Minimum Rényi Entropy Portfolios

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Abstract

Accounting for the non-normality of asset returns remains one of the main challenges in portfolio optimization. In this paper, we tackle this problem by assessing the risk of the portfolio through the "amount of randomness" conveyed by its returns. We achieve this using an objective function that relies on the exponential of *Rényi entropy*, an information-theoretic criterion that precisely quantifies the uncertainty embedded in a distribution, accounting for higher-order moments. Compared to Shannon entropy, Rényi entropy features a parameter that can be tuned to play around the notion of uncertainty. A Gram-Charlier expansion shows that it controls the relative contributions of the central (variance) and tail (kurtosis) parts of the distribution in the measure. We further rely on a non-parametric estimator of the exponential Rényi entropy that extends a robust sample-spacings estimator initially designed for Shannon entropy. A portfolio-selection application illustrates that minimizing Rényi entropy yields portfolios that outperform state-of-the-art minimum-variance portfolios in terms of risk-return-turnover trade-off. We also show how Rényi entropy can be used in risk-parity strategies.

Keywords: portfolio selection, Shannon entropy, Rényi entropy, risk measure, information theory, risk parity.

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1. Introduction

In portfolio optimization, it is well known that a high sensitivity of the optimal portfolio weights to estimation errors in parameter inputs can render otherwise sound investment strategies largely

suboptimal out of sample; see Kolm et al. (2014) and references therein. This is in particular the case of the mean-variance portfolio of Markowitz (1952): the optimal weights are very sensitive to estimation errors in the asset mean returns. To tackle this robustness issue, one can simply disregard the portfolio-mean-return constraint, leading to the risk-based-allocation framework; see, for instance, Ardia et al. (2017). *Minimum-risk portfolios* have in particular attracted investors' attention because "the minimum-variance portfolio usually performs better out of sample than any other mean-variance portfolio—even when the Sharpe ratio or other performance measures related to *both* the mean and variance are used for the comparison" (DeMiguel and Nogales 2009 p.560).

Minimum-risk portfolios are commonly built using the variance as risk measure. As the sample minimum-variance portfolio is still quite vulnerable to estimation errors, various robust alternatives have been developed; see Fabozzi et al. (2010) and Scutellà and Recchia (2013) for reviews. Shrinkage estimation, introduced by Ledoit and Wolf (2003, 2004a,b), has proved particularly appealing. Still, the problem remains that the variance is an adequate risk measure only for Gaussian distributions and is largely unaffected by increasing tail concentration (Vermorken et al. 2012). As a result, the minimum-variance portfolio does not account for the non-normality of asset returns. Two main alternative approaches can be employed to deal with non-normality. First, one can minimize a downside-risk measure, such as the minimum-VaR and CVaR portfolios. However, such portfolios coincide with a mean-risk approach (Fabozzi et al. 2010), producing robustness issues as for the mean-variance portfolio. Second, one can extend the Taylor series expansion of the utility function to include the portfolio-return third and/or fourth moments; see Harvey et al. (2010), Martellini and Ziemann (2010) and Adcock (2014). However, robustifying such higher-order portfolios is very challenging due to increased dimensionality and large sensitivity to outliers.

In this paper, we propose a new (albeit natural) way of designing robust minimum-risk portfolios that account for the non-normality of asset returns. We do so by minimizing the portfolio-return uncertainty measured via the exponential of *Rényi entropy*, estimated within a robust m-spacings framework. The optimal portfolios so obtained are called *minimum Rényi entropy portfolios*. Entropy is a well-known concept coming from information theory. It precisely aims to quantify the uncertainty/amount of randomness conveyed by a distribution, embedding all higher-order moments (Cover and Thomas 2006). As a result, it is not surprising to notice that Shannon entropy (the most standard definition of entropy) has been recognized as an appealing measure in finance (Sbuelz and Trojani 2008, Zhou et al. 2013, Ormos and Zibriczky 2014) portfolio management (Philippatos and Wilson 1972, Dionisio et al. 2006, Vermorken et al. 2012, Flores et al. 2017) and utility theory (Yang and Qiu 2005, Abbas 2006, Jose et al. 2008). However, when using entropy to *construct* optimal portfolios, the literature is so far limited to employing entropy as a penalty term besides a more standard cost function: one considers the weights as discrete probabilities, and

uses their entropy as a penalty term to shrink them toward the equally-weighted solution; see Bera and Park (2008) and Zhou et al. (2013). Instead, we use the entropy of the portfolio-return *distribution* (not that of portfolio weights) as the risk-based cost function. Searching for the portfolio weights minimizing the latter amounts to minimize the return uncertainty and thus provides the minimum-risk portfolio in the sense of information theory.

In addition, we rely on Rényi entropy, an extension of Shannon entropy. It features a parameter $\alpha \in [0, \infty]$ that allows one to trade off the minimization of central and tail uncertainty. We argue in particular in favor of setting $\alpha \in [0, 1]$ as, then, Rényi entropy has natural connections with measures of spread (as shown by the extended Chebyshev's inequality of Campbell 1966) and with a minimum-variance-kurtosis objective (as shown by a novel Gram-Charlier expansion of the measure). The empirical results support that choice as well.

Our contribution is organized as follows. Section 2 explores the theoretical properties of the exponential Rényi entropy and makes the link with the notion of risk. Section 3 follows with the minimum Rényi entropy portfolio and its connections with higher-order moments. Section 4 derives a robust m-spacings estimator of the measure and studies its consistency and robustness. We design and perform an empirical out-of-sample performance study of the proposed method in Section 5. Minimum Rényi entropy portfolios are shown to outperform standard minimum-risk portfolios in terms of risk-adjusted performance, while achieving a reasonable level of turnover for values of α close to zero. Section 6 concludes. The proofs of all results are relegated to the Supplementary Materials at the end of the paper.

2. Exponential Rényi entropy and risk measurement

We start this section by introducing the *Rényi entropy*, a flexible measure that quantifies the uncertainty of a random variable from its distribution. It encompasses the well-known *Shannon entropy*, which is recovered as a special case. We then show how its exponential transform is related to *deviation risk measures*. A discussion of the impact of Rényi's α parameter closes the section, arguing in particular that setting $\alpha \in [0, 1]$ should be favored for portfolio selection.

We denote F_X and f_X the cdf and pdf of a random variable X, respectively. In our context, X represents a random asset return. We are exclusively interested in continuous distributions.

2.1. Shannon and Rényi entropy

The entropy of a random variable X commonly refers to its Shannon entropy, first introduced by Shannon (1948), giving birth to a new scientific discipline: *information theory*. It is defined as

$$H(X) := H[f_X] := -\mathbb{E}\big(\ln f_X(X)\big). \tag{1}$$

This measure is known to quantify the amount of randomness embedded in X. For instance, when X is a continuous random variable with bounded support, this quantity is maximized for the uniform distribution, which is the most uncertain one. Shannon entropy embeds many important properties. We refer the reader to Cover and Thomas (2006) for an extended treatment.

Rényi (1961) proposed a generalization of Shannon entropy in (1) with the help of a parameter $\alpha \in \mathbb{R}^+$. The idea was to consider a generalized averaging of $-\ln f_X$, leading to the following definition:

$$H_\alpha(X) := H_\alpha[f_X] := \frac{1}{1-\alpha} \ln \mathbb{E}\big(f_X^{\alpha-1}(X)\big), \tag{2}$$

whenever the expectation exists. Shannon entropy is recovered as a special case in the sense that

$$\lim_{\alpha \to 1} H_\alpha(X) := H_1(X) = H(X).$$

Just like Shannon entropy, Rényi entropy enjoys interesting properties. However, its exponential transform has more natural properties in the context of risk. The next section is dedicated to a more detailed analysis of the exponential Rényi entropy and its connection with deviation risk measures.

2.2. Exponential Rényi entropy

We denote by $H_\alpha^{\exp}$ the exponential Rényi entropy, which, from (2), is defined as

$$H_\alpha^{\exp}(X) := \exp(H_\alpha(X)) = \left(\int (f_X(x))^\alpha dx \right)^{\frac{1}{1-\alpha}} . \tag{3}$$

This quantity was first introduced by Campbell (1966) who studied its relevance as a measure of spread of a distribution for $\alpha \in [0, 1]$. We come back to the link between $H_\alpha^{\exp}$ and measures of spread in Section 2.4. In this article, we apply this measure to the construction of minimum risk portfolios (see Section 3).

2.2.1. Properties

From the properties of Rényi entropy (Koski and Persson 1992, Johnson and Vignat 2007, Pham et al. 2008), $H_\alpha^{\exp}$ obeys the following properties (c is a real constant):

(i) Translation-invariance:

$$H_\alpha^{\exp}(X + c) = H_\alpha^{\exp}(X),$$

(ii) Scaling property:

$$H_\alpha^{\exp}(cX) = |c| H_\alpha^{\exp}(X),$$

(iii) It is non-increasing and continuous in α.

2.2.2. Connection with deviation risk measures

Quantifying uncertainty, the exponential Rényi entropy is closely connected to the class of deviation risk measures, as introduced by Rockafellar et al. (2006).

Definition 1. A *deviation risk measure* is any functional $\mathcal{D} : L^p(\Omega) \to [0, \infty]$ satisfying:[1]

(i) Positivity: $\mathcal{D}(X) > 0$ for all non-constant X, and $\mathcal{D}(X) = 0$ for any constant X,

(ii) Positive homogeneity: $\mathcal{D}(cX) = c\mathcal{D}(X)\ \forall c > 0$,

(iii) Translation-invariance: $\mathcal{D}(X + c) = \mathcal{D}(X)\ \forall c \in \mathbb{R}$,

(iv) Sub-additivity: $\mathcal{D}(X + Y) \leqslant \mathcal{D}(X) + \mathcal{D}(Y)$.

Let us show that $H_\alpha^{\exp}$ fulfills the first three properties of deviation risk measures (sub-additivity is dealt with in next section). Positive homogeneity (*ii*) and translation invariance (*iii*) result from the properties of $H_\alpha^{\exp}$ in Section 2.2.1. Regarding positivity (*i*), $H_\alpha^{\exp}(X)$ is strictly positive if X is non-constant from the positivity of the density f_X. To see that it is null if X is constant, let us compute $H_\alpha^{\exp}(k)$ where k is a constant by computing the limit of $H_\alpha^{\exp}(k + cX)$ as c tends to zero for a given random variable X of finite entropy:

$$H_\alpha^{\exp}(k) = \lim_{c \downarrow 0} H_\alpha^{\exp}(k + cX) = \lim_{c \downarrow 0} cH_\alpha^{\exp}(X) = 0.$$

2.2.3. The sub-additivity property

In this section, we begin with underlining that whereas $H_\alpha^{\exp}$ is expected to be sub-additive for most cases encountered in portfolio selection, it is not, *generally speaking*, sub-additive.

Proposition 1. *$H_\alpha^{\exp}$ is, generally speaking, not a sub-additive measure.*

To prove this Proposition, we give in the Supplementary Materials three analytical counter-examples to sub-additivity using pairs of random variables (X, Y): a pair of independent one-sided (Lévy), a pair of independent two-sided bimodal (Gaussian mixtures) based on the entropy bounds derived in Vrins et al. (2007), and a pair of comonotonic random variables.

We stress that this proposition contradicts some statements recently made in the portfolio management literature; see Flores et al. 2017.[2]

[1] $L^p(\Omega)$ is the space of random variables defined on the support set Ω having finite p^{th} moment.

[2] We are grateful to the authors of the aforementioned paper for discussion about the provided counter-examples.

It is however worth noting that those counter-examples are atypical in portfolio applications. The Lévy distribution for instance is extremely heavy-tailed: none of the moments are defined, while asset returns exhibit much lighter tails in practice (Cont 2001). Similarly, multi-modal distributions and comonotonicity are behaviours that rarely arise in portfolio management.

In fact, just like for the Value-at-Risk (Danielsson et al. 2013), sub-additivity of the exponential Rényi entropy can be reasonably assumed to hold in the specific context of portfolio optimization. For instance, the exponential Rényi entropy of $Z \sim \mathcal{N}(\mu, \sigma)$ collapses to $H_\alpha^{\exp}(Z) = \sigma\sqrt{2\pi\alpha^{1/(\alpha-1)}}$ (Koski and Persson 1992). From the stability of the Gausian distribution, the sub-additivity property is equivalent to $\sigma_{X+Y} \leqslant \sigma_X + \sigma_Y$, which in turns holds from the sub-additivity of the standard deviation (Artzner et al. 1999).

However, this particular case is quite restrictive as asset returns are typically not well described by the Gaussian distribution. A more appealing candidate to model the fat tails observed in asset returns is the general class of *elliptical* distributions, which has many applications in mathematical finance and portfolio theory; see, for instance, Chen et al. (2011). The Gaussian distribution considered above is a special instance of the class of elliptical distributions, which also comprises more realistic distributions to model asset returns such as the Student t and Laplace distributions. As the next proposition shows, the exponential Rényi entropy is sub-additive when (X, Y) is distributed according to an elliptical distribution, providing a broader and more realistic sufficient condition than the Gaussian setting.

