambda_S - \left(\frac{\sigma_{Sr}^2}{2} + \frac{\sigma_S^2}{2}\right)\right).T - \ln P(0,T)\\ &\quad + \sigma_{Sr}.\sigma_r.\int_0^T B(s,T).ds + \frac{\sigma_r^2}{2}.\int_0^T B(s,T)^2.ds\end{aligned} \qquad (9.4.7)$$

$$\begin{aligned}\mu_{P(T,T_p)}^P &= \ln P(0,T_p) - \lambda_r.\sigma_r.\int_0^T B(s,T_p).ds - \frac{\sigma_r^2}{2}\int_0^T B(s,T_p)^2.ds\\ &\quad - \ln\left(P(0,T)\right) + \sigma_{Sr}.\sigma_r.\int_0^T B(s,T).ds + \frac{\sigma_r^2}{2}.\int_0^T B(s,T)^2.ds\end{aligned} \qquad (9.4.8)$$

$$\begin{aligned}\Sigma(u_1,u_2) &= -(u_1+u_2)^2.\sigma_r^2.\int_0^T B(s,T)^2 ds - u_2.\sigma_r^2.\int_0^T B(s,T_p)^2 ds - u_1^2.\sigma_{Sr}^2.T\\ &\quad +2.(u_1+u_2).\left(u_2.\sigma_r^2.\int_0^T B(s,T).B(s,T_p).ds - u_1.\sigma_r.\sigma_{Sr}.\int_0^T B(s,T).ds\right)\\ &\quad +2.u_1.u_2.\sigma_r.\sigma_{Sr}.\int_0^T B(s,T_p).ds - u_1^2.\sigma_S^2.T\end{aligned}$$

The proof of this proposition is provided in appendix D and the detail of integrals in $\mu_{S_T}^P$, $\mu_{P(T,T_p)}^P$, $\Sigma(u_1,u_2)$ depending on $B(s,T)$, $B(s,T_p)$, are developed in appendix C.

9.5 FFT pricing.

The characteristic functions of $(\ln S_T\, , \ln P(T,T_p))$ being determined under P and F_T, we now apply the multi-factor setting of Dempster and Hong (2000) in order to compute the expected payoff:

$$V^M = E^M\left(\left[P(T,T_p) + S_T - K.e^{r_g.T}\right]_+ |\mathcal{F}_0\right) \qquad (9.5.1)$$

Where M is either the forward measure ($M = F_T$) or either the real measure ($M = P$). We adopt the following notations for logprices:

$$\begin{aligned}s &= \ln S_T\\ p &= \ln P(T,T_p)\end{aligned}$$

And the joint density of (s,p) under M is denoted $q_T^M(s,p)$. The expectation (9.5.1) may then be rewritten as follows:

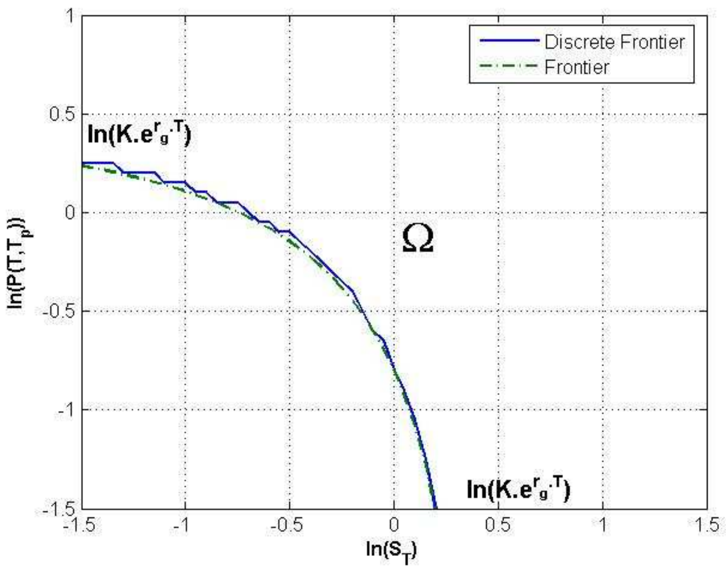
$$V^M = \int\int_\Omega \left(e^p + e^s - K.e^{r_g.T}\right).q_T^M(s,p).dp.ds \tag{9.5.2}$$

Where Ω is the domain on which the payoff is positive:

$$\Omega = \left\{(s,p) \in \mathbb{R}^2 \,|\, e^s + e^p \geq K.e^{r_g.T}\right\}$$

and its frontier is displayed on figure 9.5.1 (dotted line).

