A shortcoming of Fourier-Fick parameterisations in turbulent flows

Eric Deleersnijder, 18 November 2019

Figure 1 may be used to illustrate the inability of the classical (harmonic) diffusion operator that is inspired by Fourier's or Fick's laws to represent some observed phenomena in turbulent flows. If harmonic diffusion is at work, the shape of the plume of a passive constituent released by a point source into a steady-state flow should be close to a paraboloid of revolution. However, the plumes of Figure 1 are closer to a cone than a paraboloid of revolution[1]. Therefore, classical diffusion should be replaced by another diffusion operator. Some believe fractional diffusion[2] is a better option, for it is intrinsically non-local.

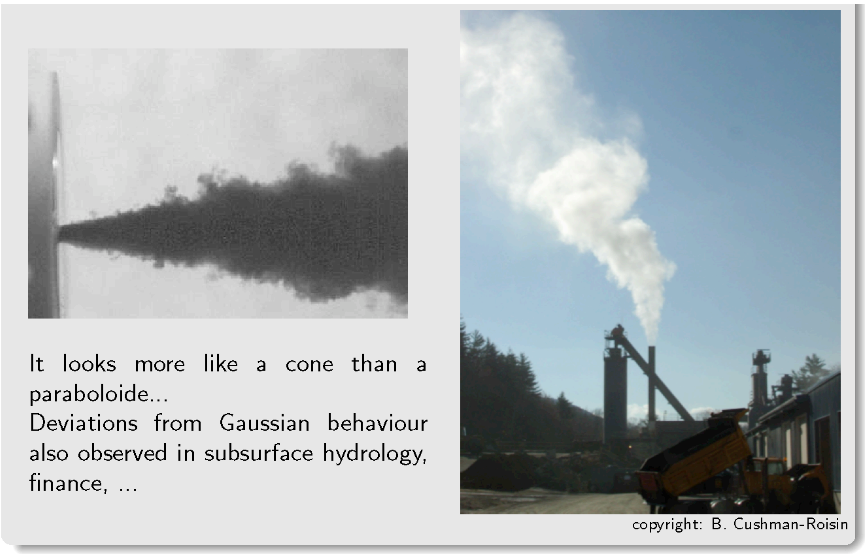

Figure 1. Cone-shaped plumes. Figure courtesy of Benoît Cushman-Roisin[3].

Let x, y and z denote Cartesian coordinates. A passive constituent point source is located at $(x,y,z)=(0,0,0)$ and its constant release rate is denoted Q, i.e. the mass released during the time interval $[t,t+\Delta t]$ is $Q\Delta t$. The flow velocity is $(U,0,0)$, where U is a positive constant. Assuming that along-flow diffusion is negligible (compared to advection) and that a steady-state situation is arrived at, the constituent concentration $C(x,y,z)$ obeys the following partial differential equation

[1] It is unclear whether the smoke released by the smokestack illustrated in the right-hand side panel of Figure 1 can be regarded as a passive constituent. Indeed, buoyancy effects are likely to be significant.

[2] Fractional diffusion is a continuous representation of non-Brownian motion — whereas Brownian motion is associated with classical diffusion

[3] This figure was presented by Emmanuel Hanert at the workshop/school TTM2011 (www.uclouvain.be/ttm2011) to illustrate the need to turn to parameterisations inspired by non-Brownian motion, i.e. fractional diffusion operators.

$$U\frac{\partial C}{\partial x} = \frac{\partial}{\partial y}\left(\kappa\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(\kappa\frac{\partial C}{\partial z}\right) . \quad (1)$$

where κ (>0) is the cross-flow constituent diffusivity. Clearly, the concentration must exhibit a cylindrical symmetry with respect to the x-axis. Therefore, the concentration may be regarded as a function of x and r only, where

$$r = \sqrt{y^2 + z^2} \quad (2)$$

is the distance to the x-axis. Next, equation (1) may be transformed to

$$U\frac{\partial C}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\kappa\frac{\partial C}{\partial r}\right) . \quad (3)$$

Let the constant ρ denote the density of the fluid. As along-flow diffusion is neglected, the constituent mass present in the sub-domain $[x, x+\Delta x]\times\Re^2$ must tend to $Q\Delta x / U$ as $\Delta x \to 0$. Therefore, for any value of x, the constituent concentration must satisfy the following constraint

$$2\pi\int_0^\infty \rho C(x,r) r\, dr = \frac{Q}{U} . \quad (4)$$

It must be underscored that this relation holds valid for any cross-flow diffusivity.