Proposition 2. *Let* $(X, Y) \sim \mathrm{El}(\mu, \Sigma, g_2)$ *with* $\mathrm{El}(\mu, \Sigma, g_2)$ *a bivariate elliptical distribution, i.e.*

$$f_{X,Y}(x) = |\Sigma|^{-1/2} g_2\big((x-\mu)'\Sigma^{-1}(x-\mu)\big),$$

where g_2 *is a non-negative density generator function and* $|\Sigma|$ *is the absolute value of the determinant of* Σ*, the scaling matrix of* (X, Y)*. Then,* $H_\alpha^{\exp}$ *is sub-additive for the pair* (X, Y)*.*

Remark 1. Proposition 2 can be extended to any dimension, meaning that, if $X = (X_1, \ldots, X_n) \sim \mathrm{El}(\mu, \Sigma, g_n)$, then $H_\alpha^{\exp}\big(\sum_{i=1}^n X_i\big) \leqslant \sum_{i=1}^n H_\alpha^{\exp}(X_i)$. Combined with the positive-homogeneity property, this means that $H_\alpha^{\exp}$ is sub-additive at the portfolio level, i.e. given $(w_1, \ldots, w_n)$ positive portfolio weights, we have $H_\alpha^{\exp}\big(\sum_{i=1}^n w_i X_i\big) \leqslant \sum_{i=1}^n w_i H_\alpha^{\exp}(X_i)$.

2.3. Exponential Rényi entropy as a flexible risk measure

This section explains how the parameter α allows one to tune the relative contributions of the central and tail parts of the distribution, leading to different definitions of risk.

To show this, consider the two extreme cases $\alpha = 0$ and $\alpha = \infty$. As shown by Koski and Persson (1992), $H_0^{\exp}(X)$ measures the spread of X, while $H_\infty^{\exp}(X)$ is given by the inverse of the supremum

of f_X:

$$H_0^{\exp}(X) := \lim_{\alpha \downarrow 0} H_\alpha^{\exp}(X) = \mathcal{L}(\Omega), \tag{4}$$

$$H_\infty^{\exp}(X) := \lim_{\alpha \to \infty} H_\alpha^{\exp}(X) = 1/\sup f_X, \tag{5}$$

where $\mathcal{L}(\Omega)$ is the Lebesgue measure of the support set of X, $\Omega := \{x : f_X(x) > 0\}$. As one can see, changing α modifies the way we measure entropy, i.e. uncertainty, and so the risk. Taking $\alpha = 0$ amounts to measure risk by the support of the distribution, while taking $\alpha = \infty$ amounts to measure risk by the maximal probability. By minimizing the portfolio-return entropy, as we propose in the next section, one can therefore minimize the density range on the x-axis with $\alpha = 0$ or maximize the density range on the y-axis with $\alpha = \infty$. $H_0^{\exp}$ focuses only on extreme values (low entropy = low distance between extreme values), while $H_\infty^{\exp}$ focuses only on the most likely outcomes (low entropy = high maximal probability), and so, in the symmetric unimodal case, on the center of the distribution.

From those two extreme cases, it results that, for portfolio selection, taking α too large is not desirable because $H_\alpha^{\exp}$ will barely be affected by tail events, which is the criticism that is made about the variance. Conversely, by decreasing α, we assign more similar "weight" to all events, hence increasing the relative importance of tail events compared to events around the mode.

Example 1. Figure 1 shows how $H_\alpha^{\exp}(X)$, $X \sim \text{Student}(\nu)$, evolves with ν for different values of α. From Zografos and Nadarajah (2005), $H_\alpha^{\exp}(X)$ expresses as

$$H_\alpha^{\exp}(X) = (\pi\nu)^{\frac{1-\alpha}{2}} \left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \right)^{\alpha} \frac{\Gamma\left(\frac{\alpha(\nu+1)}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{\alpha(\nu+1)}{2}\right)}. \tag{6}$$

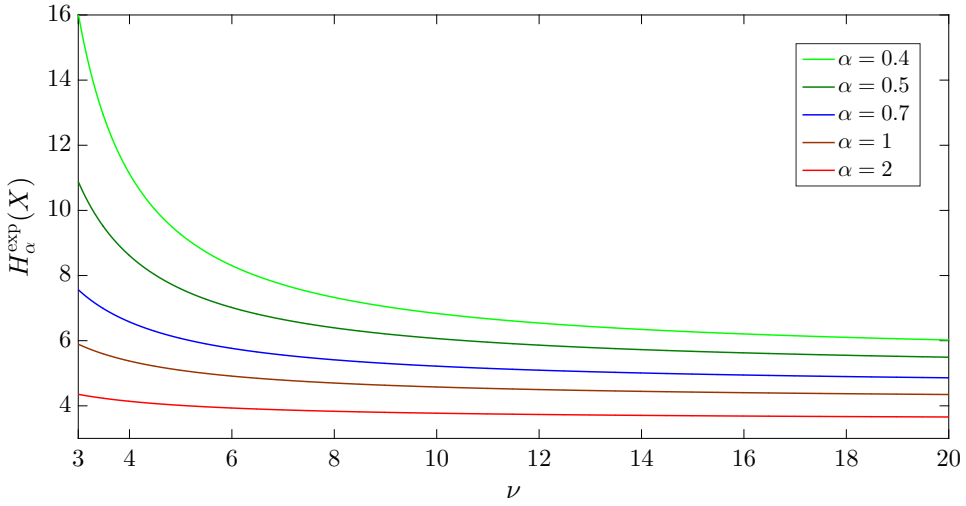
As one can see, when going from $\alpha = 2$ to $\alpha = 0.4$, the sensitivity to the increase of tail uncertainty is indeed increasingly visible.

2.4. Appeal of the $\alpha \in [0, 1]$ case in a portfolio selection context

The previous section argued how decreasing the value of α allows one to obtain a measure of entropy that is increasingly affected by tail events. In this section, we show more specifically that investors should favour setting $\alpha \in [0, 1]$, in which case $H_\alpha^{\exp}$ provides an appealing extension of the variance as a risk criterion.

First, for $\alpha \in [0, 1]$, $H_\alpha^{\exp}$ has close connections with measures of distribution spread. By minimizing the variance, investors ensure that most of the probability distribution of the portfolio return is concentrated in some small interval around the mean. This is established by Chebyshev's

Figure 1 The sensitivity of the exponential Rényi entropy $H_\alpha^{\exp}$ to tail uncertainty increases when its parameter α decreases



Notes. The figure depicts, for different values of α, the exponential Rényi entropy of the standardized t-Student distribution—$H_\alpha^{\exp}(X)$, $X \sim \text{Student}(\nu)$—as a function of the number of degrees of freedom ν. The analytical expression for $H_\alpha^{\exp}(X)$ is reported in Equation (6).

inequality which, given the set $A_k = \{x \in \Omega \mid |x - \mathbb{E}(X)| \geq k\}$, says that

$$\mathbb{P}(X \in A_k) \leq \frac{\mathbb{V}\text{ar}(X)}{k^2}.$$

Similarly, for $\alpha \in [0, 1]$, a small value of $H_\alpha^{\exp}(X)$ entails that most of the probability distribution of X is concentrated on a set of small Lebesgue measure. This is determined by Campbell (1966)'s extended Chebyshev's inequality which, given the set $A'_k = \{x \in \Omega \mid f_X(x) \leq k\}$, establishes that

$$\mathbb{P}(X \in A'_k) \leq \left(kH_\alpha^{\exp}(X)\right)^{1-\alpha}. \tag{7}$$

This inequality is more general than Chebyshev's inequality as it does not only deal with the absolute deviation around the mean, but instead relates the spread in terms of the size of the set on which most of the probability density is situated. For a unimodal random variable X with $\Omega = \mathbb{R}$ and $f_X(x) \to 0$ as $x \to \pm\infty$, which is common for asset returns, then there are only two values of x for which $f_X(x) = k$ if $k < f_X(\text{mode}(X))$. Denoting them x_k^- and x_k^+, we have $x_k^- < \text{mode}(X) < x_k^+$ and (7) means that

$$\mathbb{P}(x_k^- < X < x_k^+) \geq 1 - \left(kH_\alpha^{\exp}(X)\right)^{1-\alpha} \to 1$$

as $H_\alpha^{\exp}(X) \to 0$. In other words, if $H_\alpha^{\exp}(X)$ is small, the probability that X is concentrated on a small interval around its mode is close to one.

A second argument in favor of setting $\alpha \in [0, 1]$ is related to the Gram-Charlier expansion of Rényi entropy derived in Section 3.2. The expansion will show that, when $\alpha \in [0, 1]$, the coefficient

in front of the kurtosis of X is positive (and instead negative for $\alpha > 1$) and so that a decrease in kurtosis decreases the Rényi entropy, as desired by investors.

Third, the empirical results presented in Section 5.2 display a largely better performance of the minimum Rényi entropy portfolio when $\alpha \in [0, 1]$ as well.

3. Rényi entropy and portfolio selection

Given the good match between the theoretical properties of $H_\alpha^{\exp}$ and the desirable features of portfolio selection criteria, we use this measure as an objective function to design investment strategies. In particular, we propose to construct a minimum-risk portfolio, called the *minimum Rényi entropy (MRE) portfolio*, that minimizes the exponential Rényi entropy of the portfolio return. We denote by P the portfolio return such that $P = w'X = \sum_{i=1}^n w_i X_i$ where $w = (w_1, \ldots, w_n)'$ is the vector of portfolio weights and $X = (X_1, \ldots, X_n)'$ is the random asset-return vector.

3.1. Definition

The MRE portfolio over an investment set of n assets for a given α is defined as

$$w_\alpha^\star := \arg\min_{w \in \mathcal{W}} H_\alpha^{\exp}(P), \tag{8}$$

where $\mathcal{W}$ is a set of constraints on w, including the full investment constraint $\mathbf{1}_n' w = 1$.

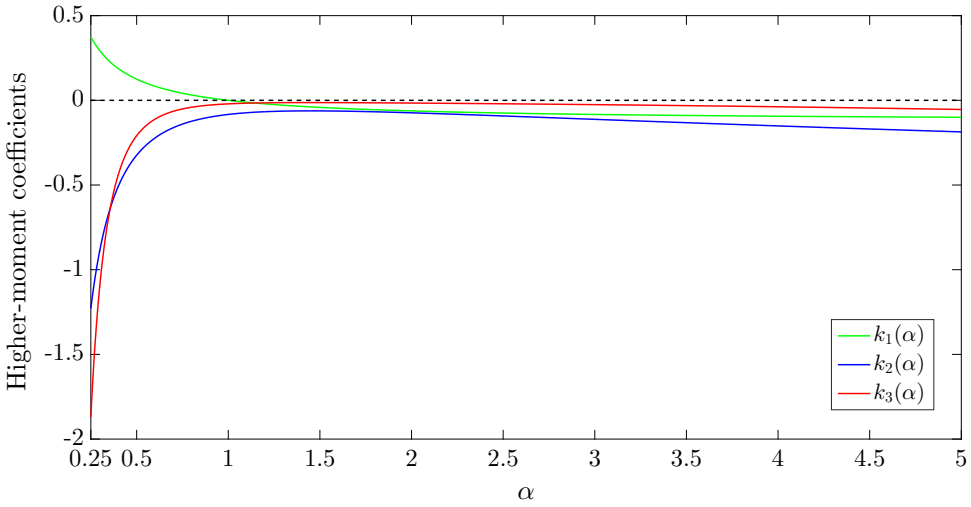
Note that because the MRE portfolio is affected by higher-order moments of the portfolio return, which are non-convex functions of the weights (Jurczenko and Maillet 2006), the optimization program in (8) may not necessarily be convex. Hence, when solving the MRE portfolio, one must ideally resort to global optimization techniques rather than standard local optimizers. We come back to this matter in Section 5.1.4.

3.2. Connection with higher-order moments: A Gram-Charlier expansion

Minimum-risk portfolios are commonly built from specific moments of the portfolio return, such as the minimum-variance portfolio, but possibly higher-order moments as well; see e.g. Martellini and Ziemann (2010), Adcock (2014) and Vanduffel and Yao (2017). We call such portfolios *higher-order portfolios*.

In the classical Markowitz Gaussian setting, the MRE portfolio coincides with the minimum-variance portfolio as there is a one-to-one correspondence between $H_\alpha^{\exp}$ and the variance for Gaussian random variables. In a more general setting however, the MRE portfolio is more appealing than the minimum-variance one because it accounts for the uncertainty coming from the higher-order

Figure 2 Coefficients of the truncated Gram-Charlier expansion of Rényi entropy



Notes. The figure depicts the coefficients $k_1(\alpha)$, $k_2(\alpha)$ and $k_3(\alpha)$ in the truncated Gram-Charlier expansion of Rényi entropy as a function of α. The expression for the expansion and the coefficients are displayed in Equations (9)–(10).

moments of the portfolio return. To see this, it is useful to derive a truncated Gram-Charlier (GC) expansion of Rényi entropy.