Figure 9.5.1: Domain



In order to approach the integral 9.5.2, the domain Ω is sliced in rectangular strips. Let N be a constant (usually a power of 2). One builds next a $N \times N$ equally spaced grid $\Lambda_s \times \Lambda_p$:

$$\Lambda_s = \{k_{1,j}\} = \left\{(j - \frac{1}{2}.N).\lambda_1 \in \mathbb{R} \,|\, 0 \leq j \leq N-1\right\}$$

$$\Lambda_p = \{k_{2,j}\} = \left\{(j - \frac{1}{2}.N).\lambda_2 \in \mathbb{R} \,|\, 0 \leq j \leq N-1\right\}$$

Furthermore, we define an index function $k_2(j)$, for $j = 0, \ldots, N-1$:

$$k_2(j) = \min_{0 \leq q \leq N-1} \left\{k_{2,q} \in \Lambda_p \,|\, e^{k_{2,q}} + e^{k_{1,j+1}} \geq K.e^{r_g.T}\right\}$$

that allows us to define a rectangular strip Ω_j :

$$\Omega_j := [k_{1,j}; k_{1,j+1}) \times [k_2(j), +\infty)$$

such that the domain Ω is approximate by $\bar{\Omega}$, the union of Ω_j :

$$\bar{\Omega} = \bigcup_{j=0}^{N-1} \Omega_j$$

An illustration of this discrete domain is provided on figure 9.5.1. Since $\bar{\Omega} \subset \Omega$ and that the payoff of (9.5.1) is positive over Ω, we have the following relation:

$$\begin{aligned} V^M &= \int\int_{\Omega} \left(e^p + e^s - K.e^{r_g.T}\right).q_T^M(s,p).dp.ds \\ &\geq \int\int_{\bar{\Omega}} \left(e^p + e^s - K.e^{r_g.T}\right).q_T^M(s,p).dp.ds \\ &= \Pi_1^M - K.e^{r_g.T}.\Pi_2^M \end{aligned} \tag{9.5.3}$$

Where

$$\begin{aligned} \Pi_1^M &= \int\int_{\bar{\Omega}} \left(e^p + e^s\right).q_T^M(s,p).dp.ds \\ \Pi_2^M &= \int\int_{\bar{\Omega}} q_T^M(s,p).dp.ds \end{aligned}$$

Before any further developments, we draw the attention of the reader on the fact that the inequality (9.5.3) means that the expected terminal payoff is bounded from below by the numerical estimation. It entails that the bonus option (the call option of eq. (9.2.1)) is then slightly underestimated whereas the price of the guarantee (the put option of eq. (9.2.1)) obtained by the put call parity (eq. (9.4.1)) is therefore slightly overestimated. The discretization infers hence a small safety margin in the calculation of the fair profit sharing rate γ. The next steps consist to calculate Π_1 and Π_2 which may be split as follows:

$$\begin{aligned} \Pi_1^M &= \sum_{j=0}^{N-1} \int\int_{\Omega_j} \left(e^p + e^s\right).q_T^M(s,p).dp.ds \\ &= \sum_{j=0}^{N-1} \left(\int_{k_{1,j}}^{\infty}\int_{k_2(j)}^{\infty} \left(e^p + e^s\right).q_T^M(s,p).dp.ds - \int_{k_{1,j+1}}^{\infty}\int_{k_2(j)}^{\infty} \left(e^p + e^s\right).q_T^M(s,p).dp.ds \right) \\ &= \sum_{j=0}^{N-1} \left(\Pi_1^M(k_{1,j}\,,\,k_2(j)) - \Pi_1^M(k_{1,j+1}\,,\,k_2(j)) \right) \end{aligned} \tag{9.5.4}$$

$$\begin{aligned} \Pi_2^M &= \sum_{j=0}^{N-1} \left(\int_{k_{1,j}}^{\infty}\int_{k_2(j)}^{\infty} q_T^M(s,p).dp.ds - \int_{k_{1,j+1}}^{\infty}\int_{k_2(j)}^{\infty} q_T^M(s,p).dp.ds \right) \\ &= \sum_{j=0}^{N-1} \left(\Pi_2^M(k_{1,j}\,,\,k_2(j)) - \Pi_2^M(k_{1,j+1}\,,\,k_2(j)) \right) \end{aligned} \tag{9.5.5}$$

Where

$$\begin{aligned} \Pi_1^M(k_1\,,\,k_2) &= \int_{k_1}^{\infty}\int_{k_2}^{\infty} \left(e^p + e^s\right).q_T^M(s,p).dp.ds \\ \Pi_2^M(k_1\,,\,k_2) &= \int_{k_1}^{\infty}\int_{k_2}^{\infty} q_T^M(s,p).dp.ds \end{aligned}$$