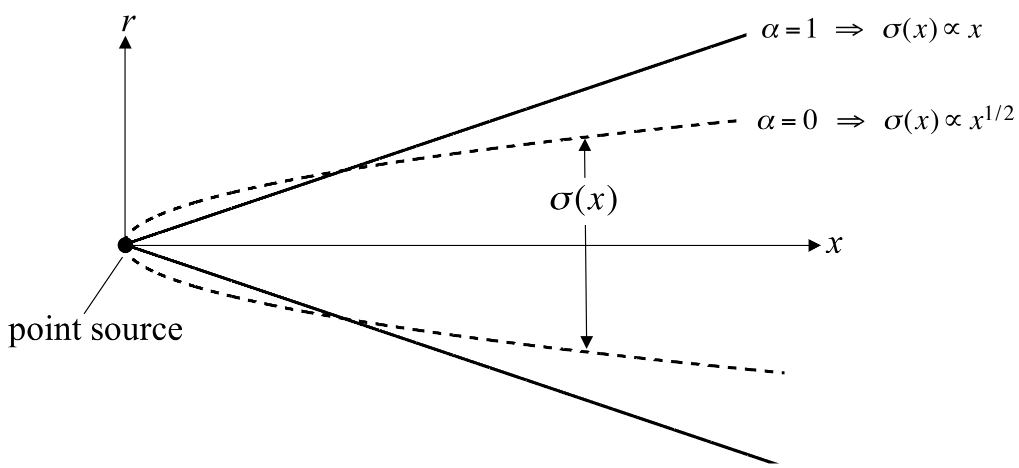

Figure 2. Width of the plume as a function of the distance to the point source. A steady state is assumed to prevail. The width of the plume is evaluated as the standard deviation of the constituent concentration distribution.

The standard deviation σ of the constituent distribution is a convenient measure of the width of the plume. The latter is the square root of the variance of the concentration:

$$\sigma^2 = \frac{2\pi\int_0^\infty r^2\, C(x,r)\, r\, dr}{2\pi\int_0^\infty C(x,r)\, r\, dr} . \quad (5)$$

Substituting (4) into (5) yields

$$\sigma^2 = \frac{2\pi\rho U}{Q}\int_0^\infty C(x,r)\, r^3\, dr \ . \tag{6}$$

If the cross-flow diffusivity is constant, i.e. $\kappa = K$ (where K is an appropriate diffusivity scale), then governing equation (3) reads

$$U\frac{\partial C}{\partial x} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) \tag{7}$$

and its solution is

$$C(x,r) = \frac{Q}{4\pi\rho K x}\exp\left(-\frac{Ur^2}{4Kx}\right) \ . \tag{8}$$

The associated standard deviation is then readily seen to be

$$\sigma(x) = \sqrt{\frac{4K}{U}}\, x^{1/2} \ . \tag{9}$$

Therefore, the plume has the shape of a paraboloid of revolution (Figure 2). As this result does not seem to be supported by the photographs displayed in Figure 1, another approach has to be sought.

Seeking inspiration in Okubo (1971, 1976) or similar studies, one may require that the diffusivity be an increasing function of the width of the plume. Accordingly, it is suggested that the cross-flow diffusivity be parameterised as follows:

$$\kappa(x) = \left(\frac{x}{\lambda}\right)^{\alpha} K \ , \tag{10}$$

where λ is a relevant length scale, while α is a non-negative constant ($\alpha \geq 0$). Clearly, a constant diffusivity is obtained by setting $\alpha = 0$.

It is appropriate to introduce a new along-flow coordinate:

$$\xi = \frac{1}{\alpha+1}\left(\frac{x}{\lambda}\right)^{\alpha} x \ . \tag{11}$$

This allows equation (7) to be transformed to

$$U\frac{\partial C}{\partial \xi} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) \ . \tag{12}$$

As (12) is similar to (7), its solution is easily derived:

$$C(\xi,r) = \frac{Q}{4\pi\rho K \xi}\exp\left(-\frac{Ur^2}{4K\xi}\right) \ . \tag{13}$$

Substituting (11) into (13) yields the concentration as a function of the along- and cross-flow coordinates, x and r, i.e.

$$C(x,r) = \frac{(\alpha+1)\lambda^{\alpha} Q}{4\pi\rho K x^{\alpha+1}}\exp\left(-\frac{(\alpha+1)U\lambda^{\alpha} r^2}{4K x^{\alpha+1}}\right) \ . \tag{14}$$

The standard deviation of this concentration distribution is

$$\sigma(x) = \sqrt{\frac{4K}{(1+\alpha)U\lambda^{\alpha}}}\, x^{(1+\alpha)/2} \tag{15}$$

implying that the plume is cone-shaped (Figure 2) if $\alpha = 1$. By setting $\alpha = 0$, a paraboloid of revolution is obtained.

Though formula (10) allows one to obtain a conical plume, there is no doubt that it is merely a quick fix. First, it is difficult to implement this method in the most popular turbulence closure schemes (in which turbulent fluxes are, however, parameterised by Fourier-Fick expressions), for the latter yield eddy coefficients that are independent of the concentration of the constituent being transported. Next, taking into account multiple point sources or distributed ones would presumably be unfeasible. Alternative approaches are needed. They are being worked out (e.g. Cushman-Roisin 2008) but they are presumably not yet sufficiently mature to be used in today's general-purpose models.

References

Cushman-Roisin B., 2008, Beyond eddy diffusivity: an alternative model for turbulent dispersion, *Environmental Fluid Mechanics*, 8, 543-549

Okubo A., 1971, Oceanic diffusion diagrams, *Deep-Sea Research*, 18, 789-802

Okubo A., 1976, Remarks on the use of 'diffusion diagrams' in modeling scale-dependent diffusion, *Deep-Sea Research*, 23, 1213-1214