Proposition 3. *Let $X \in L^4(\Omega)$ and note $\tilde{X} = (X - \mathbb{E}(X))/\sqrt{\mathbb{V}\mathrm{ar}(X)}$ its standardized copy. Define $\mathbb{S}\mathrm{kew}(X) = \mathbb{E}(\tilde{X}^3)$ and $\mathbb{K}\mathrm{urt}(X) = \mathbb{E}(\tilde{X}^4) - 3$. Then, the truncated GC expansion of $H_\alpha(X)$ is*

$$H_\alpha^{GC}(X) := H_\alpha\big[\mathcal{N}\big(0, \sqrt{\mathbb{V}\mathrm{ar}(X)}\big)\big] + k_1(\alpha)\mathbb{K}\mathrm{urt}(X) + k_2(\alpha)\mathbb{S}\mathrm{kew}(X)^2 + k_3(\alpha)\mathbb{K}\mathrm{urt}(X)^2, \tag{9}$$

with coefficients

$$\begin{aligned} k_1(\alpha) &= \frac{1-\alpha}{8\alpha}\,, \\ k_2(\alpha) &= -\frac{3\alpha^2 - 6\alpha + 5}{24\alpha^{3/2}}\,, \\ k_3(\alpha) &= -\frac{3\alpha^4 - 12\alpha^3 + 42\alpha^2 - 60\alpha + 35}{384\alpha^{5/2}}\,. \end{aligned} \tag{10}$$

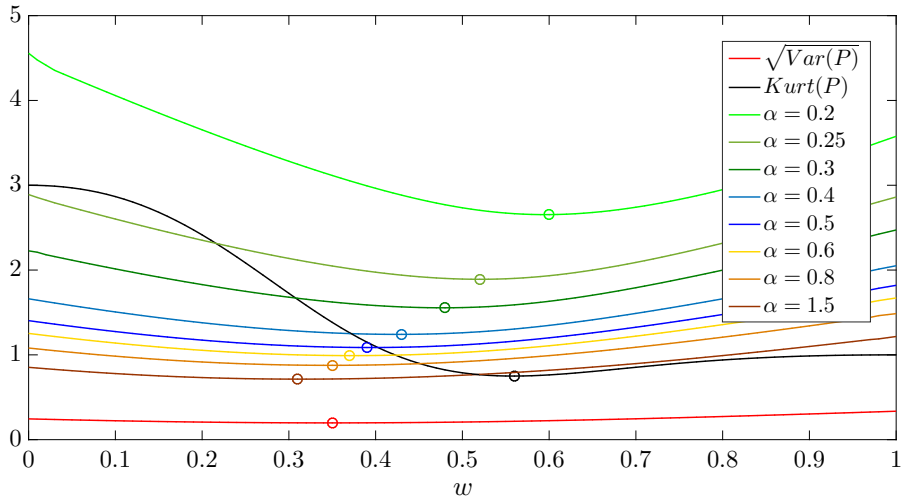
The three coefficients $k_1(\alpha)$, $k_2(\alpha)$ and $k_3(\alpha)$ are displayed in Figure 2. By setting $\alpha = 1$, we recover the GC expansion derived by Hyvärinen et al. (2001):

$$H_1^{GC}(X) = H_1\big[\mathcal{N}\big(0, \sqrt{\mathbb{V}\mathrm{ar}(X)}\big)\big] - \frac{1}{12}\mathbb{S}\mathrm{kew}(X)^2 - \frac{1}{48}\mathbb{K}\mathrm{urt}(X)^2. \tag{11}$$

As one can observe, the ability to control for kurtosis is a notable advantage of Rényi entropy over Shannon entropy, as in the latter case $k_1(1) = 0$.

The connection between the MRE and higher-order portfolios is now made explicit. We have

Figure 3 The minimum Rényi entropy portfolio balances variance and kurtosis minimization



Notes. The figure depicts the standard deviation $\sqrt{\mathbb{V}\text{ar}(P)}$, excess kurtosis $\mathbb{K}\text{urt}(P)$ and exponential Rényi entropy $H_\alpha^{\exp}(P)$ of the portfolio return $P = wX + (1-w)Y$ as a function of $w \in [0,1]$. The two asset returns $X \perp Y$ follow a zero-mean Student's t distribution with $(\sigma_X, \nu_X) = (0.3, 10)$ and $(\sigma_Y, \nu_Y) = (0.2, 6)$.

$H_\alpha\big[\mathcal{N}\big(0, \sqrt{\mathbb{V}\text{ar}(P)}\big)\big] = H_\alpha\big[\mathcal{N}(0,1)\big] + \frac{1}{2}\ln \mathbb{V}\text{ar}(P)$, which yields[3]

$$w_\alpha^\star \approx \arg\min_{w \in \mathcal{W}} \frac{1}{2}\ln \mathbb{V}\text{ar}(P) + k_1(\alpha)\mathbb{K}\text{urt}(P) + k_2(\alpha)\mathbb{S}\text{kew}(P)^2 + k_3(\alpha)\mathbb{K}\text{urt}(P)^2. \tag{12}$$

When f_P is close to a Gaussian, the main contributing higher-order term will be $k_1(\alpha)\mathbb{K}\text{urt}(P)$. When $\alpha < 1$, $k_1(\alpha) > 0$ and so the MRE portfolio is similar to a minimum-variance-kurtosis portfolio, which, as noted by Martellini and Ziemann (2010), is a well-performing higher-order portfolio as estimators for even moments are less noisy than estimators for odd moments. When $\alpha > 1$ however, $k_1(\alpha) < 0$ and so the effect is reversed. In line with investors' preferences for kurtosis, setting $\alpha \in [0,1]$ is thus more natural, as we noted in Section 2.4.

Therefore, in line with Section 2.3, by playing with α one trades off the minimization of the central (variance) and tail (kurtosis) uncertainty, i.e. of the first two even moments.

Example 2. Consider $n = 2$ assets $X_1 = X \perp X_2 = Y$ that follow a zero-mean Student's t distribution with $(\sigma_X, \nu_X) = (0.3, 10)$ and $(\sigma_Y, \nu_Y) = (0.2, 6)$. We build a portfolio $P = wX + (1-w)Y$ and evaluate $H_\alpha^{\exp}(P)$ by numerical integration. On Figure 3, we display how $H_\alpha^{\exp}(P)$, $\sqrt{\mathbb{V}\text{ar}(P)}$ and $\mathbb{K}\text{urt}(P)$ depend on w. As we can see, when α is high enough, $w^\star$ is close to the minimum-variance portfolio because $\sigma_X > \sigma_Y$ and that mostly central events matter when α is high. However, the more α decreases, the more important is the impact of the fatter tails of Y ($\nu_Y < \nu_X$) and so the more $w^\star$ approaches the minimum-kurtosis portfolio.

[3] Minimizing $H_\alpha^{\exp}(P)$ or $H_\alpha(P)$ is equivalent as $\exp(x)$ is a monotonically increasing function.

Given that $k_2(\alpha)$ and $k_3(\alpha)$ are negative for all α, the two additional terms $k_2(\alpha)\mathbb{S}\text{kew}(P)^2$ and $k_3(\alpha)\mathbb{K}\text{urt}(P)^2$ can be interpreted as driving the solution away from the Gaussian's skewness and kurtosis. This is intuitive as, under a fixed mean and variance, the Shannon entropy is maximized for the Gaussian distribution (Cover and Thomas 2006).

Finally, note that the optimization program in (8) can accommodate an additional constraint on the portfolio expected return of the form $\mathbb{E}(P) = w'\mu \geq \mu_0$ to account for the fact that investors do not only look at the risk, but also at the reward. In light of the GC expansion, such a framework would be linked to higher-moment efficient frontiers studied by e.g. Adcock (2014) and Qi et al. (2017). In the empirical study, we however concentrate on risk minimization due to the technical difficulties inherent in estimating the vector μ (see Section 1), which create significant loss in out-of-sample performance.

4. Robust m-spacings estimator of $H_\alpha^{\exp}$

In this section, we explain how, given a finite portfolio-return sample $\{P_t\}$, $P_t = \sum_{i=1}^n w_i X_{i,t}$, $1 \leqslant t \leqslant T$, one can estimate $H_\alpha^{\exp}(P)$ in a robust way. In particular, we propose to use an estimator based on sample-spacings and discuss its properties in terms of consistency and robustness.

4.1. Motivation and expression for the m-spacings estimator

To avoid making assumptions about the portfolio-return distribution, we are looking for a non-parametric estimator of $H_\alpha^{\exp}(P)$. There exists substantial research on non-parametric estimation of Shannon entropy, reviews of which can be found in Beirlant et al. (1997).

A natural way of estimating entropy is the plug-in estimate where a density estimator is plugged into the integral defining the entropy. One could for example choose the well-known Parzen (kernel) estimator. However, this estimator is known to be very sensitive to the bandwidth parameter, which can yield issues of stability for our portfolio optimization context.[4] Instead, m-spacings estimation of entropy is more reliable: Wachowiak et al. (2005) show that such estimators "are robust and accurate, compare favorably to the popular Parzen window method for estimating entropies, and, in many cases, require fewer computations than Parzen methods." Therefore, we rely on a robust m-spacings estimator of Rényi entropy that extends the Shannon entropy m-spacings estimator of Learned-Miller and Fisher (2003), a "consistent, rapidly converging and computationally efficient estimator of entropy which is robust to outliers."

A detailed derivation of the estimator is available in the Supplementary Materials. We only

[4] When applied to our empirical data in Section 5, the Parzen estimator (with Gaussian kernel) achieves a worse risk-adjusted performance than the m-spacings estimator considered here for a wide range of values of the bandwidth parameter.

report the final expression here for conciseness.

Proposition 4. *Let X be a continuous random variable of which we dispose of T i.i.d observations. Then, the m-spacings estimator of $H_\alpha^{\exp}(X)$ is given by*

$$\widehat{H}_\alpha^{\exp}(m,T) := \left(\frac{1}{T-m} \sum_{i=1}^{T-m} \left(\frac{T+1}{m} \big(X^{(i+m:T)} - X^{(i:T)} \big) \right)^{1-\alpha} \right)^{\frac{1}{1-\alpha}}, \tag{13}$$

where $X^{(1:T)} \leqslant X^{(2:T)} \leqslant \cdots \leqslant X^{(T:T)}$ are the order statistics (the observations sorted by increasing order) and $m \in [1, T-1]$ is an integer parameter.

The parameter m is a free parameter of great importance: increasing its value reduces the variance of the estimator by grouping more order statistics in each spacing $X^{(i+m:T)} - X^{(i:T)}$. As a consequence, it plays a crucial role as it determines the robustness of the estimator, and in turn of the MRE portfolio. We come back to this in Section 4.2.2.

Taking the limit where $\alpha \to 1$, we recover the exponential of the estimator of Learned-Miller and Fisher (2003):

$$\widehat{H}_1^{\exp}(m,T) := \exp\left(\frac{1}{T-m} \sum_{i=1}^{T-m} \ln\left(\frac{T+1}{m} \big(X^{(i+m:T)} - X^{(i:T)} \big) \right) \right). \tag{14}$$

4.2. Properties of the m-spacings estimator

The m-spacings estimation of entropy has attracted numerous research (see the review of Beirlant et al. 1997) and dates a while back; see e.g. Vasicek (1976). However, it has been considered mainly for Shannon entropy and, even for this specific case, only asymptotic behaviour has been studied. In the general case $\alpha \neq 1$, consistency has not been established. In this section, we first discuss the estimator's properties in terms of consistency, arguing that the asymptotic estimator bias can be ignored for the sake of our portfolio application. Second, we show how the parameter m determines the robustness of the estimator.

4.2.1. Asymptotic bias

Let us first consider the case $\alpha = 1$ and denote $\widehat{H}_1(m,T) := \ln \widehat{H}_1^{\exp}(m,T)$. van Es (1992) proved that $\widehat{H}_1(m,T)$ is asymptotically biased but, interestingly, that the bias only depends on the fixed value of m and not on the density f_X:

$$\widehat{H}_1(m,T) - H_1(X) \to \psi(m) - \ln m \quad \text{a.s.}, \tag{15}$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the digamma function. Equation (15) means that we can simply subtract the bias to get a consistent estimator. Getting back to the exponential case, this means that

$$me^{-\psi(m)} \widehat{H}_1^{\exp}(m, T) \to H_1^{\exp}(X) \quad \text{a.s.} \tag{16}$$

Interestingly, as readily seen from (16), because the asymptotic bias depends only on m when $\alpha = 1$, using the bias-corrected estimator in (16) or the biased estimator in (14) is equivalent when searching the weights that minimize the entropy in (8). Ideally, we would want the same result to hold for all α, i.e. the asymptotic bias to depend only on α and m. While such a result is not known, Hegde et al. (2005) note that "in many practical applications, [...] this bias does not affect the solution, since it is independent of the true data distribution [...]." Further, as we now show, the estimator bias for X and $\tilde{X} = (X - \mu)/\sigma$ is the same; that is, it does not depend on the specific location and scale of X.

Proposition 5. *Let* $\tilde{X} = (X - \mu)/\sigma$ *and* $\widehat{H}_\alpha(X; m, T) := \ln \widehat{H}_\alpha^{\exp}(X; m, T)$. *Then, the* m*-spacings estimator bias* $\mathbb{B}\big(\widehat{H}_\alpha(X; m, T)\big) := \mathbb{E}\big(\widehat{H}_\alpha(X; m, T)\big) - H_\alpha(X) = \mathbb{B}\big(\widehat{H}_\alpha(\tilde{X}; m, T)\big)$.

4.2.2. Robustness to outliers

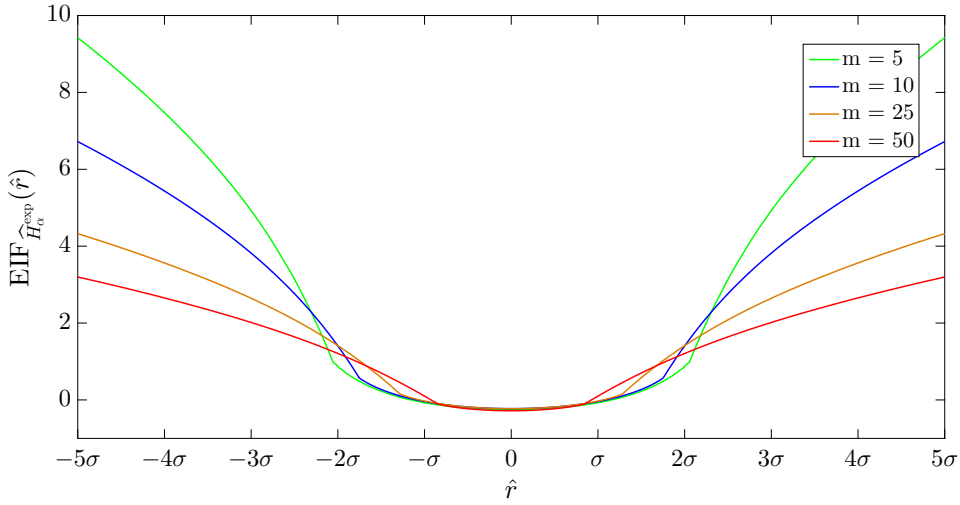
The parameter m acts as a smoothing parameter that controls the estimator variance. This section shows that increasing m makes the m-spacings estimator more robust to outliers, which is crucial to ensure a solid out-of-sample performance of the MRE portfolio. Robustness conveys that a small perturbation from the true return distribution yields only a small change in the estimated value.