The functions $\Pi_1^M(k_1,k_2)$ and $\Pi_2^M(k_1,k_2)$ don't have a finite Fourier Transform because they are not square integrable. E.g. $\Pi_1^M(k_1,k_2)$ tends to $E^M(P(T,T_p)+S_T)\geq 0$ when k_1 and k_2 tend to $-\infty$. To obtain a square integrable function, having a well defined Fourier transform, we multiply $\Pi_{1,2}^M(k_1,k_2)$ by an exponentially decaying term:

$$\begin{aligned}\pi_1^M(k_1,k_2) &= \exp(\alpha_1.k_1+\alpha_2.k_2)\Pi_1^M(k_1\,,\,k_2)\\ \pi_2^M(k_1,k_2) &= \exp(\alpha_1.k_1+\alpha_2.k_2).\Pi_2^M(k_1\,,\,k_2)\end{aligned}$$

For a range of positives values (α_1,α_2) we expect that $\pi_{1,2}(k_1,k_2)$ are well square integrable. The Fourier transform of Π_1^M is denoted $\chi_1^M(v_1,v_2)$ and may be expressed in term of the characteristic function of of $(\ln S_T\,,\,\ln P(T,T_p))$:

$$\begin{aligned}&\chi_1^M(v_1,v_2)\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{(i.(v_1.k_1+v_2.k_2))}.\pi_1^M(k_1,k_2).dk_2.dk_1\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{((\alpha_1+i.v_1)k_1+(\alpha_2+i.v_2)k_2)}\int_{k_1}^{\infty}\int_{k_2}^{\infty}(e^p+e^s)\,q_T^M(s,p).dp.ds.dk_2.dk_1\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(e^p+e^s)\,.q_T^M(s,p)\int_{-\infty}^{p}\int_{-\infty}^{s} e^{((\alpha_1+i.v_1)k_1+(\alpha_2+i.v_2)k_2)}dk_2.dk_1.dp.ds\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(e^p+e^s)\,.q_T^M(s,p).\frac{\exp\left((\alpha_1+i.v_1)\,.s+(\alpha_2+i.v_2)\,.p\right)}{(\alpha_1+i.v_1)\,.\,(\alpha_2+i.v_2)}.dp.ds\\ &= \frac{\phi_T^M\left(v_1-\alpha_1.i\,,\,v_2-(\alpha_2+1)\,.i\right)+\phi_T^M\left(v_1-(\alpha_1+1)\,.i\,,\,v_2-\alpha_2.i\right)}{(\alpha_1+i.v_1)\,.\,(\alpha_2+i.v_2)}\end{aligned}$$

Similarly, the Fourier transform of π_2^M denoted $\chi_2^M(v_1,v_2)$ is reformulated as follows:

$$\chi_2^M(v_1,v_2) = \frac{\phi_T^M\left(v_1-\alpha_1.i\,,\,v_2-\alpha_2.i\right)}{(\alpha_1+i.v_1)\,.\,(\alpha_2+i.v_2)}$$

The Fourier transforms of $\chi_{1,2}^M(v_1,v_2)$ are then easily computable since they depend on $\phi_T^M(.,.)$ whose analytical expressions are provided by propositions 9.4.1 and 9.4.2. On all $N\times N$ vertices of the grid $\Lambda_1\times\Lambda_2$, $\Pi_{1,2}^M(k_{1,j}\,,\,k_{2,l})$ are obtained by an inverse Fourier transform :

$$\Pi_1^M(k_{1,j},k_{2,l}) = \frac{e^{-\alpha_1.k_{1,j}-\alpha_2.k_{2,l}}}{(2.\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i\left(v_1.k_{1,j}+v_2.k_{2,l}\right)}\chi_1^M(v_1,v_2)dv_2dv_1 \quad (9.5.6)$$

$$\Pi_2^M(k_{1,j},k_{2,l}) = \frac{e^{-\alpha_1.k_{1,j}-\alpha_2.k_{2,l}}}{(2.\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i\left(v_1.k_{1,j}+v_2.k_{2,l}\right)}\chi_2^M(v_1,v_2)dv_2dv_1 \quad (9.5.7)$$

A naive approach to value $\Pi_{1,2}^M(k_{1,j}\,,\,k_{2,l})$ would be to discretize the double integrals present in eq. (9.5.6) and (9.5.7) and to compute them by the trapezoid rule. This way of doing is particularly inefficient and require $\mathcal{O}(n^4)$ operations. A better method is to use a two dimensional FFT that computes for any input array $\{IN(v_{1,m},v_{2,n})\,:\,m=0,\ldots,N-1\,,n=0,\ldots,N-1\}$, the following output array:

$$OUT(k_{1,j}\,,\,k_{2,l}) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-\frac{2.\pi.i}{N}(v_{1,m}.k_{1,j}+v_{2,n}.k_{2,l})}.IN(v_{1,m},v_{2,n})$$

$$\forall j=0,\ldots,N-1\ \ ,\, l=0,\ldots,N-1 \quad (9.5.8)$$

in $\mathcal{O}(N^2.\log N^2)$. In order to discretize the integrals of eq. (9.5.6) and (9.5.7), we introduce the integration steps Δ_1 and Δ_2. So as to rewrite the discrete integrals as the right term of eq. (9.5.8), we impose furthermore that:

$$\lambda_1.\Delta_1 \quad = \quad \lambda_2.\Delta_2 = \frac{2.\pi}{N}$$

If we define the following mesh points,

$$v_{1,m} := (m - \frac{N}{2}).\Delta_1 \qquad v_{2,n} := (n - \frac{N}{2}).\Delta_2$$

the discretized version of $\Pi_1^M(k_{1,j}\,,\,k_{2,l})$ is provided by:

$$\begin{aligned}
&\Pi_1^M(k_{1,j}\,,\,k_{2,l}) \\
&\quad\approx \frac{e^{-\alpha_1.k_{1,j}-\alpha_2.k_{2,l}}}{(2.\pi)^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-i.\left(v_{1,m}.k_{1,j}+v_{2,n}.k_{2,l}\right)}.\chi_1^M(v_{1,m},v_{2,n}).\Delta_1.\Delta_2 \\
&\quad= \frac{(-1)^{j+l}.e^{-\alpha_1.k_{1,j}-\alpha_2.k_{2,l}}}{(2.\pi)^2}.\Delta_1.\Delta_2\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-\frac{2.\pi.i}{N}(m.j+n.l)}.(-1)^{m+n}.\chi_1^M(v_{1,m},v_{2,n})
\end{aligned}$$

In the same way, one gets the discrete version of $\Pi_2^M(k_{1,j}\,,\,k_{2,l})$:

$$\begin{aligned}
&\Pi_2^M(k_{1,j}\,,\,k_{2,l}) \approx \\
&\quad \frac{(-1)^{j+l}.e^{-\alpha_1.k_{1,j}-\alpha_2.k_{2,l}}}{(2.\pi)^2}.\Delta_1.\Delta_2.\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-\frac{2.\pi.i}{N}(m.j+n.l)}.(-1)^{m+n}.\chi_2^M(v_{1,m},v_{2,n})
\end{aligned}$$

Once that the elements $\Pi_1^M(k_{1,j}\,,\,k_{2,l})$ and $\Pi_2^M(k_{1,j}\,,\,k_{2,l})$ are computed for all $(j = 0\ldots N-1\,,\, l = 0,\ldots,N-1)$ by the FFT algorithm, the values of Π_1^M , Π_2^M are inferred from relations (9.5.4) , (9.5.5) and V is worth:

$$\begin{aligned}
V^M &= E^M\left(\left[P(T,T_p)+S_T-K.e^{r_g.T}\right]_+|\mathcal{F}_0\right) \\
&\simeq \Pi_1^M - K.e^{r_g.T}.\Pi_2^M
\end{aligned}$$

For a given profit sharing rate γ, the price of the bonus is then:

$$\gamma.E^Q\left(e^{-\int_0^T r_s.ds}.\left[P(T,T_p)+S_T-K.e^{r_g.T}\right]_+|\mathcal{F}_0\right) \;\simeq\; \gamma.P(0,T).\left(\Pi_1^Q - K.e^{r_g.T}.\Pi_2^Q\right)$$

The put option in eq. (9.2.1), valuing the cost the guarantee, is obtained by the put-call parity:

$$\begin{aligned}
E^Q\left(e^{-\int_0^T r_s.ds}.\left[K.e^{r_g.T}-(P(T,T_p)+S_T)\right]_+|\mathcal{F}_0\right) \;&\simeq\; P(0,T).\left(\Pi_1^Q - K.e^{r_g.T}.\Pi_2^Q\right) + \\
&\quad P(0,T).K.e^{r_g.T} - P(0,T_p) - S_0
\end{aligned}$$

Finally, the expected real bonus under P is estimated by:

$$\gamma.E^P\left(\left[P(T,T_p)+S_T-K.e^{r_g.T}\right]_+|\mathcal{F}_0\right) \;=\; \gamma.\left(\Pi_1^P - K.e^{r_g.T}.\Pi_2^P\right)$$

9.6 Applications.

The restricted exponential model has been fitted to the curve of Belgian instantaneous forward rates on 31/12/2006. The parameters obtained are those presented in table 9.6.1. The other market parameters are in table 9.6.2. The insurer's time horizon, T , is set to one year and the maturity T_p of zero coupon bonds hold in portfolio is fixed to three years. The discretization parameters used to run the FFT method are provided in table 9.6.3.

Table 9.6.1: Expo model; parameters:

n	3	β_0	0.0434
θ_1	0.4443	β_1	0.0118
θ_2	0.6837	β_2	-0.0082
θ_3	0.1544	β_3	-0.0094

Table 9.6.2: Market parameters.