In assessing the robustness of an estimator, the *Empirical Influence Function* (EIF) represents a useful tool; see Hampel et al. (1986). Given an estimator $\hat{\theta}(X_1, \ldots, X_T)$ of a quantity θ based on a sample of size T, $\text{EIF}_{\hat{\theta}}(\hat{r})$ measures the sensitivity of the estimator $\hat{\theta}$ to the addition of a supplementary observation $\hat{r}$ in the sample:

$$\text{EIF}_{\hat{\theta}}(\hat{r}) := (T + 1)\big(\hat{\theta}(X_1, \ldots, X_T; \hat{r}) - \hat{\theta}(X_1, \ldots, X_T)\big). \tag{17}$$

Intuitively, the lower is $\text{EIF}_{\hat{\theta}}(\hat{r})$, the more robust is the estimator $\hat{\theta}$. Figure 4 depicts the EIF of the m-spacings estimator—$\text{EIF}_{\widehat{H}_\alpha^{\exp}}(\hat{r})$—for $T = 250$ values from $X \sim \mathcal{N}(\mu = 0, \sigma = 0.2)$. Following Hampel et al. (1986), we set $X_i = \mu + \sigma\Phi^{-1}\big(\frac{i}{T+1}\big)$ to eliminate the random sample variability. We consider $\hat{r} \in [-5\sigma, 5\sigma]$ and report the results for $\alpha = 0.5$ only as other values yield similar insights. One can indeed observe that the EIF decreases with m for large enough values of $\hat{r}$, i.e. for outliers.

Figure 4 Increasing m improves the robustness to outliers of the m-spacings estimator of the exponential Rényi entropy $\widehat{H}_\alpha^{\exp}(m,T)$



Notes. The figure depicts the empirical influence function (EIF) of the m-spacings estimator of the exponential Rényi entropy—$\mathrm{EIF}_{\widehat{H}_\alpha^{\exp}}(\hat{r})$— for $\alpha = 0.5$ and different values of m as a function of $\hat{r}$. We generate $T = 250$ observations from a Gaussian random variable $X \sim \mathcal{N}(\mu = 0, \sigma = 0.2)$ by setting $X_i = \mu + \sigma\Phi^{-1}\left(\frac{i}{T+1}\right)$ and we set $\hat{r} \in [-5\sigma, 5\sigma]$.

5. Out-of-sample empirical study

We finish the paper with an out-of-sample performance study of the MRE portfolio that aims at showing the practical interest of the proposed portfolio policy compared to several existing strategies. The study is performed on six datasets commonly used as benchmarks in the portfolio-selection literature. Section 5.1 presents the methodology, Section 5.2 reports the results, and Section 5.3 considers an extension to the risk-parity strategy.

5.1. Methodology

We begin by detailing the methodology employed to generate the out-of-sample results reported in Section 5.2.

5.1.1. Portfolio strategies

The reported results compare the MRE portfolio with $\alpha \in \{0.3, 0.5, 0.7, 1, 1.5, 2\}$ to five different minimum-variance (MV) portfolios. The first four solve the quadratic optimization program

$$w^\star = \arg\min_{w \in \mathcal{W}} w'\Sigma w \tag{18}$$

by estimating Σ with the sample covariance matrix $\widehat{\Sigma}$ and the three robust shrinkage estimators developed by Ledoit and Wolf (2003, 2004a,b):

$$\widehat{\Sigma}_{CC} := \delta^\star \widehat{F}_{CC} + (1-\delta^\star)\widehat{\Sigma} \quad , \quad \widehat{\Sigma}_{SF} := \delta^\star \widehat{F}_{SF} + (1-\delta^\star)\widehat{\Sigma} \quad , \quad \widehat{\Sigma}_{I} := \delta^\star \widehat{F}_{I} + (1-\delta^\star)\widehat{\Sigma}, \tag{19}$$

where $\delta^\star$ minimizes the Frobenius norm between the shrinkage estimator and the true matrix Σ. The three target matrices are based upon a constant correlation model ($\widehat{F}_{CC}$), a single-factor model ($\widehat{F}_{SF}$) and a multiple of the identity matrix ($\widehat{F}_{I}$).

The fifth MV portfolio is the one-step M-portfolio (MP) of DeMiguel and Nogales (2009):

$$(w^\star, \mu^\star) = \arg\min_{\boldsymbol{w}\in\mathcal{W},\, \mu} \frac{1}{T}\sum_{t=1}^{T} \rho(P_t - \mu), \tag{20}$$

where ρ is the Huber's robust loss function

$$\rho(x) := \begin{cases} x^2/2 & \text{if } |x| \leqslant c \\ c(|x| - c/2) & \text{if } |x| > c \end{cases} \quad , \quad c = 1\%. \tag{21}$$

We note that we have also implemented the robust Bayes-Stein mean-variance portfolio of Jorion (1986) as well as the minimum-VaR portfolio using the robust estimator in Boudt et al. (2008). Even though such criteria are positively affected by higher returns, we observed that they feature a lower risk-adjusted performance than the MRE portfolio due to their sensitivity to the portfolio mean return. The equally-weighted strategy has been considered as well but, while it naturally achieves the lowest turnover, it is largely outperformed by all the other strategies. It will be considered as a competitor when considering the risk-parity strategy in Section 5.3.

5.1.2. Datasets

We rely upon six monthly returns datasets from the Kenneth French library that are extensively used as benchmarks in the literature to compare portfolio strategies; see e.g. DeMiguel et al. (2009a,b), Behr et al. (2013) and Ardia et al. (2017). The datasets are listed in Table 1.

5.1.3. Dynamic rebalancing

We construct the portfolios by dynamic rebalancing. Rebalancing the weights not too frequently is important to ensure a satisfactory performance and turnover (Carroll et al. 2017), hence we set a rolling window of one year as in Behr et al. (2013). The estimation window is set to ten years, i.e. $T = 120$. We use as starting date 07/1963 as in DeMiguel et al. (2009b) and Behr et al. (2013), and rebalance the portfolios until 06/2016. This represents an out-of-sample period of 43 years.

Table 1 List of datasets considered in the empirical study

Datasets	Abb.	n	Time period
6 Fama-French portfolios of firms sorted by size and book-to-market	*6BTM*	6	07/1963 - 06/2016
25 Fama-French portfolios of firms sorted by size and book-to-market	*25BTM*	25	07/1963 - 06/2016
6 Fama-French portfolios of firms sorted by size and momentum	*6Mom*	6	07/1963 - 06/2016
25 Fama-French portfolios of firms sorted by size and momentum	*25Mom*	25	07/1963 - 06/2016
10 industry portfolios representing the US stock market	*10Ind*	10	07/1963 - 06/2016
17 industry portfolios representing the US stock market	*17Ind*	17	07/1963 - 06/2016

Notes. All datasets are made of monthly returns. The value weighting scheme is considered for the industry portfolios. Source: Kenneth French library.

5.1.4. Optimization

As pointed out in Section 3.1, the MRE optimization program is, generally speaking, not a convex problem. Therefore, to find the optimal weights, we rely on the global optimizer of Ugray et al. (2007) based on the Nelder-Mead algorithm. By doing so, we minimize the risk of getting stuck in a local minimum.

5.1.5. Choice of m

As pointed out in Section 4.2.2, the value of m for the m-spacings estimator is of paramount importance as it determines the robustness of the MRE portfolio. To illustrate this, Figure 5 reports, for the *10Ind* dataset, the time evolution of the MRE portfolio weights for $\alpha = 0.5$ and $m = 2, 8, 24$. These are unconstrained weights, i.e. corresponding to $\mathcal{W} = \{\boldsymbol{w} \in \mathbb{R}^n \mid \mathbf{1}_n'\boldsymbol{w} = 1\}$. The weights of the M-portfolio are also reported for comparison. One can clearly observe that increasing m improves the stability of the portfolio weights obtained.

As a first strategy, we have considered the *leave-one-out cross-validation* method, using as criteria maximum return, minimum variance and maximum Sharpe ratio. However, the results obtained were quite poor both in terms of performance and turnover. Allowing m to change at each rolling window seems to add instability to the procedure and thus is not recommended.

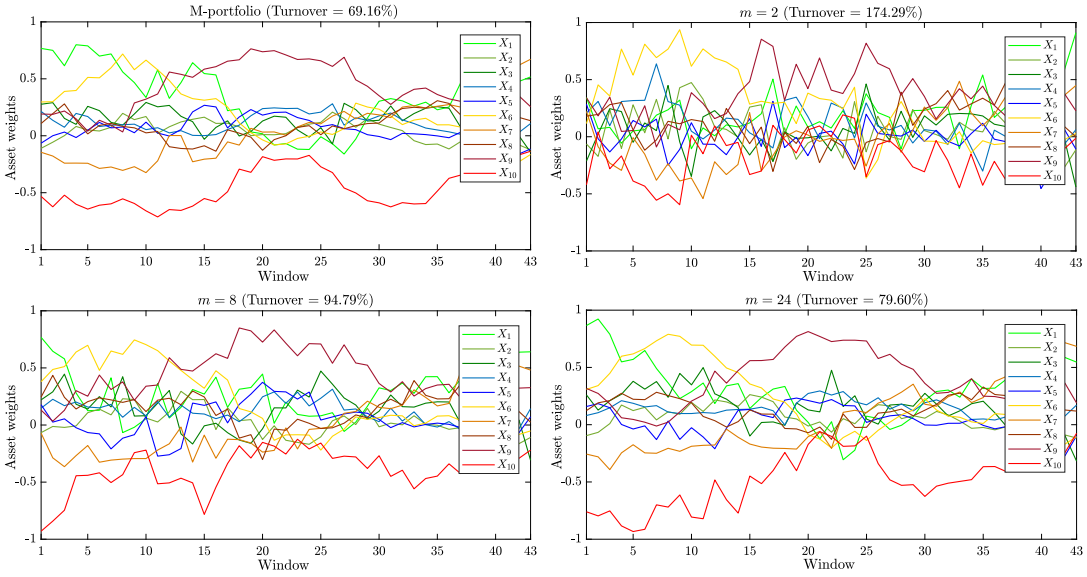
Therefore, as a second strategy, we have used the simple rule-of-thumb $m = \left[T^{2/3}\right] = 24$, which works well on the considered datasets. We observe actually that once m is high enough, the results display only a very minor sensitivity to the specific value of m that is chosen, so that one can set $m = 24$ without fearing that another different but close value would yield dissimilar results.[5]

5.1.6. Portfolio weight constraints

To alleviate the impact of estimation errors, it is common in portfolio optimization to restrict the solution space $\mathcal{W}$. This has the effect of improving the stability of the optimal weights obtained and

[5]Specifically, the results in Table 2 for $m = 18$ and $m = 35$ yield a very similar performance. The only changes are in terms of turnover, which is higher for $m = 18$ and nearly identical for $m = 35$.

Figure 5 Increasing m improves the stability of the MRE portfolio weights



Notes. The figure depicts, for the *10Ind* dataset, the time evolution of the portfolio weights for the M-portfolio of DeMiguel and Nogales (2009) and the MRE portfolio with $\alpha = 0.5$ and $m \in \{2, 8, 24\}$. The weights are unconstrained, i.e. w is restricted to $\mathcal{W} = \{w \in \mathbb{R}^n \mid \mathbf{1}_n' w = 1\}$.

in turn the portfolio out-of-sample performance. This is easily understood from Figure 5, in which all the portfolios feature a significant turnover in the unconstrained case, even though $n = 10$ is relatively low in comparison to the sample size $T = 120$.

Therefore, we optimize the different portfolios subject to a constraint on the weights. We implement the global variance-based constraint (GVBC) devised by Levy and Levy (2014):

$$\sum_{i=1}^{n} \left(w_i - \frac{1}{n} \right)^2 \frac{\sigma_i}{\bar{\sigma}} \leq \delta, \tag{22}$$

where $\sigma_i := \sqrt{\mathbb{V}\mathrm{ar}(X_i)}$ and $\bar{\sigma} := \frac{1}{n} \sum_{i=1}^{n} \sigma_i$. The underlying rationale of GVBC is "to impose more stringent constraints on stocks with relatively high standard deviations, as the estimation errors for these stocks' parameters, and hence the potential economic loss, are larger than for stocks with relatively low standard deviations" (Levy and Levy 2014 p.375). Using a U.S. industry portfolio dataset, the authors observe largely improved out-of-sample Sharpe ratios compared to several robust portfolio selection strategies for a wide range of values of δ. In particular, the results are stable for δ between 10% and 25%. In the sequel, we set δ at the higher hand, i.e. $\delta = 25\%$, because too low values make it difficult to distinguish between the different portfolios as they are too close to the equally-weighted one (corresponding to $\delta = 0$). The conclusions of the empirical study remain the same for $\delta = 20\%$ and $\delta = 15\%$, though naturally less strikingly.