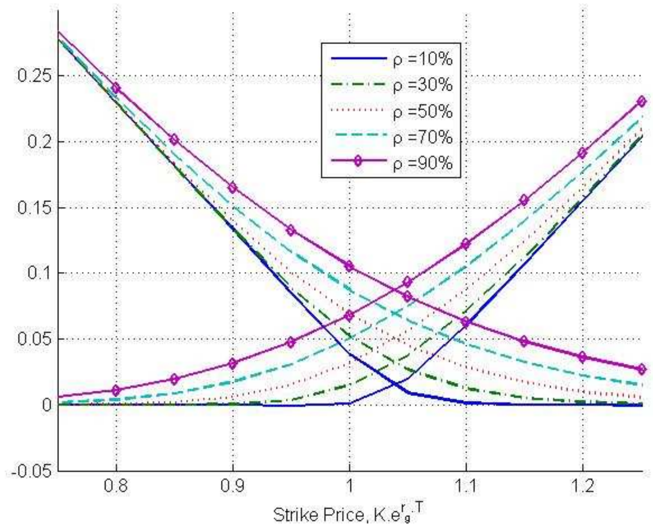
Parameters	Values
a	0.1272
σ_r	0.0175
σ_{Sr}	-0.15
σ_S	0.20
T	1 year
T_p	3 years

Table 9.6.3: FFT parameters.

Parameters	Values
N	256
α_1, α_2	0.75
λ_1, λ_2	0.05
Δ_1, Δ_2	$\frac{2.\pi}{N.\lambda_1}$

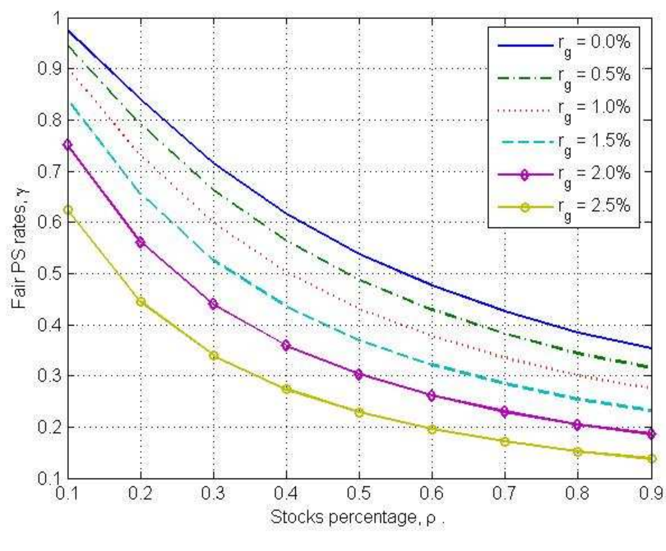
Figure 9.6.1 shows some market prices of European call and put options, of maturity 1 year, written on a basket of stocks and bonds. Those prices are displayed for different strike prices, $K.e^{r_g.T}$, and investment strategies, ρ. As we could have expected, the prices of calls and puts increase proportionally to the fraction of stocks hold in portfolio. The option prices are indeed positively correlated with the volatility of the underlying total asset, which is mainly determined by the amount of stocks.

Figure 9.6.1: Call and put options by strike price.



Once that the call and put options involved in eq. 9.2.1 are computed, we can easily determine the fair participating rate γ of surplus that should be distributed by the insurer as bonus. For this purpose, one have assumed that the premium paid in by the insured is worth $K = 1$. The graph 9.6.2 illustrates the link between the fair participating rate γ , the interest rate guarantee and the structure of asset.

Figure 9.6.2: Fair profit sharing rates by guarantee and stock ratio.



One observes that the higher is the part of stocks hold in portfolio, the lower is the participating rate. E.g. for a guarantee of 0.0%, the fair participating rate, γ , falls from 98%, for 10% of stocks, to 36%, for 90% of stocks. The profit sharing rate is also inversely proportional to the guarantee. If the portfolio counts 10% of stocks, γ goes from 98%, for $r_g = 0.0\%$, to 63%, for $r_g = 2.5\%$.

Figure 9.6.3: Fair profit sharing rates by guarantee and duration.

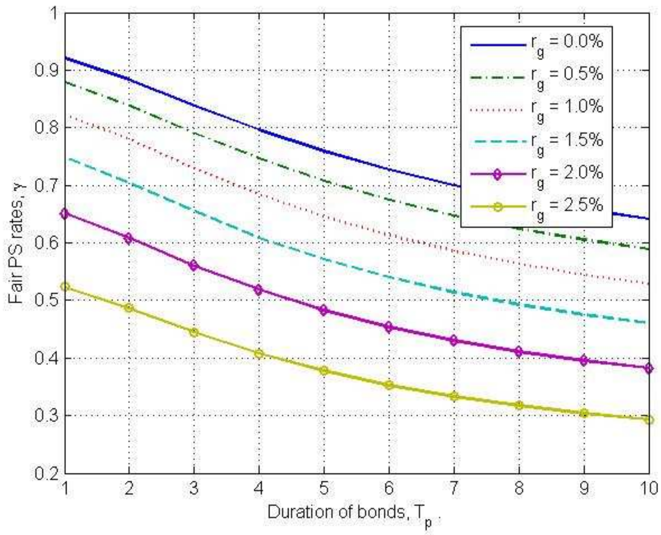


Figure 9.6.4: Expected returns by guarantee and stocks ratio.

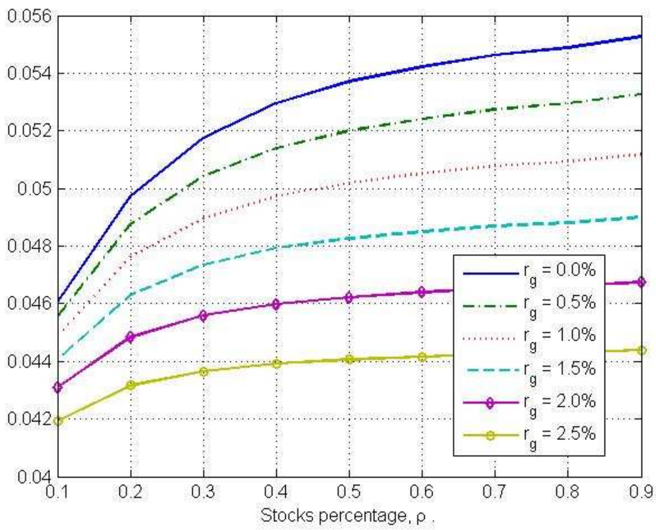


Figure 9.6.3 presents the relation between the fair PS rate γ , the interest rate guarantee

and the duration of zero coupon bonds. Increasing the duration of bonds reduces the fair participating rate, whatsoever the guarantee.

The graph 9.6.4 presents the expected total return granted to policyholders. This expected return is the sum of the guarantee and of the expected bonus under P :

$$r_g + \gamma . E^P \left(\left[P(T, T_p) + S_T - K.e^{r_g . T} \right]_+ |\mathcal{F}_0 \right)$$

The figure reveals that the insured, who accepts a lower guarantee, will be rewarded on average more than a costumer choosing a higher protection. The average total return is also proportional to the amount of stocks purchased by the insurer.

9.7 Conclusions.

In this chapter, we have developed a method to value a fair profit sharing rate for participating insurance contracts. The most novel feature of our work is to consider this issue when the insurer's asset is made up of a non-rebalanced basket of stocks and zero coupon bonds. Such modelling allows us to emphasize the dependence between the investment strategy, the guarantee and the optimal level of bonus.

The main drawback of our approach is that it doesn't provide any analytical expressions of embedded options prices, given that the total asset is not lognormally distributed. However, the characteristic function of logprices being analytically calculable, the FFT of options values may be efficiently computed by the method of Dempster and Hong.

The numerical applications reveal that the cost of the guarantee is proportional to the amount of stocks and to the duration of zero coupon bonds, hold in portfolio. One also observes that the higher is the guarantee or the quantity of stocks, the lower is the fair participating rate. Finally, if the insurer proposes a fair profit sharing rate, we have shown that an insured who accepts a lower guarantee will be rewarded on average more than a costumer choosing a higher protection. A future research could be to study the same issue in a multi-period setting.

Appendix A.

The instantaneous forward rate, $f(0,t) = -\frac{\partial}{\partial t} \log P(0,t)$ plays a crucial role for defining the mean reversion level $b(t)$ (see eq (9.3.2)). In examples presented in section 9.6 , we have fitted a continuous function (a sum of exponentials) on the observed curve of instantaneous forward rates:

$$f(t, t+s) \quad = \quad \beta_0 + \sum_{i=1}^{n} \beta_i . \exp(-\theta_i . s)$$

Where n is a constant determining the number of parameters $\beta_{0...n}$ and $\theta_{1...n}$. This model, known as the restricted exponential model was developed by Cairns (1998) and further details may be found in Cairns (2004).

Appendix B.

The Hull and White model belongs to the category of non arbitrage model. Furthermore, the price of a zero coupon bond is an affine function of interest rates:

$$P(t,T) = \exp\left(A(t,T) - B(t,T).r_t\right) \tag{9.7.1}$$

Where

$$B(t,T) = \frac{1}{a}.(1 - e^{-a.(T-t)})$$

$$A(t,T) = \log\frac{P(0,T)}{P(0,t)} + B(t,T).f(0,t) - \frac{\sigma_r^2}{4.a^3}.(1 - e^{-a.(T-t)})^2.(1 - e^{-2.a.t})$$

For further details on affine models, we refer to Duffie (2001) .

Appendix C.