5.1.7. Performance measures

We measure the out-of-sample portfolio performance and stability with three criteria:

1. The Sharpe ratio, defined as

$$SR := (\mathbb{E}(P) - r_f)/\sqrt{\mathbb{V}\text{ar}(P)} \tag{23}$$

and estimated using sample estimators, which is the most common performance measure used in the asset allocation literature. For simplicity, we assume that $r_f = 0$, as in e.g. DeMiguel et al. (2009b), i.e. we report the reciprocal of the coefficient of variation. However, given that the appeal of the MRE portfolio compared to the minimum-variance portfolio is to account for higher-order uncertainty, using the Sharpe ratio alone is not sufficient to assess the merit of our portfolio policy. Thus, the next two performance criteria also account for higher-order moments.

2. The adjusted Sharpe ratio of Pézier (2004), defined as

$$ASR := SR\bigg(1 + \frac{\mathbb{S}\text{kew}(P)}{3!}SR - \frac{\mathbb{K}\text{urt}(P)}{4!}SR^2\bigg), \tag{24}$$

that we estimate using sample moment estimators.

3. The modified Value-at-Risk introduced by Favre and Galeano (2002), which approximates the Value-at-Risk via the first four moments via a second-order Cornish-Fisher expansion. It is defined as

$$\begin{aligned} MVaR := &- \mathbb{E}(P) - \sqrt{\mathbb{V}\text{ar}(P)}\Big[q_\varepsilon + \frac{1}{6}(q_\varepsilon^2 - 1)\mathbb{S}\text{kew}(P) + \frac{1}{24}(q_\varepsilon^3 - 3q_\varepsilon)\mathbb{K}\text{urt}(P) \\ &- \frac{1}{36}(2q_\varepsilon^3 - 5q_\varepsilon)\mathbb{S}\text{kew}(P)^2\Big], \end{aligned} \tag{25}$$

where q_ε is the quantile at confidence level ε of the standard Gaussian. We fix $\varepsilon = 10\%$. It is estimated using sample moment estimators.

4. To assess the stability and associated transaction costs of the portfolios, we report the turnover, defined as

$$\text{Turnover} := \frac{1}{R-1}\sum_{t=1}^{R}\sum_{i=1}^{n}|w_{i,t+1} - w_{i,t^+}|, \tag{26}$$

where $R = 43$ is the number of rebalancing periods, $w_{i,t+1}$ is the desired weight of asset i at time $t+1$ and w_{i,t^+} is its weight before rebalancing at $t+1$.

The criteria are expressed on an annual basis, except for the $MVaR$ that is expressed on a monthly basis.

5.2. Out-of-sample results

The results are reported in Table 2. Several interesting observations can be made.

First, comparing the six MRE portfolios, one can clearly observe that $\alpha = 1.5$ and $\alpha = 2$ yield by far the worst performance. This is consistent with the Gram-Charlier expansion in (9), which shows that setting $\alpha > 1$ favors solutions with more kurtosis because as $k_1(\alpha) < 0$ when $\alpha > 1$. Therefore, both from a theoretical and empirical perspective, it is recommended to set $\alpha \leqslant 1$, as argued in Section 2.4.

Second, the turnover of the MRE portfolio systematically increases with α. It does not increase too much from $\alpha = 0.3$ to $\alpha = 0.7$ but then quickly increases dramatically. This effect can be explained by the fact that the convexity of $H_\alpha^{\exp}(P)$ (as a function of w) decreases as α increases. This is for example visible in Figure 3. This makes that the minimum is more bound to change largely from one rolling window to another because the objective function is flatter around the minimum the higher the value of α.

Third, it is appealing to observe that, for $\alpha \in \{0.3, 0.5, 0.7\}$, the performance of the MRE portfolio is very stable with respect to α. This is easily observed by looking at the average SR, ASR and $MVaR$ across the six datasets. This parameter robustness is an appealing behaviour for the decision-maker. For $\alpha = 1$, the SR and MSR remain very similar, but the $MVaR$ is quite substantially worse. Combined with the fact that the turnover increases with α, this means that choosing α quite low, in this case around $\alpha = 0.3$, yields the best trade-off between risk, return and turnover.[6]

Fourth, let us compare the MRE and MV portfolios, focusing only on $\alpha \in \{0.3, 0.5, 0.7\}$ following the above observations. One can definitely observe that the MRE portfolios improve upon the MV portfolios in terms of SR, ASR and $MVaR$. Indeed, averaging across the six datasets, each MRE portfolio displays a better SR, ASR and $MVaR$ than *all* the MV portfolios. In terms of turnover however, all the MRE portfolios display less stability than the MV portfolios. This is to be expected because the MRE portfolio is sensitive to the higher-order moments, which are more affected by outliers than the variance. That said, for $\alpha = 0.3$ and $\alpha = 0.5$, the increase in turnover remains quite modest (around four percentage points on average for $\alpha = 0.3$).

Therefore, we conclude that Rényi entropy provides a superior risk criterion than the variance in an asset-allocation context, especially for low values of α, specifically around $\alpha = 0.3$ for the datasets considered here.

[6] For completeness, we have checked the results for $\alpha \in \{0.05, 0.1, 0.2\}$ as well. The turnover barely decreases compared to $\alpha = 0.3$, and the performance measures remain nearly identical.

Table 2 Out-of-sample performance of minimum-variance and minimum Rényi entropy portfolios

	Sharpe ratio (SR)										
	MRE portfolios						*MV portfolios*				
	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\widehat{\Sigma}$	$\widehat{\Sigma}_{CC}$	$\widehat{\Sigma}_{SF}$	$\widehat{\Sigma}_{I}$	MP
6BTM	0.844	0.845	0.846	0.841	0.832	0.815	0.835	0.821	0.833	0.836	0.841
25BTM	0.985	0.980	0.973	0.961	0.937	0.906	0.953	0.913	0.951	0.957	0.956
6Mom	0.768	0.763	0.753	0.750	0.716	0.700	0.738	0.741	0.738	0.737	0.731
25Mom	0.936	0.948	0.957	0.960	0.954	0.928	0.902	0.920	0.904	0.903	0.916
10Ind	0.995	1.001	1.014	1.013	0.971	0.936	0.977	0.970	0.983	0.973	0.976
17Ind	0.938	0.947	0.936	0.965	0.905	0.891	0.936	0.924	0.935	0.935	0.918
Average	**0.911**	**0.914**	**0.913**	**0.915**	**0.886**	**0.863**	**0.890**	**0.882**	**0.891**	**0.890**	**0.890**
	Adjusted Sharpe ratio (ASR)										
	MRE portfolios						*MV portfolios*				
	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\widehat{\Sigma}$	$\widehat{\Sigma}_{CC}$	$\widehat{\Sigma}_{SF}$	$\widehat{\Sigma}_{I}$	MP
6BTM	0.829	0.830	0.831	0.826	0.816	0.800	0.822	0.809	0.821	0.824	0.826
25BTM	0.969	0.963	0.956	0.943	0.919	0.889	0.940	0.906	0.939	0.944	0.940
6Mom	0.759	0.753	0.744	0.741	0.708	0.694	0.730	0.733	0.729	0.729	0.723
25Mom	0.921	0.934	0.943	0.947	0.943	0.918	0.889	0.909	0.892	0.890	0.902
10Ind	0.990	0.995	1.005	1.004	0.965	0.927	0.973	0.966	0.979	0.968	0.972
17Ind	0.950	0.959	0.945	0.975	0.911	0.901	0.950	0.940	0.950	0.949	0.929
Average	**0.903**	**0.906**	**0.904**	**0.906**	**0.877**	**0.855**	**0.884**	**0.877**	**0.885**	**0.884**	**0.882**
	Modified Value-at-Risk (MVaR)										
	MRE portfolios						*MV portfolios*				
	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\widehat{\Sigma}$	$\widehat{\Sigma}_{CC}$	$\widehat{\Sigma}_{SF}$	$\widehat{\Sigma}_{I}$	MP
6BTM	3.71%	3.71%	3.75%	3.78%	3.91%	3.90%	3.73%	3.75%	3.73%	3.73%	3.74%
25BTM	3.12%	3.13%	3.17%	3.33%	3.49%	3.52%	3.23%	3.70%	3.19%	3.22%	3.17%
6Mom	3.97%	4.00%	4.07%	4.10%	4.09%	4.03%	3.94%	4.01%	3.95%	3.94%	4.11%
25Mom	3.37%	3.33%	3.23%	3.16%	3.35%	3.61%	3.47%	3.51%	3.43%	3.47%	3.38%
10Ind	3.16%	3.19%	3.09%	3.16%	3.28%	3.38%	3.18%	3.34%	3.19%	3.21%	3.10%
17Ind	2.98%	2.95%	3.12%	3.34%	3.25%	3.26%	2.99%	3.15%	3.04%	3.11%	3.10%
Average	**3.39%**	**3.39%**	**3.41%**	**3.48%**	**3.56%**	**3.62%**	**3.42%**	**3.58%**	**3.42%**	**3.45%**	**3.43%**
	Turnover										
	MRE portfolios						*MV portfolios*				
	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\widehat{\Sigma}$	$\widehat{\Sigma}_{CC}$	$\widehat{\Sigma}_{SF}$	$\widehat{\Sigma}_{I}$	MP
6BTM	0.184	0.182	0.183	0.189	0.218	0.266	0.171	0.167	0.170	0.168	0.168
25BTM	0.499	0.535	0.616	0.966	1.356	1.502	0.448	0.464	0.443	0.445	0.478
6Mom	0.167	0.171	0.180	0.191	0.247	0.284	0.168	0.168	0.167	0.168	0.178
25Mom	0.437	0.444	0.523	0.635	0.860	1.200	0.417	0.413	0.410	0.416	0.422
10Ind	0.348	0.350	0.378	0.450	0.600	0.776	0.283	0.276	0.274	0.286	0.321
17Ind	0.526	0.568	0.685	0.825	1.033	1.246	0.449	0.422	0.433	0.452	0.467
Average	**0.360**	**0.375**	**0.427**	**0.542**	**0.719**	**0.879**	**0.323**	**0.318**	**0.316**	**0.322**	**0.339**

Notes. The table reports the Sharpe ratio, adjusted Sharpe ratio, modified Value-at-Risk and turnover of the MRE and MV portfolios on the six datasets. The portfolios are constructed with the methodology detailed in Section 5.1.

5.3. Extension to the risk-parity strategy

As a final test of the appeal of Rényi entropy in portfolio selection, we apply it to the risk-parity strategy, which is an alternative risk-based strategy to the minimum-risk strategy considered in this paper. The risk-parity portfolio has been rigorously studied for the first time by Maillard et al. (2010), followed by many other studies. The underlying rationale of risk parity is to find the portfolio for which all the assets in the portfolio contribute equally to the portfolio-return risk.

To implement this portfolio, we need to compute the asset risk contributions to the total portfolio-return risk. For positive-homogeneous risk measures, it can be computed with the Euler risk contribution, which defines the risk contribution of the ith asset as w_i times the partial derivative of the portfolio-return risk with respect to w_i. However, without distributional assumptions, this partial derivative can not be computed analytically for the exponential Rényi entropy. Thus, we rely on an alternative more tractable approach proposed by Tasche (2008) called the *marginal risk contribution*. It is defined by starting from the following risk contribution:

$$H_\alpha^{\exp}(X_i|P) := H_\alpha^{\exp}(P) - H_\alpha^{\exp}(P - w_i X_i). \tag{27}$$

As shown by Tasche (2008), these risk contributions add up to less than the total portfolio-return risk for continuously differentiable, sub-additive and positive homogeneous risk measures. To circumvent this problem, Tasche (2008) proposes to normalize them, leading to the marginal contribution of the ith asset to $H_\alpha^{\exp}(P)$ defined as

$$MH_\alpha^{\exp}(X_i|P) := \frac{H_\alpha^{\exp}(X_i|P)}{\sum_{j=1}^n H_\alpha^{\exp}(X_j|P)}, \tag{28}$$

which fulfills

$$\sum_{i=1}^n MH_\alpha^{\exp}(X_i|P) = 1. \tag{29}$$

Then, the *Rényi entropy parity* (REP) portfolio is defined as the solution to

$$\min_{w\in\mathcal{W}} \sum_{i=1}^n \left(MH_\alpha^{\exp}(X_i|P) - 1/n\right)^2. \tag{30}$$

We report in Table 3 the out-of-sample performance of the REP portfolio with the same methodology than in Section 5.1. We have tested for values of $\alpha \in \{0.3, 0.5, 0.7, 1, 1.5, 2\}$, and observe that the performance criteria remain very stable with the choice of α but, as for the MRE portfolio, that the turnover systematically increases with α. Thus, for the sake of conciseness, we report the results for $\alpha = 0.3$ only. As shown by Maillard (2010), the risk-parity portfolio achieves a trade-off between the equally weighted (EW) portfolio and the minimum-risk portfolio in terms of risk. Thus,

Table 3 Out-of-sample performance of equally weighted and Rényi entropy parity portfolios

	6BTM	*25BTM*	*6Mom*	*25Mom*	*10Ind*	*17Ind*	***Average***
Sharpe ratio (SR)							
REP portfolio ($\alpha = 0.3$)	0.725	0.730	0.657	0.682	0.799	0.851	**0.741**
EW portfolio	0.708	0.708	0.637	0.654	0.749	0.803	**0.710**
Adjusted Sharpe ratio (ASR)							
REP portfolio ($\alpha = 0.3$)	0.713	0.717	0.649	0.674	0.788	0.848	**0.731**
EW portfolio	0.696	0.696	0.631	0.648	0.739	0.800	**0.702**
Modified Value-at-Risk (MVaR)							
REP portfolio ($\alpha = 0.3$)	4.26%	4.45%	4.61%	4.75%	3.57%	3.76%	**4.23%**
EW portfolio	4.43%	4.61%	4.77%	4.97%	3.89%	4.00%	**4.45%**
Turnover							
REP portfolio ($\alpha = 0.3$)	5.88%	6.89%	7.03%	7.58%	8.86%	10.33%	**7.76%**
EW portfolio	6.02%	6.67%	6.99%	7.63%	8.56%	9.68%	**7.59%**

Notes: The table reports the Sharpe ratio, adjusted Sharpe ratio, modified Value-at-Risk and turnover of the EW and REP portfolios on the six datasets. The portfolios are constructed with the methodology presented in Sections 5.1 and 5.3.

in Table 3, we also compare the performance of the REP portfolio with that of the EW portfolio.

We make two main conclusions. First, the REP portfolio clearly outperforms the EW portfolio on all three performance criteria, while being subject to a turnover that is only slightly larger (7.76% versus 7.59% on average over the six datasets). Second, comparing the results in Tables 2 and 3, one can observe that the appeal of the REP portfolio compared to the MRE strategy is not in terms of performance: the REP portfolio is outperformed on all three performance criteria. Instead, the appeal of the REP portfolio is in terms of turnover, which is very substantially reduced: for $\alpha = 0.3$, the average turnover over the six datasets is 36.02% for the MRE portfolio, versus 7.76% for the REP portfolio. The greater stability of the REP portfolio strategy makes that it may be preferred by investors facing non-negligible transaction costs.

6. Conclusion

Many studies from the wide scientific literature suggest that minimum-risk portfolios exhibit solid out-of-sample performances in spite of the fact that there is no portfolio-mean-return constraint. Whereas variance—initially introduced by Markowitz—is a natural risk measure in a Gaussian framework, it fails to capture extreme events that arguably arise in real applications. In order to take this reality into account, various alternative risk measures have been put forward.

In this paper, we have proposed a natural uncertainty measure—the exponential Rényi entropy—as a higher-moment criterion for portfolio selection. Rényi entropy generalizes Shannon entropy, yielding a set of uncertainty measures. Its parameter α enables to tune the relative contributions of the central and tail parts of the distribution in the measure. Its exponential transform fulfills desirable properties as it is closely related to the class of deviation risk measures,

as well as to measures of spread for $\alpha \in [0, 1]$.

Minimizing this measure yields the minimum Rényi entropy portfolio. A truncated Gram-Charlier expansion shows that this portfolio represents a higher-moment extension of the minimum-variance portfolio, with α controlling the trade-off between variance and kurtosis minimization.

In practical settings, the empirical study has demonstrated that the minimum Rényi entropy portfolio fares better out of sample compared to state-of-the-art robust minimum-variance portfolios in terms of trading off risk, return and turnover, especially for values of α close to zero. We have also exemplified how the exponential Rényi entropy can be used for other portfolio-selection strategies such as the popular risk-parity approach.

In this paper, we have concentrated on the single-period portfolio-selection problem. One important direction for future research is to extend the minimum Rényi entropy portfolio to the multiperiod setting. To that end, using the mathematical formulation proposed in Li and Ng (2010), one can consider an investor facing a finite number of future time periods during which he can invest, and who looks for the investment in the risky assets at each of the future time periods so as to minimize the exponential Rényi entropy of his terminal portfolio wealth. This portfolio would extend the multiperiod minimum-variance portfolio studied by Li and Ng (2010) and Pun (2018), among others. Without specific distributional assumptions, this problem is analytically intractable, just like for the single-period case considered in this paper. However, using our m-spacings estimator, it can be readily be solved using standard stochastic programming models, such as backward induction.

Beyond our application, this paper demonstrates the potential appeal of Rényi entropy in various operations-research problems as a powerful way of capturing higher-moment uncertainty, and of using entropy as an optimization criterion rather than simply an ad hoc evaluation measure. In the particular case of portfolio selection, Rényi entropy has been shown to be a powerful alternative to the variance as risk criterion, opening the door to other applications in higher-moment portfolio selection.

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References

Abbas, A. (2006). Maximum entropy utility. *Operations Research*, 54(2), 277–290.

Adcock, C. (2014). Mean–variance–skewness efficient surfaces, Stein's lemma and the multivariate extended skew-student distribution. *European Journal of Operational Research*, 234(2), 392–401.

Ardia, D., Bolliger, G., Boudt, K., & Gagnon-Fleury, J. (2017). The impact of covariance misspecification in risk-based portfolios. *Annals of Operations Research*, 254(1-2), 1–16.

Artzner, P., Delbaen, F., Eber, J., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3), 203–228.

Behr, P., Guettler, A., & Miebs, F. (2013). On portfolio optimization: Imposing the right constraints. *Journal of Banking and Finance*, 37, 1232–1242.

Beirlant, J., Dudewicz, E., Gyofi, L., & van der Meulen, E. (1997). Non-parametric entropy estimation: An overview. *International Journal of Mathematical and Statistical Sciences*, 6(1), 17–39.

Bera, A., & Park, S. (2008). Optimal portfolio diversification using the maximum entropy principle. *Econometric Reviews*, 27(4–6), 484–512.

Boudt, K., Peterson, B., & Croux, C. (2008). Estimation and decomposition of downside risk for portfolios with non-normal returns. *Journal of Risk*, 11, 79–103.

Campbell, L. (1966). Exponential entropy as a measure of extent of a distribution, *Z. Wahrsch.*, 5, 217–225.

Carroll, R., Conlon, T., Cotter, J., & Salvador, E. (2017). Asset allocation with correlation: A composite trade-off. *European Journal of Operational Research*, 262(3), 1164–1180.

Chen, L., He, S., & Zhang, S. (2011). When all risk-adjusted performance measures are the same: In praise of the Sharpe ratio. *Quantitative Finance*, 11(10), 1439–1447.

Cont, R. (2001). Empirical properties of asset returns: Stylized facts and statistical issues. *Quantitative Finance*, 1, 223–236.

Cover, T., & Thomas, J. (2006). *Elements of information theory (2nd ed.)*. New Jersey: Wiley.

Daníelsson, J., Jorgensenb, B., Samorodnitskyc, G., Sarma, M., & de Vries, C. (2013). Fat tails, VaR and subadditivity. *Journal of Econometrics*, 172, 283–291.

DeMiguel, V., Garlappi, L., & Uppal, R. (2009a). Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? *The Review of Financial Studies*, 22(5), 1915–1953.

DeMiguel, V., Garlappi, L., Nogales, F., & Uppal, R. (2009b). A generalized approach to portfolio optimization: Improving performance by constraining portfolio norms. *Management Science*, 55(5), 798–812.

DeMiguel, V., & Nogales, F. (2009). Portfolio selection with robust estimation. *Operations Research*, 57, 560–577.

Dionisio, A., Menezes, R., & Mendes, A. (2006). An econophysics approach to analyse uncertainty in financial markets: An application to the Portuguese stock market. *The European Physical Journal B*, 50, 161–164.

van Es, B. (1992). Estimating functionals related to a density by a class of statistics based on spacings. *Scandinavian Journal of Statistics*, 19(1), 61–72.

Fabozzi, F., Huang, D., & Zhou, G. (2010). Robust portfolios: Contributions from operations research and finance. *Annals of Operations Research*, 176(1), 191–220.

Favre, L., & Galeano, J. (2002) Mean-modified value-at-risk optimization with hedge funds. *Journal of Alternative Investments*, 5(2), 21–25.

Flores, Y., Bianchi, R., Drew, M., & Trück, S. (2017). The diversification delta: A different perspective. *Journal of Portfolio Management*, 43(4), 112–124.

Hampel, F., Ronchetti, E., Rousseeuw, P., & Stahel, W. (1986). *Robust Statistics: The Approach Based on Influence Functions*. New York: Wiley.

Harvey, C., Liechty, J., Liechty, M., & Müller, P. (2010). Portfolio selection with higher moments. *Quantitative Finance*, 10(5), 469–485.

Hegde, A., Lan, T., & Erdogmus, D. (2005). Order statistics based estimator for Rényi entropy. *IEEE Workshop on Machine Learning for Signal Processing*, 335–339.

Hyvärinen, A., Karhunen, J., & Oja, E. (2001). *Independent Component Analysis*. New York: Wiley.

Johnson, O., & Vignat, C. (2007). Some results concerning maximum Rényi entropy distributions. *Annales de l'Institut Henri-Poincaré (B) Probab. Statist.*, 43(3), 339–351.

Jorion, P. 1986. Bayes-Stein estimation for portfolio analysis. *Journal of Financial and Quantitative Analysis*, 21(3), 279–292.

Jose, V., Nau, R., & Winkler, R. (2008). Scoring rules, generalized entropy, and utility maximization. *Operations Research*, 56(5), 1146–1157.

Jurczenko, E., & Maillet, B. (2006). *Multi-Moment Asset Allocation and Pricing Models*. West Sussex: Wiley.

Kolm, P., Tütüncü, R., & Fabozzi, F. (2014). 60 years of portfolio optimization: Practical challenges and current trends. *European Journal of Operational Research*, 234(2), 356–371.

Koski, T., & Persson, L. (1992). Some properties of generalized exponential entropies with application to data compression. *Information Sciences*, 62, 103–132.

Ledoit, O., & Wolf, M. (2003). Improved estimation of the covariance matrix of stock returns with an application to portfolio selection. *Journal of Empirical Finance*, 10, 603–621.

Ledoit, O., & Wolf, M. (2004a). A well-conditioned estimator for large-dimensional covariance matrices. *Journal of Multivariate Analysis*, 88, 365–411.

Ledoit, O., & Wolf, M. (2004b). Honey, I shrunk the sample covariance matrix. *Journal of Portfolio*

Management, 30, 110–119.

Levy, H., & Levy, M. (2014). The benefits of differential variance-based constraints in portfolio optimization. *European Journal of Operational Research*, 234, 372–381.

Li, D., & Ng., W. (2000). Optimal dynamic portfolio selection: Multiperiod mean-variance formulation. *Mathematical Finance*, 10(3), 387–406.

Maillard, S., Roncalli, T., & Teiletche, J. (2010). On the properties of equally-weighted risk contributions portfolios. *Journal of Portfolio Management*, 36(4), 60–70.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7(1), 77–91.

Martellini, L., & Ziemann, V. (2010). Improved estimates of higher-order comoments and implications for portfolio selection. *Review of Financial Studies*, 23(4), 1467–1502.

Learned-Miller, E., & Fisher, J. (2003). ICA using spacings estimates of entropy. *Journal of Machine Learning Research*, 4, 1271–1295.

Ormos, M., & Zibriczky D. (2014). Entropy-based financial asset pricing. *Plos One*, 9(12).

Pézier, J. 2004. Risk and risk aversion. In Alexander, C., & Sheedy, E. *The Professional Risk Managers' Handbook*. Wilmington DE: PRMIA Publications.

Pham, D., Vrins, F., & Verleysen, M. (2008). On the risk of using Rényi's entropy for blind source separation. *IEEE Transactions on Signal Processing*, 56(10), 4611–4620.

Philippatos, G., & Wilson, C. (1972). Entropy, market risk, and the selection of efficient portfolios. *Applied Economics*, 4(3), 209–220.

Pun, C. S. (2018). Time-consistent mean-variance portfolio selection with only risky assets. *Economic Modelling*, 75, 281–292.

Qi, Y., Steuer, R., & Wimmer, M. (2017). An analytical derivation of the efficient surface in portfolio selection with three criteria. *Annals of Operations Research*, 251(1), 161–177.

Rényi, A. (1961). On measures of entropy and information. *Fourth Berkeley Symposium on Mathematical Statistics and Probability*, 547–561.

Rockafellar, R., Uryasev, S., & Zabarankin, M. (2006). Generalized deviations in risk analysis. *Finance and Stochastics*, 10, 51–74.

Sbuelz, A., & Trojani, F. (2008). Asset prices with locally constrained-entropy recursive multiple-priors utility. *Journal of Economic Dynamics and Control*, 32(11), 3695–3717.

Scutellà, M., & Recchia, R. (2013). Robust portfolio asset allocation and risk measures. *Annals of Operations Research*, 204(1), 145–169.

Shannon, C. (1948). A mathematical theory of communication. *Bell Systems Technical Journal*, 27, 379–423 and 623–656.

Tasche, D. (2008). Capital allocation to business units and sub-portfolios: the Euler principle. In Resti, A. *Pillar II in the New Basel Accord: The Challenge of Economic Capital*. Risk Books,

423–453.

Ugray, Z., Lasdon, L., Plummer, J., Glover, F., Kelly, J., & Martí, R. (2007). Scatter search and local NLP solvers: A multistart framework for global optimization. *INFORMS Journal on Computing*, 19(3), 328–340.

Vanduffel, S., & Yao, J. (2017). A stein type lemma for the multivariate generalized hyperbolic distribution. *European Journal of Operational Research*, 261(2), 606–612.

Vasicek, O. (1976). A test for normality based on entropy. *Journal of the Royal Statistical Society Series B (Methodological)*, 38(1), 54–59.

Vermorken, M., Medda, F., & Schroder, T. (2012). The diversification delta: A higher-moment measure for portfolio diversification. *Journal of Portfolio Management*, 39(1), 67–74.

Vrins, F., Pham, D., & Verleysen, M. (2007). Mixing and non-mixing local minima of the entropy contrast for blind source separation. *IEEE Transactions on Information Theory*, 53(3), 1030–1042.

Wachowiak, M., Smolikova, R., Tourassi, G., & Elmaghraby, A. (2005). Estimation of generalized entropies with sample spacing. *Pattern Analysis and Applications*, 8, 95–101.

Yang, J., & Qiu, W. (2005). A measure of risk and a decision-making model based on expected utility and entropy. *European Journal of Operational Research*, 164(3), 792–799.

Zhou, R., Cai, R., & Tong, G. (2013). Applications of entropy in finance: A review. *Entropy*, 15, 4909–4931.