This appendix proves the proposition 9.4.1. Under the forward measure F_T, the following processes:

$$\begin{aligned} dW_t^{S,F} &= dW_t^{S,Q} \\ dW_t^{r,F} &= dW_t^{r,Q} + \sigma_r.B(t,T).dt \end{aligned}$$

are Brownian motions. Under F_T, the dynamics of assets are such that:

$$\begin{aligned} \frac{dS_t}{S_t} &= \left(r_t - \sigma_{Sr}.\sigma_r.B(t,T)\right).dt + \sigma_{Sr}.dW_t^{r,F} + \sigma_S.dW_t^{S,F} \\ \frac{dP(t,T_p)}{P(t,T_p)} &= \left(r_t + \sigma_r^2.B(t,T_p).B(t,T)\right)dt - \sigma_r.B(t,T_p).dW_t^{r,F} \end{aligned}$$

And by the Itô's lemma, we infer the following dynamics of logprices:

$$\begin{aligned} d\ln S_t &= \left(r_t - \sigma_{Sr}.\sigma_r.B(t,T) - \frac{\sigma_{Sr}^2}{2} - \frac{\sigma_S^2}{2}\right).dt + \sigma_{Sr}.dW_t^{r,F} + \sigma_S.dW_t^{S,F} \\ d\ln P(t,T_p) &= \left(r_t + \sigma_r^2.B(t,T_p).B(t,T) - \frac{\sigma_r^2}{2}.B(t,T_p)^2\right)dt - \sigma_r.B(t,T_p).dW_t^{r,F} \end{aligned}$$

By integration, we get that logprices at time T are given by:

$$\begin{aligned} \ln S_T &= \ln S_0 + \int_0^T \left(r_s - \sigma_{Sr}.\sigma_r.B(s,T) - \frac{\sigma_{Sr}^2}{2} - \frac{\sigma_S^2}{2}\right).ds \\ &\quad + \int_0^T \sigma_{Sr}.dW_s^{r,F} + \int_0^T \sigma_S.dW_s^{S,F} \end{aligned} \tag{9.7.2}$$

$$\begin{aligned} \ln P(T,T_p) &= \ln P(0,T_p) + \int_0^T \left(r_s + \sigma_r^2.B(s,T_p).B(s,T) - \frac{\sigma_r^2}{2}.B(s,T_p)^2\right)ds \\ &\quad -\sigma_r.\int_0^T B(s,T_p).dW_s^{r,F} \end{aligned} \tag{9.7.3}$$

Logprices depend on the integral of r_t , which may also be written as a stochastic integral with respect to $W_s^{r,F}$. According to proposition 9.7.1 proved in appendix E, we can establish the following decompositions of logprices:

$$\begin{aligned}
\ln S_T &= \mu_{S_T}^{F_T} + \int_0^T (\sigma_r.B(s,T) + \sigma_{Sr}).dW_s^{r,F} + \int_0^T \sigma_S.dW_s^{S,F} \\
\ln P(T,T_p) &= \mu_{P(T,T_p)}^{F_T} + \sigma_r.\int_0^T (B(s,T) - B(s,T_p)).dW_s^{r,F}
\end{aligned}$$

Where $\mu_{S_T}^{F_T}$ and $\mu_{P(T,T_p)}^{F_T}$ are respectively defined by expressions (9.4.3) and (9.4.4). The logprices being normal random variables, the characteristic function of the random vector $(\ln S_T\,,\,\ln P(T,T_p))$ is given by:

$$\begin{aligned}
\phi_T^{F_T}(u_1,u_2) &= E^{F_T}\left(\exp\left(i.u_1.\ln S_T + i.u_2.\ln P(T,T_p)\right)|\mathcal{F}_0\right) \\
&= \exp\left(i.u_1.\mu_{S_T}^{F_T} + i.u_2.\mu_{P(T,T_p)}^{F_T} + \frac{1}{2}.\Sigma(u_1,u_2)\right)
\end{aligned}$$

Where

$$\Sigma(u_1,u_2) = \int_0^T ((i.u_1 + i.u_2).\sigma_r.B(s,T) - i.u_2.\sigma_r.B(s,T_p) + i.u_1.\sigma_{Sr})^2 + (i.u_1.\sigma_S)^2\,ds$$

The integrals depending on $B(s,T)$, $B(s,T_p)$, present in eq. (9.4.3) (9.4.4) and (9.4.5) are given by:

$$\begin{aligned}
&\int_0^T B(s,T).ds = \frac{1}{a}.(T - B(0,T)) \\
&\int_0^T B(s,T)^2.ds = \frac{1}{a^2}.\left(T - B(0,T) - \frac{1}{2}.a.B(0,T)^2\right) \\
&\int_0^T B(s,T_p).ds = \frac{1}{a}.T - \frac{1}{a^2}.\left(e^{-a.(T_p-T)} - e^{-a.T_p}\right) \\
&\int_0^T B(s,T_p)^2.ds = \frac{1}{a^2}.T - \frac{2}{a^3}.\left(e^{-a.(T_p-T)} - e^{-a.T_p}\right) \\
&\qquad + \frac{1}{2.a^3}.\left(e^{-2.a.(T_p-T)} - e^{-2.a.T_p}\right) \\
&\int_0^T B(s,T).B(s,T_p).ds = \frac{1}{a^2}.T - \frac{1}{a^3}.\left(e^{-a.(T_p-T)} - e^{-a.T_p}\right) \\
&\qquad - \frac{1}{a^3}.\left(1 - e^{-a.T}\right) + \frac{1}{2.a^3}.\left(e^{-a.(T_p-T)} - e^{-a.(T_p+T)}\right)
\end{aligned}$$

Appendix D.

The proof of proposition 9.4.2 is based upon the results of proposition 9.4.1. Given that the following processes:

$$\begin{aligned}
dW_t^{S,P} &= dW_t^{S,F} - \lambda_r.\sigma_{Sr}.dt - \lambda_S.\sigma_S.dt \\
dW_t^{r,P} &= dW_t^{r,F} - \lambda_r.\sigma_r.dt - \sigma_r.B(t,T).dt
\end{aligned}$$

are Brownian motion under P and according relations (9.7.2) and (9.7.3), we obtain the following decompositions of logprices:

$$\begin{aligned}\ln S_T &= \mu^P_{S_T} + \int_0^T (\sigma_r.B(s,T) + \sigma_{Sr}).dW_s^{r,P} + \int_0^T \sigma_S.dW_s^{S,P} \\ \ln P(T,T_p) &= \mu^P_{P(T,T_p)} + \sigma_r.\int_0^T (B(s,T) - B(s,T_p)).dW_s^{r,P}\end{aligned}$$

Where $\mu^P_{S_T}$ and $\mu^P_{P(T,T_p)}$ are respectively defined by expressions (9.4.7) and (9.4.8). The logprices being normal random variables, the result of proposition 9.4.2 follows.

Appendix E.

In this appendix, we establish the expressions of $\int_0^T r_s.ds$ under the real and forward measures.

Proposition 9.7.1. *The integral of short term interest rates is a Gaussian random variable under P and F_T:*

$$\begin{aligned}\int_0^T r_s.ds &= -\log(P(0,T)) - \frac{\sigma_r^2}{2}.\int_0^T B(s,T)^2.ds \\ &\quad + \int_0^T \sigma_r.B(s,T).dW_s^{r,F}\end{aligned}$$

$$\begin{aligned}\int_0^T r_s.ds &= -\log(P(0,T)) + \frac{\sigma_r^2}{2}.\int_0^T B(s,T)^2.ds \\ &\quad + \lambda_r.\sigma_r.\int_0^T B(s,T).ds + \int_0^T \sigma_r.B(s,T).dW_s^{r,P}\end{aligned}$$

Proof. As mentioned in section 9.3 , the dynamics of risk free rate under the risk neutral measure is given by:

$$dr_s = a.(b(s) - r_s).ds + \sigma_r.dW_s^{r,Q} \tag{9.7.4}$$

Consider a process Z_u defined by:

$$Z_s = e^{a.s}.(b(s) - r_s) \tag{9.7.5}$$

Taking into account (9.7.4), the differential of Z_s is so that:

$$\begin{aligned}dZ_s &= a.e^{a.s}.(b(s) - r_s).ds + e^{a.s}.b^{'}(s).du - e^{a.s}.dr_s \\ &= e^{a.s}.b^{'}(s).ds - e^{a.s}.\sigma_r.dW_s^{r,Q}\end{aligned}$$

And then the process Z_s may be rewritten as the following sum of integrals:

$$Z_s = Z_0 + \int_0^s e^{a.u}.b^{'}(u).du - \int_0^s e^{a.u}.\sigma_r.dW_u^{r,Q} \tag{9.7.6}$$

From relation (9.7.5), we know that

$$r_s = b(s) - e^{-a.s}.Z_s \qquad\qquad Z_0 = (b(0) - r_0) \tag{9.7.7}$$

It suffices therefore to combine (9.7.6) (9.7.7) to get that

$$r_s \quad = \quad e^{-a.s}.r_0 + \int_0^s a.e^{-a.(s-u)}.b(u).du + \int_0^s e^{-a.(s-u)}.\sigma_r.dW_u^{r,Q} \tag{9.7.8}$$

The short term rate r_s is hence Gaussian under Q and

$$\int_0^s a.e^{-a.(s-u)}.b(u).du \quad = \quad \left[e^{-a.(s-u)}.f(0,u)\right]_{u=0}^{u=s} + \int_0^s \frac{\sigma_r^2}{2.a}.e^{-a.(s-u)}.(1-e^{-2.a.u}).du$$

Integrating expression (9.7.8) and taking into account that

$$\begin{aligned} dW_u^{r,Q} &= dW_u^{r,P} + \lambda_r.du \\ dW_u^{r,Q} &= dW_u^{r,F} - \sigma_r.B(u,T).du \\ r_0 &= f(0,0) \end{aligned}$$

lead to the desired result.

□