Zografos, K., & Nadarajah, S. (2003). Formulas for Rényi information and related measures for univariate distributions. *Information Sciences*, 155(1-2), 119–138.

Zografos, K., & Nadarajah, S. (2005). Expressions for Rényi and Shannon entropies for multivariate distributions. *Statistics & Probability Letters*, 71(1), 71–84.

Supplementary Materials to "Minimum Rényi Entropy Portfolios" Forthcoming in *Annals of Operations Research*

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We report here additional materials to the main paper. First, the three counter-examples to sub-additivity that prove Proposition 2. Second, the proofs of Propositions 3, 6 and 8. Third, the derivation of the m-spacings estimator in Proposition 7.

1. Counter-examples to sub-additivity of $H_\alpha^{\exp}$

In this section, we report the three counter-examples to sub-additivity as mentioned in Proposition 2 of the paper. The counter-examples are provided for the special case $\alpha = 1$.

1.1. Lévy distributions

Proposition *$H_1^{\exp}$ is not sub-additive for a pair (X, Y) of independent Lévy-distributed random variables.*

Proof. The pdf of $Z \sim \text{Lévy}(\mu, \sigma)$ is given by $f_Z(x) = \sqrt{\frac{\sigma}{2\pi}} \frac{e^{-\frac{\sigma}{2(x-\mu)}}}{(x-\mu)^{3/2}}$ and is strictly positive for $x > \mu$. Its exponential entropy is known in closed-form (Zografos and Nadarajah 2003): $H_1^{\exp}(Z) = 4\sigma\sqrt{\pi} e^{\frac{1+2\gamma}{2}}$, where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. We note the parameters of X, Y as (μ_X, σ_X) and (μ_Y, σ_Y), with $\sigma_X, \sigma_Y > 0$. As Lévy is a stable law, the sum $X + Y$ is again Lévy-distributed with parameters $\mu_{X+Y} = \mu_X + \mu_Y$ and $\sigma_{X+Y} = \sigma_X + \sigma_Y + 2\sqrt{\sigma_X \sigma_Y}$. Sub-additivity is thus equivalent to $\sigma_{X+Y} \leqslant \sigma_X + \sigma_Y \Leftrightarrow 2\sqrt{\sigma_X}\sqrt{\sigma_Y} \leqslant 0$, which never holds when $\sigma_X, \sigma_Y > 0$. □

1.2. Bimodal distributions

Proposition *Consider (Z_X, Z_Y) a pair of independent standard Normal variables and (U_X, U_Y) a pair of independent Bernoulli variables of parameter $1/2$, independent from both Z_Y, Z_Y. Define $X := (2\mu_X U_X - \mu_X) + \sigma Z_X, Y := (2\mu_Y U_Y - \mu_Y) + \sigma Z_Y$ with constants μ_X, μ_Y and $\sigma > 0$. Then, $H_1^{\exp}$ is not sub-additive for the pair (X, Y) whenever e.g. $(\mu_X, \mu_Y) = (1, 2)$ and $\sigma < 0.3918$.*

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Proof. Noting $\phi(x)$ the standard Gaussian density, the marginal densities of X, Y are given by the Gaussian mixtures

$$\begin{aligned} f_X(x) &= \frac{1}{2\sigma}\Big(\phi\Big(\frac{x+\mu_X}{\sigma}\Big) + \phi\Big(\frac{x-\mu_X}{\sigma}\Big)\Big), \\ f_Y(x) &= \frac{1}{2\sigma}\Big(\phi\Big(\frac{x+\mu_Y}{\sigma}\Big) + \phi\Big(\frac{x-\mu_Y}{\sigma}\Big)\Big). \end{aligned} \tag{1}$$

It is easy to show (Pham and Vrins 2005) that the density of $X+Y$ is

$$f_{X+Y}(x) = \frac{1}{4\sigma\sqrt{2}}\Big(\phi\Big(\frac{x+\mu_X+\mu_Y}{\sigma\sqrt{2}}\Big)+\phi\Big(\frac{x+\mu_X-\mu_Y}{\sigma\sqrt{2}}\Big)+\phi\Big(\frac{x-\mu_X+\mu_Y}{\sigma\sqrt{2}}\Big)+\phi\Big(\frac{x-\mu_X-\mu_Y}{\sigma\sqrt{2}}\Big)\Big). \tag{2}$$

Vrins et al. (2007) show that, for a random variable Z whose density can be written in the form $f_Z(x) = \sum_{n=1}^N \pi_n K_n(x)$ with positive weights π_n summing to 1 and Gaussian kernels $K_n(x) = \frac{1}{\sigma_n}\phi\big(\frac{x-\mu_n}{\sigma_n}\big)$, then $H_1(Z)$ can be bounded below and above. More explicitly,

$$\underline{H}_1(Z) \leqslant H_1(Z) \leqslant \overline{H}_1(Z), \tag{3}$$

with

$$\begin{aligned} \overline{H}_1(Z) &:= \sum_{n=1}^N \pi_n H_1[K_n] + h(\boldsymbol{\pi}), \\ \underline{H}_1(Z) &:= \overline{H}_1(Z) - \sum_{n=1}^N \pi_n \epsilon'_n - \sum_{n=1}^N \pi_n \left[\ln\left(\frac{s}{\pi_n s_n}\right) + 1\right]\epsilon_n, \end{aligned} \tag{4}$$

where $h(\boldsymbol{\pi}) := -\sum_{n=1}^N \pi_n \ln \pi_n$, $s_n := \max_x K_n(x) = \big(\sqrt{2\pi}\sigma_n\big)^{-1}$ and $s := \max_n s_n$. Rearranging the μ_n by increasing order and defining $d_n := \min(\mu_n - \mu_{n-1}, \mu_{n+1} - \mu_n)$ with $\mu_0 := -\infty$, $\mu_{N+1} := \infty$ by convention, we have

$$\epsilon_n := \mathrm{Erfc}\left(\frac{d_n}{2\sqrt{2}\sigma_n}\right), \quad \epsilon'_n := \frac{1}{2}\epsilon_n + \frac{d_n}{2\sqrt{2\pi}\sigma_n} e^{-\left(\frac{d_n}{2\sqrt{2}\sigma_n}\right)^2}, \tag{5}$$

with the complementary error function $\mathrm{Erfc}(x) := 2\Phi(-x\sqrt{2})$ and Φ the standard Gaussian cdf.

Using these lower and upper bounds, the $H_1^{\exp}$ operator fails to be sub-additive for the pair (X, Y) if

$$\exp(\underline{H}_1(X+Y)) > \exp(\overline{H}_1(X)) + \exp(\overline{H}_1(Y)). \tag{6}$$

Indeed, the LHS is a lower-bound to $H_1^{\exp}(X+Y)$ while the RHS is an upper-bound to $H_1^{\exp}(X) + H_1^{\exp}(Y)$. Setting $(\mu_X, \mu_Y) = (\mu, 2\mu)$, the bounds in (4) applied to the densities in (1)-(2) read as

$$\begin{aligned} \overline{H}_1(X) = \overline{H}_1(Y) &= \ln\big(2\sigma\sqrt{2\pi e}\big), \\ \underline{H}_1(X+Y) &= \ln\big(8\sigma\sqrt{\pi e}\big) - \mathrm{Erfc}(\mu/2\sigma)(3/2 + \ln(4)) \\ &\quad - \frac{\mu}{2\sigma\sqrt{\pi}} e^{-(\mu/2\sigma)^2}, \end{aligned} \tag{7}$$

from which we find for example that, setting $\mu = 1$, (6) holds as long as $0 < \sigma < 0.3918$. □

1.3. Comonotonic random variables

Proposition *$H_1^{\exp}$ is not sub-additive for a comonotonic pair (X, Y), with $Y = F(X)$, $F'(X) \sim Exp(1)$.*

Proof. Two random variables X, Y are comonotonic when Y can be written as $F(X)$ where F is a continuous strictly increasing function. Denote $G(x) := x + F(x)$, which is also strictly increasing and so invertible, and denote $H(x)$ its inverse. Then, the cdf of $X + Y = G(X)$ is given by

$$F_{X+Y}(x) = \mathbb{P}(G(X) \leqslant x) = \mathbb{P}(X \leqslant H(x)) = F_X(H(x)),$$

and its pdf reads

$$f_{X+Y}(x) = \frac{f_X(H(x))}{G'(H(x))}.$$

As a result, $H_1^{\exp}(X + Y)$ becomes

$$H_1^{\exp}(X+Y) = \exp\left(-\int \frac{f_X(H(x))}{G'(H(x))} \ln \frac{f_X(H(x))}{G'(H(x))}\, dx \right).$$

A change of variable $z = H(x)$ and algebraic manipulations lead to

$$H_1^{\exp}(X+Y) = H_1^{\exp}(X) \exp\big(\mathbb{E}\big(\ln(1 + F'(X))\big)\big).$$

Based on a similar reasoning, we can show that

$$H_1^{\exp}(Y) = H_1^{\exp}(X) \exp\big(\mathbb{E}\big(\ln F'(X)\big)\big),$$

meaning that sub-additivity amounts to showing that

$$\exp\big(\mathbb{E}\big(\ln(1 + F'(X))\big)\big) \leqslant 1 + \exp\big(\mathbb{E}\big(\ln F'(X)\big)\big). \tag{8}$$

Let us now show instead a counter-example to (8) where the left-hand side is higher than the right-hand side, i.e. where $H_1^{\exp}$ is super-additive. Because $F'(X)$ has to be a positive random variable (F is strictly increasing), we consider $F'(X) := \zeta \sim Exp(1)$. Define $W := \ln(1 + \zeta)$ and $Z := \ln \zeta$. To have super-additivity, we have to show that

$$e^{\mathbb{E}(W)} > 1 + e^{\mathbb{E}(Z)}. \tag{9}$$

One can find the pdf of W and Z to be given by

$$f_W(x) = e^{1+x-e^x}\ ,\ x \geqslant 0, \tag{10}$$

$$f_Z(x) = e^{x-e^x} \ , \ x \in \mathbb{R}. \tag{11}$$

We can now compute the expectations. From (10), $\mathbb{E}(W)$ is given by the following integral:

$$\mathbb{E}(W) = e \int_0^\infty x e^{x-e^x} dx.$$

A change of variable $z = e^x$ and integration by parts yields

$$\mathbb{E}(W) = e\,\Gamma(0,1) \approx 0.596,$$

where $\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt$ is the upper incomplete Gamma function. A similar derivation yields $\mathbb{E}(Z) = -\gamma \approx -0.577$, i.e. minus the Euler-Mascheroni constant. Finally, we have in agreement with (9) that

$$e^{\mathbb{E}(W)} = e^{e\,\Gamma(0,1)} \approx 1.815 > 1.561 \approx 1 + e^{-\gamma} = 1 + e^{\mathbb{E}(Z)},$$

hence providing a counter-example to sub-additivity. □

2. Proofs of propositions

2.1. Proposition 3

Proof. We set $\mu = 0$ without loss of generality as $H_\alpha^{\exp}$ is translation-invariant. The proof relies on the convolution properties of elliptical distributions; see Fang and Zhang (1990). In particular, any linear combination of an elliptical random vector remains elliptical, meaning that we can write

$$X \sim \mathrm{El}(\sigma_X^2, g_1) \ \ , \ \ Y \sim \mathrm{El}(\sigma_Y^2, g_1) \ \ , \ \ X + Y \sim \mathrm{El}(e'\Sigma e, g_1),$$

where $e = (1,1)'$, $\Sigma = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$ and thus $e'\Sigma e = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y$. Moreover, any elliptical distribution X can be written as $X = \mu + AY$, where $AA' = \Sigma$ and Y is a spherical distribution, i.e. an elliptical distribution with $\Sigma = I$, the identity matrix. Applied to our case, this means that we can write $X = \sigma_X Z$, $Y = \sigma_Y Z$ and $X + Y = \sqrt{e'\Sigma e}Z$, with $Z \sim \mathrm{El}(1, g_1)$. Finally, the sub-additivity condition for $H_\alpha^{\exp}$ reduces to

$$\begin{aligned} & H_\alpha^{\exp}(X+Y) \leqslant H_\alpha^{\exp}(X) + H_\alpha^{\exp}(Y) \\ \Leftrightarrow\ & \sqrt{e'\Sigma e} H_\alpha^{\exp}(Z) \leqslant \sigma_X H_\alpha^{\exp}(Z) + \sigma_Y H_\alpha^{\exp}(Z) \\ \Leftrightarrow\ & \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y} \leqslant \sigma_X + \sigma_Y, \end{aligned}$$

which is satisfied for any $\rho \in [-1, 1]$. □

2.2. Proposition 6

Proof. The truncated Gram-Charlier (GC) expansion of the pdf of $\tilde{X}$ is given by

$$f_{\tilde{X}}(x) \approx \phi(x)\left(1 + \mathbb{S}\text{kew}(X)\frac{H_3(x)}{3!} + \mathbb{K}\text{urt}(X)\frac{H_4(x)}{4!}\right). \tag{12}$$

The proof relies on special properties of the Hermite polynomials H_i's. Those are defined in relation with the derivatives of the standard Gaussian pdf ϕ:

$$\frac{\partial^i \phi(x)}{\partial x^i} = (-1)^i H_i(x)\phi(x).$$

The first four polynomials are given by $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$ and $H_4(x) = x^4 - 6x^2 + 3$. They form an orthonormal system in the sense that

$$\int H_i(x)H_j(x)\phi(x)dx = \begin{cases} i! & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

To find the GC expansion of $H_\alpha(X)$, we want to have a similar system for the α-th power of ϕ. One can check that $\phi^\alpha(x) = k\phi(\sqrt{\alpha}x)$ with $k = (2\pi)^{\frac{1-\alpha}{2}}$ and that

$$\frac{\partial^i \phi(\sqrt{\alpha}x)}{\partial x^i} = (-1)^i \widetilde{H}_i(x)\phi(\sqrt{\alpha}x),$$

with $\widetilde{H}_1(x) = \alpha x$, $\widetilde{H}_2(x) = \alpha^2 x^2 - \alpha$, $\widetilde{H}_3(x) = \alpha^3 x^3 - 3\alpha^2 x$ and $\widetilde{H}_4(x) = \alpha^4 x^4 - 6\alpha^3 x^2 + 3\alpha^2$. Hence, in relation with the original polynomials H_i's, $\phi(\sqrt{\alpha}x)$ forms the system

$$\int H_i(x)H_j(x)\phi(\sqrt{\alpha}x)dx = \begin{cases} C_i(\alpha) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Following algebraic manipulations, the first four coefficients $C_i(\alpha)$'s express as

$$\begin{aligned} C_1(\alpha) &= \alpha^{-3/2}, \\ C_2(\alpha) &= \frac{(\alpha-2)\alpha + 3}{\alpha^{5/2}}, \\ C_3(\alpha) &= \frac{3(3(\alpha-2)\alpha + 5)}{\alpha^{7/2}}, \\ C_4(\alpha) &= \frac{3(3\alpha^4 - 12\alpha^3 + 42\alpha^2 - 60\alpha + 35)}{\alpha^{9/2}}. \end{aligned}$$

Let us now first derive the GC expansion of $I_\alpha(\tilde{X}) := \int (f_{\tilde{X}}(x))^\alpha dx$. Using the results above and the second-order Taylor expansion $(1+\varepsilon)^\alpha \approx 1 + \alpha\varepsilon + \frac{\alpha(\alpha-1)}{2}\varepsilon^2$, $I_\alpha(\tilde{X})$ approximates as

$$I_\alpha(\tilde{X}) \approx I\big[\mathcal{N}(0,1)\big] + \frac{3k\alpha(\alpha-1)^2}{4!\alpha^{5/2}}\mathbb{K}\text{urt}(X) + \frac{k\alpha(\alpha-1)C_3(\alpha)}{2\times 3!^2}\mathbb{S}\text{kew}(X)^2 + \frac{k\alpha(\alpha-1)C_4(\alpha)}{2\times 4!^2}\mathbb{K}\text{urt}(X)^2.$$

Note that there is no $\mathbb{S}\mathrm{kew}(X)$ term left because $\int \phi(\sqrt{\alpha}x)H_3(x)dx = 0$. Now, to finish, we need to get back to $H_\alpha(X) = \frac{1}{1-\alpha}\ln I_\alpha(X)$. We apply the Taylor expansion

$$\ln\big(I\big[\mathcal{N}(0,1)\big] + \varepsilon\big) \approx \ln I\big[\mathcal{N}(0,1)\big] + I\big[\mathcal{N}(0,1)\big]^{-1}\varepsilon,$$

where $I\big[\mathcal{N}(0,1)\big] = \sqrt{(2\pi)^{1-\alpha}/\alpha}$, which finally yields

$$H_\alpha(X) \approx\ H_\alpha\big[\mathcal{N}(0, \mathbb{V}\mathrm{ar}(X))\big] + k_1(\alpha)\mathbb{K}\mathrm{urt}(X) + k_2(\alpha)\mathbb{S}\mathrm{kew}(X)^2 + k_3(\alpha)\mathbb{K}\mathrm{urt}(X)^2,$$

where the functions $k_1(\alpha)$, $k_2(\alpha)$ and $k_3(\alpha)$ are given in the paper. □

2.3. Proposition 8

Proof. From Royston (1982), the density of $X^{(r:T)}$, the rth order statistics of X, writes as

$$f_{X^{(r:T)}}(x) = (1 - F_X(x))^{r-1}(F_X(x))^{T-r} f_X(x)\ . \tag{13}$$

As we have $F_X(x) = F_{\tilde{X}}\big(\frac{x-\mu}{\sigma}\big)$ and $f_X(x) = f_{\tilde{X}}\big(\frac{x-\mu}{\sigma}\big)/\sigma$, we find from (13) that the density of $X^{(r:T)}$ is given by

$$f_{X^{(r:T)}}(x) = \frac{1}{\sigma} f_{\tilde{X}^{(r:T)}}\Big(\frac{x-\mu}{\sigma}\Big),$$

meaning that

$$X^{(r:T)} \sim \mu + \sigma X_0^{(r:T)}. \tag{14}$$

Replacing (14) in the expression of $\widehat{H}_\alpha(X; m, T)$, we can write

$$\widehat{H}_\alpha(X; m, T) = \ln\sigma + \widehat{H}_\alpha(X_0; m, T). \tag{15}$$

Moreover, as $H_\alpha^{\exp}(X) = \sigma H_\alpha^{\exp}(\tilde{X})$, we have $\ln\sigma = H_\alpha(X) - H_\alpha(\tilde{X})$. Replacing this in (15) yields

$$\begin{aligned}
&\widehat{H}_\alpha(X; m, T) = H_\alpha(X) - H_\alpha(\tilde{X}) + \widehat{H}_\alpha(\tilde{X}; m, T)\\
\Leftrightarrow\ &\widehat{H}_\alpha(X; m, T) - H_\alpha(X) = \widehat{H}_\alpha(\tilde{X}; m, T) - H_\alpha(\tilde{X})\\
\Leftrightarrow\ &\mathbb{B}\big(\widehat{H}_\alpha(X; m, T)\big) = \mathbb{B}\big(\widehat{H}_\alpha(\tilde{X}; m, T)\big),
\end{aligned}$$

which completes the proof. □

3. Derivation of m-spacings estimator of $H_\alpha^{\exp}$

This section derives the m-spacings estimator of $H_\alpha^{\exp}$ of Proposition 7.

Consider i.i.d. copies $X^1, X^2, \ldots, X^T$ of a continuous random variable X. We denote $X^{(1:T)} \leqslant X^{(2:T)} \leqslant \cdots < X^{(T:T)}$ the corresponding order statistics and define the associated m-spacings ($1 \leqslant m < T$) as the sequence of non-negative differences $X^{(i+m:T)} - X^{(i:T)}$, for $1 \leqslant i \leqslant T - m$.

In a first step, we build a 1-spacing estimator of $H_\alpha^{\exp}(X)$ because the case $m = 1$ has a natural

relation to a sample-spacings estimator of the density f_X.

First, recall that the order statistics $Y^{(1:T)}, \ldots, Y^{(T:T)}$ of a uniform $\mathcal{U}(0,1)$ random variable Y follow a Beta distribution (Arnold et al. 1992). In particular,

$$\mathbb{E}\big(Y^{(i:T)}\big) = \frac{i}{T+1}.$$

Let us now map $X^1, \ldots, X^T$ through F_X to obtain T $\mathcal{U}(0,1)$ i.i.d. random variables $Y^i := F_X(X^i)$. Obviously, the sequence $F_X(X^{(1:T)}), \ldots, F_X(X^{(T:T)})$ agrees with the order statistics $Y^{(1:T)}, \ldots, Y^{(T:T)}$, leading to:

$$\mathbb{E}\big(Y^{(i:T)}\big) = \mathbb{E}\big(F_X\big(X^{(i:T)}\big)\big) = \mathbb{P}\big(X \leqslant X^{(i:T)}\big) = \frac{i}{T+1}.$$

Hence, the expected probability mass between two order statistics $X^{(i:T)} \leqslant X^{(i+1:T)}$ is

$$\mathbb{E}\left(\int_{X^{(i:T)}}^{X^{(i+1:T)}} f_X(x)dx\right) = \frac{1}{T+1}. \tag{16}$$

One can use this key observation to obtain an estimator $\widehat{f}_X$ of f_X being told T order statistics. Indeed, one can thus approximate $f_X(x)$ between two successive order statistics $X^{(i:T)}, X^{(i+1:T)}$ by a constant k_i such that the corresponding probability mass

$$\int_{X^{(i:T)}}^{X^{(i+1:T)}} \widehat{f}_X(x)dx = \int_{X^{(i:T)}}^{X^{(i+1:T)}} k_i dx = k_i\big(X^{(i+1:T)} - X^{(i:T)}\big)$$

agrees with the expected probability mass in (16). Denoting $X^{(0:T)} := \inf X$ and $X^{(T+1:T)} := \sup X$, this yields

$$k_i = \frac{1}{(T+1)(X^{(i+1:T)} - X^{(i:T)})}$$

for $X^{(i:T)} < x \leqslant X^{(i+1:T)}$. As the $T+1$ spacings form a partition of $\big[X^{(0:T)}, X^{(T+1:T)}\big]$, one can approximate the density f_X by

$$\widehat{f}_X(x) = \sum_{i=0}^{T} \mathbb{1}_{\{X^{(i:T)} < x \leqslant X^{(i+1:T)}\}}\, k_i. \tag{17}$$

This estimator corresponds to the histogram composed of $T+1$ bins with bounds $[X^{(i:T)}, X^{(i+1:T)}]$, $0 \leqslant i \leqslant T$, and with height such that the area of each bin is equal to $1/(T+1)$.

From this density estimator, one can derive a 1-spacing plug-in estimator of $H_\alpha^{\exp}$ as follows.

Proposition *Let the density f_X of a continuous random variable X be approximated by* (17)*, then*

the 1-spacing plug-in estimator of $H_\alpha^{\exp}(X)$ is given by

$$\left(\frac{1}{T+1}\sum_{i=0}^{T}\Big((T+1)\big(X^{(i+1:T)}-X^{(i:T)}\big)\Big)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}. \tag{18}$$

Proof. The 1-spacing estimator of $\int (f_X(x))^\alpha dx$ becomes

$$\begin{aligned}\int (\widehat{f}_X(x))^\alpha dx &= \sum_{i=0}^{T}\int_{X^{(i:T)}}^{X^{(i+1:T)}} (\widehat{f}_X(x))^\alpha dx \\ &= \sum_{i=0}^{T}\frac{1}{\big((T+1)(X^{(i+1)}-X^{(i)})\big)^\alpha}\int_{X^{(i:T)}}^{X^{(i+1:T)}} dx \\ &= \frac{1}{T+1}\sum_{i=0}^{T}\Big((T+1)\big(X^{(i+1:T)}-X^{(i:T)}\big)\Big)^{1-\alpha},\end{aligned}$$

resulting in (18). □

The estimator in (18) can not be used as such because, in general, we do not know $X^{(0:T)}$ and $X^{(T+1:T)}$, i.e. the true support of X. Following Learned-Miller and Fisher (2003), we therefore disregard the values below $X^{(1:T)}$ and above $X^{(T:T)}$, and compensate this by a factor $\frac{T+1}{T-1}$, yielding the final approximation

$$\widehat{H}_\alpha^{\exp}(1,T) := \left(\frac{1}{T-1}\sum_{i=1}^{T-1}\Big((T+1)\big(X^{(i+1:T)}-X^{(i:T)}\big)\Big)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}.$$

As detailed by Learned-Miller and Fisher (2003) in the specific case of Shannon entropy, the 1-spacing estimator suffers from high variance. To reduce the asymptotic variance, one can consider a m-spacings estimator where the m-spacings overlap. The counterpart of k_i becomes

$$k_i(m) := \frac{m}{(T+1)(X^{(i+m:T)}-X^{(i:T)})}$$

for $X^{(i:T)} < x \leqslant X^{(i+m:T)}$. However, because the m-spacings overlap, they do not form a partition of $\big[X^{(0:T)}, X^{(T+1:T)}\big]$ anymore (the same x can fall in more than one m-spacing), hence we lose the correspondence with the density estimator as a weighted sum of indicators in (17). Still, from the definition of $k_i(m)$, we can consider this extension of $\widehat{H}_\alpha^{\exp}(1,T)$:

$$\widehat{H}_\alpha^{\exp}(m,T) := \left(\frac{1}{T-m}\sum_{i=1}^{T-m}\left(\frac{T+1}{m}\big(X^{(i+m:T)}-X^{(i:T)}\big)\right)^{1-\alpha}\right)^{\frac{1}{1-\alpha}},$$

which corresponds to Equation (14) in the paper. Taking the limit $\alpha \to 1$, we recover the estimator

in Equation (15) of the paper:

$$\widehat{H}_1^{\exp}(m,T) := \exp\left(\frac{1}{T-m}\sum_{i=1}^{T-m}\ln\left(\frac{T+1}{m}\big(X^{(i+m:T)}-X^{(i:T)}\big)\right)\right).$$

References

Arnold, B., Balakrishnan, N., & Nagaraja, H. (1992). *A First Course in Order Statistics*. New York: Wiley.

Fang, K., & Zhang, Y. (1990). *Generalized Multivariate Analysis*. New York: Springer.

Learned-Miller, E., & Fisher, J. (2003). ICA using spacings estimates of entropy. *Journal of Machine Learning Research*, 4, 1271–1295.

Pham, D., & Vrins, F. (2005) Local minima of information-theoretic criteria in blind source separation. *IEEE Signal Processing Letters*, 12(11), 788–791.

Royston, J. (1982). Expected normal order statistics (exact and approximate). *Journal of the Royal Statistical Society Series C (Applied Statistics)*, 31(2), 161–165.

Vrins, F., Pham, D., & Verleysen, M. (2007). Mixing and non-mixing local minima of the entropy contrast for blind source separation. *IEEE Transactions on Information Theory*, 53(3), 1030–1042.

Zografos, K., & Nadarajah, S. (2003). Formulas for Rényi information and related measures for univariate distributions. *Information Sciences*, 155(1-2), 119–138.